Contraction: A Unified Perspective of Correlation Decay and Zero-Freeness of 2-Spin Systems

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Abstract

We study complex zeros of the partition function of 2-spin systems, viewed as a multivariate polynomial in terms of the edge interaction parameters and the uniform external field. We obtain new zero-free regions in which all these parameters are complex-valued. Crucially based on the zero-freeness, we are able to extend the existence of correlation decay to these complex regions from real parameters. As a consequence, we obtain an FPTAS for computing the partition function of 2-spin systems on graphs of bounded degree for these parameter settings. We introduce the *contraction* property as a *unified* sufficient condition to devise FPTAS via either Weitz's algorithm or Barvinok's algorithm. Our main technical contribution is a very simple but general approach to extend any *real* parameter of which the 2-spin system exhibits correlation decay to its *complex* neighborhood where the partition function is zero-free and correlation decay still exists. This result formally establishes the inherent connection between two distinct notions of phase transition for 2-spin systems: the existence of correlation decay and the zero-freeness of the partition function via a unified perspective, contraction.

2012 ACM Subject Classification Mathematics of computing → Approximation algorithms

Keywords and phrases 2-Spin system, Correlation decay, Zero-freeness, Phase transition, Contraction

Digital Object Identifier 10.4230/LIPIcs.ICALP.2020.96

Category Track A: Algorithms, Complexity and Games

Related Version A full version of this paper is available at `https://arxiv.org/abs/1909.04244`.

Funding The authors are supported by NSF CCF-1714275.

Acknowledgements We want to thank Professor Jin-Yi Cai, the advisor of the first author, for many inspiring discussions and valuable comments on a preliminary version of this paper. Despite his support, he generously declined our invitation for co-authorship. We also want to thank Professor Alex Scott, Dr. Jingcheng Liu and anonymous reviewers for their helpful comments.

1 Introduction

Spin systems originated from statistical physics to model interactions between neighbors on graphs. In this paper, we focus on 2-state spin (2-spin) systems. Such a system is specified by two edge interaction parameters β and γ, and a uniform external field λ. An instance is a graph $G = (V, E)$. A configuration σ is a mapping $\sigma : V \to \{+, -\}$ which assigns one of the two spins $+$ and $-$ to each vertex in V. The weight $w(\sigma)$ of a configuration σ is given by $w(\sigma) = \beta^{m_+(\sigma)}\gamma^{m_-(\sigma)}\lambda^{n_+(\sigma)}$, where $m_+(\sigma)$ denotes the number of $(+,+)$ edges under the configuration σ, $m_-(\sigma)$ denotes the number of $(-,-)$ edges, and $n_+(\sigma)$ denotes the number of vertices assigned to spin $+$. The partition function $Z_G(\beta, \gamma, \lambda)$ of the system parameterized by (β, γ, λ) is defined to be the sum of weights over all configurations, i.e.,

47th International Colloquium on Automata, Languages, and Programming (ICALP 2020).
Editors: Artur Czumaj, Anuj Dawar, and Emanuela Merelli; Article No. 96; pp. 96:1–96:15
Leibniz International Proceedings in Informatics
LIPICS Schloss Dagstuhl – Leibniz-Zentrum für Informatik, Dagstuhl Publishing, Germany

$$Z_G(\beta, \gamma, \lambda) = \sum_{\sigma: V \to \{+,-\}} w(\sigma).$$

It is a sum-of-product computation. If a 2-spin system is restricted to graphs of degree bounded by Δ, we say such a system is Δ-bounded.

In classical statistical mechanics the parameters (β, γ, λ) are usually non-negative real numbers, and such 2-spin systems are divided into the *ferromagnetic* case ($\beta\gamma > 1$) and the *antiferromagnetic* case ($\beta\gamma < 1$). The case $\beta\gamma = 1$ is degenerate. When (β, γ, λ) are non-negative numbers and they are not all zero, the partition function can be viewed as the normalizing factor of the Gibbs distribution, which is the distribution where a configuration σ is drawn with probability $\Pr_{G;\beta,\gamma,\lambda}(\sigma) = \frac{w(\sigma)}{Z_G(\beta,\gamma,\lambda)}$. However, it is meaningful to consider parameters of complex values. By analyzing the location of *complex* zeros of the partition function, the phenomenon of *phase transitions* was discovered by statistical physicists. One of the first and also the best known result is the Lee-Yang theorem [21] for the *Ising model*, a special case of 2-spin systems. This result was later extended to more general models by several people [1, 34, 37, 29, 24]. In this paper, we view the partition function $Z_G(\beta, \gamma, \lambda)$ as a *multivariate* polynomial over these three *complex* parameters (β, γ, λ). We study the zeros of this polynomial and the relation to the approximation of the partition function.

Partition functions encode rich information about the macroscopic properties of 2-spin systems. They are not only of significance in statistical physics, but also are well-studied in computer science. Computing the partition function of 2-spin systems given an input graph G can be viewed as the most basic case of Counting Graph Homomorphisms (#GH) [11, 5, 14, 8] and Counting Constraint Satisfaction Problems (#CSP) [10, 9, 6, 12, 7], which are two very well studied frameworks for counting problems. Many natural combinatorial problems can be formulated as 2-spin systems. For example, when $\beta = \gamma$, such a system is the famous *Ising model*. And when $\beta = 0$ and $\gamma = 1$, $Z_G(0, 1, \lambda)$ is the independence polynomial of the graph G (also known as the *hard-core model* in statistical physics); it counts the number of independent sets of the graph G when $\lambda = 1$.

Related work

For exact computation of $Z_G(\beta, \gamma, \lambda)$, the problem is proved to be #P-hard for all complex valued parameters but a few very restricted trivial settings [2, 8, 9]. So the main focus is to approximate $Z_G(\beta, \gamma, \lambda)$. This is an area of active research, and many inspiring algorithms are developed. The pioneering algorithm developed by Jerrum and Sinclair gives a *fully polynomial-time randomized approximation scheme* (FPRAS) for the ferromagnetic Ising model [19]. This FPRAS is based on the *Markov Chain Monte Carlo* (MCMC) method which devises approximation counting algorithms via random sampling. Later, it was extended to general ferromagnetic 2-spin systems [15, 26]. The MCMC method can only handle non-negative parameters as it is based on probabilistic sampling.

The *correlation decay* method developed by Weitz [43] was originally used to devise *deterministic fully polynomial-time approximation schemes* (FPTAS) for the hardcore model up to the uniqueness threshold of the infinite regular tree. It turns out to be a very powerful tool for devising FPTAS for antiferromagnetic 2-spin systems [44, 22, 23, 39]. Combining with hardness results [40, 13], an exact threshold of computational complexity transition of antiferromagnetic 2-spin systems is identified and the only remaining case is at the critical point. On the other hand, for ferromagnetic 2-spin systems, limited results [44, 17] have been obtained via the correlation decay method. Although correlation decay is usually analyzed in 2-spin systems of non-negative parameters, it can be adapted to complex parameters. An FPTAS was obtained for the hard-core model in the Shearer's region (a disc in the complex plane) via correlation decay in [18].

Recently, a new method developed by Barvinok [3], and extended by Patel and Regts [30] is the *Taylor polynomial interpolation* method that turns complex zero-free regions of the partition function into FPTAS of corresponding complex parameters. Suppose that the partition function $Z_G(\beta, \gamma, \lambda)$ has no zero in a complex region containing an easy computing point, e.g., $\lambda = 0$. It turns out that, probably after a change of coordinates, $\log Z_G(\beta, \gamma, \lambda)$ is well approximated in a slightly smaller region by a low degree Taylor polynomials which can be efficiently computed. This method connects the long-standing study of complex zeros to algorithmic studies of the partition function of physical systems. Motivated by this, more recently some complex zero-free regions have been obtained for hard-core models [4, 32], Ising models [27, 31], and general 2-spin systems [16].

Our contribution

In this paper, we obtain new zero-free regions of the partition function of 2-spin systems. Crucially based on the zero-freeness, we are able to extend the existence of correlation decay to these *complex* regions from real parameters. As a consequence, we obtain an FPTAS for computing the partition function of bounded 2-spin systems for these parameter settings. Our result gives the first zero-free regions in which all three parameters (β, γ, λ) are complex-valued and new correlation decay results for bounded ferromagnetic 2-spin systems. Our main technical contribution is a very simple but general approach to extend any real parameter of which the bounded 2-spin system exhibits correlation decay to its complex neighborhood where the partition function is zero-free and correlation decay still exists. We show that for bounded 2-spin systems, the *real contraction*[1] property that ensures correlation decay exists for certain real parameters directly implies the zero-freeness and the existence of correlation decay of corresponding complex neighborhoods.

