Node-Connectivity Terminal Backup, Separately-Capacitated Multiflow, and Discrete Convexity

Hiroshi Hirai
Department of Mathematical Informatics,
Graduate School of Information Science and Technology,
The University of Tokyo, Japan
hirai@mist.i.u-tokyo.ac.jp

Motoki Ikeda
Department of Mathematical Informatics,
Graduate School of Information Science and Technology,
The University of Tokyo, Japan
motoki_ikeda@mist.i.u-tokyo.ac.jp

Abstract

The *terminal backup problems* [Anshelevich and Karagiozova, 2011] form a class of network design problems: Given an undirected graph with a requirement on terminals, the goal is to find a minimum cost subgraph satisfying the connectivity requirement. The *node-connectivity terminal backup problem* requires a terminal to connect other terminals with a number of node-disjoint paths. This problem is not known whether is NP-hard or tractable. Fukunaga (2016) gave a 4/3-approximation algorithm based on LP-rounding scheme using a general LP-solver.

In this paper, we develop a combinatorial algorithm for the relaxed LP to find a half-integral optimal solution in $O(m \log(mUA) \cdot \mathrm{MF}(kn, m + k^2 n))$ time, where m is the number of edges, k is the number of terminals, A is the maximum edge-cost, U is the maximum edge-capacity, and $\mathrm{MF}(n', m')$ is the time complexity of a max-flow algorithm in a network with n' nodes and m' edges. The algorithm implies that the 4/3-approximation algorithm for the node-connectivity terminal backup problem is also efficiently implemented. For the design of algorithm, we explore a connection between the node-connectivity terminal backup problem and a new type of a multiflow, called a *separately-capacitated multiflow*. We show a min-max theorem which extends Lovász–Cherkassky theorem to the node-capacity setting. Our results build on discrete convex analysis for the node-connectivity terminal backup problem.

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1 Introduction

Network design problems are central problems in combinatorial optimization. A large number of basic combinatorial optimization problems are network design problems. Examples are spanning tree, matching, TSP, and Steiner networks. They admit a typical formulation of a network design problem: Find a minimum-cost network satisfying given connectivity requirements. The present paper addresses a relatively new class of network design problems, called *terminal backup problems*. The problem is to find a cheapest subnetwork in which each terminal can send a specified amount of flows to other terminals, i.e., the data in each terminal can be backed up, possibly in a distributed manner, in other terminals.

A mathematical formulation of the terminal backup problem is given as follows. Let $((V, E), S, u, c, a, r)$ be an undirected network, where (V, E) is a simple undirected graph, $S \subseteq V$ $(|S| \geq 3)$ is a set of *terminals*, $u : E \to \mathbb{Z}_+$ is a nonnegative edge-capacity function, $c : V \setminus S \to \mathbb{Z}_+$ is a nonnegative node-capacity function, $a : E \to \mathbb{Z}_+$ is a nonnegative edge-cost function, and $r : S \to \mathbb{Z}_+$ is a nonnegative requirement function on terminals. The goal is to find a feasible edge-capacity function x of minimum cost $\sum_{e \in E} a(e)x(e)$. Here an edge-capacity function x is said to be *feasible* if $0 \leq x \leq u$ and each terminal $s \in S$ has a flow from s to $S \setminus \{s\}$, an $\{s\}$–$(S \setminus \{s\})$ flow, of total flow-value $r(s)$ in the network $((V, E), S, x, c)$ capacitated by the edge-capacity x and the node-capacity c.

The original formulation, due to Anshelevich and Karagiozova [1], is uncapacitated (i.e., u, c are infinity), requires x to be integer-valued, and assumes $r(s) = 1$ for all $s \in S$. They showed that an optimal solution can be obtained in polynomial time. Bernáth et al. [2] extended this polynomial time solvability to an arbitrary integer-valued requirement r. For the setting of general edge-capacity (and infinite node-capacity), which we call the *edge-connectivity terminal backup problem (ETB)*, it is unknown whether ETB is NP-hard or tractable.

Fukunaga [8] considered the above setting including both edge-capacity and node-capacity, which we call the *node-connectivity terminal backup problem (NTB)*, and explored intriguing features of its fractional relaxation. The *fractional ETB (FETB)* and *fractional NTB (FNTB)* are LP-relaxations obtained from ETB and NTB, respectively, by relaxing solution x to be real-valued. Fukunaga showed the half-integrality property of FNTB, that is, there always exists an optimal solution that is half-integer-valued. Based on this property, he developed a 4/3-approximation algorithm for NTB by rounding a half-integral (extreme) optimal solution. Moreover, he noticed a useful relationship between FETB and *multicommodity flow (multiflow)*. In fact, a solution of FETB is precisely the edge-support of a multiflow consisting of the $r(s)$ amount of $\{s\}$–$(S \setminus \{s\})$ flow for each $s \in S$. This is a consequence of Lovász–Cherkassky theorem [5, 21] in multiflow theory. In particular, FETB is equivalent to a minimum-cost multiflow problem, which is a variant of the one studied by Karzanov [19, 20] and Goldberg and Karzanov [10].

Utilizing this connection, Hirai [12] developed a combinatorial polynomial time algorithm for FETB and the corresponding multiflow problem. This algorithm uses a max-flow algorithm as a subroutine, and brings a combinatorial implementation of Fukunaga's 4/3-approximation algorithm for ETB, where he used a generic LP-solver (e.g., the ellipsoid method) to obtain a half-integral extreme optimal solution.

Our first contribution is the extension of this result to the NTB setting, implying that the 4/3-approximation algorithm for NTB is also efficiently implemented.

▶ **Theorem 1.** *A half-integral optimal solution of FNTB can be obtained in* $O(m \log(mUA) \cdot \mathrm{MF}(kn, m + k^2 n))$ *time.*

Here $n := |V|$, $m := |E|$, $k := |S|$, $U := \max_{e \in E} u(e)$, and $A := \max_{e \in E} a(e)$, and $\mathrm{MF}(n', m')$ is the time complexity of an algorithm for solving the max-flow problem in the network with n' nodes and m' edges.

As in the ETB case, we explore and utilize a new connection between NTB and a multiflow problem. We introduce a new notion of a *free multiflow with separate node-capacity constraints* or simply a *separately-capacitated multiflow*. Instead of the usual node-capacity constraints, this multiflow should satisfy the separate node-capacity constraints: For each terminal $s \in S$ and each node $i \in V$, the total flow-value of flows connecting s to the other terminals and flowing into i is at most the node capacity $c(i)$.

Our second contribution is a min-max theorem for separately-capacitated multiflows, which extends Lovász–Cherkassky theorem to the node-capacitated setting and implies that a solution of FNTB is precisely the edge-support of a separately-capacitated multiflow. This answers Fukunaga's comment: *how the computation should proceed in the node capacitated setting remains elusive* [8, p. 799].

▶ **Theorem 2.** *The maximum flow-value of a separately-capacitated multiflow is equal to* $(1/2)\sum_{s\in S}\nu_s$, *where* ν_s *is the minimum capacity of an* $\{s\}$–$(S \setminus \{s\})$ *cut. Moreover, a half-integral maximum multiflow exists, and it can be found in* $O(n \cdot \mathrm{MF}(kn, m + k^2 n))$ *time.*

Here, a T–T' *cut* is a union of an edge-subset $F \subseteq E$ and a node-subset $X \subseteq V \setminus (T \cup T')$ such that removing those subsets disconnects T and T', and its capacity is defined as $u(F) + c(X)$.

Our algorithm for Theorem 1 builds on the ideas of *Discrete Convex Analysis (DCA) beyond* $\mathbb{Z}^n$ – a theory of discrete convex functions on special graph structures generalizing $\mathbb{Z}^n$ (the grid graph), which has been recently differentiated from the original DCA [23] and has been successfully applied to algorithm design for well-behaved classes of multiflow and related network design problems [12, 13, 14, 16]. Indeed, the algorithm in [12] for FETB was designed as: Formulate the dual of FETB as a minimization of an *L-convex function* on the (Cartesian) product of trees, apply the framework of the *steepest descent algorithm (SDA)*, and show that it is implemented by using a max-flow algorithm as a subroutine.

We formulate the dual of FNTB as an optimization problem on the product of the spaces of all subtrees of a fixed tree (Section 2.1). We develop a simple cut-descent algorithm for this optimization problem (Sections 2.2 and 2.3). Then we prove that this coincides with SDA for an L-convex function defined on the graph structure on the space of all subtrees (Section 3). Then the number of descents is estimated by a general theory of SDA, and the cost-scaling method is naturally incorporated to derive the time complexity (Section 2.4). Theorem 2 is obtained as a byproduct of these arguments. Due to the space limitation, we omit most of technical proofs, which are given in the full version.

Related work

ETB is a *survivable network design problem (SND)* with a special skew-supermodular function, and NTB is a node connectivity version (NSND) with a special skew-supermodular biset function. In his influential paper [18], Jain devised the iterative rounding method, and obtains a 2-approximation algorithm for SND, provided that an extreme optimal solution of the LP-relaxation of SND (with modified skew-supermodular functions) is available. Fleischer, Jain, and Williamson [7] and Cheriyan, Vempala, and Vetta [4] extended this iterative rounding 2-approximation algorithm to some classes of NSND. One of important open problems in the literature is a design of a combinatorial 2-approximation algorithm for (V)SND with the skew-supermodular (biset) function associated with connectivity requirements. One approach is to devise a combinatorial polynomial time algorithm to find an extreme optimal solution of its LP-relaxation; the currently known only polynomial time algorithm is a general LP-solver (e.g., the ellipsoid method). Our algorithm for FNTB, though it is the LP-relaxation of a very special NSND, may give an insight on such a research direction. On this direction, Feldmann, Könemann, Pashkovich, and Sanità [6] gave a $(2+\epsilon)$-approximation algorithm for SND with a proper function by solving the LP-relaxation approximately via the multiplicative weights method [9].

The notion of a separately-capacitated multiflow, introduced in this paper, is a new variation of S-paths packing. As seen in [24, Chapter 73], S-paths packing is one of the well-studied subjects in combinatorial optimization. Recent work [17] developed a fast algorithm for half-integral *nonzero S-paths packing problem on a group-valued graph* (with unit-capacity). Our derivation of Theorem 2 is different with flow-augmenting arguments such as Cherkassky's T-operation or those in [17]. It is a future research to develop such an algorithm for a separately-capacitated multiflow. Also, exploring an integer version of Theorem 2, an analogue of Mader's theorem [22], is an interesting future direction.

