Roundtrip Spanners with $(2k-1)$ Stretch

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Abstract

A roundtrip spanner of a directed graph G is a subgraph of G preserving roundtrip distances approximately for all pairs of vertices. Despite extensive research, there is still a small stretch gap between roundtrip spanners in directed graphs and undirected graphs. For a directed graph with real edge weights in $[1, W]$, we first propose a new deterministic algorithm that constructs a roundtrip spanner with $(2k-1)$ stretch and $O(kn^{1+1/k}\log(nW))$ edges for every integer $k > 1$, then remove the dependence of size on W to give a roundtrip spanner with $(2k-1)$ stretch and $O(kn^{1+1/k}\log n)$ edges. While keeping the edge size small, our result improves the previous $2k+\epsilon$ stretch roundtrip spanners in directed graphs [Roditty, Thorup, Zwick'02; Zhu, Lam'18], and almost matches the undirected $(2k-1)$-spanner with $O(n^{1+1/k})$ edges [Althöfer et al. '93] when k is a constant, which is optimal under Erdös conjecture.

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1 Introduction

A t-spanner of a graph G is a subgraph of G in which the distance between every pair of vertices is at most t times their distance in G, where t is called the stretch of the spanner. Sparse spanner is an important choice to implicitly representing all-pair distances [19], and spanners also have application backgrounds in distributed systems (see [14]). For undirected graphs, $(2k-1)$-spanner with $O(n^{1+1/k})$ edges is proposed and conjectured to be optimal [2, 17]. However, directed graphs may not have sparse spanners with respect to the normal distance measure. For instance, in a bipartite graph with two sides U and V, if there is a directed edge from every vertex in U to every vertex in V, then removing any edge (u, v) in this graph will destroy the reachability from u to v, so its only spanner is itself, which has $O(n^2)$ edges. To circumvent this obstacle, one can approximate the optimal spanner in terms of edge size (e.g. in [9, 3]), or one can define directed spanners on different distance measures. This paper will study directed sparse spanners on roundtrip distances.

Roundtrip distance is a natural metric with good property. Cowen and Wagner [7, 8] first introduce it into directed spanners. Formally, roundtrip distance between vertices u, v in G is defined as $d_G(u \leftrightarrows v) = d_G(u \to v) + d_G(v \to u)$, where $d_G(u \to v)$ is the length of shortest path from u to v in G. For a directed graph $G = (V, E)$, a subgraph $G' = (V, E')$ $(E' \subseteq E)$ is called a t-roundtrip spanner of G if for all $u, v \in G$, $d_{G'}(u \leftrightarrows v) \le t \cdot d_G(u \leftrightarrows v)$, where t is called the stretch of the roundtrip spanner.

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In a directed graph $G=(V,E)$ $(n=|V|, m=|E|)$ with real edge weights in $[1,W]$, Roditty et al. [16] give a $(2k+\epsilon)$-spanner of $O(\min\{(k^2/\epsilon)n^{1+1/k}\log(nW), (k/\epsilon)^2 n^{1+1/k}(\log n)^{2-1/k}\})$ edges. Recently, Zhu and Lam [18] derandomize it and improve the size of the spanner to $O((k/\epsilon)n^{1+1/k}\log(nW))$ edges, while the stretch is also $2k+\epsilon$. We make a step further based on these works and reduce the stretch to $2k-1$. Formally, we state our main results in the following theorems.

▶ **Theorem 1.** *For any directed graph G with real edge weights in $[1,W]$ and integer $k \geq 1$, there exists a $(2k-1)$-roundtrip spanner of G with $O(kn^{1+1/k}\log(nW))$ edges, which can be constructed in $\tilde{O}(kmn\log W)$ time*[1]*.*

By a similar scaling method in [16], we can make the size of the spanner independent of the maximum edge weight W to obtain a $(2k-1)$-spanner with strongly subquadratic space.

▶ **Theorem 2.** *For any directed graph G with real edge weights in $[1,W]$ and integer $k \geq 1$, there exists a $(2k-1)$-roundtrip spanner of G with $O(kn^{1+1/k}\log n)$ edges, which can be constructed in $\tilde{O}(kmn\log W)$ time.*

Actually, our result almost matches the lower bound following girth conjecture. The girth conjecture, implicitly mentioned by Erdös [11], says that for any k, there exists a graph with n vertices and $\Omega(n^{1+1/k})$ edges whose girth (minimum cycle) is at least $2k+2$. This conjecture implies that no algorithm can construct a spanner of $O(n^{1+1/k})$ size and less than $2k-1$ stretch for all undirected graph with n vertices [17]. This lower bound also holds for roundtrip spanners on directed graphs.