We formally describe our main result. We use $\boldsymbol{\zeta} \in \mathbb{C}^3$ to denote the parameter vector (β, γ, λ). Since the case $\beta = \gamma = 0$ is trivial, by symmetry we always assume $\gamma \neq 0$.

▶ **Theorem 1.** *Fix $\Delta \in \mathbb{N}$. If $\boldsymbol{\zeta}_0 \in \mathbb{R}^3$ satisfies real contraction for Δ, then there exists a $\delta > 0$ such that for any $\boldsymbol{\zeta} \in \mathbb{C}^3$ where $\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_0\|_\infty < \delta$, we have*
- *$Z_G(\boldsymbol{\zeta}) \neq 0$ for every graph[2] G of degree at most Δ;*
- *the Δ-bounded 2-spin system specified by $\boldsymbol{\zeta}$ exhibits correlation decay.*

As a consequence, there is an FPTAS for computing $Z_G(\boldsymbol{\zeta})$.

This result formally establishes the inherent connection between two distinct notions of phase transition for bounded 2-spin systems: the existence of correlation decay and the zero-freeness of the partition function, via a unified perspective, *contraction*. The connection from the existence of correlation decay of real parameters to the zero-freeness of corresponding complex neighborhoods was already observed for the hard-core model [32] and the Ising model without external field [27]. In this paper, we extend it to general 2-spin systems, and furthermore we establish the connection from the zero-freeness of complex neighborhoods back to the existence of correlation decay of such complex regions.

Now, we give our zero-free regions. We first identify the sets of real parameters of which bounded 2-spin systems exhibit correlation decay.

[1] See Definition 11. In many cases, the existence of correlation decay boils down to this property.
[2] This is true even if G contains some vertices pinned by a *feasible configuration* (Definition 7).

▶ **Definition 2.** *Fix integer $\Delta \geq 3$. We define the following four real correlation decay sets.*
1. $\mathcal{S}_1^\Delta = \{\boldsymbol{\zeta} \in \mathbb{R}^3 \mid \frac{\Delta-2}{\Delta} < \sqrt{\beta\gamma} < \frac{\Delta}{\Delta-2}, \beta, \gamma > 0$ *and* $\lambda \geq 0\}$,
2. $\mathcal{S}_2^\Delta = \{\boldsymbol{\zeta} \in \mathbb{R}^3 \mid \beta\gamma < 1, \beta \geq 0, \gamma > 0, \lambda \geq 0,$ *and* $\boldsymbol{\zeta}$ *is up-to-*Δ *unique (Definition 14)*$\}$,
3. $\mathcal{S}_3^\Delta = \{\boldsymbol{\zeta} \in \mathbb{R}^3 \mid \beta\gamma > \frac{\Delta}{\Delta-2}, \beta, \gamma > 0$ *and* $0 \leq \lambda < \frac{\gamma}{t^{\Delta-1}[(\Delta-2)\beta\gamma-\Delta]}\}$ *where* $t = \max\{1, \beta\}$,
4. *and* $\mathcal{S}_4^\Delta = \{\boldsymbol{\zeta} \in \mathbb{R}^3 \mid \beta\gamma > \frac{\Delta}{\Delta-2}, \beta, \gamma > 0$ *and* $\lambda > \frac{(\Delta-2)\beta\gamma-\Delta}{\beta r^{\Delta-1}}\}$ *where* $r = \min\{1, 1/\gamma\}$.

When context is clear, we omit the superscript Δ.

The set $\mathcal{S}_1^\Delta$ was given in [44] and $\mathcal{S}_2^\Delta$ was given in [23]. To our best knowledge, $\mathcal{S}_1^\Delta$ and $\mathcal{S}_2^\Delta$ cover all *non-negative* parameters of which *bounded* 2-spin systems are known to exhibit correlation decay. The sets $\mathcal{S}_3^\Delta$ and $\mathcal{S}_4^\Delta$ are obtained in this paper[3]. They give new correlation decay results and hence FPTAS for bounded ferromagnetic 2-spin systems. When $\beta < \gamma$ and λ is sufficiently large, it is known that approximating the partition function of ferromagnetic 2-spin systems over general graphs is #BIS-hard [26]. Our result $\mathcal{S}_4^\Delta$ shows that there is an FPTAS for such a problem when restricted to graphs of bounded degree. When $\beta < 1 < \gamma$, the FPTAS obtained from $\mathcal{S}_3^\Delta$ is covered by [17].

▶ **Theorem 3.** *Fix integer $\Delta \geq 3$. For every $\boldsymbol{\zeta}_0 \in \mathcal{S}_i^\Delta$ $(i \in [4])$, there exists a $\delta > 0$ such that for any $\boldsymbol{\zeta} \in \mathbb{C}^3$ where $\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_0\|_\infty < \delta$, we have*
- *$Z_G(\boldsymbol{\zeta}) \neq 0$ for every graph G of degree at most Δ; (G may contain a feasible configuration.)*
- *the Δ-bounded 2-spin system specified by $\boldsymbol{\zeta}$ exhibits correlation decay.*

Then via either Weitz's algorithm or Barvinok's algorithm, there is an FPTAS for computing the partition function $Z_G(\boldsymbol{\zeta})$.

▶ Remark 4. The choice of δ does not depend on the size of the graph, only on Δ and $\boldsymbol{\zeta}_0$.

Organization

This paper is organized as follows. In Section 2, we briefly describe Weitz's algorithm [43]. We introduce real contraction as a sufficient condition for the existence of correlation decay of real parameters, and we show that sets $\mathcal{S}_i^\Delta (i \in [4])$ satisfy it. In Section 3, we briefly describe Barvinok's algorithm [3]. We introduce complex contraction as a generalization of real contraction, and we show that it gives a unified sufficient condition for both the zero-freeness of the partition function and the existence of correlation decay of complex parameters. Finally, in Section 4, we prove our main result that real contraction implies complex contraction. This finishes the proof of Theorem 3. We use the following diagram (Figure 1) to summarize our approach to establish the connection between correlation decay and zero-freeness. We expect it to be further explored for other models.

Independent work

After a preliminary version [36] of this manuscript was posted, we learned that based on similar ideas, Liu simplified the proofs of [32] and [27], and generalized them to antiferromagnetic Ising models ($\beta = \gamma < 1$) in chapter 3 of his Ph.D. thesis [25], where similar zero-freeness results (a complex neighborhood of $\mathcal{S}_2^\Delta$ restricted to $\beta = \gamma$) were obtained. We mention that by using the unique analytic continuation and the inverse function theorem, our main technical result (Theorem 24) is generic; it does not rely on a particularly chosen potential function. Thus, in our approach we can work with *any* existing potential function based

[3] Since we do *not* assume $\beta \leqslant \gamma$ or $\beta \geqslant \gamma$, $\mathcal{S}_3^\Delta$ and $\mathcal{S}_4^\Delta$ are essentially the same by swapping β and γ and replacing λ with $1/\lambda$. However, if one restrict to $\beta \leqslant \gamma$, then $\mathcal{S}_3^\Delta$ is no longer the same as $\mathcal{S}_4^\Delta$.

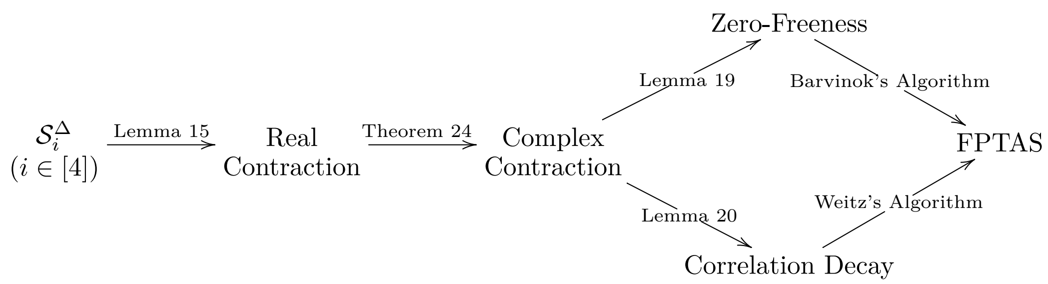

Figure 1 The structure of our approach.

argument for correlation decay even if the potential function does not have an explicit expression, for instance, the one used in [23] when $\beta \neq \gamma$. Furthermore, we mention also that based on the zero-freeness, we obtain new correlation decay results for complex parameters (Lemma 20). Note that Barvinok's algorithm requires an entire region in which the partition function is zero-free and there is an easy computing point in this region. However, our correlation decay results show that one can always devise an FPTAS for these parameter settings via Weitz's algorithm, even if Barvinok's algorithm fails.