Notations

Let $\mathbb{Z}, \mathbb{Z}_+, \mathbb{R}, \mathbb{R}_+$ be the set of integers, nonnegative integers, reals, and nonnegative reals, respectively. Let $\mathbb{Z}^*, \mathbb{Z}^*_+$ be the set of half-integers and nonnegative half-integers, respectively, i.e., $\mathbb{Z}^* := \mathbb{Z}/2$. Let $\overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ and $\underline{\mathbb{R}} := \mathbb{R} \cup \{-\infty\}$. Let denote $(a)^+ := \max\{a, 0\}$ for $a \in \mathbb{R}$. For a finite set V, we often identify a function on V with a vector in $\mathbb{R}^V$. For $i \in V$, its characteristic function $\chi_i : V \to \mathbb{R}$ is defined by $\chi_i(j) = 1$ if $j = i$ and $\chi_i(j) = 0$ otherwise. For a function f on V and a subset $U \subseteq V$, we denote $f(U) := \sum_{i \in U} f(i)$.

In this paper, all graphs are simple and connected unless otherwise specified. For an undirected graph on nodes V, the set of edges connecting U_1 and U_2 ($U_1, U_2 \subseteq V$) is denoted by $\delta(U_1, U_2)$. If $U_2 = V \setminus U_1$, we simply denote it by δU_1. If U_1 is a singleton, i.e., $U_1 = \{i\}$, then we denote $\delta\{i\}$ by δi. An edge connecting i and j is denoted by ij.

2 Node-Connectivity Terminal Backup Problem

Let $((V, E), S, u, c, a, r)$ be a network. Assume that $S = \{1, \ldots, k\} \subseteq V = \{1, \ldots, n\}$. By a perturbation technique, we may assume that a is positive; see Remark 3.

A sufficient and necessity condition for the feasibility of NTB is easily derived from the Menger's theorem as follows. A *biset* is a pair of node subsets $X, X^+ \subseteq V$ with $X \subseteq X^+$. We write $\hat{X} = (X, X^+)$ for a biset. Let $\Gamma(\hat{X}) := X^+ \setminus X$, and let $\delta(\hat{X}) := \delta(X, V \setminus X^+)$. For $s \in S$, define a family $\mathcal{C}_s$ of bisets by

$$\mathcal{C}_s := \{(X, X^+) \mid \{s\} \subseteq X \subseteq X^+ \subseteq V \setminus (S \setminus \{s\})\}.$$

Let $\mathcal{C} := \bigcup_{s \in S} \mathcal{C}_s$. Then an edge-capacity $x : E \to \mathbb{Z}_+$ is feasible if and only if

$$x(\delta(\hat{X})) + c(\Gamma(\hat{X})) \geq r(s) \quad (\hat{X} \in \mathcal{C}_s,\ s \in S). \tag{1}$$

We assume that u satisfies (1) throughout the paper (otherwise NTB is infeasible).

Fukunaga [8] developed an approximation algorithm for NTB via the following relaxation problem FNTB:

$$\begin{aligned} \text{(FNTB)} \quad \text{Minimize} \quad & \sum_{e \in E} a(e)x(e) \\ \text{subject to} \quad & x(\delta\hat{X}) + c(\Gamma(\hat{X})) \geq r(s) \quad (s \in S,\ \hat{X} \in \mathcal{C}_s), && (2) \\ & 0 \leq x(e) \leq u(e) \quad (e \in E). && (3) \end{aligned}$$

From the assumption, the polytope defined by (2) and (3) is nonempty. Also, it is known [8, Corollary 3.3] that the polytope is half-integral. Thus FNTB has a half-integral optimal solution. This can be obtained by a general LP solver [8, Lemma 4.4].

▶ Remark 3. If $Z := \{e \in E \mid a(e) = 0\}$ is nonempty, then we use the following perturbation technique based on [10, 20]. Recall that U is the maximum edge capacity. Define a positive edge-cost a' by $a'(e) := 1$ for $e \in Z$ and $a'(e) := (2U|Z| + 1)a(e)$ for $e \notin Z$. Let x^* be a half-integral optimal solution of FNTB under the edge-cost a' (it exists by the half-integrality). We prove that x^* is also optimal under the original edge-cost a. It suffices to show that $\sum_{e \in E} a(e)x^*(e) \leq \sum_{e \in E} a(e)x(e)$ for any feasible half-integral edge-capacity x. It holds that $(2U|Z| + 1)(\sum_{e \in E} a(e)x^*(e) - \sum_{e \in E} a(e)x(e)) = \sum_{e \in E} a'(e)x^*(e) - \sum_{e \in E} a'(e)x(e) - x^*(Z) + x(Z) \leq U|Z|$ and thus $\sum_{e \in E} a(e)x^*(e) - \sum_{e \in E} a(e)x(e) \leq U|Z|/(2U|Z| + 1) < 1/2$. By the half-integrality, we obtain $\sum_{e \in E} a(e)x^*(e) - \sum_{e \in E} a(e)x(e) \leq 0$.

2.1 Combinatorial Duality for FNTB

We introduce a combinatorial duality theory for FNTB. For each $s \in S$, consider an infinite path graph P_s with one endpoint. Glue those $k\,(= |S|)$ endpoints, and denote the resulting graph by $\mathbb{T}$. We denote the set of nodes of P_s and $\mathbb{T}$ also by P_s and $\mathbb{T}$, respectively. We give length $1/2$ for each edge in $\mathbb{T}$. The glued endpoint is denoted by 0, and the point in P_s ($s \in S$) having the distance l from 0 is denoted by (l, s). We denote the set of all subtrees of $\mathbb{T}$ by $\mathbb{S} = \mathbb{S}(\mathbb{T})$. If a subtree T does not contain 0, then it is contained in some P_s. Such a subtree T is said to be of *s-type* and is denoted by $[l, l']_s$, where (l, s) and (l', s) are the closest and farthest nodes from 0 in T, respectively. If a subtree T contains 0, then it is said to be of *0-type* and is denoted by $[l_1, l_2, \ldots, l_k] = [l_s]_{s \in S}$, where (l_s, s) is the node in $T \cap P_s$ farthest from 0 for each $s \in S$. We identify a node on $\mathbb{T}$ with a subtree consisting of this node only.

For a 0-type subtree $T = [l_s]_{s \in S} \in \mathbb{S}$, let $\mathrm{size}_s(T) := l_s$ for $s \in S$, and $\mathrm{size}(T) := \sum_{s=1}^{k} \mathrm{size}_s(T)$. For an s-type subtree $T = [l, l']_s \in \mathbb{S}$, let $\mathrm{size}(T) := l' - l$. For two subtrees $T, T' \in \mathbb{S}$, we denote the minimum distance between T and T' on $\mathbb{T}$ by $\mathrm{dist}(T, T')$, i.e., $\mathrm{dist}(T, T') := \min\{d_{\mathbb{T}}(v, v') \mid v \in T,\ v' \in T'\}$, where $d_{\mathbb{T}}$ is the shortest distance on $\mathbb{T}$.

We formulate a dual of FNTB as a problem of assigning a subtree for each node $i \in V$. That is, subtrees are viewed as node-potentials. So we use p_i and $p : V \to \mathbb{S}$ for denoting a subtree assigned for node $i \in V$ and a potential function, respectively. Formally, let us consider the following maximization problem DTB.

$$\begin{aligned} \text{(DTB)} \quad \text{Maximize} \quad & \sum_{s \in S} r_s \,\mathrm{dist}(0, p_s) - \sum_{i \in V \setminus S} c_i \,\mathrm{size}(p_i) - \sum_{ij \in E} u_{ij} (\mathrm{dist}(p_i, p_j) - a_{ij})^+ \\ \text{subject to} \quad & p : V \to \mathbb{S}, \\ & p_s \in P_s \quad (s \in S). \end{aligned} \tag{4}$$

It turns out in the proof of Proposition 4 below that this seemingly strange formulation of DTB is essentially the LP-dual of FTB. If $p : V \to \mathbb{S}$ satisfies (4), then it is called a *potential*. See Figure 1 for an intuition for a subtree-valued potential p. A potential p is said to be *proper* if any p_i for $i \in V$ is contained in the minimal subtree that contains all p_s ($s \in S$).

▶ **Proposition 4.** *The optimum value of FNTB is at least that of DTB. Moreover, there exists a proper optimal potential for DTB.*

Proof. Let $p : V \to \mathbb{S}$ be any potential (not necessarily proper). For each $s \in S$, suppose that p_s is written as $p_s = (M_s, s)$ for $M_s \in \mathbb{Z}_+^*$. Define a new proper potential $p' : V \to \mathbb{S}$ by

$$p'_i := \begin{cases} [\min\{l, M_s\}, \min\{l', M_s\}]_s & \text{if } p_i = [l, l']_s, \\ [\min\{l_1, M_1\}, \ldots, \min\{l_k, M_k\}] & \text{if } p_i = [l_1, \ldots, l_k]. \end{cases}$$

Then the objective function value of p' does not decrease. This implies the latter part of the statement.

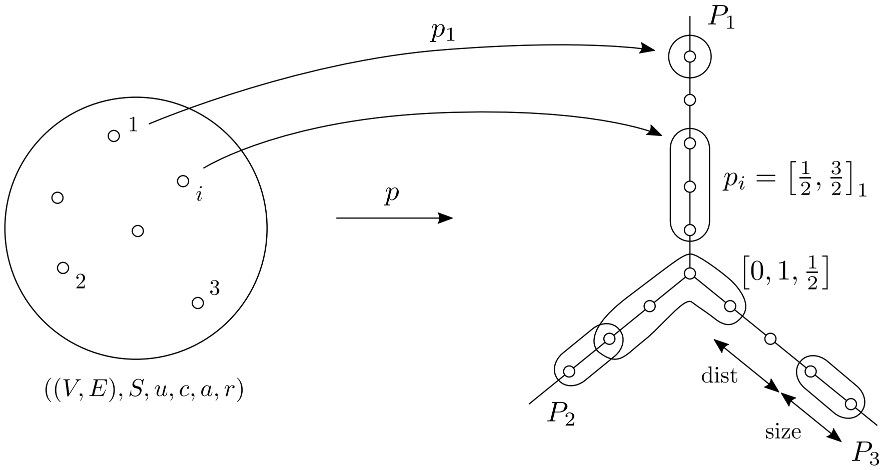

Figure 1 A subtree-valued potential p.