Our approach is based on the scaling constructions of the $(2k+\epsilon)$-stretch roundtrip spanners in [16, 18]. To reduce the stretch, we construct inward and outward shortest path trees from vertices in a hitting set [1, 10] of size $O(n^{1/k})$, and carefully choose the order to process vertices in order to make the stretch exactly $2k-1$. To further make the size of the spanner strongly subquatratic, we use a similar approach as in [16] to contract small edges in every scale, and treat vertices with different radii of balls of size $n^{1-1/k}$ differently.

1.1 Related Works

The construction time in this paper is $\tilde{O}(kmn\log W)$. However, there exist roundtrip spanners with $o(mn)$ construction time but larger stretches. Pachoci et al. [13] proposes an algorithm which can construct $O(k\log n)$-roundtrip spanner with $O(n^{1+1/k}\log^2 n)$ edges. Its construction time is $O(mn^{1/k}\log^5 n)$, which breaks the cubic time barrier. Very recently, Chechik et al. [6] give an algorithm which constructs $O(k\log\log n)$-roundtrip spanners with $\tilde{O}(n^{1+1/k})$ edges in $\tilde{O}(m^{1+1/k})$ time.

For spanners defined with respect to normal directed distance, researchers aim to approximate the k-spanner with minimum number of edges. Dinitz and Krauthgamer [9] achieve $\tilde{O}(n^{2/3})$ approximation in terms of edge size, and Bermen et al. [3] improves the approximation ratio to $\tilde{O}(n^{1/2})$.

Another type of directed spanners is transitive-closure spanner, introduced by Bhattacharyya et al. [5]. In this setting the answer may not be a subgraph of G, but a subgraph of the transitive closure of G. In other words, selecting edges outside the graph is permitted. The tradeoff is between diameter (maximum distance) and edge size. One of Bhattacharyya et al.'s results is spanners with diameter k and $O((n\log n)^{1-1/k})$ approximation of optimal edge size [5], using a combination of linear programming rounding and sampling. Berman et al. [4] improves the approximation ratio to $O(n^{1-1/\lceil k/2\rceil}\log n)$. We refer to Raskhodnikova [15] as a review of transitive-closure spanners.

[1] $\tilde{O}(\cdot)$ hides $\log n$ factors.

1.2 Organization

In Section 2, the notations and basic concepts used in this paper will be discussed. In Section 3 we describe the construction of the $(2k-1)$-roundtrip spanner with $O(kn^{1+1/k}\log(nW))$ edges, thus proving Theorem 1. Then in Section 4 we improve the size of the spanner to $O(kn^{1+1/k}\log n)$ and still keep the stretch to $(2k-1)$, thus proving Theorem 2. The conclusion and further direction are discussed in Section 5.

2 Preliminaries

In this paper we consider a directed graph $G=(V,E)$ with non-negative real edge weights w where $w(e)\in[1,W]$ for all $e\in E$. Denote $G[U]$ to be the subgraph of G induced by $U\subseteq V$, i.e. $G[U]=(U,E\cap(U\times U))$. A roundtrip path between nodes u and v is a cycle (not necessarily simple) passing through u and v. The roundtrip distance between u and v is the minimum length of roundtrip paths between u and v. Denote $d_U(u\leftrightarrows v)$ to be the roundtrip distance between u and v in $G[U]$. (Sometimes we may also use $d_U(u\leftrightarrows v)$ to denote a roundtrip shortest path between u,v in $G[U]$.) It satisfies:

- For $u,v\in U$, $d_U(u\leftrightarrows u)=0$ and $d_U(u\leftrightarrows v)=d_U(v\leftrightarrows u)$.
- For $u,v\in U$, $d_U(u\leftrightarrows v)=d_U(u\to v)+d_U(v\to u)$.
- For $u,v,w\in U$, $d_U(u\leftrightarrows v)\le d_U(u\leftrightarrows w)+d_U(w\leftrightarrows v)$.

Here $d_U(u\to v)$ is the one-way distance from u to v in $G[U]$. We use $d(u\leftrightarrows v)$ to denote the roundtrip distance between u and v in the original graph $G=(V,E)$.

In G, a t-roundtrip spanner of G is a subgraph H of G on the same vertex set V such that the roundtrip distance between any pair of $u,v\in V$ in H is at most $t\cdot d(u\leftrightarrows v)$. t is called the *stretch* of the spanner.