2 Weitz's Algorithm

In this section, we describe Weitz's algorithm and introduce real contraction. We first consider positive parameters $\boldsymbol{\zeta} \in \mathbb{R}_+^3$. An obvious but important fact about $\boldsymbol{\zeta}$ being positive is that $Z_G(\boldsymbol{\zeta}) \neq 0$ for any graph G. This is true even if G contains arbitrary number of vertices pinned to spin $+$ or $-$. Then, the partition function can be viewed as the normalizing factor of the Gibbs distribution.

2.1 Notations and definitions

Let $\boldsymbol{\zeta} \in \mathbb{R}_+^3$. We use $p_v(\boldsymbol{\zeta})$ to denote the marginal probability of v being assigned to spin $+$ in the Gibbs distribution, i.e., $p_v(\boldsymbol{\zeta}) = \frac{Z_{G,v}^{+}(\boldsymbol{\zeta})}{Z_G(\boldsymbol{\zeta})}$, where $Z_{G,v}^{+}(\boldsymbol{\zeta})$ is the contribution to $Z_G(\boldsymbol{\zeta})$ over all configurations with v being assigned to spin $+$. We know that $p_v(\boldsymbol{\zeta})$ is well-defined since $Z_G(\boldsymbol{\zeta}) \neq 0$. (Later, we will extend the definition of $p_v(\boldsymbol{\zeta})$ to complex parameters $\boldsymbol{\zeta}$.)

Let $\sigma_\Lambda \in \{0,1\}^\Lambda$ be a configuration of some subset $\Lambda \subseteq V$. We allow Λ to be the empty set. We call vertices in Λ *pinned* and other vertices *free*. We use $p_v^{\sigma_\Lambda}(\boldsymbol{\zeta})$ to denote the marginal probability of a free vertex v ($v \notin \Lambda$) being assigned to spin $+$ conditioning on the configuration σ_Λ of Λ, i.e., $p_v^{\sigma_\Lambda}(\boldsymbol{\zeta}) = \frac{Z_{G,v}^{\sigma_\Lambda,+}(\boldsymbol{\zeta})}{Z_G^{\sigma_\Lambda}(\boldsymbol{\zeta})}$, where $Z_G^{\sigma_\Lambda}(\boldsymbol{\zeta})$ is the weight over all configurations where vertices in Λ are pinned by the configuration σ_Λ, and $Z_{G,v}^{\sigma_\Lambda,+}(\boldsymbol{\zeta})$ is the contribution to $Z_G^{\sigma_\Lambda}(\boldsymbol{\zeta})$ with v being assigned to spin $+$. Correspondingly, we can define $Z_{G,v}^{\sigma_\Lambda,-}(\boldsymbol{\zeta})$. Let $R_{G,v}^{\sigma_\Lambda}(\boldsymbol{\zeta}) := \frac{Z_{G,v}^{\sigma_\Lambda,+}(\boldsymbol{\zeta})}{Z_{G,v}^{\sigma_\Lambda,-}(\boldsymbol{\zeta})} = \frac{p_v^{\sigma_\Lambda}(\boldsymbol{\zeta})}{1-p_v^{\sigma_\Lambda}(\boldsymbol{\zeta})}$ be the ratio between the two probabilities that the free vertex v is assigned to spin $+$ and $-$, while imposing some condition σ_Λ. Since $Z_G(\boldsymbol{\zeta}) \neq 0$ for any graph G with arbitrary number of pinned vertices, both $p_v^{\sigma_\Lambda}(\boldsymbol{\zeta})$ and $R_{G,v}^{\sigma_\Lambda}(\boldsymbol{\zeta})$ are well-defined. When context is clear, we write $p_v(\boldsymbol{\zeta})$, $p_v^{\sigma_\Lambda}(\boldsymbol{\zeta})$ and $R_{G,v}^{\sigma_\Lambda}(\boldsymbol{\zeta})$ as p_v, $p_v^{\sigma_\Lambda}$ and $R_{G,v}^{\sigma_\Lambda}$ for convenience.

Since computing the partition function of 2-spin systems is self-reducible, if one can compute p_v for any vertex v, then the partition function can be computed via telescoping [20]. The goal of Weitz's algorithm is to estimate $p_v^{\sigma_\Lambda}$, which is equivalent to estimating $R_{G,v}^{\sigma_\Lambda}$. For the case that the graph is a tree T, $R_{T,v}^{\sigma_\Lambda}$ can be computed by recursion. Suppose that a free vertex v has d children, and s_1 of them are pinned to $+$, s_2 are pinned to $-$, and k are free ($s_1+s_2+k=d$). We denote these k free vertices by $v_i (i \in [k])$ and let T_i be the corresponding subtree rooted at v_i. We use σ_Λ^i to denote the configuration σ_Λ restricted to T_i. Since all subtrees are independent, it is easy to get the following recurrence relation,

$$R_{T,v}^{\sigma_\Lambda} = \frac{Z_{T,v}^{\sigma_\Lambda,+}(\boldsymbol{\zeta})}{Z_{T,v}^{\sigma_\Lambda,-}(\boldsymbol{\zeta})} = \frac{\lambda^{1+s_1}\beta^{s_1}\prod_{i=1}^{k}\left(\beta Z_{T_i,v_i}^{\sigma_\Lambda^i,+}(\boldsymbol{\zeta}) + Z_{T_i,v_i}^{\sigma_\Lambda^i,-}(\boldsymbol{\zeta})\right)}{\lambda^{s_1}\gamma^{s_2}\prod_{i=1}^{k}\left(Z_{T_i,v_i}^{\sigma_\Lambda^i,+}(\boldsymbol{\zeta}) + \gamma Z_{T_i,v_i}^{\sigma_\Lambda^i,-}(\boldsymbol{\zeta})\right)} = \frac{\lambda\beta^{s_1}}{\gamma^{s_2}}\prod_{i=1}^{k}\left(\frac{\beta R_{T_i,v_i}^{\sigma_\Lambda^i}+1}{R_{T_i,v_i}^{\sigma_\Lambda^i}+\gamma}\right).$$

▶ **Definition 5** (Recursion function). *Let* $\mathbf{s} = (s_1, s_2, k) \in \mathbb{N}^3$ *(including 0). A recursion function* $F_{\mathbf{s}}$ *for 2-spin systems is defined to be*

$$F_{\mathbf{s}}(\boldsymbol{\zeta}, \mathbf{x}) := \lambda\beta^{s_1}\gamma^{-s_2}\prod_{i=1}^{k}\left(\frac{\beta x_i+1}{x_i+\gamma}\right),$$

where $\boldsymbol{\zeta} = (\beta,\gamma,\lambda) \in \mathbb{C}\times(\mathbb{C}\backslash\{0\})\times\mathbb{C}$ *and* $\mathbf{x} = (x_1,\ldots,x_k) \in (\mathbb{C}\backslash\{-\gamma\})^k$. *We define* $F_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x}) := F_{\mathbf{s}}(\boldsymbol{\zeta},\mathbf{x})$ *for fixed* $\boldsymbol{\zeta}$ *with* $\gamma \neq 0$, *and* $F_{\mathbf{x},\mathbf{s}}(\boldsymbol{\zeta}) := F_{\mathbf{s}}(\boldsymbol{\zeta},\mathbf{x})$ *for fixed* $\mathbf{x}$.

▶ Remark 6. Every recursion function is analytic on its domain.

For a general graph G, Weitz reduced computing $R_{G,v}^{\sigma_\Lambda}$ to that in a tree T, called the self-avoiding walk (SAW) tree, and Weitz's theorem [43] states that $R_{G,v}^{\sigma_\Lambda} = R_{T,v}^{\sigma_\Lambda}$. (Please see [43], [17] or the full paper for more details.) We want to generalize Weitz's theorem to complex parameters $\boldsymbol{\zeta} \in \mathbb{C}^3$. First, we need to make sure that $R_{G,v}^{\sigma_\Lambda}$ and $p_v^{\sigma_\Lambda}$ are well-defined for any vertex $v \notin \Lambda$. This requires that $Z_G^{\sigma_\Lambda}(\boldsymbol{\zeta}) \neq 0$ for any graph G and any configuration σ_Λ. Now, $p_v^{\sigma_\Lambda}$ no longer has a probabilistic meaning. It is just a ratio of two complex numbers. However, one can easily observe that for some special parameters, there are trivial configurations such that $Z_{G,v}^{\sigma_\Lambda}(\boldsymbol{\zeta}) = 0$. We will rule these cases out as they are *infeasible*.