We next show the former part, i.e., the weak duality. The LP dual of FNTB is written as

$$\text{Maximize} \quad \sum_{s \in S} \sum_{\hat{X} \in \mathcal{C}_s} (r_s - c(\Gamma(\hat{X})))\pi(\hat{X}) - \sum_{e \in E} u_e \left(\sum_{\hat{X} \in \mathcal{C}: e \in \delta \hat{X}} \pi(\hat{X}) - a_e \right)^+$$

$$\text{subject to} \quad \pi : \mathcal{C} \to \mathbb{R}_+.$$

We show that for any proper potential $p : V \to \mathbb{S}$, we can construct $\pi : \mathcal{C} \to \mathbb{R}_+$ such that

$$\sum_{\hat{X} \in \mathcal{C}_s} \pi(\hat{X}) = \mathrm{dist}(0, p_s) \quad (s \in S), \tag{5}$$

$$\sum_{\hat{X} \in \mathcal{C}: i \in \Gamma(\hat{X})} \pi(\hat{X}) = \mathrm{size}(p_i) \quad (i \in V \setminus S), \tag{6}$$

$$\sum_{\hat{X} \in \mathcal{C}: e \in \delta \hat{X}} \pi(\hat{X}) = \mathrm{dist}(p_i, p_j) \quad (ij \in E). \tag{7}$$

Then by $\sum_{\hat{X} \in \mathcal{C}} c(\Gamma(\hat{X}))\pi(\hat{X}) = \sum_{\hat{X} \in \mathcal{C}} \sum_{i \in \Gamma(\hat{X})} c_i \pi(\hat{X}) = \sum_{i \in V \setminus S} c_i \sum_{\hat{X} \in \mathcal{C}: i \in \Gamma(\hat{X})} \pi(\hat{X})$, the weak duality follows.

Let e be an edge in $\mathbb{T}$. We define a biset (X_e, X_e^+) as follows. When we remove e from $\mathbb{T}$, there appear two connected components. Let T_e be the component which does not contain $0\,(\in \mathbb{T})$. Define $X_e, X_e^+ \subseteq V$ by

$$X_e := \{i \in V \mid p_i \text{ is contained in } T_e\}, \quad X_e^+ := X_e \cup \{i \in V \mid p_i \text{ contains } e\}.$$

Observe that if e is an edge in P_s and $X_e \neq \emptyset$, then $(X_e, X_e^+) \in \mathcal{C}_s$. Then a potential function $\pi : \mathcal{C} \to \mathbb{R}_+$ defined by

$$\pi(\hat{X}) := \frac{1}{2} |\{e \mid \hat{X} = (X_e, X_e^+)\}| \quad (\hat{X} \in \mathcal{C})$$

satisfies (5)–(7). ◀

We remark that the technique used in the above proof is based on a tree representation of a laminar biset family; see also [11] for the relating argument that maps to each node a subtree as a potential. We also note that our algorithm below will give an algorithmic proof of the strong duality.

We next derive from Proposition 4 the complementary slackness condition. Let $p : V \to \mathbb{S}$ be a proper potential. By p, we decompose V into $S \cup V_0 \cup \bigcup_{s \in S} V_s$, where

$$V_0 := \{i \in V \setminus S \mid p_i \text{ is of 0-type}\},$$
$$V_s := \{i \in V \setminus S \mid p_i \text{ is of } s\text{-type}\} \quad (s \in S).$$

In the next lemma, we see that it is sufficient to only consider edges $ij \in E$ with $\mathrm{dist}(p_i, p_j) \geq a_{ij}$. Let denote the set of such edges by

$$E^* := \{ij \in E \mid \mathrm{dist}(p_i, p_j) \geq a_{ij}\}.$$

For $i \in V_0$ and $s \in S$, we denote a set of edges in E^* connecting i and V_s by

$$E^{i,s} := \{ij \in E^* \mid j \in V_s\} \quad (i \in V_0,\ s \in S).$$

By the positivity of a, we see that $(E^{i,1}, E^{i,2}, \dots, E^{i,k})$ is a partition of $E^* \cap \delta i$. For $i \in V_s$ $(s \in S)$, there appear two connected components when we remove p_i from $\mathbb{T}$. Let $T_{i,0}$ be the component which includes $0\,(\in \mathbb{T})$, and let $T_{i,+}$ be the other component. Then we define the sets of edges $E^{i,0}$ and $E^{i,+}$ by

$$E^{i,0} := \{ij \in E^* \mid p_j \text{ is contained in } T_{i,0}\},$$
$$E^{i,+} := \{ij \in E^* \mid p_j \text{ is contained in } T_{i,+}\}.$$

By the positivity of a, we see that $(E^{i,0}, E^{i,+})$ is a partition of $E^* \cap \delta i$.

▶ **Lemma 5.** *Let $x : E \to \mathbb{R}_+$ be an edge-capacity function with $0 \leq x \leq u$, and let $p : V \to \mathbb{S}$ be a proper potential. If x and p satisfy the following conditions (A1–5), then x and p are optimal solutions for FNTB and DTB, respectively:*

(A1) *For each $ij \in E$, if $\mathrm{dist}(p_i, p_j) > a_{ij}$, then $x_{ij} = u_{ij}$.*
(A2) *For each $ij \in E$, if $\mathrm{dist}(p_i, p_j) < a_{ij}$, then $x_{ij} = 0$.*
(A3) *For each $i \in \bigcup_{s \in S} V_s$, it holds $x(E^{i,0}) = x(E^{i,+}) \leq c_i$. If $\mathrm{size}(p_i) > 0$, then $x(E^{i,0}) = x(E^{i,+}) = c_i$.*
(A4) *For each $i \in V_0$ and $s \in S$, it holds $x(E^{i,s}) \leq c_i$ and $x(E^{i,s}) \leq \sum_{s' \neq s} x(E^{i,s'})$. If $\mathrm{size}_s(p_i) > 0$, then $x(E^{i,s}) = c_i$.*
(A5) *For each $s \in S$, it holds $x(\delta s) \geq r_s$. If $\mathrm{dist}(0, p_s) > 0$, then $x(\delta s) = r_s$.*

Proof. Let x and p satisfy (A1–5). For the feasibility of x, it is sufficient to show that, for each $s \in S$, there exists a flow satisfying the capacities x and c that connects s and $S \setminus \{s\}$ with flow-value r_s. To prove this, we decompose x into a separately-capacitated multiflow. An *S-path* is a path connecting distinct terminals. Consider the following algorithm, which takes x as an input and outputs a function $\lambda : \mathcal{P} \to \mathbb{R}_+$, where $\mathcal{P}$ is a set of S-paths:

0. Let $\mathcal{P} = \emptyset$.
1. Take $s \in S$ and an edge sj satisfying $x(sj) > 0$. If such a pair does not exist, then stop the algorithm; output $(\mathcal{P}, \lambda)$. Otherwise, let $j_0 \leftarrow s,\ j_1 \leftarrow j,\ \mu \leftarrow x(sj),\ t \leftarrow 1$.
2. If j_t is a terminal, then add $P = (j_0, j_1, \dots, j_t)$ to $\mathcal{P}$ and let $\lambda(P) := \mu > 0$. Update $x(e) \leftarrow x(e) - \mu$ on each edge e in P, and return to Step 1. Otherwise go to Step 3.
3. If $j_t \in \bigcup_{s \in S} V_s$, then $j_{t-1}j_t \in E^{j_t,+}$ or $j_{t-1}j_t \in E^{j_t,0}$ by (A2) and $x(j_{t-1}j_t) > 0$. In the former case, take $j_tj_{t+1} \in E^{j_t,0}$ with $x(j_tj_{t+1}) > 0$. Such an edge exists by the former part of (A3). In the latter case, take $j_tj_{t+1} \in E^{j_t,+}$ with $x(j_tj_{t+1}) > 0$. Update $\mu \leftarrow \min\{\mu, x(j_tj_{t+1})\},\ t \leftarrow t + 1$, and return to Step 2.

If $j_t \in V_0$, then $j_{t-1}j_t \in E^{j_t,s}$ (as we will show). Take $s' \neq s$ with maximum $x(E^{j_t,s'})$ ($>$ 0), and take $j_t j_{t+1} \in E^{j_t,s'}$ with $x(j_t j_{t+1}) > 0$. Such an edge exists by $x(j_{t-1}j_t) > 0$ and the former part of (A4). Update

$$\mu \leftarrow \min\left\{\mu, x(j_t j_{t+1}), \frac{\min\left\{\sum_{s'''\neq s''} x(E^{j_t,s'''}) - x(E^{j_t,s''}) \mid s'' \neq s, s'\right\}}{2}\right\},$$

and $t \leftarrow t+1$. Note that $\mu > 0$ by the maximality of $x(E^{j_t,s'})$. Return to Step 2.