For a subset of vertices $U\subseteq V$, given a center $u\in U$ and a radius R, define roundtrip ball $Ball_U(u,R)$ to be the set of vertices whose roundtrip distance on $G[U]$ to center u is strictly smaller than the radius R. Formally, $Ball_U(u,R)=\{v\in U: d_U(u\leftrightarrows v)<R\}$. Then the size of the ball, denoted by $|Ball_U(u,R)|$, is the number of vertices in it. Similarly we define $\overline{Ball}_U(u,R)=\{v\in U: d_U(u\leftrightarrows v)\le R\}$. Subroutine $\texttt{InOutTrees}(U,u,R)$ calculates the edge set of an inward and an outward shortest path tree centered at u spanning vertices in $Ball_U(u,R)$ on $G[U]$. (That is, the shortest path tree from u to all vertices in $Ball_U(u,R)$ and the shortest path tree from all vertices in $Ball_U(u,R)$ to u.) It is easy to see that the shortest path trees will not contain vertices outside $Ball_U(u,R)$:

▶ **Lemma 3.** *The inward and outward shortest path trees returned by* $\texttt{InOutTrees}(U,u,R)$ *only contain vertices in* $Ball_U(u,R)$*.*

Proof. For any $v\in Ball_U(u,R)$, let C be a cycle containing u and v such that the length of C is less than R. Then for any vertex $w\in C$, $d_U(u\leftrightarrows w)<R$, so w must be also in the trees returned by $\texttt{InOutTrees}(U,u,R)$. ◀

For all notations above, we can omit the subscript V when the roundtrip distance is considered in the original graph $G=(V,E)$. Our algorithm relies on the following well-known theorem to calculate hitting sets deterministically.

▶ **Theorem 4** (Cf. Aingworth et al. [1], Dor et al. [10])**.** *For universe V and its subsets $S_1,S_2,\dots,S_n$, if $|V|=n$ and the size of each S_i is greater than p, then there exists a hitting set $H\subseteq V$ intersecting all S_i, whose size $|H|\le(n\ln n)/p$, and such a set H can be found in $O(np)$ time deterministically.*

3 A $(2k-1)$-Roundtrip Spanner Algorithm

In this section we introduce our main algorithm constructing a $(2k-1)$-roundtrip spanner with $O(kn^{1+1/k}\log(nW))$ edges for any G. We may assume $k \geq 2$ in the following analysis, since the result is trivial for $k=1$.

Our approach combines the ideas of [16] and [18]. In [18], given a length L, we pick an arbitrary vertex u and find the smallest integer h such that $|\overline{Ball}(u,(h+1)L)| < n^{1/k}|\overline{Ball}(u,h\cdot L)|$, then we include the inward and outward shortest path tree centered at u spanning $\overline{Ball}(u,(h+1)L)$ and remove vertices in $\overline{Ball}(u,h\cdot L)$ from V. We can see that $h \leq k$, so the stretch is $2k$ for u,v with roundtrip distance L, and by a scaling approach the final stretch is $2k+\epsilon$. We observe that if $h=k-1$, $|\overline{Ball}(u,(k-1)L)| \geq n^{(k-1)/k}$, so by Theorem 4 we can preprocess the graph by choosing a hitting set H with size $O(n^{1/k}\log n)$ and construct inward and outward shortest path trees centered at all vertices in H, then we do not need to include the shortest path trees spanning $\overline{Ball}(u,k\cdot L)$. The stretch can then be decreased to $2k-1+\epsilon$. To make the stretch equal $2k-1$, instead of arbitrarily selecting u each time, we carefully define the order to select u.

3.1 Preprocessing

We first define a radius $R(u)$ for each vertex u. It is crucial for the processing order of vertices.

▶ **Definition 5.** *For all $u \in V$, we define $R(u)$ to be the maximum length R such that $|Ball(u,R)| < n^{1-1/k}$, that is, if we sort the vertices by their roundtrip distance to u in G by increasing order, $R(u)$ is the roundtrip distance from u to the $\lceil n^{1-1/k}\rceil$-th vertex.*

For any $u \in V$, $|\overline{Ball}(u,R(u))| \geq n^{1-1/k}$. By Theorem 4, we can find a hitting set H intersecting all sets in $\{\overline{Ball}(u,R(u)) : u \in V\}$, such that $|H| = O(n^{1/k}\log n)$. For all $t \in H$, we build an inward and an outward shortest path tree of G centered at t, and denote the set of edges of these trees by E_0 and include them in the final spanner. This step generates $O(n^{1+1/k}\log n)$ edges in total, and it is easy to obtain the following statement:

▶ **Lemma 6.** *For $u,v \in V$ such that $d(u \leftrightarrows v) \geq R(u)/(k-1)$, the roundtrip distance between u and v in the graph (V,E_0) is at most $(2k-1)d(u \leftrightarrows v)$.*

Proof. Find the vertex $t \in H$ such that $t \in \overline{Ball}(u,R(u))$, that is, $d(u \leftrightarrows t) \leq R(u)$. Then the inward and outward shortest path trees from t will include $d(u \leftrightarrows t)$ and $d(t \leftrightarrows v)$. By $R(u) \leq (k-1)d(u \leftrightarrows v)$, we have $d(u \leftrightarrows t) \leq (k-1)d(u \leftrightarrows v)$ and $d(t \leftrightarrows v) \leq d(t \leftrightarrows u) + d(u \leftrightarrows v) \leq k \cdot d(u \leftrightarrows v)$. So the roundtrip distance of u and v in E_0 is at most $d(u \leftrightarrows t) + d(t \leftrightarrows v) \leq (2k-1)d(u \leftrightarrows v)$. ◀