▶ **Definition 7** (Feasible configuration). *Let* $\boldsymbol{\zeta} \in \mathbb{C}^3$. *Given a graph* $G = (V, E)$ *of the 2-spin system specified by* $\boldsymbol{\zeta}$, *a configuration* σ_Λ *on some vertices* $\Lambda \subseteq V$ *is feasible if*

- σ_Λ *does not assign any vertex in* G *to spin* $+$ *when* $\lambda = 0$, *and*
- σ_Λ *does not assign any two adjacent vertices in* G *both to spin* $+$ *when* $\beta = 0$.

▶ Remark 8. Let σ_Λ be a feasible configuration. If we further pin one vertex $v \notin \Lambda$ to spin $-$, and get the configuration $\sigma_{\Lambda'}$ on $\Lambda' = \Lambda \cup \{v\}$, then $\sigma_{\Lambda'}$ is still a feasible configuration. Thus, given $\boldsymbol{\zeta} \in \mathbb{C}^3$, if $Z_G^{\sigma_\Lambda}(\boldsymbol{\zeta}) \neq 0$ for any graph G and any arbitrary feasible configuration σ_Λ on G, then both $p_v^{\sigma_\Lambda}$ and $R_{G,v}^{\sigma_\Lambda}$ are well-defined.

Given $R_{G,v}^{\sigma_\Lambda}$ is well-defined for some $\boldsymbol{\zeta} \in \mathbb{C}^3$, we can still compute it by recursion via SAW tree. We first consider the case that $\lambda \neq 0$. Let σ_Λ be a feasible configuration. It is easy to verify that the corresponding configuration on the SAW tree is also feasible and Weitz's theorem still holds. For the case that $\lambda = 0$, it is obvious that $R_{G,v}^{\sigma_\Lambda} \equiv 0$ for any graph G, any free vertex v and any feasible configuration σ_Λ. This is equal to the value of recursion functions $F_{\mathbf{s}}(\boldsymbol{\zeta}, \mathbf{x})$ at $\lambda = 0$. We agree that $R_{G,v}^{\sigma_\Lambda}$ can be computed by recursion functions when $\lambda = 0$, although Weitz's theorem does not hold for this case. For the case that $\beta = 0$, we have $R_{G,v}^{\sigma_\Lambda} = 0$ if one of the children of v is pinned to $+$. Then, we may view v as it is pinned to $-$. Thus, for $\beta = 0$, we only consider recursion functions $F_{\mathbf{s}}$ where $s_1 = 0$.

$$\begin{tikzcd}
\mathbf{y} \arrow[r, shift left, "\varphi"] \arrow[d, "F"] & \mathbf{x} \arrow[l, shift left, "\varphi^{-1}"] \arrow[d, "F^{\varphi}"] \\
F(\mathbf{y}) \arrow[r, shift left, "\varphi"] & F^{\varphi}(\mathbf{x}) \arrow[l, shift left, "\varphi^{-1}"]
\end{tikzcd}$$

Figure 2 Commutative diagram between F and F^{φ}.

2.2 Correlation decay and real contraction

The SAW tree may be exponentially large in size of G. In order to get a polynomial time approximation algorithm, we may run the tree recursion at logarithmic depth and hence in polynomial time, and plug in some arbitrary values at the truncated boundary. We have the following notion of *strong spatial mixing* (SSM) to bound the error caused by arbitrary guesses. It was originally introduced for non-negative parameters. Here, we extend it to complex parameters.

▶ **Definition 9** (Strong spatial mixing). *A 2-spin system specified by $\boldsymbol{\zeta} \in \mathbb{C}^3$ on a family $\mathcal{G}$ of graphs is said to exhibit strong spatial mixing if for any graph $G = (V, E) \in \mathcal{G}$, any $v \in V$, and any feasible configurations $\sigma_{\Lambda_1} \in \{0,1\}^{\Lambda_1}$ and $\tau_{\Lambda_2} \in \{0,1\}^{\Lambda_2}$ where $v \notin \Lambda_1 \cup \Lambda_2$, we have*

1. *$Z_G^{\sigma_{\Lambda_1}}(\boldsymbol{\zeta}) \neq 0$ and $Z_G^{\tau_{\Lambda_2}}(\boldsymbol{\zeta}) \neq 0$, and*
2. *$\left|p_v^{\sigma_{\Lambda_1}} - p_v^{\tau_{\Lambda_2}}\right| \leq \exp(-\Omega(\mathrm{dist}(v, S)))$.*

Here, $S \subseteq \Lambda_1 \cup \Lambda_2$ is the subset on which σ_{Λ_1} and τ_{Λ_2} differ (If a vertex v is free in one configuration but pinned in the other, we say that these two configurations differ at v), and $\mathrm{dist}_G(v, S)$ is the shortest distance from v to any vertex in S.

▶ Remark 10. When $\boldsymbol{\zeta} \in \mathbb{R}_+^3$, condition 1 is always satisfied. Condition 2 is a stronger form of SSM of real parameters (see Definition 5 of [23]). For real values, by monotonicity one need to consider only the case that $\Lambda_1 = \Lambda_2$ (the two configurations are on the same set of vertices). Here, we need to consider the case that $\Lambda_1 \neq \Lambda_2$. This is *necessary* to extend Weitz's algorithm to complex parameters.

In statistical physics, SSM implies correlation decay. If SSM holds, then the error caused by arbitrary boundary guesses at logarithmic depth of the SAW tree is polynomially small. Hence, Weitz's algorithm gives an FPTAS. A main technique that has been widely used to establish SSM is the *potential method* [33, 22, 23, 38, 17]. Instead of bounding the rate of decay of recursion functions directly, we use a potential function $\varphi(x)$ to map the original recursion to a new domain (See Figure 2 for the commutative diagram).

Let $F_{\mathbf{s}}(\boldsymbol{\zeta}, \mathbf{y})$ be a recursion function ($\mathbf{s} = (s_1, s_2, k) \in \mathbb{N}^3$). We use $F_{\mathbf{s}}^{\varphi}(\boldsymbol{\zeta}, \mathbf{x})$ to denote the composition $\varphi(F_{\mathbf{s}}(\boldsymbol{\zeta}, \boldsymbol{\varphi}^{-1}(\mathbf{x})))$ where $\mathbf{y} = \boldsymbol{\varphi}^{-1}(\mathbf{x})$ denotes the vector $(\varphi^{-1}(x_1), \dots, \varphi^{-1}(x_k))$. Correspondingly, we define $F_{\boldsymbol{\zeta},\mathbf{s}}^{\varphi}(\mathbf{x})$ for fixed $\boldsymbol{\zeta}$, and $F_{\mathbf{x},\mathbf{s}}^{\varphi}(\boldsymbol{\zeta})$ for fixed $\mathbf{x}$. We will specify the domain on which $F_{\mathbf{s}}^{\varphi}$ is well-defined per each φ that will be used. For positive $\boldsymbol{\zeta}$, a sufficient condition for the bounded 2-spin system of $\boldsymbol{\zeta}$ exhibiting SSM is that there exists a "good" potential function φ such that $F_{\boldsymbol{\zeta},\mathbf{s}}^{\varphi}$ satisfies the following contraction property.

▶ **Definition 11** (Real contraction). *Fix $\Delta \in \mathbb{N}$. We say that $\boldsymbol{\zeta} \in \mathbb{R}^3$ satisfies real contraction for Δ if there is a real compact interval $J \subseteq \mathbb{R}$ where $\lambda \in J$, $-\gamma \notin J$ and $-1 \notin J$, and a real analytic function $\varphi : J \to I$ where $\varphi'(x) \neq 0$ for all $x \in J$, such that*

1. *$F_{\boldsymbol{\zeta},\mathbf{s}}(J^k) \subseteq J$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$ and $-1 \notin F_{\boldsymbol{\zeta},\mathbf{s}}(J^k)$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 = \Delta$;*
2. *there exists $\eta > 0$ s.t. $\left\|\nabla F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x})\right\|_1 \leq 1 - \eta$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$ and all $\mathbf{x} \in I^k$.*

We say φ defined on J is a good potential function for $\boldsymbol{\zeta}$.

▶ Remark 12. Since φ is analytic and $\varphi'(x) \neq 0$ for all $x \in J$, the function φ is invertible and the inverse $\varphi^{-1} : I \to J$ is also analytic by the inverse function theorem (Theorem 22). Also for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$, since $F_{\boldsymbol{\zeta},\mathbf{s}}(J^k) \subseteq J$ and $-\gamma \notin J$, the function $F_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x})$ is analytic on J^k. Thus, $F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x})$ is well-defined and analytic on I^k, and then $\nabla F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x})$ is well-defined on I^k. Note that I is also a real compact interval since J is a real compact interval and φ is a real analytic function.