Suppose that we add $(j_0, j_1, \ldots, j_\ell)$ to $\mathcal{P}$ in Step 2. Observe that j_{t+1} is at a side opposite to j_{t-1} based on j_t for each $t = 1, \ldots, \ell-1$. By the positivity of a and (A2), $\{j_{t-1}, j_t, j_{t+1}\}$ are distinct and

$$\mathrm{dist}(p_{j_{t-1}}, p_{j_{t+1}}) = \mathrm{dist}(p_{j_{t-1}}, p_{j_t}) + \mathrm{size}(p_{j_t}) + \mathrm{dist}(p_{j_t}, p_{j_{t+1}})$$

if $j_t \in \bigcup_{s\in S} V_s$, and

$$\mathrm{dist}(p_{j_{t-1}}, p_{j_{t+1}}) = \mathrm{dist}(p_{j_{t-1}}, p_{j_t}) + \mathrm{size}_s(p_{j_t}) + \mathrm{size}_{s'}(p_{j_t}) + \mathrm{dist}(p_{j_t}, p_{j_{t+1}})$$

if $j_t \in V_0$, where $j_{t-1} \in V_s$ and $j_{t+1} \in V_{s'}$ $(s \neq s')$. Since $\mathbb{T}$ is a tree, we can show

$$\mathrm{dist}(p_{j_0}, p_{j_\ell}) = \sum_{t=0}^{\ell-1} \mathrm{dist}(p_{j_t}, p_{j_{t+1}}) + \sum_{1\leq t\leq \ell-1,\ t\neq t'} \mathrm{size}(p_{j_t}) + \mathrm{size}_{j_0}(p_{j_{t'}}) + \mathrm{size}_{j_\ell}(p_{j_{t'}}) \quad (8)$$

by an induction, where $j_{t'} \in V_0$ (if exists); see also [12, Lemma 3.9]. Hence $(j_0, j_1, \ldots, j_\ell)$ is a "shortest path on $\mathbb{T}$" from j_0 to j_ℓ, and $j_0, \ldots, j_\ell$ are distinct.

Thus after $|V|$ executions of Step 3, the algorithm adds a path P to $\mathcal{P}$ in Step 2. Also the algorithm keeps (A2) and the former parts of (A3–4). To see it for (A4), suppose that the algorithm adds a path $(j_0, j_1, \ldots, j_t, \ldots, j_\ell)$ to $\mathcal{P}$ in Step 2, where $j_0 = s \in S$, $j_t \in V_0$ and $j_\ell = s' \in S$. By the above argument, such t is uniquely determined (if exists). Then for all $s'' \neq s, s'$, we have $\sum_{s'''\neq s''} x(E^{j_t,s'''}) - x(E^{j_t,s''}) \geq 2\mu$. Thus after the decrease of the value of x along with P, it satisfies that $\sum_{s'''\neq s''} x(E^{j_t,s'''}) - x(E^{j_t,s''}) \geq 0$.

After the decrease of the value of x along with a path, it becomes $x(e) = 0$ for at least one edge $e \in E$, or becomes $\sum_{s'\neq s} x(E^{i,s'}) - x(E^{i,s}) = 0$ for at least one pair of $i \in V_0$ and $s \in S$. The algorithm keeps those values to be zero in the remaining execution, implying that it terminates after adding at most $O(m + kn)$ paths to $\mathcal{P}$. To see it, suppose that after the decrease of the value of x along with a path, it becomes $\sum_{s'\neq s} x(E^{i,s'}) - x(E^{i,s}) = 0$ for $i \in V_0$ and $s \in S$. If the algorithm chooses a path $(j_0, \ldots, j_t = i, \ldots, j_\ell)$ for adding to $\mathcal{P}$ in the remaining execution, then by the maximality of $x(E^{i,s})$, it should satisfy that $j_{t-1}j_t \in E^{i,s}$ or $j_t j_{t+1} \in E^{i,s}$. Thus $\sum_{s'\neq s} x(E^{i,s'}) - x(E^{i,s})$ does not change by the decrease of the value of x along with $(j_0, \ldots, j_\ell)$.

We have shown the algorithm always terminates in finite steps. For the output $f = (\mathcal{P}, \lambda)$, let $f(e) := \sum_{P\in\mathcal{P}: e\in P} \lambda(P)$ for $e \in E$, and let $f(i) := \sum_{P\in\mathcal{P}: i\in P} \lambda(P)$ for $i \in V$. Also let $\mathcal{P}_s \subseteq \mathcal{P}$ be the subset of paths connecting s to other terminals, and let $f_s = (\mathcal{P}_s, \lambda_s)$ for $s \in S$. Clearly, it holds that $f(e) \leq x(e) \leq u(e)$ for $e \in E$. For $i \in V_s$ $(s \in S)$, if a path $P \in \mathcal{P}$ goes through i, then P must be contained in $\mathcal{P}_s$. Thus by the former part of (A3), $f_s(i) = f(i) \leq x(E^{i,0})$ $(= x(E^{i,+})) \leq c(i)$. Also, $f_{s'}(i) \leq f_s(i) \leq c(i)$ for any $s' \neq s$. On the other hand, for $i \in V_0$, if a path in $\mathcal{P}_s$ $(s \in S)$ goes through i, then it must include an edge contained in $E^{i,s}$. Thus by the former part of (A4), we have $f_s(i) \leq x(E^{i,s}) \leq c(i)$. Therefore f is a separately-capacitated multiflow. Moreover, f_s satisfies the requirement r by the former part of (A5). Thus x is a feasible solution of FNTB.

We next show the optimality of x and p. First observe that when the algorithm terminates, all edges $e \in E$ satisfy $x(e) = 0$. In fact, if there exists an edge $e \in E$ with $x(e) > 0$, then we can construct an S-path with edges having positive x-values by repeating to apply the former parts of (A3–4). Thus $f(e) = x(e)$ $(e \in E)$ for the original input x. We see that

$$
\begin{aligned}
&\sum_{ij \in E} a_{ij} x_{ij} - \sum_{s \in S} r_s \operatorname{dist}(0, p_s) + \sum_{i \in V \setminus S} c_i \operatorname{size}(p_i) + \sum_{ij \in E} u_{ij} (\operatorname{dist}(p_i, p_j) - a_{ij})^+ \\
&= \sum_{ij \in E} (\operatorname{dist}(p_i, p_j) - a_{ij})^+ (u_{ij} - x_{ij}) + \sum_{ij \in E} (a_{ij} - \operatorname{dist}(p_i, p_j))^+ x_{ij} + \sum_{ij \in E} x_{ij} \operatorname{dist}(p_i, p_j) \\
&\quad + \sum_{i \in V \setminus S} c_i \operatorname{size}(p_i) - \sum_{s \in S} r_s \operatorname{dist}(0, p_s) \\
&= \sum_{ij \in E} (\operatorname{dist}(p_i, p_j) - a_{ij})^+ (u_{ij} - x_{ij}) + \sum_{ij \in E} (a_{ij} - \operatorname{dist}(p_i, p_j))^+ x_{ij} \\
&\quad + \sum_{s \in S} \sum_{i \in V_s} (c_i - f(i)) \operatorname{size}(p_i) + \sum_{i \in V_0} \sum_{s \in S} (c_i - f_s(i)) \operatorname{size}_s(p_i) + \sum_{s \in S} (f(s) - r_s) \operatorname{dist}(0, p_s), \qquad (9)
\end{aligned}
$$

where we use $a + (d-a)^+ = d + (a-d)^+$ for $a, d \in \mathbb{R}$ and

$$
\begin{aligned}
&\sum_{ij \in E} f(ij) \operatorname{dist}(p_i, p_j) + \sum_{s \in S} \sum_{i \in V_s} f(i) \operatorname{size}(p_i) + \sum_{i \in V_0} \sum_{s \in S} f_s(i) \operatorname{size}_s(p_i) \\
&= \sum_{ij \in E} \sum_{P \in \mathcal{P}, ij \in E(P)} \lambda(P) \operatorname{dist}(p_i, p_j) \\
&\qquad + \sum_{s \in S} \sum_{i \in V_s} \sum_{P \in \mathcal{P}, i \in V(P)} \lambda(P) \operatorname{size}(p_i) + \sum_{i \in V_0} \sum_{s \in S} \sum_{P \in \mathcal{P}_s, i \in V(P)} \lambda_s(P) \operatorname{size}_s(p_i) \\
&= \sum_{st} \sum_{P \in \mathcal{P} : P \text{ connects } st} \lambda(P) \operatorname{dist}(p_s, p_t) = \sum_{s \in S} f(s) \operatorname{dist}(0, p_s)
\end{aligned}
$$

by (8). We see $f(i) = x(E^{i,0})$ $(= x(E^{i,+}))$ for $i \in \bigcup_{s \in S} V_s$, and $f_s(i) = x(E^{i,s})$ for $i \in V_0$ and $s \in S$. Also $f(s) = x(\delta s)$ for $s \in S$. Then (9) is zero by (A1–2) and the latter parts of (A3–5). By Proposition 4, we conclude that x and p are both optimal. ◀

▶ Remark 6. Suppose the input edge-capacity x satisfies $x(\delta i) \in \mathbb{Z}_+$ for any $i \in V$. Then μ is always half-integral, and the integrality of $x(\delta i)$ is also kept in the execution of the algorithm. Thus the output multiflow is half-integer-valued. This argument will be used for proving a min-max theorem (Theorem 2) for a separately-capacitated multiflow later.

The decomposition algorithm is based on [11, Lemma 4.5]; see also [14, Lemma 3.3].

The existence of an edge-capacity x satisfying (A1–5) can be checked by solving the *undirected circulation problem*. This fact leads a simple descent algorithm for DTB and FNTB. Notice that a potential $p : V \to \mathbb{S}$ can be identified with a vector in $\mathbb{S}^n$. For brevity we write $p \in \mathbb{S}^n$ below. Let $h_a = h : \mathbb{S}^n \to \overline{\mathbb{R}}$ be a function defined by

$$
h(p) := - \sum_{s \in S} r_s \operatorname{dist}(0, p_s) + \sum_{i \in V \setminus S} c_i \operatorname{size}(p_i) + \sum_{ij \in E} u_{ij} (\operatorname{dist}(p_i, p_j) - a_{ij})^+ \qquad (10)
$$

if $p \in \mathbb{S}^n$ is a potential and $h(p) := \infty$ otherwise. Then DTB is precisely a minimization of h over $\mathbb{S}^n$. Consider the following algorithm DESCENT:

Algorithm 1 DESCENT.

0. Initialize $p \equiv 0$ (i.e., $p(i) = 0$ for any $i \in V$).
1. Check the sufficiency of the optimality of p by searching x satisfying (A1–5).
2. If x is found, then x and p are optimal; stop.
3. Otherwise find $q \in \mathbb{S}^n$ with $h(q) < h(p)$; update p by q and go to Step 1.

We give more details of DESCENT in Section 2.3. As for Step 1, we can also do Step 3 by the undirected circulation problem; q is computed by the certificate of the nonexistence of x. In the following subsections, we introduce the undirected circulation problem and discuss how to find x or q in each case.