3.2 Approximating a Length Interval

Instead of approximating all roundtrip distances at once, we start with an easier subproblem of approximating all pairs of vertices whose roundtrip distances are within an interval $[L/(1+\epsilon), L)$. Parameter ϵ is a real number in $(0, 1/(2k-2)]$. The procedure $\texttt{Cover}(G,k,L,\epsilon)$ described in Algorithm 1 will return a set of edges which gives a $(2k-2)(1+\epsilon)$-approximation of roundtrip distance $d(u \leftrightarrows v)$ if $R(u)/(k-1) > d(u \leftrightarrows v)$, for $d(u \leftrightarrows v) \in [L/(1+\epsilon), L)$.

Note that in Algorithm 1, initially $U=V$ and the balls are considered in $G[U]=G$. In the end of every iteration we remove a ball from U, and the following balls are based on the roundtrip distances in $G[U]$. However, $R(u)$ does not need to change during the algorithm and can still be based on roundtrip distances in the original graph G. The analysis for the size of the returned set $\hat{E}$ and the stretch are as follows.

Algorithm 1 Cover$(G(V, E), k, L, \epsilon)$.

```
1: U ← V, Ê = ∅
2: while U ≠ ∅ do
3:     u ← arg max_{u∈U} R(u)
4:     step ← min{R(u)/(k − 1), L}
5:     h ← minimum positive integer satisfying |Ball_U(u, h · step)| < n^{h/k}
6:     Add InOutTrees(U, u, h · step) to Ê
7:     Remove \overline{Ball}_U(u, (h − 1)step) from U
8: end while
9: return Ê
```

▶ **Lemma 7.** *The returned edge set of* `Cover`(G, k, L, ϵ) *has* $O(n^{1+1/k})$ *size.*

Proof. When processing a vertex u, by the selection of h in line 5, $|Ball_U(u, h \cdot step)| < n^{h/k}$ and $|\overline{Ball}_U(u, (h-1)step)| \geq n^{(h-1)/k}$. When $h \geq 2$ it is because of h's minimality, and when $h = 1$ it is because $u \in \overline{Ball}_U(u, 0)$. So each time `InOutTrees` is called, the size of ball to build shortest path trees is no more than $n^{1/k}$ times the size of ball to remove. During an execution of `Cover`(G, k, L, ϵ), each vertex is removed once from U. Therefore the total number of edges added in $\hat{E}$ is $O(n^{1+1/k})$. ◀

We can also see that if the procedure `Cover`$(G[U], k, L, \epsilon)$ is run on a subgraph $G[U]$ induced on a subset $U \subseteq V$, then the size of $\hat{E}$ is bounded by $O(|U|n^{1/k})$. It is also easy to see that h is at most $k - 1$:

▶ **Lemma 8.** *The h selected at line 5 in* `Cover`(G, k, L, ϵ) *satisfies* $h \leq k - 1$.

Proof. In $G[U]$, the ball $Ball_U(u, (k-1)step)$ must have size no greater than $Ball(u, (k-1)step)$ since the distances in $G[U]$ cannot decrease while some vertices are removed. Since $|Ball(u, R(u))| < n^{1-1/k}$ and $step \leq R(u)/(k-1)$, we get $|Ball_U(u, (k-1)step)| \leq |Ball(u, (k-1)step)| < n^{1-1/k}$, thus $h \leq k - 1$. ◀

Next we analyze the roundtrip distance stretch in $\hat{E}$. Note that in order to make the final stretch $2k - 1$, for the roundtrip distance approximated by edges in $\hat{E}$ we can make the stretch $(2k-2)(1+\epsilon)$, but for the roundtrip distance approximated by E_0 we need to make the stretch at most $2k - 1$ as E_0 stays the same.

▶ **Lemma 9.** *For any pair of vertices u, v such that $d(u \leftrightarrows v) \in [L/(1+\epsilon), L)$, either* `Cover`$(G, k, L, \epsilon)$*'s returned edge set $\hat{E}$ can form a cycle passing through u, v with length at most $(2k-2)(1+\epsilon)d(u \leftrightarrows v)$, or $R(u) \leq (k-1)d(u \leftrightarrows v)$, in which case the E_0 built in Section 3.1 can form a detour cycle with length at most $(2k-1)d(u \leftrightarrows v)$ by Lemma 6.*

Proof. Consider any pair of vertices u, v with roundtrip distance $d = d(u \leftrightarrows v) \in [L/(1+\epsilon), L)$, and a shortest cycle P going through u, v with length d.