Since $-1 \notin J$, $F_{\boldsymbol{\zeta},\mathbf{s}}(J^k) \subseteq J$ implies that $-1 \notin F_{\boldsymbol{\zeta},\mathbf{s}}(J^k)$. Thus, real contraction implies that $-1 \notin F_{\boldsymbol{\zeta},\mathbf{s}}(J^k)$ for all $\|\mathbf{s}\|_1 \leq \Delta$. The reason why we require $F_{\boldsymbol{\zeta},\mathbf{s}}(J^k) \subseteq J$ for $\|\mathbf{s}\|_1 \leq \Delta-1$, but only require $-1 \notin F_{\boldsymbol{\zeta},\mathbf{s}}(J^k)$ for $\|\mathbf{s}\|_1 = \Delta$ is that in a tree of degree at most Δ, only the root node may have Δ many children, while other nodes have at most $\Delta - 1$ many children.

▶ **Lemma 13.** *If $\boldsymbol{\zeta} \in \mathbb{R}^3_+$ satisfies real contraction for Δ, then the Δ-bounded 2-spin system of $\boldsymbol{\zeta}$ exhibits SSM. Thus there is an FPTAS for computing the partition function $Z_G(\boldsymbol{\zeta})$.*

Proof. The proof directly follows from the argument of the potential method, see [23, 17]. The FPTAS follows from Weitz's algorithm. ◀

Now, we give the sets of non-negative parameters that satisfy real contraction.

▶ **Definition 14** (Uniqueness condition [23]). *Let $\boldsymbol{\zeta} \in \mathbb{R}^3$ be antiferromagnetic ($\beta\gamma < 1$) with $\beta \geq 0$, $\gamma > 0$ and $\lambda \geq 0$, and $f_d(x) = \lambda \left(\frac{\beta x + 1}{x + \gamma}\right)^d$. We say $\boldsymbol{\zeta}$ is up-to-Δ unique, if $\lambda = 0$ or $\lambda > 0$ and there exists a constant $0 < c < 1$ such that for every integer $1 \leq d \leq \Delta - 1$,*

$$|f_d'(\hat{x}_d)| = \frac{d(1-\beta\gamma)\hat{x}_d}{(\beta\hat{x}_d + 1)(\hat{x}_d + \gamma)} \leq c,$$

where $\hat{x}_d$ is the unique positive fixed point of the function $f_d(x)$.

Let $\mathcal{S}_i^{\Delta} (i \in [4])$ be the correlation decay sets defined in Definition 2. The set $\mathcal{S}_1^{\Delta}$ was given in [44] and $\mathcal{S}_2^{\Delta}$ was given in [23]. Directly following their proofs, it is easy to verify that both sets satisfy real contraction . The sets $\mathcal{S}_3^{\Delta}$ and $\mathcal{S}_4^{\Delta}$ are obtained in this paper, and we show that they also satisfy real contraction.

▶ **Lemma 15.** *Fix $\Delta \geq 3$. For every $\boldsymbol{\zeta} \in \mathcal{S}_i^{\Delta} (i \in [4])$, it satisfies real contraction for Δ.*

Proof. We only give a proof for sets $\mathcal{S}_3^{\Delta}$ and $\mathcal{S}_4^{\Delta}$. For a proof of sets $\mathcal{S}_1^{\Delta}$ and $\mathcal{S}_2^{\Delta}$, please refer to the full paper. The case that $\lambda = 0$ is easy to check. We only consider that $\lambda \neq 0$.

Since $\beta\gamma > \frac{\Delta}{\Delta-2} > 1$, we have $1/\gamma < \beta$. We pick the interval $J = [\lambda r^{\Delta-1}, \lambda t^{\Delta-1}]$ where $r = \min\{1, 1/\gamma\}$ and $t = \max\{1, \beta\}$, and the potential function $\varphi = \log(x)$. Clearly, φ is analytic on J and $\varphi'(x) \neq 0$ for all $x \in J$. Also, we know that $\lambda \in J$, $-\gamma \notin J$ and $-1 \notin J$, and $-1 \notin F_{\boldsymbol{\zeta},\mathbf{s}}(J^k)$ for every $\|\mathbf{s}\|_1 = \Delta$. Since $\beta > 0$ and $\gamma > 0$, for all $x > 0$,

$$r \leq \min\{\beta, 1/\gamma\} \leq \frac{\beta x + 1}{x + \gamma} \leq \max\{\beta, 1/\gamma\} \leq t.$$

Thus, for all $\mathbf{x} \in J^k$,

$$F_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x}) = \lambda \beta^{s_1} \gamma^{-s_2} \prod_{i=1}^{k} \left(\frac{\beta x_i + 1}{x_i + \gamma} \right) \in \left[\lambda r^{\|\mathbf{s}\|_1}, \lambda t^{\|\mathbf{s}\|_1} \right] \subseteq \left[\lambda r^{\Delta-1}, \lambda t^{\Delta-1} \right].$$

Hence, $F_{\boldsymbol{\zeta},\mathbf{s}}(J^k) \subseteq J$ for every $\|\mathbf{s}\|_1 \le \Delta - 1$. Condition 1 of real contraction is satisfied.

Let $I = \varphi(J)$. Consider the gradient $\nabla F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x})$ for every $\|\mathbf{s}\|_1 \le \Delta - 1$ and all $\mathbf{x} \in I^k$. Note that $F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x}) = \log \lambda + s_1 \log \beta - s_2 \log \gamma + \sum_{i=1}^{k} \log \left(\frac{\beta e^{x_i} + 1}{e^{x_i} + \gamma} \right)$, and $e^{x_i} = \varphi^{-1}(x_i) \in J$ and $e^{-x_i} \in [\frac{1}{\lambda t^{\Delta-1}}, \frac{1}{\lambda r^{\Delta-1}}]$ when $x_i \in I$.

If $\boldsymbol{\zeta} \in \mathcal{S}_3^{\Delta}$, then $(\Delta-2)\beta\gamma - \Delta < \frac{\gamma}{\lambda t^{\Delta-1}}$. Thus,

$$\left| \frac{\partial F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}}{\partial x_i} \right| = \frac{\beta\gamma - 1}{\beta e^{x_i} + \gamma e^{-x_i} + 1 + \beta\gamma} \le \frac{\beta\gamma - 1}{\frac{\gamma}{\lambda t^{\Delta-1}} + 1 + \beta\gamma} < \frac{\beta\gamma - 1}{(\Delta-2)\beta\gamma - \Delta + 1 + \beta\gamma} = \frac{1}{\Delta - 1}.$$

Otherwise, $\boldsymbol{\zeta} \in \mathcal{S}_4^{\Delta}$ and then $\lambda\beta r^{\Delta-1} > (\Delta-2)\beta\gamma - \Delta$. Thus,

$$\left| \frac{\partial F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}}{\partial x_i} \right| = \frac{\beta\gamma - 1}{\beta e^{x_i} + \gamma e^{-x_i} + 1 + \beta\gamma} \le \frac{\beta\gamma - 1}{\beta\lambda r^{\Delta-1} + 1 + \beta\gamma} < \frac{\beta\gamma - 1}{(\Delta-2)\beta\gamma - \Delta + 1 + \beta\gamma} = \frac{1}{\Delta - 1}.$$

Thus, in both cases, there exists some $\eta > 0$ such that $\left\| \nabla F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x}) \right\|_1 \le 1 - \eta$ for every $\|\mathbf{s}\|_1 \le \Delta - 1$ and all $\mathbf{x} \in I^k$. Condition 2 of real contraction is satisfied. ◀

In order to generalize the correlation decay technique to complex parameters, we need to ensure that the partition function is zero-free. Now, let us first take a detour to Barvinok's algorithm which crucially relies on the zero-free regions of the partition function. After we carve out our new zero-free regions, we will come back to the existence of correlation decay of complex parameters.

3 Barvinok's Algorithm

In this section, we describe Barvinok's algorithm and introduce complex contraction. Let $I = [0, t]$ be a closed real interval. We define the δ-strip of I to be $\{z \in \mathbb{C} \mid |z - z_0| < \delta, z_0 \in I\}$, denoted by I_δ. It is a complex neighborhood of I. Suppose a graph polynomial $P(z) = \sum_{i=0}^{n} a_i z^i$ of degree n is zero-free in I_δ. Barvinok's method [3] roughly states that for any $z \in I_\delta$, $P(z)$ can be $(1 \pm \varepsilon)$-approximated using coefficients $a_0, \dots, a_k$ for some $k = O(e^{\Theta(1/\delta)} \log(n/\varepsilon))$, via truncating the Taylor expansion of the logarithm of the polynomial. For the partition function of 2-spin systems, these coefficients can be computed in polynomial time [30, 28]. For the purpose of obtaining an FPTAS, we will view the partition function as a univariate polynomial $Z_{G;\beta,\gamma}(\lambda)$ in λ and fix β and γ. The following result is known.

