From Holant to Quantum Entanglement and Back

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Abstract

Holant problems are intimately connected with quantum theory as tensor networks. We first use techniques from Holant theory to derive new and improved results for quantum entanglement theory. We discover two particular entangled states $|\Psi_6\rangle$ of 6 qubits and $|\Psi_8\rangle$ of 8 qubits respectively, that have extraordinary closure properties in terms of the Bell property. Then we use entanglement properties of constraint functions to derive a new complexity dichotomy for all real-valued Holant problems containing a signature of odd arity. The signatures need not be symmetric, and no auxiliary signatures are assumed.

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1 Introduction

1.1 Holant problems

Holant problems are a broad class of sum-of-products computations. It generalizes other frameworks such as counting constraint satisfaction problems (#CSP) and counting graph homomorphisms (#GH). Both have been well studied and full complexity dichotomies have been established [9, 26, 7, 13, 11, 25, 8, 31, 12]. On the other hand, the understanding of Holant problems, even restricted to the Boolean domain, is still limited. In this paper, we focus on Holant problems defined over the Boolean domain.

Holant problems are parameterized by a set of constraint functions, also called signatures. A signature (over the Boolean domain) of arity $n > 0$ is a map $\mathbb{Z}_2^n \to \mathbb{C}$. Let $\mathcal{F}$ be any fixed set of signatures. A signature grid $\Omega = (G, \pi)$ over $\mathcal{F}$ is a tuple, where $G = (V, E)$ is a graph without isolated vertices, π labels each $v \in V$ with a signature $f_v \in \mathcal{F}$ of arity $\deg(v)$, and labels the incident edges $E(v)$ at v with input variables of f_v. We consider all 0-1 edge assignments σ, and each gives an evaluation $\prod_{v\in V} f_v(\sigma|_{E(v)})$, where $\sigma|_{E(v)}$ denotes the restriction of σ to $E(v)$.

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▶ **Definition 1** (Holant problems). *The input to the problem* $\text{Holant}(\mathcal{F})$ *is a signature grid* $\Omega = (G, \pi)$ *over* $\mathcal{F}$*. The output is the partition function*

$$\text{Holant}_\Omega = \sum_{\sigma: E(G) \to \{0,1\}} \prod_{v \in V(G)} f_v(\sigma|_{E_{(v)}}).$$

Bipartite Holant problems $\text{Holant}(\mathcal{F} \mid \mathcal{G})$ *are Holant problems over bipartite graphs* $H = (U, V, E)$*, where each vertex in* U *or* V *is labeled by a signature in* $\mathcal{F}$ *or* $\mathcal{G}$ *respectively. We say* $\mathcal{F}$ *is on the left hand side* (LHS) *and* $\mathcal{G}$ *is on the right hand side* (RHS).

Weighted #CSP is a special class of Holant problems. So are all weighted #GH. Other problems expressible as Holant problems include counting matchings and perfect matchings [43], counting weighted Eulerian orientations [38, 15], computing the partition functions of six-vertex models [41, 16] and eight-vertex models [4, 14], and a host of other vertex models from statistical physics [5]. It is proved that counting perfect matchings cannot be expressed by #GH [28, 17]. Thus, Holant problems are provably more expressive.

Progress has been made in the complexity classification of Holant problems. When all signatures are restricted to be symmetric, a full dichotomy is proved [18]. When asymmetric signatures are allowed, some dichotomies are proved for special families of Holant problems by assuming that certain auxiliary signatures are available, e.g., Holant^*, Holant^+ and Holant^c [20, 2, 22, 3]. Without assuming auxiliary signatures a Holant dichotomy is established only for non-negative real-valued signatures [37].

1.2 Quantum entanglement theory

Holant problems can be viewed as tensor networks in quantum theory. The partition function Holant_Ω can be used in a (strong) simulation of quantum circuits [44]. A signature grid is just a tensor network, where each signature is a tensor with its inputs associated with its incident edges and the Holant value of the signature grid is obtained by contracting all edges. In this sense, a signature of arity n represents a state of n *qubits*. In quantum theory, the basic component of a system is a qubit. The (pure) state $|\Psi\rangle$ of n qubits is described by a vector in $\mathbb{C}^{2^n}$. (The standard notion requires quantum states to have norm 1, but in this paper, normalization by a nonzero scalar makes no difference for complexity, so we work with states having arbitrary nonzero norms.) A nonzero n-ary signature f is synonymous with an n-qubit state $|f\rangle = \sum_{x \in \{0,1\}^n} f(x)|x\rangle$. In this paper, we use them interchangeably. When f is a zero signature (i.e., $f \equiv 0$), we agree that $|f\rangle$ is a null state, denoted by $\mathfrak{N}$.

A core concept in quantum theory is *entanglement*. It is perhaps the most distinguishing characteristic feature separating quantum and classical physics.