During `Cover`(G, k, L, ϵ), consider the vertices on P that are first removed from U. Suppose w is one of the first removed vertices, and w is removed as a member of $\overline{Ball}_{U_c}(c, (h_c - 1)step_c)$ centered at c. This is to say $d_{U_c}(c \leftrightarrows w) \leq (h_c - 1)step_c$.

Case 1: $step_c > d$. Then

$$d_{U_c}(c \leftrightarrows u) \leq d_{U_c}(c \leftrightarrows w) + d_{U_c}(w \leftrightarrows u) \leq (h_c - 1)step_c + d < h_c step_c,$$

and $u \in Ball_{U_c}(c, h_c step_c)$. The second inequality holds because U_c is the remaining vertex set before removing w, so by definition of w, all vertices on P are in U_c. Symmetrically

$v \in Ball_{U_c}(c, h_c step_c)$. `InOutTrees`$(U_c, c, h_c step_c)$ builds a detour cycle passing through u, v with length $< 2h_c step_c$. By Lemma 8, we have $h_c \le k-1$. Also $step_c \le L \le (1+\epsilon)d$, therefore we build a detour of length $< 2(k-1)step_c \le (2k-2)(1+\epsilon)d$ in $\hat{E}$.

Case 2: $step_c \le d$. Because $d < L$, this case can only occur when $step_c = R(c)/(k-1)$. Because c is chosen before u, $R(u) \le R(c) = (k-1)step_c \le (k-1)d$. By Lemma 6, E_0 can give a $(2k-1)$-approximation of d. ◀

3.3 Main Construction

Now we can proceed to prove the main theorem based on a scaling on lengths of the cycles from 1 to $2nW$.

▶ **Theorem 10.** *For any directed graph G with real edge weights in $[1, W]$, there exists a polynomial time constructible $(2k-1)$-roundtrip spanner of G with $O(kn^{1+1/k}\log(nW))$ edges.*

Proof. Note that the roundtrip distance between any pair of vertices must be in the range $[1, 2(n-1)W]$. First do the preprocessing in Section 3.1. Then divide the range of roundtrip distance $[1, 2nW)$ into intervals $[(1+\epsilon)^{p-1}, (1+\epsilon)^p)$, where $\epsilon = 1/(2k-2)$. Call `Cover`$(G, k, (1+\epsilon)^p, \epsilon)$ for $p = 0, \cdots, \lfloor \log_{1+\epsilon}(2nW) \rfloor + 1$, and merge all returned edges with E_0 to form a spanner.

First we prove that the edge size is $O(kn^{1+1/k}\log(nW))$. Preprocessing adds $O(n^{1+1/k} \cdot \log n)$ edges. `Cover`$(G, k, (1+\epsilon)^p, \epsilon)$ is called for $\log_{1+1/(2k-2)}(2nW) = O(k\log(nW))$ times. By Lemma 7, each call generates $O(n^{1+1/k})$ edges. So the total number of edges in the roundtrip spanner is $O(kn^{1+1/k}\log(nW))$.

Next we prove the stretch is $2k-1$. For any pair of vertices u, v with roundtrip distance d, let $p = \lfloor \log_{1+\epsilon} d \rfloor + 1$, then $d \in [(1+\epsilon)^{p-1}, (1+\epsilon)^p)$. By Lemma 9, either the returned edge set of `Cover`$(G, k, (1+\epsilon)^p, \epsilon)$ can form a detour cycle passing through u, v of length at most $(2k-2)(1+\epsilon)d = (2k-1)d$, or the edges in E_0 can form a detour cycle passing through u, v of length at most $(2k-1)d$.

In conclusion this algorithm can construct a $(2k-1)$-roundtrip spanner with $O(kn^{1+1/k} \cdot \log(nW))$ edges. ◀

3.4 Construction Time

The running time of the algorithm in the proof of Theorem 10 is $O(kn(m+n\log n)\log(nW))$. It is also easy to see that the algorithm is deterministic. Next we analyze construction time in detail.

In preprocessing, for any $u \in V$, $R(u)$ can be calculated by running Dijkstra searches with Fibonacci heap [12] starting at u, so calculating $R(\cdot)$ takes $O(n(m+n\log n))$ time. Finding H takes $O(n^{2-1/k})$ time by Theorem 4. Building E_0 takes $O(n^{1/k}\log n \cdot (m+n\log n))$ time.

A `Cover` call's while loop runs at most n times since each time at least one node is removed. In a loop, u can be found in $O(n)$ time, and all other operations regarding roundtrip balls can be done in $O(m+n\log n)$ time by Dijkstra searches starting at u on $G[U]$. Therefore a `Cover` call takes $O(n(m+n\log n))$ time.

`Cover` is called $O(k\log(nW))$ times. Combined with the preprocessing time, the total construction time is $O(kn(m+n\log n)\log(nW))$.