▶ **Lemma 16.** *Fix $\beta, \gamma \in \mathbb{C}$ and $\Delta \in \mathbb{N}$. Let G be a graph of degree at most Δ. If $Z_{G;\beta,\gamma}(\lambda) \neq 0$ lies in a δ-strip I_δ of $I = [0, t]$, then there is an FPTAS for computing $Z_{G;\beta,\gamma}(\lambda)$ for $\lambda \in I_\delta$.*

Proof. This lemma is a generalization of Lemma 4 in [16], where β and γ are both real. The generalization to complex valued parameters directly follows from the argument in [28]. ◀

3.1 Zero-freeness and complex contraction

With Lemma 16 in hand, the main effort is to obtain zero-free regions of the partition function. For this purpose, we will still view $Z_G(\boldsymbol{\zeta})$ as a multivariate polynomial in (β, γ, λ). A main and widely used approach to obtain zero-free regions is the *recursion* method [41, 35, 4, 32, 27]. This method is related to the correlation decay method.

Assuming $Z^-_{G,v}(\boldsymbol{\zeta}) \neq 0$ for some vertex v, then $Z_G(\boldsymbol{\zeta}) \neq 0$ is equivalent to $R_{G,v} = \frac{Z^+_{G,v}(\boldsymbol{\zeta})}{Z^-_{G,v}(\boldsymbol{\zeta})} \neq -1$. As pointed above, the ratio $R_{G,v}$ can be computed by recursion via the SAW tree in which v is the root. Roughly speaking, the key idea of the recursion method is to construct a *contraction* region $Q \subseteq \mathbb{C}$ where $\lambda \in Q$ and $-1 \notin Q$ such that for all recursion functions $F_{\boldsymbol{\zeta},\mathbf{s}}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$, $F_{\boldsymbol{\zeta},\mathbf{s}}(Q^k) \subseteq Q$, and for all $F_{\boldsymbol{\zeta},\mathbf{s}}$ with $\|\mathbf{s}\|_1 = \Delta$, $-1 \notin F_{\boldsymbol{\zeta},\mathbf{s}}(Q^k)$. This condition guarantees that with the initial value $R_{G,v_\ell} = \lambda$ where v_ℓ is a free leaf node in the SAW tree of which the degree is bounded by Δ, the recursion will never achieve -1. Hence, we have $Z_G(\boldsymbol{\zeta}) \neq 0$ by induction. Again, we may use a potential function $\varphi : Q \to P$ to change the domain, and we prove $F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}(P^k) \subseteq P$.

Now, we introduce the following *complex* contraction property as a generalization of real contraction. This property gives a sufficient condition for the zero-freeness of the partition function.

▶ **Definition 17** (Complex contraction). *Fix $\Delta \in \mathbb{N}$. We say that $\boldsymbol{\zeta} \in \mathbb{C}^3$ satisfies complex contraction for Δ if there is a closed and bounded complex region $Q \subseteq \mathbb{C}$ where $\lambda \in Q$, $-\gamma \notin Q$ and $-1 \notin Q$, and an analytic and invertible function $\varphi : Q \to P$ where the inverse $\varphi^{-1} : P \to Q$ is also analytic and P is convex, such that*

1. *$F_{\boldsymbol{\zeta},\mathbf{s}}(Q^k) \subseteq Q$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$ and $-1 \notin F_{\boldsymbol{\zeta},\mathbf{s}}(Q^k)$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 = \Delta$;*
2. *there exists $\eta > 0$ s.t. $\left\|\nabla F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x})\right\|_1 \leq 1 - \eta$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$ and all $\mathbf{x} \in P^k$.*

▶ Remark 18. Similar to the remark of Definition 11, the function $F^{\varphi}_{\boldsymbol{\zeta},\mathbf{s}}(\mathbf{x})$ is well-defined and analytic on P^k. Here, we directly assume that the inverse φ^{-1} is analytic instead of $\varphi'(x) \neq 0$ for the sake of simplicity of our proof.

▶ **Lemma 19.** *If $\boldsymbol{\zeta} \in \mathbb{C}^3$ satisfies complex contraction for Δ, then $Z^{\sigma_\Lambda}_G(\boldsymbol{\zeta}) \neq 0$ for any graph G of degree at most Δ and any feasible configuration σ_Λ.*

Please refer to the full paper for a proof of Lemma 19. Such a proof only uses condition 1 of complex contraction. However, condition 2 combining with the zero-freeness result of Lemma 19 gives a sufficient condition for bounded 2-spin systems of *complex* parameters exhibiting correlation decay. This is a generalization of Lemma 13. Also, please refer to the full paper for a proof.

▶ **Lemma 20.** *If $\boldsymbol{\zeta} \in \mathbb{C}^3$ satisfies complex contraction for Δ, then the Δ-bounded 2-spin system of $\boldsymbol{\zeta}$ exhibits SSM. Thus, there is an FPTAS for computing $Z_G(\boldsymbol{\zeta})$ via Weitz's algorithm.*

4 From Real Contraction to Complex Contraction

In this section, we prove our main result. We first give some preliminaries in complex analysis. The main tools are the unique analytic continuation and the inverse function theorem. Here, we slightly modify the statements to fit for our settings. Please refer to [42] for the proofs.

▶ **Theorem 21** (Unique analytic continuation). *Let $f(x)$ be a (real) analytic function defined on a compact real interval $I \subseteq \mathbb{R}$. Then, there exists a complex neighborhood $\widetilde{I} \subseteq \mathbb{C}$ of I, and a (complex) analytic function $\widetilde{f}(x)$ defined on $\widetilde{I}$ such that $\widetilde{f}(x) \equiv f(x)$ for all $x \in I$. Moreover, if there is another (complex) analytic function $\widetilde{g}(x)$ also defined on $\widetilde{I}$ such that $\widetilde{g}(x) \equiv \widetilde{f}(x)$ for all $x \in I$ and the measure $\mathfrak{m}(I) \neq 0$, then $\widetilde{g}(x) \equiv \widetilde{f}(x)$ for all $x \in \widetilde{I}$. We call $\widetilde{f}(x)$ the unique analytic continuation of $f(x)$ on $\widetilde{I}$.*

▶ **Theorem 22** (Inverse function theorem). *For a real analytic function φ defined on a real interval $J \subseteq \mathbb{R}$, if $\varphi'(x) \neq 0$ for all $x \in J$, then φ is invertible on J and the inverse φ^{-1} is also analytic on $\varphi(J)$. For a complex analytic function ψ defined on $U \subseteq \mathbb{C}$, if $\psi'(z) \neq 0$ for some $z \in U$, then there exists a complex neighborhood D of z such that ψ is invertible on D and the inverse is also analytic.*

Combining the above theorems, we have the following result.

▶ **Lemma 23.** *Let $\varphi : J \to I$ be a real analytic function, and $\varphi'(x) \neq 0$ for all $x \in J$ where J and I are real compact intervals. Then, there exists an analytic continuation $\widetilde{\varphi}$ on a complex neighborhood $\widetilde{J}$ of J such that $\widetilde{\varphi}$ is invertible on $\widetilde{J}$ and the inverse $\widetilde{\varphi}^{-1}$ is also analytic.*

Proof. If $\mathfrak{m}(J) = 0$, i.e., $J = \{x\}$, then by Theorem 21 there exists an analytic continuation $\widetilde{\varphi}$ of φ. Since $\widetilde{\varphi}'(x) = \varphi'(x) \neq 0$, by Theorem 22, there is a neighborhood of x on which $\widetilde{\varphi}$ is invertible and the inverse $\widetilde{\varphi}^{-1}$ is analytic.

Otherwise, $\mathfrak{m}(J) \neq 0$. Since $\varphi(x)$ is analytic and $\varphi'(x) \neq 0$ for all $x \in J$, the function φ is invertible and by Theorem 22, the inverse $\varphi^{-1} : I \to J$ is analytic on I. By Theorem 21, there exists an analytic continuation $\widetilde{\varphi^{-1}}$ of φ^{-1} defined on a neighborhood $\widetilde{I}_1$ of I. Similarly, there exists an analytic continuation $\widetilde{\varphi}$ of φ defined on a neighborhood $\widetilde{J}$ of J. We use $\widetilde{I}$ to denote the image $\widetilde{\varphi}(\widetilde{J})$. Since $\widetilde{\varphi}$ is analytic, by the open mapping theorem $\widetilde{I}$ is an open set in the complex plane. Clearly, we have $\varphi(J) = I \subseteq \widetilde{I}$. We can pick $\widetilde{J}$ small enough while still keeping $J \subseteq \widetilde{J}$ such that the image $\widetilde{I} = \widetilde{\varphi}(\widetilde{J}) \subseteq \widetilde{I}_1$ and still $I \subseteq \widetilde{I}$. Thus, the composition $\widetilde{\varphi^{-1}} \circ \widetilde{\varphi}$ is a well-defined analytic function on $\widetilde{J}$. Clearly, we have that

$$\widetilde{\varphi^{-1}} \circ \widetilde{\varphi}(x) = \varphi^{-1} \circ \varphi(x) \equiv x \text{ for all } x \in J.$$

Since $\mathfrak{m}(J) \neq 0$, by Theorem 21, we have that $\widetilde{\varphi^{-1}} \circ \widetilde{\varphi}(x) \equiv x$ for all $x \in \widetilde{J}$.