▶ **Definition 2** (Quantum entanglement). *A state of* n *qubits (*$n > 1$*, representing a multiple system) is* entangled *if it cannot be decomposed as a tensor product of single-qubit states (individual systems). It is* genuinely entangled *if it cannot be decomposed as a tensor product of states of proper subsystems. It exhibits* multipartite entanglement *if it involves a genuinely entangled state of subsystem of more than two qubits (i.e., it cannot be decomposed as a tensor product of single-qubit states and 2-qubit states).*

Today, entanglement is recognized as an important resource in quantum computing and quantum information theory. It has been shown that quantum computing speedups essentially depend on unbounded entanglement [34]. While in quantum information theory, an entangled state is shared by several parties, one can perform operations on a subsystem locally without access to the other subsystems. This set-up is commonly used in quantum teleportation and quantum key distribution [27, 6]. For different information-theoretic tasks, different types of entanglement can be used [40]. The classification of them under *stochastic local operation with classical communication* (SLOCC) equivalence was proposed in 2000 by Dür et al. [24], and is an area of active research [46, 39, 36, 35, 1, 30]. Yet so far, even the classification of entangled 4-qubit states is not completely settled. For more about quantum entanglement theory, we refer to the survey [33].

1.3 Existing dichotomies inspired by entanglement theory

There are many natural connections between Holant problems and quantum theory. The introduction of Holant problems is inspired by holographic transformations [45]. Such a holographic transformation applied separately on each qubit i with a matrix A_i is just a SLOCC in quantum theory. Also, many known P-time computable signature sets for Holant problems can be clearly described in the quantum literature [19, 2] and they correspond directly to sets of states that are of independent interest in quantum theory [23, 32].

Going beyond that, Backens recently applied knowledge from the theory of quantum entanglement, directly to the study of Holant problems and derived new dichotomy results [2, 3]. We give a short description for these results in this subsection. We use $\langle\Phi|$ to denote the Hermitian adjoint (complex conjugate) of $|\Phi\rangle$, and $\langle\Phi|\Psi\rangle$ to denote the (complex) inner product of two n-qubit states.

▶ **Definition 3** (Projection). *The projection of the i-th qubit of an n-qubit ($n \geqslant 2$) state $|\Psi\rangle$ onto a single-qubit state $|\theta\rangle = a|0\rangle + b|1\rangle$ is defined as $\langle\theta|_i|\Psi\rangle = \bar{a}|\Psi_i^0\rangle + \bar{b}|\Psi_i^1\rangle$ where $\bar{a}$ and $\bar{b}$ are complex conjugates of a and b, and $|\Psi_i^0\rangle$ and $|\Psi_i^1\rangle$ are states of the remaining $n-1$ qubits when the i-th qubit of $|\Psi\rangle$ is set to 0 and 1 respectively.*

▶ **Theorem 4** ([42, 29]). *Let $|\Psi\rangle$ be a genuinely entangled n-qubit ($n \geqslant 3$) state. For any two qubits of $|\Psi\rangle$, there exist projections of the other $n-2$ qubits onto $n-2$ many single-qubit states that result in an entangled 2-qubit state.*

This result was presented to show that any pure entangled multipartite quantum state violates some Bell's inequality [42]. The original proof [42] was flawed and was corrected recently [29]. Theorem 4 shows that two particle entanglement can be realized via performing local projections on a genuinely entangled multiparticle state. It is observed in [2] that the theorem holds even when restricted to only local projections onto computational or Hadamard basis states, i.e., $|0\rangle$, $|1\rangle$, $|+\rangle = |0\rangle + |1\rangle$ and $|-\rangle = |0\rangle - |1\rangle$.

Based on Theorem 4 and the inductive entanglement classification under SLOCC equivalence [36, 35, 1], Backens showed that beyond entangled 2-qubit states, genuinely entangled 3-qubit states can be realized via local projections onto computational or Hadamard basis states (Theorem 12 in [2]). This theorem is equivalent to the following inductive statement.

▶ **Theorem 5** ([2]). *Let $|\Psi\rangle$ be an n-qubit ($n \geqslant 4$) state exhibiting multipartite entanglement. Then, there exists some i and some $|\theta\rangle \in \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\}$ such that $\langle\theta|_i|\Psi\rangle$ exhibits multipartite entanglement.*

▶ Remark 6. This result shows that multipartite entanglement of an n-qubit ($n \geqslant 4$) state can be *preserved* under projections onto states $|0\rangle$, $|1\rangle$, $|+\rangle$ and $|-\rangle$.

The $\operatorname{Holant}^+(\mathcal{F})$ problem is defined as $\operatorname{Holant}(\mathcal{F} \cup \{|0\rangle, |1\rangle, |+\rangle, |-\rangle\})$, where single qubit states $|0\rangle, |1\rangle, |+\rangle$ and $|-\rangle$ represent unary signatures $\Delta_0 = (1, 0)$, $\Delta_1 = (0, 1)$, $\Delta_+ = (1, 1)$ and $\Delta_- = (1, -1)$ in the Holant framework. According to Theorem 5, we know that in the framework of Holant^+ problems, a genuinely entangled 3-qubit state can always be realized from an n-qubit ($n \geqslant 4$) state exhibiting multipartite entanglement. Then, using a genuinely entangled 3-qubit state, a full dichotomy was proved for Holant^+ problems [2]. Later, it was generalized to Holant^c problems [22, 3] where $\operatorname{Holant}^c(\mathcal{F})$ is defined as $\operatorname{Holant}(\mathcal{F} \cup \{|0\rangle, |1\rangle\})$.