2.2 Checking the Optimality

Let (U, F) be an undirected graph, and let $\underline{b} : F \to \underline{\mathbb{R}}$ and $\overline{b} : F \to \overline{\mathbb{R}}$ be lower and upper capacity functions satisfying $\underline{b}(e) \le \overline{b}(e)$ for each $e \in F$. The graph (U, F) may contain self-loops but no multiedges. The *circulation problem* on $((U, F), \underline{b}, \overline{b})$ is the problem of finding an edge-weight $y : F \to \mathbb{R}$ satisfying $\underline{b}(e) \le y(e) \le \overline{b}(e)$ for each $e \in F$ and $\sum_{ij \in F} y(ij) = 0$ for each $i \in U$. Such a y is called a *circulation*.

Let 3^U denote the set of pairs (Y, Z) of two subsets $Y, Z \subseteq U$ with $Y \cap Z = \emptyset$. For $(Y, Z) \in 3^U$, let $\chi_{Y,Z} := \sum_{i \in Y} \chi_i - \sum_{i \in Z} \chi_i \in \mathbb{R}^U$. Define the *cut function* $\kappa : 3^U \to \underline{\mathbb{R}}$ by

$$\kappa(Y, Z) := \sum_{ij \in F} \{(\chi_{Y,Z}(\{i, j\}))^+ \underline{b}(ij) - (\chi_{Z,Y}(\{i, j\}))^+ \overline{b}(ij)\} \quad ((Y, Z) \in 3^U).$$

It is well-known that the feasibility of the circulation problem is characterized via the cut function. We can show it by reducing to Hoffman's circulation theorem. A cut $(Y, Z) \in 3^U$ with $\kappa(Y, Z) > 0$ is called *violating*, and is called *maximum violating* if it attains the maximum $\kappa(Y, Z)$ among all violating cuts.

▶ **Lemma 7** (see, e.g., [16, Theorems 2.4, 2.7])**.** *Let $((U, F), \underline{b}, \overline{b})$ be an undirected network.*
(1) *The circulation problem is feasible if and only if $\kappa(Y, Z) \le 0$ for any $(Y, Z) \in 3^U$.*
(2) *If $\underline{b}$ and $\overline{b}$ are integer-valued, then there exists a feasible half-integer-valued circulation $y : E \to \mathbb{Z}^*_+$.*
(3) *Under the same assumption, we can obtain a feasible half-integer-valued circulation or a maximum violating cut in $O(\mathrm{MF}(|U|, |F|))$ time.*

Let us return to our problem. For a given proper potential $p \in \mathbb{S}^n$, the existence of $x : E \to \mathbb{R}_+$ satisfying (A1–5) reduces to the undirected circulation problem on the following network $\mathcal{N}_p := ((U, F), \underline{c}, \overline{c})$. See Figure 2 for the following construction.

For each $i \in \bigcup_{s \in S} V_s$, divide i into two nodes $U_i := \{i^0, i^+\}$, and connect nodes by an edge $i^0 i^+$. For representing (A3), let $\underline{c}(i^0 i^+) := -c_i$, and let $\overline{c}(i^0 i^+) := 0$ if $\mathrm{size}(p_i) = 0$ and $\overline{c}(i^0 i^+) := -c_i$ if $\mathrm{size}(p_i) > 0$. For each $i \in V_0$, divide i into $2k$ nodes $U_i := U_i^0 \cup U_i^+$, where $U_i^0 := \{i^{1,0}, i^{2,0}, \dots, i^{k,0}\}$ and $U_i^+ := \{i^{1,+}, i^{2,+}, \dots, i^{k,+}\}$, and connect them by edges $i^{s,0} i^{s,+}$ for $s \in S$ and $i^{s,0} i^{s',0}$ for distinct $s, s' \in S$. For representing (A4), let $\underline{c}(i^{s,0} i^{s,+}) := -c_i$, and let $\overline{c}(i^{s,0} i^{s,+}) := 0$ if $\mathrm{size}_s(p_i) = 0$ and $\overline{c}(i^{s,0} i^{s,+}) := -c_i$ if $\mathrm{size}_s(p_i) > 0$. Also let $\underline{c}(i^{s,0} i^{s',0}) := 0$ and $\overline{c}(i^{s,0} i^{s',0}) := \infty$. For each $s \in S$, let $s^0 := s$ and $U_s := \{s^0\}$, and add a self-loop $s^0 s^0$. For representing (A5), let $\underline{c}(s^0 s^0) := -\infty$ if $\mathrm{dist}(0, p_s) = 0$ and $\underline{c}(s^0 s^0) := -r_s$ if $\mathrm{dist}(0, p_s) > 0$, and let $\overline{c}(s^0 s^0) := -r_s$.

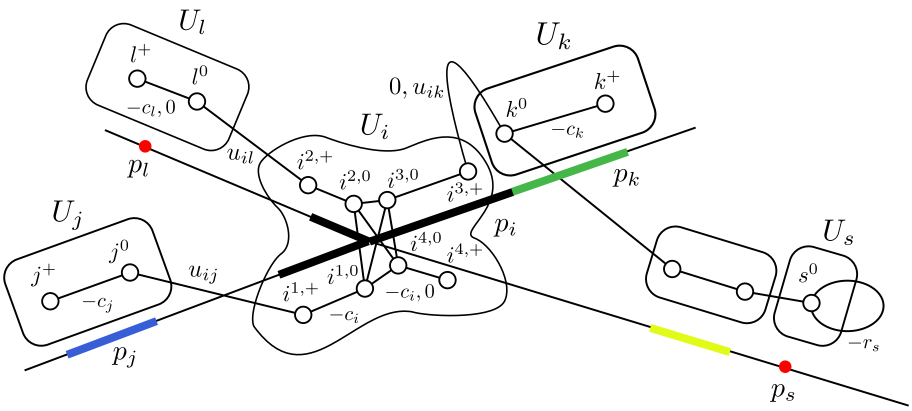

Figure 2 The undirected network $\mathcal{N}_p$.

For each edge $ij \in E$, if $\mathrm{dist}(p_i, p_j) < a_{ij}$, then $x_{ij} = 0$ by (A2). Thus we remove those edges. Let $E_>$ be the set of edges $ij \in E$ with $\mathrm{dist}(p_i, p_j) > a_{ij}$, and let $E_=$ be the set of edges $ij \in E$ with $\mathrm{dist}(p_i, p_j) = a_{ij}$. We replace endpoints of each edge $ij \in E_> \cup E_=$. If $i \in V_0$ and $j \in V_s$, then replace ij with $i^{s,+}j^0$. If $i \in V_s$ and $j \in V_{s'}$ $(s \neq s')$, then replace ij with i^0j^0. If $i, j \in V_s$ and p_i is closer to 0 than p_j, i.e., $\mathrm{dist}(0, p_i) < \mathrm{dist}(0, p_j)$, then replace ij with i^+j^0. We identify those replaced edges with the original edges. Let $\underline{c}(ij) := 0$ if $ij \in E_=$ and $\underline{c}(ij) := u_{ij}$ if $ij \in E_>$, and let $\overline{c}(ij) := u_{ij}$. U and F are defined as the union of all nodes and edges in the above, respectively.

▶ **Theorem 8.** *Let $\mathcal{N}_p = ((U, F), \underline{c}, \overline{c})$ be the undirected network constructed from a proper potential $p \in \mathbb{S}^n$. If it has a (half-integer-valued) circulation $y : F \to \mathbb{R}$, then an edge-capacity function $x : E \to \mathbb{R}_+$ defined by*

$$x(e) := \begin{cases} y(e) & \text{if } e \in E_> \cup E_=, \\ 0 & \text{otherwise } (e = ij \text{ with } \mathrm{dist}(p_i, p_j) < a_{ij}) \end{cases}$$

satisfies (A1–5).

Proof. We can obtain (A1–5) from definitions immediately. For example, the former part of (A4) follows from $x(E^{i,s}) = -y(i^{s,0}i^{s,+}) \leq -\underline{c}(i^{s,0}i^{s,+}) = c_i$ and

$$x(E^{i,s}) = -y(i^{s,0}i^{s,+}) = \sum_{s' \neq s} y(i^{s,0}i^{s',0}) \leq \sum_{s' \neq s} -y(i^{s',0}i^{s',+}) = \sum_{s' \neq s} x(E^{i,s'}),$$

and the latter part of (A4) follows from $-y(i^{s,0}i^{s,+}) \geq -\overline{c}(i^{s,0}i^{s,+}) = c_i$ for $i \in V_0$ and $s \in S$ with $\mathrm{size}_s(p_i) > 0$. ◀

2.3 Finding a Descent Direction

If the algorithm in Lemma 7 outputs a circulation in $\mathcal{N}_p$, then an optimal edge-capacity is computed from the circulation, and p is optimal by Lemma 5 and Theorem 8. Otherwise the algorithm outputs a maximum violating cut. We show that we can find $q \in \mathbb{S}^n$ with $h(q) < h(p)$ using the maximum violating cut. A basic idea is to modify each subtree p_i, according to the intersection pattern of the maximum violating cut with U_i, so that the objective function h decreases. This implies the necessity of Lemma 5 and the strong duality of Proposition 4.

We begin with introducing the notion of *basic moves* for a subtree. For an s-type subtree $T = [l, l']_s$, we denote its endpoints by $v_0(T) := (l, s) \in \mathbb{T}$ and $v_+(T) := (l', s) \in \mathbb{T}$. When we remove T from $\mathbb{T}$, there appear two connected components. Let T'_0 be the component containing $0\,(\in \mathbb{T})$, and T'_+ be the other. We can expand the subtree T by adding a node next to T. There are two nodes next to T, one is contained in T'_0 and the other is contained in T'_+. The 0-*expansion* is the operation to add that node contained in $T_{i,0}$ to T, and the $+$-*expansion* is the operation to add that node contained in $T_{i,+}$ to T. If T satisfies $\mathrm{size}(T) > 0$, then we can shrink T by removing $v_0(T)$ or $v_+(T)$ from T. The 0-*shrinkage* is the operation to remove $v_0(T)$ from T, and the $+$-*shrinkage* is the operation to remove $v_+(T)$ from T.