Thus, $\widetilde{\varphi}$ is invertible on $\widetilde{J}$ and the inverse $\widetilde{\varphi}^{-1} = \widetilde{\varphi^{-1}}$ is analytic. ◀

Now, we are ready to prove our main result.

▶ **Theorem 24.** *If $\boldsymbol{\zeta}_0$ satisfies real contraction for Δ, then there exists a $\delta > 0$ such that for every $\boldsymbol{\zeta} \in \mathbb{C}^3$ with $\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_0\|_\infty < \delta$, $\boldsymbol{\zeta}$ satisfies complex contraction for Δ.*

Proof. Let $\varphi : J \to I$ be a good potential function for $\boldsymbol{\zeta}_0$. By Definition 11 and Lemma 23, there exists a neighborhood $\widetilde{J}$ of J such that the analytic continuation $\widetilde{\varphi} : \widetilde{J} \to \widetilde{I}$ of φ on $\widetilde{J}$ is invertible. Here $\widetilde{I} = \widetilde{\varphi}(\widetilde{J})$ is a neighborhood of I, and the inverse $\widetilde{\varphi}^{-1}$ is also analytic on $\widetilde{I}$. We use $\mathcal{B}_\delta := \{\mathbf{z} \in \mathbb{C}^3 \mid \|\mathbf{z} - \boldsymbol{\zeta}_0\|_\infty < \delta\}$ to denote the 3-dimensional complex ball around $\boldsymbol{\zeta}_0$ of radius δ in terms of the infinity norm. Recall that we define $I_\varepsilon = \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon, z_0 \in I\}$. Given a set $U \subseteq \mathbb{C}^k$, we use $\overline{U}$ to denote its closure.

We first show that we can pick a pair of $(\delta_1, \varepsilon_1)$ such that for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$, the composition

$$F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}, \mathbf{x}) = \widetilde{\varphi}(F_{\mathbf{s}}(\boldsymbol{\zeta}, \widetilde{\boldsymbol{\varphi}}^{-1}(\mathbf{x}))) \text{ is well-defined and analytic on } \mathcal{B}_{\delta_1} \times I_{\varepsilon_1}^k.$$

Given some $\mathbf{s}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$, we consider the function $F_{\mathbf{s}}(\boldsymbol{\zeta}, \mathbf{x})$. We know that it is analytic on a neighborhood of $\{\boldsymbol{\zeta}_0\} \times J^k$ and by real contraction we have $F_{\mathbf{s}}(\boldsymbol{\zeta}_0, J^k) \subseteq J$. Then, we can pick some $\delta_{\mathbf{s}}$ and a neighborhood $\widetilde{J}_{\mathbf{s}}$ of J that are small enough such that $F_{\mathbf{s}}(\boldsymbol{\zeta}, \mathbf{x})$ is analytic on $\mathcal{B}_{\delta_{\mathbf{s}}} \times \widetilde{J}_{\mathbf{s}}^k$, and $F_{\mathbf{s}}(\mathcal{B}_{\delta_{\mathbf{s}}}, \widetilde{J}_{\mathbf{s}}^k) \subseteq \widetilde{J}$. Let

$$\delta_1 = \min_{\|\mathbf{s}\|_1 \leq \Delta - 1} \{\delta_{\mathbf{s}}\} \quad \text{and} \quad \widetilde{J}_1 = \bigcap_{\|\mathbf{s}\|_1 \leq \Delta - 1} \widetilde{J}_{\mathbf{s}}.$$

Since there is only a finite number of $\mathbf{s}$ with $\|\mathbf{s}\|_1 \le \Delta - 1$, we have that $\delta_1 > 0$, and $\widetilde{J_1}$ is open and it is a neighborhood of J. Then, $F_{\mathbf{s}}(\mathcal{B}_{\delta_1}, \widetilde{J_1}) \subseteq \widetilde{J}$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \le \Delta - 1$. Since $\widetilde{\varphi}^{-1}$ is analytic on $\widetilde{I}$ and $\widetilde{\varphi}^{-1}(I) = J$, similarly we can pick a small enough neighborhood $\widetilde{I_1}$ of I where $\widetilde{I_1} \subseteq \widetilde{I}$ such that $\widetilde{\varphi}^{-1}(\widetilde{I_1}) \subseteq \widetilde{J_1}$. For every $z_0 \in I$, we can pick an ε_{z_0} such that the disc $B_{z_0,\varepsilon_{z_0}} := \{z \in \mathbb{C} \mid |z - z_0| < \varepsilon_{z_0}\}$ is in $\widetilde{I_1}$. Recall that I is a compact real interval, by the finite cover theorem, we can uniformly pick an ε_1 such that $I \subseteq I_{\varepsilon_1} \subseteq \widetilde{I_1}$. Thus, $F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}, \mathbf{x})$ is well-defined and analytic on $\mathcal{B}_{\delta_1} \times I_{\varepsilon_1}^k$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \le \Delta - 1$. In fact, $F_{\mathbf{s}}^{\widetilde{\varphi}}$ is a (multivariate) analytic continuation of $F_{\mathbf{s}}^{\varphi}$. Since I is a compact interval, in the following when we pick a neighborhood $\widetilde{I}$ of I, without loss of generality, we may always pick $\widetilde{I}$ as an ε-strip I_ε of I.

Then, we show that we can pick a pair of $(\delta_2, \varepsilon_2)$ where $\delta_2 < \delta_1$ and $\varepsilon_2 < \varepsilon_1$, a constant $M > 0$ and a constant $\eta > 0$ such that for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \le \Delta - 1$,

$$\left\|\nabla F_{\boldsymbol{\zeta},\mathbf{s}}^{\widetilde{\varphi}}(\mathbf{x})\right\|_1 \le 1 - \eta \qquad \text{and} \qquad \left\|\nabla F_{\mathbf{x},\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta})\right\|_1 \le M$$

for all $\boldsymbol{\zeta} \in \overline{\mathcal{B}_{\delta_2}}$ and all $\mathbf{x} \in \overline{I_{\varepsilon_2}^k}$. By real contraction, there is an $\eta' > 0$ such that $\left\|\nabla F_{\boldsymbol{\zeta}_0,\mathbf{s}}^{\widetilde{\varphi}}(\mathbf{x})\right\|_1 \le 1 - \eta'$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \le \Delta - 1$ and all $\mathbf{x} \in I^k$. Given some $\mathbf{s}$ with $\|\mathbf{s}\|_1 \le \Delta - 1$, since $F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}, \mathbf{x})$ is analytic on $\mathcal{B}_{\delta_1} \times I_{\varepsilon_1}^k$, by continuity we can pick some $\delta_{\mathbf{s}} < \delta_1$ and $\varepsilon_{\mathbf{s}} < \varepsilon_1$ such that $\left\|\nabla F_{\boldsymbol{\zeta},\mathbf{s}}^{\widetilde{\varphi}}(\mathbf{x})\right\|_1 \le 1 - \frac{\eta'}{2}$ for all $\boldsymbol{\zeta} \in \overline{\mathcal{B}_{\delta_{\mathbf{s}}}}$ and all $\mathbf{x} \in \overline{I_{\varepsilon_{\mathbf{s}}}^k}$. In addition, let

$$M_{\mathbf{s}} = \sup_{\boldsymbol{\zeta} \in \overline{\mathcal{B}_{\delta_{\mathbf{s}}}}, \mathbf{x} \in \overline{I_{\varepsilon_{\mathbf{s}}}^k}} \left\|\nabla F_{\mathbf{x},\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta})\right\|_1,$$

and we know that $M_{\mathbf{s}} < +\infty$ since $F^{\widetilde{\varphi}}$ is analytic on $\overline{\mathcal{B}_{\delta_{\mathbf{s}}}} \times \overline{I_{\varepsilon_{\mathbf{s}}}^k}$ which is closed and bounded. Finally, let

$$\eta = \frac{\eta'}{2}, \quad \delta_2 = \min_{\|\mathbf{s}\|_1 \le \Delta - 1} \{\delta_{\mathbf{s}}\}, \quad \varepsilon_2 = \min_{\|\mathbf{s}\|_1 \le \Delta - 1} \{\varepsilon_{\mathbf{s}}\}, \quad \text{and} \quad M = \max_{\|\mathbf{s}\|_1 \le \Delta - 1} \{M_{\mathbf{s}}\}.$$

These choices will satisfy our requirement.