1.4 Our results

In this paper, we consider when multipartite entanglement can be preserved under projections onto only computational basis states, i.e., $|0\rangle$ or $|1\rangle$. We have the following result.

▶ **Theorem 7.** *Let $|\Psi\rangle$ be an n-qubit ($n \geqslant 4$) state exhibiting multipartite entanglement and $\langle 0^n|\Psi\rangle \neq 0$. If $n \geqslant 5$ and $|\Psi\rangle$ is not of the form $a|0^n\rangle + b|1^n\rangle$, or $n = 4$ and $|\Psi\rangle$ is not of the form $a|0000\rangle + b|1111\rangle + c|0011\rangle + d|1100\rangle$ (up to a permutation of the four qubits) where a, b, c and d can possibly be zero, then there exists some i such that $|\Psi_i^0\rangle$ or $|\Psi_i^1\rangle$ exhibits multipartite entanglement.*

Under SLOCC equivalence, without loss of generality, we may assume that $\langle 0^n|\Psi\rangle \neq 0$. The other conditions are all necessary to ensure the preservation of multipartite entanglement under projections to $|0\rangle$ and $|1\rangle$. Thus Theorem 7 is a strengthening of Theorem 5. More importantly, our approach is in the opposite direction to Backens'. While Backens proved results in quantum entanglement theory to apply it to the complexity classification of Holant problems, we prove new results in quantum entanglement theory by employing the machinery from Holant problems. We prove Theorem 7 using a technique developed for Holant problems called the interplay between the *unique prime factorization* of signatures and *gadget constructions*. This technique is at the heart of a standard approach (arity reduction) to build inductive arguments for Holant problems [15]. The new result in quantum entanglement theory sheds light on the classification of entanglement under SLOCC equivalence.

Going one step further, we ask whether we can restrict projections onto only one state $|0\rangle$, while multipartite entanglement is still preserved. The answer is no. Then, one way to salvage the situation is to consider the self-loop gadget using one of the Bell states, $|\phi^+\rangle = |00\rangle + |11\rangle$ together with projections onto $|0\rangle$.

▶ **Definition 8** (Self-loop). *The self-loop on the i-th and j-th qubits of a state $|\Psi\rangle$ by the Bell state $|\phi^+\rangle = |00\rangle + |11\rangle$ is defined as $\langle \phi^+|_{ij}|\Psi\rangle = |\Psi_{ij}^{00}\rangle + |\Psi_{ij}^{11}\rangle$, where $|\Psi_{ij}^{00}\rangle$ and $|\Psi_{ij}^{11}\rangle$ are states of $n-2$ qubits when setting the i-th and j-th qubits of $|\Psi\rangle$ to 00 and 11 respectively.*

▶ **Lemma 9.** *Let $|\Psi\rangle$ be an n-qubit ($n \geqslant 4$) state exhibiting multipartite entanglement. There exists some choice of three or four of the n qubits such that by performing self-loops by $|\phi^+\rangle$ and projections onto $|0\rangle$ of the other qubits, we get*

- *a 3-qubit state exhibiting multipartite entanglement, or*
- *a GHZ type 4-qubit state, i.e., $|\mathrm{GHZ}_4\rangle = |0000\rangle + |1111\rangle$, or*
- *the state $|1\rangle$.*

Why do we consider $|\phi^+\rangle$ and $|0\rangle$? The state $|\phi^+\rangle$ is synonymous with the binary EQUALITY signature $=_2$. It is always available in the Holant framework as it means *merging* two dangling edges in a graph. Moreover, we can show that $|0\rangle$ is realizable from any state of odd number of qubits under some mild assumptions. Then, we can apply Lemma 9 to get a new dichotomy for Holant problems where at least one signature of odd arity is present.

▶ **Theorem 10.** *Let $\mathcal{F}$ be a set of real-valued signatures containing at least one signature of odd arity. If $\mathcal{F}$ satisfies the tractability condition* (T) *in Theorem 21, then* $\operatorname{Holant}(\mathcal{F})$ *is polynomial-time computable; otherwise,* $\operatorname{Holant}(\mathcal{F})$ *is #P-hard.*

▶ Remark 11. Theorem 7 and Lemma 9 hold for complex-valued n-qubit states. However, Theorem 10 is restricted to real-valued signatures, in which the Hermitian conjugate and the complex inner product can be represented by a *mating* gadget in the Holant framework.