4 Removing the Dependence on W

In this section we prove Theorem 2. The size of the roundtrip spanner in Section 3 is dependent on the maximum edge weight W. In this section we remove the dependence by designing the scaling approach more carefully. Our idea is similar to that in [16]. When we consider the roundtrip distances in the range $[L/(1+\epsilon), L)$, all cycles with length $\leq L/n^3$ have little effect so we can contract them into one node, and all edges with length $> (2k-1)L$ cannot be in any $(2k-1)L$ detour cycles, so they can be deleted. Thus, an edge with length l can only be in $O(\log_{1+\epsilon} n)$ iterations for L between $l/(2k-1)$ and $l \cdot n^3$ (based on the girth of this edge). However, the stretch will be a little longer if we directly apply the algorithm in Section 3 on the contracted graph.

To overcome this obstacle, we only apply the vertex contraction when $R(u)$ is large (larger than $2(k-1)L$). By making the "step" a little larger than L and ϵ smaller, when $d < L < step$, the stretch is still bounded by $(2k-1)$. When $R(u) \leq 2(k-1)L$, we first delete all node v with $R(v) < L/8$, then simply apply the algorithm in Section 3 in the original graph. Since every node u can only be in the second part when $R(u)/2(k-1) \leq L \leq 8R(u)$, the number of edges added in the second part is also strongly polynomial.

First we define the girth of an edge:

▶ **Definition 11.** *We define the* girth of an edge e *in* G *to be the length of shortest directed cycle containing* e*, and denote it by* $g(e)$*.*

It is easy to see that for $e = (u, v)$, $d(u \leftrightarrows v) \leq g(e)$. In $O(n(m + n \log n))$ time we can compute $g(e)$ for all edges e in G [12].

Algorithm 2 approximates roundtrip distance $d(u \leftrightarrows v) \in [L/(1+\epsilon), L)$. In the p-th iteration of the algorithm, $G_p[U_p]$ is always the subgraph contracted from the subgraph $G[U]$. Given $v_p \in U_p$, let $C(v_p)$ be the set of vertices in U that are contracted into v_p. We can see the second part of this algorithm (after line 12) is the same as Algorithm 1 in Section 3.

For the contracted subgraph $G_p[U_p]$, we give new definitions for balls and `InOutTrees`. Given two vertices $u_p, v_p \in U_p$, define

$$\hat{d}_{U_p}(u_p, v_p) = \min_{u \in C(u_p), v \in C(v_p)} d_U(u, v)$$

Balls in $G_p[U_p]$ are defined as follows.

$$Ball_{U_p}(u_p, r) = \{v_p \in U_p : \hat{d}_{U_p}(u_p, v_p) < r\}$$

$$\overline{Ball}_{U_p}(u_p, r) = \{v_p \in U_p : \hat{d}_{U_p}(u_p, v_p) \leq r\}$$

In Line 9, `NewInOutTrees`$(U_p, u_p, h \cdot step)$ is formed by only keeping the edges between different contracted vertices in `InOutTrees`$(U, u, h \cdot step)$ from u (see Line 6). In the inward tree or outward tree of `InOutTrees`$(U, u, h \cdot step)$, if after contraction there are multiple edges from or to a contracted vertex, respectively, only keep one of them. We can see the number of edges added to $\hat{E}$ is bounded by $O(|Ball_{U_p}(u_p, h \cdot step)|)$. Also in U_p, the roundtrip distance from u_p to vertices in $Ball_{U_p}(u_p, h \cdot step)$ by edges in `NewInOutTrees`$(U_p, u_p, h \cdot step)$ is at most $h \cdot step$.

In line 3, we can delete long edges since obviously they cannot be included in $\hat{E}$.

The main algorithm is shown in Algorithm 3.

▶ **Lemma 12.** *For* $k \leq n$ *and* $n \geq 12$*, Algorithm* Spanner(G, k) *constructs a* $(2k-1)$*-roundtrip spanner of* G*.*

Algorithm 2 Cover2(G, k, p, ϵ).