For the case that $\|\mathbf{s}\|_1 = \Delta$, we show that we can pick a pair of $(\delta_3, \varepsilon_3)$ where $\delta_3 < \delta_1$ and $\varepsilon_3 < \varepsilon_1$ such that for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 = \Delta$, we have $-1 \notin F_{\mathbf{s}}(\overline{\mathcal{B}_{\delta_3}}, \widetilde{J_2}^k)$ where $\widetilde{J_2} = \widetilde{\varphi}^{-1}(\overline{I_{\varepsilon_3}})$ is a closed neighborhood of J. Since $F_{\mathbf{s}}$ is analytic, by real contraction, $-1 \notin F_{\boldsymbol{\zeta}_0,\mathbf{s}}(J^k)$ which is closed. Again by continuity we can pick some $(\delta_3, \varepsilon_3)$ that satisfy our requirement.

Since $\boldsymbol{\zeta}_0 = (\beta_0, \gamma_0, \lambda_0)$ satisfies real contraction, we have $\lambda_0 \in J$, $-\gamma_0 \notin J$ and $-1 \notin J$. Recall that $J = \widetilde{\varphi}^{-1}(I)$. Again, since $\widetilde{\varphi}^{-1}$ is analytic, by continuity we can pick some $\varepsilon \le \min\{\varepsilon_2, \varepsilon_3\}$ such that $\lambda_0 \in \widetilde{\varphi}^{-1}(I_\varepsilon)$ (an open set), $-\gamma_0 \notin \widetilde{\varphi}^{-1}(\overline{I_\varepsilon})$ (a closed set) and $-1 \notin \widetilde{\varphi}^{-1}(\overline{I_\varepsilon})$. Moreover, we can pick some δ_4 small enough such that the disc $B_{\lambda_0,\delta_4} := \{z \in \mathbb{C} \mid |z - \lambda_0| < \delta_4\}$ is in $\widetilde{\varphi}^{-1}(I_\varepsilon)$, and the disc $B_{-\gamma_0,\delta_4} := \{z \in \mathbb{C} \mid |z - (-\gamma_0)| < \delta_4\}$ is disjoint with $\widetilde{\varphi}^{-1}(\overline{I_\varepsilon})$. Let $P = \overline{I_\varepsilon}$ and $Q = \widetilde{\varphi}^{-1}(\overline{I_\varepsilon})$. Clearly, P is convex. For every $\boldsymbol{\zeta}$ with $\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_0\|_\infty < \delta$, we have $\lambda \in Q$, $-\gamma \notin Q$ and $-1 \notin Q$. In addition, we know that Q is closed and bounded since P is closed and bounded and $\widetilde{\varphi}^{-1}$ is analytic on P. Finally, let $\delta = \min\{\delta_2, \delta_3, \delta_4, \frac{\varepsilon\eta}{M}\}$. We show that for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \le \Delta - 1$, $F_{\mathbf{s}}^{\widetilde{\varphi}}(\mathcal{B}_\delta, P^k) \subseteq P$, which implies that $F_{\mathbf{s}}(\mathcal{B}_\delta, Q^k) \subseteq Q$.

Consider some $\mathbf{x} \in P^k$. By the definition of P, there exists an $\mathbf{x}_0 \in I^k$ such that $\|\mathbf{x} - \mathbf{x}_0\|_\infty \leq \varepsilon$. Also, consider some $\boldsymbol{\zeta} \in \mathcal{B}_\delta$, and we have $\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_0\|_\infty < \delta$. Then, for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$, consider $F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}, \mathbf{x}) - F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}_0, \mathbf{x}_0)$. We have

$$\begin{aligned}
&\left|F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}, \mathbf{x}) - F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}_0, \mathbf{x}_0)\right| \\
\leq &\left|F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}, \mathbf{x}) - F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}_0, \mathbf{x})\right| + \left|F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}_0, \mathbf{x}) - F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}_0, \mathbf{x}_0)\right| \\
\leq &\sup_{\boldsymbol{\zeta}' \in \mathcal{B}_\delta} \left\|\nabla F_{\mathbf{x},\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}')\right\|_1 \cdot \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_0\|_\infty + \sup_{\mathbf{x}' \in P^k} \left\|\nabla F_{\boldsymbol{\zeta}_0,\mathbf{s}}^{\widetilde{\varphi}}(\mathbf{x}')\right\|_1 \cdot \|\mathbf{x} - \mathbf{x}_0\|_\infty \\
\leq &M\delta + (1 - \eta) \cdot \varepsilon \leq \varepsilon.
\end{aligned}$$

The second inequality above uses the fact that both $\mathcal{B}_\delta$ and P^k are convex, which ensures that the line between $\boldsymbol{\zeta}_0$ and $\boldsymbol{\zeta}$ is in $\mathcal{B}_\delta$ and the line between $\mathbf{x}_0$ and $\mathbf{x}$ is in P^k. By real contraction, we know that $F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}_0, \mathbf{x}_0) \in I$ since $\mathbf{x}_0 \in I^k$. Thus, we have $F_{\mathbf{s}}^{\widetilde{\varphi}}(\boldsymbol{\zeta}, \mathbf{x}) \in P$. Thus, for every $\boldsymbol{\zeta}$ with $\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_0\|_\infty < \delta$, we have that $\lambda \in Q$, $-\gamma \notin Q$ and $-1 \notin Q$, and

1. $F_{\boldsymbol{\zeta},\mathbf{s}}(Q^k) \subseteq Q$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$ and $-1 \notin F_{\boldsymbol{\zeta},\mathbf{s}}(Q^k)$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 = \Delta$;
2. there exists $\eta > 0$ s.t. $\left\|\nabla F_{\boldsymbol{\zeta},\mathbf{s}}^{\varphi}(\mathbf{x})\right\|_1 \leq 1 - \eta$ for every $\mathbf{s}$ with $\|\mathbf{s}\|_1 \leq \Delta - 1$ and all $\mathbf{x} \in P^k$.

The function $\widetilde{\varphi} : Q \to P$ is a good potential function for $\boldsymbol{\zeta}$. ◀

Combining Lemmas 15, 19, 20 and Theorem 24, we have the following result.

▶ **Theorem 25.** *Fix $\Delta \geq 3$. For every $\boldsymbol{\zeta}_0 \in \mathcal{S}_i^\Delta$ ($i \in [4]$), there exists a $\delta > 0$ such that for any $\boldsymbol{\zeta} \in \mathbb{C}^3$ where $\|\boldsymbol{\zeta} - \boldsymbol{\zeta}_0\|_\infty < \delta$, we have*

- *$Z_G^{\sigma_\Lambda}(\boldsymbol{\zeta}) \neq 0$ for every graph G of degree at most Δ and every feasible configuration σ_Λ;*
- *the Δ-bounded 2-spin system specified by $\boldsymbol{\zeta}$ exhibits correlation decay.*

Then via either Weitz's or Barvinok's algorithm, there is an FPTAS for computing $Z_G(\boldsymbol{\zeta})$.

▶ Remark 26. The choice of δ does not depend on the size of the graph, only on Δ and $\boldsymbol{\zeta}_0$. In particular, let D be a compact set in $\mathcal{S}_i^\Delta$ for some $i \in [4]$. Then there is a uniform δ such that for all $\boldsymbol{\zeta}$ in a complex neighborhood D_δ of radius δ around D, i.e., $\boldsymbol{\zeta} \in D_\delta := \{\mathbf{z} \in \mathbb{C}^3 \mid \|\mathbf{z} - \mathbf{z}_0\|_\infty < \delta, \mathbf{z}_0 \in D\}$, $Z_G(\boldsymbol{\zeta}) \neq 0$ for every graph G of degree at most Δ. In addition, in order to apply Barvinok's algorithm, by Lemma 16, we need to make sure that the zero-free regions contain $\lambda = 0$ (an easy computing point). This is true for $\mathcal{S}_1^\Delta$, $\mathcal{S}_2^\Delta$ and $\mathcal{S}_3^\Delta$. For parameters in $\mathcal{S}_4^\Delta$, we will reduce the problem to a case in $\mathcal{S}_3^\Delta$ by swapping β and γ and replacing λ by $1/\lambda$. Then, one can apply Barvinok's algorithm.

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