1.5 Surprising discovery of two extraordinary quantum states

What about signature sets containing only signatures of even arity, in which $|0\rangle$ cannot be realized. Since $|\phi^+\rangle$ is always available, we consider whether multipartite entanglement is preserved under self-loops by $|\phi^+\rangle$ alone. Given an n-qubit ($n \geqslant 6$ is even) state $|\Psi\rangle$ exhibiting multipartite entanglement, are there some i and j such that performing a self-loop by $|\phi^+\rangle$ on the i-th and j-th qubits of $|\Psi\rangle$ results in an $(n-2)$-qubit state exhibiting multipartite entanglement? (By the definition of multipartite entanglement, for even n it must be $n \geq 6$.)

The answer is no. Here we made a quite surprising discovery: There *exist* genuinely entangled 6-qubit and 8-qubit states such that multipartite entanglement is *not* preserved under self-loops. Furthermore, it is not preserved under self-loops not only by $|\phi^+\rangle$, but also by all four Bell states, $|\phi^+\rangle, |\psi^+\rangle = |01\rangle + |10\rangle, |\phi^-\rangle = |00\rangle - |11\rangle$, and $|\psi^-\rangle = |01\rangle - |10\rangle$. The self loop of the i-th and j-th qubits of $|\Psi\rangle$ by $|\psi^+\rangle$ is defined as $\langle\psi^+|_{ij}|\Psi\rangle = |\Psi^{01}_{ij}\rangle + |\Psi^{10}_{ij}\rangle$. Similarly, we can define $\langle\phi^-|_{ij}|\Psi\rangle$ and $\langle\psi^-|_{ij}|\Psi\rangle$.

▶ **Definition 12** (Bell property). *Let $|\Psi\rangle$ be a genuinely entangled state. We say that it satisfies the Bell property if for any two qubits i and j of $|\Psi\rangle$ and any Bell state $|\phi\rangle$, $\langle\phi|_{ij}|\Psi\rangle$ is a tensor product of Bell states. It satisfies the strong Bell property if for any i and j and any Bell state $|\phi\rangle$, $\langle\phi|_{ij}|\Psi\rangle$ is a tensor product of the Bell state $|\phi\rangle$, i.e., $\langle\phi|_{ij}|\Psi\rangle = |\phi\rangle \otimes \cdots \otimes |\phi\rangle$.*

▶ **Theorem 13.** *There exist genuinely entangled 6-qubit states that satisfy the Bell property, and genuinely entangled 8-qubit states that satisfy the strong Bell property.*

We first give an 8-qubit state $|\Psi_8\rangle$ that satisfies the strong Bell property.

$$|\Psi_8\rangle = |00000000\rangle + |00001111\rangle + |00110011\rangle + |00111100\rangle + |01010101\rangle + |01011010\rangle + |10011001\rangle + |10010110\rangle$$
$$+ |01101001\rangle + |01100110\rangle + |10100101\rangle + |10101010\rangle + |11000011\rangle + |11001100\rangle + |11110000\rangle + |11111111\rangle.$$

$|\Psi_8\rangle$ can be represented by an 8-ary signature Ψ_8. Let $\mathcal{S}(\Psi_8)$ be the support of Ψ_8, i.e., $\mathcal{S}(\Psi_8) = \{\alpha \in \mathbb{Z}_2^8 \mid \Psi_8(\alpha) \neq 0\}$. $\mathcal{S}(\Psi_8)$ has the following structure: the sums of the first four variables, and the last four variables are both even; the assignment of the first four variables are either identical to, or complement of the assignment of the last four variables. While it is not obvious from this description that the support set is an affine subspace of $\mathbb{Z}_2^8$, but *it is*.

$$\mathcal{S}(\Psi_8) = \big\{(x_1, x_2, \ldots, x_8) \in \mathbb{Z}_2^8 \mid x_1 + x_2 + x_3 + x_4 \equiv 0,\ x_1 + x_2 + x_5 + x_6 \equiv 0,$$
$$x_1 + x_3 + x_5 + x_7 \equiv 0,\ x_2 + x_3 + x_5 + x_8 \equiv 0,\ \mathrm{mod} 2\big\}.$$

In other words, take 4 variables x_1, x_2, x_3, x_5, (these are not the first 4 variables in the description above), then on the support the remaining 4 variables are mod 2 sums of $\binom{4}{3}$ subsets of $\{x_1, x_2, x_3, x_5\}$. This will imply that $|\Psi_8\rangle$ is genuinely entangled. Also, one can check that $\langle\phi|_{12}|\Psi_8\rangle = |\phi\rangle^{\otimes 3}$ for any Bell state $|\phi\rangle$. Due to the symmetry of $|\Psi_8\rangle$, the same result holds by replacing $\{1, 2\}$ with any $\{i, j\}$. Thus, $|\Psi_8\rangle$ satisfies the strong Bell property.