1: $L \leftarrow (1+\epsilon)^p$
2: Contract all edges e with $g(e) \le L/n^3$ in G to form a graph G_p, let its vertex set be V_p
3: (Delete edges e with $g(e) > 2(k-1)L$ from G_p)
4: $U \leftarrow V, U_p \leftarrow V_p, \hat{E} \leftarrow \emptyset$
5: **while** $U \neq \emptyset$ and $\max_{u \in U} R(u) \ge 2(k-1)L$ **do**
6: $\quad u \leftarrow \arg\max_{u \in U} R(u)$, let u_p be the corresponding vertex in U_p
7: $\quad \mathit{step} \leftarrow (1 + 1/n^2)L$
8: $\quad h \leftarrow$ minimum positive integer satisfying $|\mathit{Ball}_{U_p}(u_p, h \cdot \mathit{step})| < n^{h/k}$
9: $\quad$ Add NewInOutTrees$(U_p, u_p, h \cdot \mathit{step})$ to $\hat{E}$
10: $\quad$ Remove $\overline{\mathit{Ball}}_{U_p}(u_p, (h-1)\mathit{step})$ from U_p, remove corresponding vertices from U
11: **end while**
12: Remove all vertices u from U with $R(u) < L/8$
13: **while** $U \neq \emptyset$ **do**
14: $\quad u \leftarrow \arg\max_{u \in U} R(u)$
15: $\quad \mathit{step} \leftarrow \min\{R(u)/(k-1), L\}$
16: $\quad h \leftarrow$ minimum positive integer satisfying $|\mathit{Ball}_U(u, h \cdot \mathit{step})| < n^{h/k}$
17: $\quad$ Add InOutTrees$(U, u, h \cdot \mathit{step})$ to $\hat{E}$
18: $\quad$ Remove $\overline{\mathit{Ball}}_U(u, (h-1)\mathit{step})$ from U
19: **end while**
20: **return** $\hat{E}$

Algorithm 3 Spanner$(G(V, E), k)$.

Do the preprocessing in Section 3.1. Let E_0 be the added edges
$\epsilon \leftarrow \frac{1}{4(k-1)}$.
$\hat{E} \leftarrow E_0$
for $p \leftarrow 0$ to $\lfloor \log_{1+\epsilon}(2nW) \rfloor + 1$ **do**
$\quad \hat{E} \leftarrow \hat{E} \cup$ Cover2(G, k, p, ϵ)
end for
return $H(V, \hat{E})$

Proof. For any pair of vertices u, v with roundtrip distance $d = d(u \leftrightarrows v)$ on G, there exists a p, such that $d \in [(1+\epsilon)^{p-1}, (1+\epsilon)^p)$. Let $L = (1+\epsilon)^p$. If $R(u) \le (k-1)d$ or $R(v) \le (k-1)d$, by Lemma 6, E_0 contains a roundtrip cycle between u and v with length at most $(2k-1)d$. So we assume $R(u) > (k-1)d$ and $R(v) > (k-1)d$. Also, if there is a vertex w on the shortest cycle containing u and v with $R(w) < L/8$, then there will be a vertex $t \in H$ so that $d(w \leftrightarrows t) < L/8$, so the roundtrip distance in E_0 will be $d(u \leftrightarrows t) + d(t \leftrightarrows v) < L/4 + 2d \le (1+\epsilon)d/4 + 2d < (2k-1)d$ for $k \ge 2$, so Line 12 cannot impact the correctness.

Consider the iteration p of Algorithm 2, let u_p, v_p be the contracted vertices of u, v respectively. Let P be a shortest cycle going through u, v in G and P' be the contracted cycle going through u_p, v_p in G_p. It is easy to see that each vertex on P corresponds to some vertex on P'. Similar as Lemma 8, in Line 8 and Line 16, we have $(k-1) \cdot \mathit{step} \le R(u)$. It is easy to see that $|\mathit{Ball}_{U_p}(u_p, (k-1) \cdot \mathit{step})| \le |\mathit{Ball}_U(u, (k-1) \cdot \mathit{step})| < n^{1-1/k}$, which implies $h \le k-1$.

We prove it by the induction on p. When p is small, there is no contracted vertex in G_p. By the same argument as in Lemma 9, either $\texttt{Cover2}(G, k, L, \epsilon)$'s returned edge set $\hat{E}$ contains a roundtrip cycle between u and v with length at most

$$2h_c \cdot \mathit{step}_c \leq 2(k-1)(1+1/n^2)(1+\epsilon)d = (2k-3/2)(1+1/n^2)d \leq (2k-1)d$$

($k \geq 2$, $k \leq n$ and $n \geq 12$) since $\mathit{step}_c \leq (1+1/n^2)L$ in Line 7 and Line 15 and $h_c \leq k-1$, or E_0 contains a cycle between u and v with length at most $(2k-1)d$. Next we assume that vertices of G contracted in the same vertex in G_p are already connected in $\hat{E}$, and has the $(2k-1)$-stretch.