For a 0-type subtree $T = [l_s]_{s \in S}$, we denote its endpoints by $v_s(T) := (l_s, s) \in \mathbb{T}$ for $s \in S$. When we remove T from $\mathbb{T}$, there appear k $(= |S|)$ connected components. For $s \in S$, let T'_s be the component which is contained in P_s. As above, we can expand the subtree T by adding a node next to T. There are k nodes next to T, and each T'_s $(s \in S)$ contains exactly one such a node. The $(s, +)$-*expansion* for $s \in S$ is the operation to add that node contained in T'_s to T. If T satisfies $\mathrm{size}_s(T) > 0$ for $s \in S$, then we can shrink T by removing $v_s(T)$ from T. The $(s, +)$-*shrinkage* for $s \in S$ with $\mathrm{size}_s(T) > 0$ is the operation to remove $v_s(T)$ from T. For $s \in S$, if $\mathrm{size}_{s'}(T) = 0$ for any other $s' \in S$, then we can shrink T by removing $0\,(\in \mathbb{T})$ from T. The $(s, 0)$-*shrinkage* for such $s \in S$ is the operation to remove 0 from T. We call these expansion and shrinkages *basic moves*.

Let $(Y, Z) \in 3^U$ be a cut. From (Y, Z), the modification $p^{Y,Z}$ of p is defined as follows. For $s \in S$, do:

- If $s^0 \in Y$, then 0-expand and $+$-shrink p_s.
- If $s^0 \in Z$, then $+$-expand and 0-shrink p_s.

For $i \in \bigcup_{s \in S} V_s$, do:

- If $i^0 \in Y$, then 0-expand p_i. If $i^0 \in Z$, then 0-shrink p_i.
- If $i^+ \in Y$, then $+$-expand p_i. If $i^+ \in Z$, then $+$-shrink p_i.

For $i \in V_0$, do:

- If $U_i^0 \cap (Y \cup Z) = \emptyset$, then we do the following for each $s \in S$:
  - If $i^{s,+} \in Y$, then $(s, +)$-expand p_i. If $i^{s,+} \in Z$, then $(s, +)$-shrink p_i.
- If $i^{s,0} \in Z$ for some $s \in S$, then $(s, 0)$-shrink p_i. Also do the following:
  - If $i^{s,+} \in Y$, then $(s, +)$-expand p_i. If $i^{s,+} \in Z$, then $(s, +)$-shrink p_i.

There may exists $i \in V$ that such a move cannot be defined, e.g., $i \in \bigcup_{s \in S} V_s$ with $\mathrm{size}(p_i) \leq 1/2$ and $\{i^0, i^+\} \subseteq Z$, or $j \in V_0$ with $\{j^{s,0}, j^{s',0}\} \subseteq Z$. If the moves can be defined for all $i \in V$, then the cut (Y, Z) is called *movable*. For a movable cut $(Y, Z) \in 3^U$, we denote the modified potential by $p^{Y,Z}$.

We cal a node $(l, s) \in \mathbb{T}$ *even* if the number of edges between (l, s) and 0 is even, and *odd* otherwise. A basic move is said to be *upward* if the added node is even or the removed node is odd. A basic move is said to be *downward* if the added node is odd or the removed node is even. A movable cut $(Y, Z) \in 3^U$ is *upward-movable* (resp. *downward-movable*) if all basic moves occurring in the modification from p to $p^{Y,Z}$ are basic upward moves (resp. basic downward moves). Let denote the sets of all upward-movable cuts and downward-movable cuts by $\mathcal{M}^{\uparrow}$ and $\mathcal{M}^{\downarrow}$, respectively.

▶ **Lemma 9.** *For $(Y, Z) \in \mathcal{M}^{\uparrow} \cup \mathcal{M}^{\downarrow}$, it holds $h(p^{Y,Z}) - h(p) = -\kappa(Y, Z)/2$.*

Thus we are motivated to obtain an upward- or downward-movable cut (Y, Z) with a positive $\kappa(Y, Z)$ value. The following lemma says that we can do this efficiently given a maximum violating cut.

▶ **Lemma 10.** *Given a maximum violating cut, we can obtain an upward-movable cut* $(Y, Z) \in \mathcal{M}^{\uparrow}$ *and a downward-movable cut* $(Y', Z') \in \mathcal{M}^{\downarrow}$ *satisfying*

$$\kappa(Y, Z) = \max_{(Y'',Z'')\in\mathcal{M}^{\uparrow}} \kappa(Y'', Z''), \quad \kappa(Y', Z') = \max_{(Y'',Z'')\in\mathcal{M}^{\downarrow}} \kappa(Y'', Z'') \tag{11}$$

in $O(kn)$ *time. Moreover, at least one of* $\kappa(Y, Z)$ *and* $\kappa(Y', Z')$ *is positive.*

▶ **Theorem 11.** *Let* $\mathcal{N}_p := ((U, F), \underline{c}, \overline{c})$ *be the undirected network constructed from a proper potential* $p \in \mathbb{S}^n$*. Suppose that the instance is infeasible. Given a maximum violating cut, we can obtain a proper potential* $q \in \mathbb{S}^n$ *with* $h(q) < h(p)$ *in* $O(kn)$ *time.*

Proof. By Lemma 10, we can obtain an upward-movable cut $(Y, Z) \in \mathcal{M}^{\uparrow}$ and a downward-movable cut $(Y', Z') \in \mathcal{M}^{\downarrow}$ satisfying (11) in $O(kn)$ time. Let (Y'', Z'') be the cut that attains maximum κ-value among $\{(Y, Z), (Y', Z')\}$, and let $q := p^{Y'',Z''}$. Then $h(q) < h(p)$ by Lemmas 9 and 10. We can make q proper by the procedure given in the first part of the proof of Proposition 4. ◀

Now we are ready to present the details of DESCENT. First construct $\mathcal{N}_p$ from the current proper potential $p \in \mathbb{S}^n$, and run the algorithm given in Lemma 7 to solve the circulation problem; this corresponds to Step 1 given in the procedure at the end of Section 2.1. If a feasible half-integer-valued circulation is obtained, then a half-integral optimal edge-capacity x is computed by Theorem 8; this corresponds to Step 2. Otherwise a maximum violating cut is obtained, and then a proper potential $q \in \mathbb{S}^n$ with $h(q) < h(p)$ is computed by Theorem 11; this corresponds to Step 3. One iteration of this algorithm can be done in $O(\mathrm{MF}(kn, m + k^2n))$ time.

The value $-h(p)$ is at most mUA (by Proposition 4) and $-h(p) \in \mathbb{Z}^*_+$. Thus the number of iterations is at most $O(mUA)$. Actually, this analysis of the time complexity is not tight. In fact, the number of iterations can be evaluated as $O(nA)$.

If a potential $q \in \mathbb{S}^n$ is obtained from a potential p by a modification defined by a movable cut on $\mathcal{N}_p$, then we say that q is a *neighbor* of p, that is, there exists a movable cut $(Y', Z') \in 3^U$ such that $q = p^{Y',Z'}$. For $p, q \in \mathbb{S}^n$, define a distance $\tilde{d}_{\mathbb{S}^n}(p, q)$ by the minimum length of a sequence $(p = p_0, p_1, \ldots, p_\ell = q)$ such that p_t is a neighbor of p_{t-1} for all $t = 1, \ldots, \ell$. Let $\mathrm{opt}(h)$ denote the set of minimizers of h, and let $\tilde{d}_{\mathbb{S}^n}(p, \mathrm{opt}(h)) := \min_{q\in\mathrm{opt}(h)} \tilde{d}_{\mathbb{S}^n}(p, q)$.

▶ **Lemma 12.** *Starting with an initial potential* $p_0 \in \mathbb{S}^n$*, DESCENT finds an optimal potential at most* $\tilde{d}_{\mathbb{S}^n}(p_0, \mathrm{opt}(h)) + 2$ *iterations.*

Lemma 12 can be shown by using *DCA beyond* $\mathbb{Z}^n$. We will discuss it in Section 3.

▶ **Lemma 13.** *There exists an optimal potential* $p \in \mathrm{opt}(h)$ *satisfying that for any* $i \in V$*,* p_i *is contained in* $(2nA, 2nA, \ldots, 2nA) \in \mathbb{S}$.

▶ **Theorem 14.** *DESCENT solves FNTB in* $O(nA \cdot \mathrm{MF}(kn, m + k^2n))$ *time.*

Proof. We can only consider the potentials satisfying the condition in Lemma 13. Any pair of such potentials $p, q \in \mathbb{S}$ satisfies $\tilde{d}_{\mathbb{S}^n}(p, q) = O(nA)$. Then the statement follows from Lemma 12. ◀

We note that Theorem 14 is shown under the positivity assumption of the edge-cost a. We prove Theorem 2 using Theorem 14.

Proof of Theorem 2. Let $f = (\mathcal{P}, \lambda)$ be a separately-capacitated multiflow. Recall that $f_s = (\mathcal{P}_s, \lambda|_{\mathcal{P}_s})$, where $\mathcal{P}_s \subseteq \mathcal{P}$ is a subset of paths connecting s to other terminals. Let $\operatorname{val} f := \sum_{P \in \mathcal{P}} \lambda(P)$ and $\operatorname{val} f_s := \sum_{P \in \mathcal{P}_s} \lambda(P)$ for $s \in S$. Then $\operatorname{val} f_s$ is at most the capacity of any $\{s\}$–$(S \setminus \{s\})$ cut. Thus $\operatorname{val} f = (1/2) \sum_{s \in S} \operatorname{val} f_s \leq (1/2) \sum_{s \in S} \nu_s$.

Consider an instance $((V, E), S, u, c, a, r)$ of FNTB, where $a \equiv 1$ and $r_s := \nu_s$ for each $s \in S$. Since u clearly satisfies (1), this instance is feasible. Then DESCENT outputs a half-integral optimal edge-capacity x and an optimal potential p. Since x and p satisfy the conditions (A1–5), we can apply the decomposition algorithm in the proof of Lemma 5 for x, and obtain a separately-capacitated multiflow f. Then $\operatorname{val} f = (1/2) \sum_{s \in S} f(s) \geq (1/2) \sum_{s \in S} r_s = (1/2) \sum_{s \in S} \nu_s$. Moreover, since x comes from a half-integral circulation (Theorem 8), x satisfies $x(\delta i) \in \mathbb{Z}_+$ for any $i \in V \setminus S$. In fact, for $i \in \bigcup_{s \in S} V_s$, it is observed from $x(\delta i) = -2y(i^0 i^+)$, and for $i \in V_0$, it is observed from $x(\delta i) = \sum_{s \in S} -y(i^{s,0} i^{s,+}) = 2 \sum_{s < s'} y(i^{s,0}, i^{s',0})$. Then by Remark 6, the decomposition algorithm outputs a half-integer-valued multiflow.