The 6-qubit state $|\Psi_6\rangle$ satisfying the Bell property has 32 nonzero coefficients. We give it in the signature form.

$$\Psi_6(x_1, \ldots, x_6) = \chi_{\mathcal{S}(\Psi_6)} \cdot (-1)^{x_1x_4 + x_2x_5 + x_3x_6 + x_4x_5 + x_5x_6 + x_4x_6},$$

where $\chi_{\mathcal{S}(\Psi_6)}$ is the indicator function on the support $\mathcal{S}(\Psi_6) = \{(x_1, \ldots, x_6) \in \mathbb{Z}_2^6 \mid \sum_{i=1}^6 x_i = 0 \bmod 2\}$ (even parity). Such a support will imply that $|\Psi_6\rangle$ is genuinely entangled. We can write Ψ_6 as the following 8-by-8 matrix where the assignment of the first three variables in lexicographic order (from 000 to 111) is the row index and the assignment of the last three variables in lexicographic order is the column index.

$$M_{123,456}(\Psi_6) = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

By the symmetry of $|\Psi_6\rangle$, one can check that $|\Psi_6\rangle$ satisfies the Bell property by verifying $\langle\phi|_{12}|\Psi_6\rangle$, $\langle\phi|_{45}|\Psi_6\rangle$ and $\langle\phi|_{14}|\Psi_6\rangle$ are tensor products of Bell states for any bell state $|\phi\rangle$.

We can use Pauli operations to generate more states satisfying the Bell property. Consider the following four Pauli operators

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -\mathfrak{i} \\ \mathfrak{i} & 0 \end{bmatrix} \quad \text{and} \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A Pauli operation on an n-qubit state $|\Psi\rangle$ is defined as $P_1 \otimes P_2 \otimes \ldots \otimes P_n|\Psi\rangle$ (which produces another n-qubit state) where each P_i is a Pauli operator. Let $|\Psi_6\rangle$ and $|\Psi_8\rangle$ be states described above. Let $\mathfrak{P}_6$ and $\mathfrak{P}_8$ denote the sets of states realized by performing Pauli operations on $|\Psi_6\rangle$ and $|\Psi_8\rangle$ respectively.

▶ **Theorem 14.** *Every state in $\mathfrak{P}_6$ or $\mathfrak{P}_8$ satisfies the Bell property.*

Due to the existence of these 6-qubit and 8-qubit states with such extraordinary properties, it remains as a difficult task to achieve a full dichotomy for real-valued Holant problems. On the other hand, we hope such states can be further investigated and perhaps applied to quantum computing or quantum information theory.

The paper is organized as follows. In Section 2, we give a proof of our main quantum entanglement result (Theorem 7) by using the theory of signatures. Then we give some preliminaries for Holant problems in Section 3, and a proof sketch for our dichotomy result (Theorem 10) in Section 4.

2 Preservation of Multipartite Entanglement under Projections

We use the theory of signatures to prove Theorem 7. Recall that by our definition, a signature always has arity at least one. A nonzero signature g *divides* f denoted by $g \mid f$, if there is a signature h such that $f = g \otimes h$ (with possibly a permutation of variables) or there is a constant λ such that $f = \lambda \cdot g$. In the latter case, if $\lambda \neq 0$, then we also have $f \mid g$ since $g = \frac{1}{\lambda} \cdot f$. For nonzero signatures f and g, we use $f \sim g$ to denote both $g \mid f$ and $f \mid g$. A nonzero signature f is irreducible if f cannot be written as $g \otimes h$ for some signatures g and h. This is equivalent to saying that $|f\rangle$ is a genuinely entangled state of multiple qubits or $|f\rangle$ is a single-qubit state. Let $\mathcal{T}_1$ denote the set of tensor products of unary signatures and $\mathcal{T}$ denote the set of tensor products of unary and binary signatures. Then a state $|f\rangle$ of multiple qubits is entangled iff $f \notin \mathcal{T}_1$, and $|f\rangle$ exhibits multipartite entanglement iff $f \notin \mathcal{T}$. In terms of the above division relation, the unique prime factorization (UPF) of signatures is established (see Lemma 2.13 of [15]). The following result is a direct corollary.

▶ **Corollary 15.** *Let f be a nonzero n-ary signature. Suppose that there are two irreducible signatures g on variables in $A \subseteq [n]$ and h on variables in $B \subseteq [n]$ such that $g \mid f$ and $h \mid f$. Then, either A is disjoint with B, or $A = B$ and $g \sim h$.*

We use f_i^0 and f_i^1 to denote the signature forms of quantum states $|f_i^0\rangle$ and $|f_i^1\rangle$ realized by projections onto $|0\rangle$ and $|1\rangle$. In the Holant framework, these signatures are realized by a *pinning* gadget, i.e., connecting the variable x_i of f with unary signatures $\Delta_0 = (1, 0)$ and $\Delta_1 = (0, 1)$ respectively. We may further pick a variable x_j of f_i^c and pin it to the value d $(c, d \in \{0, 1\})$. Obviously, the pinning gadgets on different variables x_i and x_j commute. Thus, we have $(f_i^c)_j^d = (f_j^d)_i^c$. We denote it by f_{ij}^{cd}.

Suppose that $g \mid f$ where g is on variables in A. Then for any variable x_i of f that is not in A, we have $g \mid f_i^0$ and $g \mid f_i^1$ (By definition, the division relation holds even if f_i^0 or f_i^1 is a zero signature). Thus, the division relation is unchanged under pinning gadgets on variables out of A. The following lemma shows that a stronger converse is also true.