During $\texttt{Cover2}(G, k, p, \epsilon)$, if some vertices in P' are removed from U_p in Line 10, like Lemma 9, suppose $w_p \in U_p$ is one of the first removed vertices, and w_p is removed as a member of $\overline{Ball}_{U_c}(c, (h_c-1)\mathit{step}_c)$ centered at c. Let $w' \in C(w_p)$ be one vertex on P, since there are at most n original vertices contracted and $\mathit{step}_c = (1+1/n^2)L$, we have $d_U(c \leftrightarrows u) \leq d_{U_p}(c \leftrightarrows w_p) + d_U(w' \leftrightarrows u) + n \cdot L/n^3 \leq (h_c-1)\mathit{step}_c + d_U(u \leftrightarrows v) + L/n^2 < (h_c-1)\mathit{step}_c + L + L/n^2 = h_c \mathit{step}_c$, and symmetrically $d_U(c \leftrightarrows v) < h_c \mathit{step}_c$. Thus $\texttt{NewInOutTrees}(U_c, c, h_c \mathit{step}_c)$ builds a roundtrip cycle passing through u_p, v_p of length $< 2h_c \mathit{step}_c$ in current contracted graph. It follows that $d_{G_p[\hat{E}]}(u_p, v_p) < 2h_c \mathit{step}_c \leq 2(k-1)(1+1/n^2)L$. Since there are at most n contracted vertices in the roundtrip cycle between u_p and v_p, and $w(e) \leq g(e)$ for every contracted edge e, we have

$$d_{G[\hat{E}]}(u, v) \leq 2(k-1)(1+1/n^2)L + n \cdot (2k-1)L/n^3 \leq (2k-3/2)(1+3/n^2)d \leq (2k-1)d.$$

($k \geq 2$, $k \leq n$ and $n \geq 12$.)

If there is no vertex in P' removed from U_p in Line 10 and Line 12, then all vertices w in P have $L/8 \leq R(w) < 2(k-1)L$. By the same argument as in Lemma 9, the second part of Algorithm 2 also ensures that $\hat{E} \cup E_0$ contains a roundtrip cycle passing through u, v with length at most $(2k-1)d$. ◀

▶ **Lemma 13.** *The subgraph returned by algorithm* Spanner(G, k) *has* $O(kn^{1+1/k} \log n)$ *edges.*

Proof. Preprocessing adds $O(n^{1+1/k} \log n)$ edges as in Section 3.1. The edges added in Line 17 is bounded as follows. Consider Algorithm 2, after Line 12, the subgraph only consists of vertices with $R(u) \in [L/8, 2(k-1)L]$, so each vertex belongs to at most $\log_{1+\epsilon} 16k$ such iterations. Thus the total number of edges added after Line 12 is at most $n^{1+1/k} \log_{1+\epsilon} 16k = O(kn^{1+1/k} \log k)$ edges. Next we count the edges added in Line 9.

We remove the directions of all edges in G to get an undirected graph G', and remove the directions of all edges in every G_p to get an undirected graph G'_p, but define the weight of an edge e in G' and every G'_p to be the girth $g(e)$ in G. Let F be a minimum spanning forest of G' w.r.t. the girth $g(e)$. We can see that in iteration p, if we remove edges in F with $g(e) > 2(k-1)(1+\epsilon)^p$ and contract edges e with $g(e) \leq (1+\epsilon)^p/n^3$ in F, then the connected components in F will just be the connected components in G'_p, which are the strongly connected components in G_p. This is because of the cycle property of MST: If an edge $e = (u, v)$ in G'_p has $g(e) \leq (1+\epsilon)^p/n^3$, then in F all edges f on the path connecting u, v have $g(f) \leq (1+\epsilon)^p/n^3$, thus u, v are already contracted in F; If an edge $e = (u, v)$ in G'_p has $g(e) \leq 2(k-1)(1+\epsilon)^p$, then in F all edges f on the path connecting u, v have $g(f) \leq 2(k-1)(1+\epsilon)^p$, so u, v are in the same component in F.

So the total size of connected components $\{C : |C| \geq 2\}$ in G'_p is at most 2 times the number of edges e in F with $(1+\epsilon)^p/n^3 < g(e) \leq 2(k-1)(1+\epsilon)^p$, and every edge in F can be in at most $\log_{1+\epsilon} 2(k-1)n^3 = O(k \log n)$ number of different G'_p. Thus, the total size of connected components with size at least 2 in all G'_p is bounded by $O(kn \log n)$. By a similar argument of Lemma 7, in each call of `Cover2`(G, k, p, ϵ), line 9 will add $|C|n^{1/k}$ new edges to $\hat{E}$, for every connected component C with $|C| \geq 2$ in G'_p. Thus the total number of edges in the subgraph returned by `Spanner`(G, k) is bounded by $O(kn^{1+1/k} \log n)$. ◀

Construction Time

The analysis of `Spanner`'s running time is similar to Section 3.4. Compared with `Cover`, `Cover2` adds operations of building G_p. We also need to calculate $g(\cdot)$ in preprocessing, which can done by n Dijkstra searches. G_p can be built in $O(m)$ time. `Cover2` is called $\log_{1+\epsilon'}(2nW) = O(k \log(nW))$ times. Therefore the total construction time is still $O(kn(m + n \log n) \log(nW))$.

5 Conclusion

In this paper we discuss the construction of $(2k-1)$-roundtrip spanners with $O(kn^{1+1/k} \log n)$ edges. An important and interesting further direction is whether we can find truly subcubic algorithm constructing such spanners.

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