The time complexity result follows from that FNTB can be solved in $O(n \cdot \mathrm{MF}(kn, m + k^2 n))$ time by Theorem 14, and the decomposition algorithm runs in $O((m + kn)n)$ time. ◀

2.4 Scaling Algorithm

The time complexity of DESCENT is pseudo-polynomial. We improve it by combining with a (cost-)scaling method.

Let $\gamma \in \mathbb{Z}_+$ be an integer such that $2^\gamma \geq A$. The scaling algorithm consists of $\gamma + 1$ phases. In t-th phase, solve DTB with an edge-cost $a_t : E \to \mathbb{Z}_+$ defined by $a_t(e) := \lceil a(e)/2^t \rceil$ $(e \in E)$, i.e., minimize h_{a_t}. (Recall h_a is defined by (10).) Here $\lceil \cdot \rceil$ is the round-up operator. Note that all $a_t(e)$ are positive. Begin with $t = \mu$, and decrease t one-by-one. Then, when $t = 0$, the problem coincides with the original DTB. In each t-phase, we use DESCENT to minimize h_{a_t}. At the initial phase $t = \mu$, we run DESCENT with the starting point $p_0 \in \mathbb{S}^n$, where $(p_0)_i = 0$ for all $i \in V$. For t-phase with $t \leq \mu - 1$, the starting point is determined from the obtained optimal potential in the previous phase. Let $2[l, l']_s := [2l, 2l']_s$ and $2[l_s]_{s \in S} := [2l_s]_{s \in S}$. For a potential $p \in \mathbb{S}^n$, define a new potential $2p \in \mathbb{S}^n$ by $(2p)_i := 2p_i$ for $i \in V$.

▶ **Lemma 15.** *Let $p \in \mathbb{S}^n$ be an optimal potential for t-phase ($t = 1, \ldots, \mu$). Then the potential $2p \in \mathbb{S}^n$ is optimal for DTB with an edge-cost $2a_t$.*

Proof. By the strong duality of Proposition 4, there exists a solution $x : E \to \mathbb{R}$ for FNTB, such that $\sum_{e \in E} a_t(e) x(e) = -h_{a_t}(p)$. Then $\sum_{e \in E} 2a_t(e) x(e) = -h_{2a_t}(2p)$ holds, which implies the optimality of $2p$ by (the weak duality of) Proposition 4. ◀

Observe that $a_{t-1} = 2a_t - \sum_{e \in F} \chi_e$, where $F := \{e \in E \mid a_{t-1}(e) \text{ is odd}\}$. The key property is the following sensitivity result.

▶ **Lemma 16.** *Let $a : E \to \mathbb{Z}_+$ be a positive edge-cost. Let $e \in E$ be an edge satisfying $a(e) \geq 2$, and $a' := a - \chi_e$. Let $p \in \operatorname{opt}(h_a)$. Then $\tilde{d}_{\mathbb{S}^n}(p, \operatorname{opt}(h_{a'})) \leq 2$.*

We prove Lemma 16 in Section 3.3 using the notion of discrete convexity.

Proof of Theorem 1. For the initial phase $t = \mu$, an optimal potential can be obtained in $O(n)$ iterations of DESCENT by Lemmas 12 and 13. For each remaining phase, an optimal potential can be obtained in $O(m)$ iterations of DESCENT by Lemmas 12, 15 and 16. Thus $O(n + m \log A) = O(m \log A)$ iterations of DESCENT are sufficient. Recall that we assume the positivity of the edge-cost a. When a is not positive, the perturbation (Remark 3) is needed. Thus the maximum of edge-costs is $O(mUA)$. Then the theorem follows. ◀

3 Discrete Convex Analysis for Node-Connectivity Terminal Backup

The theory of DCA beyond $\mathbb{Z}^n$ gives an algorithm, called the steepest descent algorithm (SDA), for minimizing L-convex functions on certain graph structures. We first introduce the L-convexity and SDA, and next show that DESCENT is precisely SDA for an L-convex function. Then Lemma 12 immediately follows. Finally, we discuss a sensitivity argument, which shows Lemma 16.

3.1 A General Theory

In this subsection, we briefly introduce a theory of discrete convexity on graph structures specialized to median graphs. See [15] for further details.

We use basic terminologies of poset and lattice. Let $\mathcal{L}$ be a poset (partially ordered set) with a partial order $\preceq$. The *principal filter* $\mathcal{F}_x$ and the *principal ideal* $\mathcal{I}_x$ of $x \in \mathcal{L}$ are defined as $\{y \in \mathcal{L} \mid y \succeq x\}$ and $\{y \in \mathcal{L} \mid y \preceq x\}$, respectively. For $x, y \in \mathcal{L}$ with $x \preceq y$, the *interval* $[x, y]$ is defined as the set of $z \in \mathcal{L}$ satisfying $x \preceq z \preceq y$. We consider a (meet-)semilattice having the minimum element. A *median semilattice* $\mathcal{L}$ is a semilattice that every principal ideal is a distributive lattice and for any $x, y, z \in \mathcal{L}$, the join $x \vee y \vee z$ exists if $x \vee y$, $y \vee z$, and $z \vee x$ exist. A *Boolean semilattice* is a median semilattice that every principal ideal is a Boolean lattice.

Let G be a (possibly infinite) undirected graph. We denote the set of nodes also by G. Let $d = d_G$ be the shortest path metric on G. The *(metric) interval* $I(u, v)$ of $u, v \in G$ is the set of $w \in G$ satisfying $d(u, v) = d(u, w) + d(w, v)$. A *median* graph G is a graph that for any $u, v, w \in G$, $I(u, v) \cap I(v, w) \cap I(w, u)$ is a singleton.

We consider an *orientation* on edges of a median graph G, that takes $u \searrow v$ or $u \swarrow v$ on each edge uv. An orientation is *admissible* if for any 4-cycle (u_1, u_2, u_3, u_4), $u_1 \searrow u_2$ implies $u_4 \searrow u_3$. It is known [13, Lemma 2.4] that an admissible orientation on a median graph is acyclic. Thus we can define a poset on G by the admissible orientation, i.e., if an edge uv is oriented as $u \swarrow v$, then $u \preceq v$. G with an admissible orientation is *well-oriented* if $[u, v]$ is a Boolean lattice for any u, v with $u \preceq v$. In a well-oriented median graph G, it is known [15, Proposition 2] that every principal filter of G is a Boolean semilattice, and every principal ideal of G is a Boolean semilattice with the reversed order.

We can define an L-convex function on a well-oriented median graph G. For a function $f : G \to \overline{\mathbb{R}}$, define the *effective domain* of f as $\{u \in G \mid f(u) < \infty\}$ and denote by $\operatorname{dom} f$. If a sequence of nodes $(u = u_0, u_1, \ldots, u_\ell = v)$ satisfies that for any $i = 1, \ldots, \ell$, there exist $u', v' \in G$ with $u' \preceq v'$ such that $\{u_{i-1}, u_i\} \subseteq [u', v']$, then the sequence is said to be a Δ-*path* connecting u and v. A subset $X \subseteq G$ is Δ-*connected* if for any $u, v \in X$, there exists a Δ-path in X connecting u and v. A function $f : G \to \overline{\mathbb{R}}$ is called *L-convex* if $\operatorname{dom} f$ is Δ-connected and the restrictions of f to every principal filter and ideal are submodular. Here the *submodularity* on a median semilattice is a rather complicated notion; we give a formal definition in the full version.

The global optimality of an L-convex function f can be characterized by a *local* condition; $u \in \operatorname{dom} f$ is a minimizer of f if and only if u is a minimizer of f restricted to $\mathcal{F}_u \cup \mathcal{I}_u$. This induces a natural minimization algorithm, called the *steepest descent algorithm* (SDA):

Algorithm 2 SDA.

---

0. Initialize $u \in G$ with $f(u) < \infty$.
1. Find a local minimizer $v \in \mathcal{F}_u \cup \mathcal{I}_u$ of f.
2. If $f(v) = f(u)$, then stop; output u. Otherwise update u by v and go to Step 1.

The number of iterations of SDA is bounded by the Δ-distance from the initial point u and minimizers of f. Here the Δ-*distance* $d^{\Delta}(u, v)$ of $u, v \in G$ is the minimum length of a Δ-path connecting u to v. Let $\mathrm{opt}(f)$ denote the set of minimizers of f, and let $d^{\Delta}(u, \mathrm{opt}(f)) := \min_{v \in \mathrm{opt}(f)} d^{\Delta}(u, v)$.

▶ **Theorem 17** ([15, Theorem 4.3]). *The number of iterations of SDA with the initial point $u \in G$ is at most $d^{\Delta}(u, \mathrm{opt}(f)) + 2$.*

3.2 Discrete Convexity in Node-Connectivity Terminal Backup

We show that the dual objective function h defined in (10) is actually an L-convex function, and the algorithm DESCENT is precisely SDA. Define a graph on $\mathbb{S}$ by connecting two nodes (subtrees) $T, T' \in \mathbb{S}$ such that T and T' can transform to each other by a basic move. If we can move T to T' by a basic downward-move (equivalently, we can move T' to T by a basic upward-move), we give an orientation $T \searrow T'$. The graph $\mathbb{S}$ is a median graph, but not well-oriented. To make the graph well-oriented, we add a virtual subtree connecting to nodes (l, s) and $(l + 1/2, s)$ for each $l \in \mathbb{Z}_+^*$ and $s \in S$. We denote such a virtual subtree by $[l + 1/2, l]_s$. Give a natural orientation to each added edge. Let $\overline{\mathbb{S}} := \mathbb{S} \cup \{[l + 1/2, l]_s \mid l \in \mathbb{Z}_+^*, s \in S\}$. Extend h to be a function on $\overline{\mathbb{S}}^n$ by $h(p) := \infty$ if there exists $i \in V$ such that $p_i \in \overline{\mathbb{S}} \setminus \mathbb{S}$.