▶ **Lemma 16.** *Let f be a signature of arity $n \geqslant 2$. If there exists a signature g on variables in $A \subseteq [n]$ and some x_i not in A such that $g \mid f_i^0$ and $g \mid f_i^1$, then $g \mid f$.*

Now, we are ready to prove Theorem 7. We restate it in terms of signatures. We use 0^n and 1^n to denote the n-bit all-0 and all-1 strings, and $\mathcal{S}(f)$ to denote the support of f.

▶ **Theorem 17.** *Let f be an n-ary $(n \geqslant 4)$ signature, $f \notin \mathcal{T}$ and $f(0^n) \neq 0$. If $n \geqslant 5$ and $\mathcal{S}(f) \not\subseteq \{0^n, 1^n\}$, or $n = 4$ and $\mathcal{S}(f) \not\subseteq \{0000, 1111, 0011, 1100\}$ up to any permutation of four variables, then there exists some i such that f_i^0 or f_i^1 is not in $\mathcal{T}$.*

Proof. Since $f(0^n) \neq 0$, we have $f_i^0 \not\equiv 0$ and $f_{ij}^{00} \not\equiv 0$ (not identically 0) for all indices i and j. Also, since the support $\mathcal{S}(f) \not\subseteq \{0^n, 1^n\}$, there exist some s and t such that $f_{st}^{01} \not\equiv 0$. For a contradiction, we assume $f_i^0, f_i^1 \in \mathcal{T}$ for all i. We consider the following two possible cases.

Case 1. For all indices i, $f_i^0 \in \mathcal{T}_1$ (i.e., tensor product of unary signatures).

We will show that in this case, there is a unary signature $a(x_u)$ on some variable x_u, such that $a(x_u) \mid f$. This will lead to a contradiction.

Recall that there exist some s and t such that $f_{st}^{01} \not\equiv 0$. Then, clearly $f_t^1 \not\equiv 0$. Since $f_t^1 \in \mathcal{T}$, in the UPF of f_t^1, the variable x_s may appear in a unary signature or an irreducible binary signature. In both cases, since f has arity at least 4, we can pick a variable x_u such that x_u and x_s appear in two distinct irreducible signatures in the UPF of f_t^1 (i.e., x_u and x_s are not entangled in f_t^1). Then, we show that x_u must appear in a unary signature in the UPF of f_t^1. Otherwise, there is an irreducible binary signature $b(x_u, x_v)$ such that $b(x_u, x_v) \mid f_t^1$. Since x_u is not entangled with x_s in f_t^1, we have $v \neq s$. Then, $b(x_u, x_v) \mid f_{st}^{01}$. On the other hand, we consider f_s^0. By our assumption, $f_s^0 \in \mathcal{T}_1$ and hence there exists some unary signature $a'(x_u)$ such that $a'(x_u) \mid f_s^0$. Then, $a'(x_u) \mid f_{st}^{01}$. Since $f_{st}^{01} \not\equiv 0$, by Corollary 15, $b(x_u, x_v) \sim a'(x_u)$. Contradiction. Thus, there exists some $a(x_u)$ such that $a(x_u) \mid f_t^1$.

Now we show that $a(x_u) \mid f_t^0$. First, we show that $a(x_u) \mid f_s^0$. Since $f_s^0 \in \mathcal{T}_1$, there exists some unary signature $a'(x_u)$ such that $a'(x_u) \mid f_s^0$, and then $a'(x_u) \mid f_{st}^{01}$. Also, we have $a(x_u) \mid f_{st}^{01}$ since $a(x_u) \mid f_t^1$. Since $f_{st}^{01} \not\equiv 0$, by Corollary 15, we have $a(x_u) \sim a'(x_u)$. Thus, $a(x_u) \mid f_s^0$. Since $f_t^0 \in \mathcal{T}_1$, there exists a unary signature $a''(x_u)$ such that $a''(x_u) \mid f_t^0$, and then $a''(x_u) \mid f_{st}^{00}$. Also, we have $a(x_u) \mid f_{st}^{00}$ since $a(x_u) \mid f_s^0$. Remember that $f_{st}^{00} \not\equiv 0$. Then, by Corollary 15, we have $a(x_u) \sim a''(x_u)$. Thus, $a(x_u) \mid f_t^0$.

Since $a(x_u) \mid f_t^0$ and $a(x_u) \mid f_t^1$, by Lemma 16, we have $a(x_u) \mid f$. In other words, $f = a(x_u) \otimes g$ where g is a nonzero signature of arity $n - 1$ on variables other than x_u. Since $f \notin \mathcal{T}$, we have $g \notin \mathcal{T}$. Consider f_u^0. We know that it is a nonzero signature and hence $f_u^0 \sim g$. Thus, $f_u^0 \notin \mathcal{T}$. We have reached a contradiction.

Case 2. There exists some index k and an irreducible binary signature $b(x_v, x_w)$ such that $b(x_v, x_w) \mid f_k^0$.

We will show that in this case, $b(x_v, x_w) \mid f$. First, we show that $b(x_v, x_w) \mid f_i^0$ for all $i \notin \{v, w\}$. We already have $b(x_v, x_w) \mid f_k^0$. Consider f_i^0 for all indices $i \notin \{v, w, k\}$. Since $f_i^0 \in \mathcal{T}$ and $f_i^0 \not\equiv 0$, there is either a unary signature $a(x_v)$ or an irreducible binary signature $b'(x_v, x_{w'})$ for some $w' \notin \{i, v\}$ that appears in the UPF of f_i^0, i.e., $a(x_v) \mid f_i^0$ or $b'(x_v, x_{w'}) \mid f_i^0$. In the former case, we have $a(x_v) \mid f_{ik}^{00}$. In the latter case and if $w' \neq k$, we have $b'(x_v, x_{w'}) \mid f_{ik}^{00}$. In the latter case and if $w' = k$, then let $a'(x_v)$ be the unary signature realized from $b'(x_v, x_{w'})$ by pinning $x_{w'} = x_k$ to 0, we get $a'(x_v) \mid f_{ik}^{00}$. On the other hand, since $b(x_v, x_w) \mid f_k^0$, we have $b(x_v, x_w) \mid f_{ik}^{00}$. Since $f_{ik}^{00} \not\equiv 0$, by Corollary 15, we know that the two cases that $a(x_v) \mid f_{ik}^{00}$ and $a'(x_v) \mid f_{ik}^{00}$ cannot occur. Thus, $w' \neq k$ and $b'(x_v, x_{w'}) \mid f_{ik}^{00}$. By Corollary 15, $w' = w$, and $b(x_v, x_w) \sim b'(x_v, x_{w'})$. Thus, $b(x_v, x_w) \mid f_i^0$ for all $i \notin \{v, w\}$.

Then we want to show that there exists some $j \notin \{v, w\}$ such that $b(x_v, x_w) \mid f_j^1$.

- We first consider the case that there exist some indices i and j where $\{i, j\}$ is disjoint with $\{v, w\}$ such that $f_{ij}^{01} \not\equiv 0$. We show that $b(x_v, x_w) \mid f_j^1$. Since $b(x_v, x_w) \mid f_i^0$, we have $b(x_v, x_w) \mid f_{ij}^{01}$. By assumption $f_{ij}^{01} \not\equiv 0$, and then clearly $f_j^1 \not\equiv 0$. Recall that $f_j^1 \in \mathcal{T}$. Again, there is either a unary signature $a(x_v)$ or an irreducible binary signature $b'(x_v, x_{w'})$ that appears in the UPF of f_j^1, i.e., $a(x_v) \mid f_j^1$ or $b'(x_v, x_{w'}) \mid f_j^1$. In the first case since $i \neq v$, we can pin x_i of f_j^1 to 0, and we get $a(x_v) \mid f_{ij}^{01}$. In the second case and if $w' = i$, again we can get $a'(x_v) \mid f_{ij}^{01}$, where $a'(x_v) = b'(x_v, 0)$, obtained from pinning x_i to 0. But $f_{ij}^{01} \not\equiv 0$ and $b(x_v, x_w) \mid f_{ij}^{01}$. Then, in the UPF of f_{ij}^{01}, it does not have a unary signature on x_v as a factor. Thus, it must be the case that $b'(x_v, x_{w'}) \mid f_j^1$ where $w' \neq i$. Then, we have $b'(x_v, x_{w'}) \mid f_{ij}^{01}$. Since $b(x_v, x_w) \mid f_{ij}^{01}$ and $f_{ij}^{01} \not\equiv 0$, by Corollary 15, $w' = w$ and $b'(x_v, x_{w'}) \sim b(x_v, x_w)$, and thus $b(x_v, x_w) \mid f_j^1$. Then, by Lemma 16, we have $b(x_v, x_w) \mid f$. In other words, $f = b(x_v, x_w) \otimes h$ where h is a nonzero signature of arity $n-2$ on variables other than x_v and x_w. Since $f \notin \mathcal{T}$, we have $h \notin \mathcal{T}$. Then consider f_v^0. We know that it is a nonzero signature and $h \mid f_v^0$. Thus, $f_v^0 \notin \mathcal{T}$. Contradiction.
- Then we consider the case that $f_{ij}^{01} \equiv 0$ for all indices $\{i, j\}$ that are disjoint with $\{v, w\}$. Consider an n-bit input α of f. We write α as $\alpha_v \alpha_w \beta$ where α_v is the input on variable x_v, α_w is the input on variable x_w, and β is the input on the other $n-2$ variables. Then, $f(\alpha) = 0$ if β is not the all-0 or all-1 bit string in $\{0,1\}^{n-2}$. It follows that f has at most eight nonzero entries. We list all its entries by the following 4-by-2^{n-2} matrix $M_{vw}(f)$ with $(x_v, x_w) \in \{0,1\}^2$ as the row index (in the order 00, 01, 10, 11) and the assignment of the other variables in lexicographic order as the column index.