▶ **Proposition 18.**
(1) *$\overline{\mathbb{S}}$ is a well-oriented median graph, and so is $\overline{\mathbb{S}}^n$.*
(2) *h is an L-convex function on $\overline{\mathbb{S}}^n$.*
(3) *For $p, q \in \mathbb{S}^n$, $\tilde{d}_{\mathbb{S}^n}(p, q) = d^{\Delta}(p, q)$.*
(4) *The map $(Y, Z) \mapsto p^{Y,Z}$ is a bijection between $\mathcal{M}^{\uparrow}$ and $\mathcal{F}_p \cap \mathrm{dom}\, h$, and $\mathcal{M}^{\downarrow}$ and $\mathcal{I}_p \cap \mathrm{dom}\, h$.*

Proof of Lemma 12. By Lemma 9 and Proposition 18 (4), the cuts (Y, Z) and (Y', Z') in Lemma 10 are minimizers of h on $\mathcal{F}_p$ and $\mathcal{I}_p$, respectively. Therefore DESCENT is precisely SDA for h. Thus the number of iterations can be evaluated by Theorem 17, and the statement follows from Proposition 18 (3). ◀

3.3 Sensitivity

To prove Lemma 16, we transform the instance $((V, E), S, u, c, a, r)$ of FNTB to an edge-uncapacitated one by a standard technique: Divide each edge $e \in E$ into two edges e_1, e_2, and add a new node v_e into the middle of these two edges. Let the edge-costs of e_1 and e_2 be the same as the original edge-cost of e, and let the edge-capacities of e_1 and e_2 be ∞. Let the node-capacity of the added node be $u(e)$. The number of vertices in the new instance is $|V| + |E| = n + m$, and the number of edges is $2|E| = 2m$. We denote the new instance by $((\bar{V}, \bar{E}), S, \bar{u}, \bar{c}, \bar{a}, r)$.

We consider the dual problem DTB for the edge-uncapacitated instance. In this case, we say that $\bar{p} \in \mathbb{S}^{n+m}$ is a *potential* for an edge-cost $\bar{a}$ if it satisfies (4) and $\mathrm{dist}(\bar{p}_i, \bar{p}_j) \leq \bar{a}_{ij}$ for any $ij \in \bar{E}$. Then DTB is a minimization of a function $h_a : \mathbb{S}^n \to \overline{\mathbb{R}}$ defined by

$$\bar{h}_{\bar{a}}(\bar{p}) := -\sum_{s \in S} r_s \,\mathrm{dist}(0, \bar{p}_s) + \sum_{i \in \bar{V} \setminus S} \bar{c}_i \,\mathrm{size}(\bar{p}_i) \tag{12}$$

if $\bar{p}$ is a potential for $\bar{a}$ and $\bar{h}_{\bar{a}}(\bar{p}) := \infty$ otherwise.

Let $p \in \mathbb{S}^n$ be a potential for the original instance. We can extend p to a potential $\bar{p}$ for the edge-uncapacitated instance as follows: For $v = i \in V$, let $\bar{p}_v := 2p_i$. For $v = v_{ij}$ ($ij \in E$), we have two cases $\mathrm{dist}(p_i, p_j) \leq a_{ij}$ and $\mathrm{dist}(p_i, p_j) > a_{ij}$. For the former case, let $\bar{p}_v$ be any point in $\mathbb{T}$ (i.e., $\mathrm{size}(\bar{p}_v) = 0$) satisfying $\mathrm{dist}(\bar{p}_i, \bar{p}_v) \leq a_{ij}$ and $\mathrm{dist}(\bar{p}_v, \bar{p}_j) \leq a_{ij}$. For the latter case, let $\bar{p}_v$ satisfy $\mathrm{dist}(\bar{p}_i, \bar{p}_v) = a_{ij}$, $\mathrm{dist}(\bar{p}_v, \bar{p}_j) = a_{ij}$ and $\mathrm{size}(\bar{p}_v) = 2(\mathrm{dist}(\bar{p}_v, \bar{p}_j) - a_{ij}) > 0$.

▶ **Proposition 19.** *Let $p \in \mathbb{S}^n$ be an optimal potential for the original instance. Then the extended potential $\bar{p} \in \mathbb{S}^{n+m}$ defined above is optimal for the edge-uncapacitated instance.*

We first show Lemma 16 for an edge-uncapacitated instance. For brevity, we assume that the original instance $((V, E), S, u, c, a, r)$ is already an edge-capacitated instance. By Proposition 18 (3), the following is equivalent to Lemma 16.

▶ **Lemma 20.** *Let $a : E \to \mathbb{Z}_+$ be a positive edge-cost. Let $ij \in E$ be an edge satisfying $a(ij) \geq 2$, and $a' := a - \chi_{ij}$. Then for any $p \in \mathrm{opt}(h_a)$, it holds $d^\Delta(p, \mathrm{opt}(h_{a'})) \leq 2$.*

We prove Lemma 20 via the notion of *normal* Δ*-paths*. Let G be an oriented median graph. For nodes $u, v \in G$ with $d^\Delta(u, v) = 1$, let $\langle\langle u, v\rangle\rangle$ be the minimum interval $[u', v']$ such that $\{u, v\} \subseteq [u', v']$. A Δ-path $(u = u_0, u_1, \ldots, u_\ell = v)$ is the *normal* Δ*-path* from u to v if for any $t = 1, \ldots, \ell - 1$ and any interval $[u', v']$ with $\{u_{t-1}, u_t\} \subseteq [u', v']$ it holds $[u', v'] \cap \langle\langle u_t, u_{t+1}\rangle\rangle = \{u_t\}$. The normal Δ-path from u to v is uniquely determined, and the length ℓ equals to ${d_G}^\Delta(u, v)$ [3, Theorem 6.24]. Let $u \to v$ denote u_1, and let $u \twoheadrightarrow v$ denote $u_{\ell-1}$. Also Let $u \to^t v$ denote u_t for $t = 0, \ldots, \ell$.

▶ **Lemma 21.** *Let $p, q \in \mathrm{dom}\, h_a$. Then*

$$h_a(p) + h_a(q) \geq h_a(p \to q) + h_a(q \to p), \tag{13}$$

$$h_a(p) + h_a(q) \geq h_a(p \twoheadrightarrow q) + h_a(q \twoheadrightarrow p). \tag{14}$$

▶ **Lemma 22.** *Let $p, q \in \overline{\mathbb{S}}^n$ and $i, j \in V$. Suppose that $\mathrm{dist}(q_i, q_j) < \mathrm{dist}((q \to p)_i, (q \to p)_j)$ and $\mathrm{dist}(q_i, q_j) < \mathrm{dist}((p \twoheadrightarrow q)_i, (p \twoheadrightarrow q)_j)$. Then for any $t = 1, \ldots, d^\Delta(p, q)$, it holds $\mathrm{dist}((p \to^t q)_i, (p \to^t q)_j) + 1/2 \leq \mathrm{dist}((p \to^{t-1} q)_i, (p \to^{t-1} q)_j)$.*

Proof of Lemma 20. If p is a potential for a', then $p \in \mathrm{opt}(h_{a'})$. Suppose that p is not a potential for a'. Take $q \in \mathrm{opt}(h_{a'})$ having the minimum Δ-distance from p. Then $q \in \mathrm{dom}\, h_a$. Thus by (13) and $p \in \mathrm{opt}(h_a)$, we have $h_a(q) \geq h_a(q \to p)$. If $(q \to p) \in \mathrm{dom}\, h_{a'}$, then $h_{a'}(q \to p) = h_a(q \to p) \leq h_a(q) = h_{a'}(q)$ and thus $(q \to p) \in \mathrm{opt}(h_{a'})$; a contradiction to the minimality of q. Hence $(q \to p) \notin \mathrm{dom}\, h_{a'}$, and $\mathrm{dist}((q \to p)_i, (q \to p)_j) \geq a'_{ij} + 1/2 > a'_{ij} \geq \mathrm{dist}(q_i, q_j)$ (by the half-integrality of $\mathrm{dist}(\cdot, \cdot)$). Similarly we have $\mathrm{dist}((p \twoheadrightarrow q)_i, (p \twoheadrightarrow q)_j) > \mathrm{dist}(q_i, q_j)$. Then we can apply Lemma 22 and obtain

$$\begin{aligned}\mathrm{dist}(p_i, p_j) &\geq \mathrm{dist}((p \to q)_i, (p \to q)_j) + 1/2 \\ &\geq \mathrm{dist}((p \to^2 q)_i, (p \to^2 q)_j) + 2/2 \\ &\geq \cdots \geq \mathrm{dist}((p \twoheadrightarrow q)_i, (p \twoheadrightarrow q)_j) + (d^\Delta(p, q) - 1)/2.\end{aligned}$$

By $\mathrm{dist}(p_i, p_j) \leq a_{ij}$ and $\mathrm{dist}((p \twoheadrightarrow q)_i, (p \twoheadrightarrow q)_j) \geq a'_{ij} + 1/2 = a_{ij} - 1/2$, we have

$$d^\Delta(p, q) \leq 1 + 2(\mathrm{dist}(p_i, p_j) - \mathrm{dist}((p \twoheadrightarrow q)_i, (p \twoheadrightarrow q)_j)) \leq 2.$$ ◀

We give a sketch of a proof of Lemma 16 for an edge-capacitated instance. First construct the edge-uncapacitated instance $((\bar{V}, \bar{E}), S, \bar{u}, \bar{c}, \bar{a}, r)$ as above. Then an optimal potential $\bar{p} \in \mathbb{S}^{n+m}$ is obtained from p by Proposition 19, and $e \in E$ is divided into two edges

$e_1, e_2 \in \bar{E}$. By Lemma 20 for e_1 and e_2, there exists an optimal potential $\bar{p}'$ for the edge-uncapacitated instance with $d^{\Delta}(\bar{p}, \bar{p}') \leq 4$. By halving $\bar{p}'$, a "quarter-integral" optimal potential $p' \in \mathrm{opt}(h_{a'})$ is obtained. Lemma 16 is then shown by rounding quarter-integral components to half-integral.

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