$$M_{vw}(f) = \begin{bmatrix} c_1 & 0 & \dots & \dots & 0 & c_2 \\ c_3 & 0 & \dots & \dots & 0 & c_4 \\ c_5 & 0 & \dots & \dots & 0 & c_6 \\ c_7 & 0 & \dots & \dots & 0 & c_8 \end{bmatrix}.$$

Here, $c_1 = f(0^n) \neq 0$. Consider signatures f_v^0 and f_v^1. They have the following matrix forms with the variable $x_w \in \{0,1\}$ as the row index.

$$M_w(f_v^0) = \begin{bmatrix} c_1 & 0 & \dots & \dots & 0 & c_2 \\ c_3 & 0 & \dots & \dots & 0 & c_4 \end{bmatrix} \quad \text{and} \quad M_w(f_v^1) = \begin{bmatrix} c_5 & 0 & \dots & \dots & 0 & c_6 \\ c_7 & 0 & \dots & \dots & 0 & c_8 \end{bmatrix}.$$

Also consider signatures f_w^0 and f_w^1. They have the following matrix forms with the variable $x_v \in \{0,1\}$ as the row index.

$$M_v(f_w^0) = \begin{bmatrix} c_1 & 0 & \dots & \dots & 0 & c_2 \\ c_5 & 0 & \dots & \dots & 0 & c_6 \end{bmatrix} \quad \text{and} \quad M_v(f_w^1) = \begin{bmatrix} c_3 & 0 & \dots & \dots & 0 & c_4 \\ c_7 & 0 & \dots & \dots & 0 & c_8 \end{bmatrix}.$$

Consider f_v^0. Since f has arity at least 4, f_v^0 has arity at least 3. Since $f_v^0 \in \mathcal{T}$, the variable x_w either appears in a unary factor $a(x_w)$ of f_v^0 or an irreducible binary factor $b(x_w, x_{w'})$ of f_v^0. In the latter case, we can pick another variable x_r of f_v^0 where $r \neq w$ or w', and we consider f_{vr}^{00}. We know that $f_{vr}^{00} \not\equiv 0$ since $c_1 \neq 0$ and $b(x_w, x_{w'}) \mid f_{vr}^{00}$ since $b(x_w, x_{w'}) \mid f_v^0$. Notice that the column with c_2 and c_4 does not appear in $M_w(f_{vr}^{00})$. Thus, the signature f_{vr}^{00} is of the form $(c_1, c_3) \otimes (1, 0)^{\otimes(n-2)}$ which is a tensor product of unary signatures. Contradiction. Thus, there is a unary signature $a(x_w)$ such that $a(x_w) \mid f_v^0$. Then, we have $c_1c_4 = c_2c_3$. Similarly by considering f_w^0, we have $c_1c_6 = c_2c_5$. Now, we consider f_v^1, and prove $c_5c_8 = c_6c_7$. If $c_5 = c_7 = 0$, then clearly we have $c_5c_8 = c_6c_7 = 0$. Otherwise, for any $r \neq w$ or v, we have $f_{vr}^{10} = (c_5, c_7) \otimes (1, 0)^{\otimes(n-2)} \not\equiv 0$ which is a tensor product of unary signatures. If there is a binary signature $b(x_w, x_{w'})$ such that $b(x_w, x_{w'}) \mid f_v^1$, then we can find some $r \neq w, w'$ such that $b(x_w, x_{w'}) \mid f_{vr}^{10}$. Contradiction. Thus, there is a unary signature $a(x_w)$ such that $a(x_w) \mid f_v^1$. Then, we have $c_5c_8 = c_6c_7$. Similarly by considering f_w^1, we have $c_3c_8 = c_4c_7$.

- Suppose $n \geqslant 5$. Then f_v^0 has arity at least 4. We first show that $c_2 = 0$. We consider $f_{vw}^{00} = [c_1, 0, \ldots\ldots, 0, c_2]$. Since $f_v^0 \in \mathcal{T}$, we have $f_{vw}^{00} \in \mathcal{T}$. Note that f_{vw}^{00} has arity at least 3. Since $c_1 \neq 0$, the only possible value of c_2 to make $f_{vw}^{00} \in \mathcal{T}$ is 0. Thus, $c_2 = 0$. Since $c_1c_4 = c_2c_3 = 0$ and $c_1 \neq 0$, we have $c_4 = 0$. Also, since $c_1c_6 = c_2c_5 = 0$ and $c_1 \neq 0$, we have $c_6 = 0$. If $c_8 = 0$, then $f = b(x_v, x_w) \otimes (1, 0)^{\otimes(n-2)} \in \mathcal{T}$. A contradiction with $f \notin \mathcal{T}$. Thus, we have $c_8 \neq 0$. Since $c_5c_8 = c_6c_7 = 0$ and $c_8 \neq 0$, we have $c_5 = 0$. Also since $c_3c_8 = c_4c_7 = 0$ and $c_8 \neq 0$, we have $c_3 = 0$. Consider $f_{vw}^{11} = [c_7, 0, \ldots\ldots, 0, c_8]$. Since $f_{vw}^{11} \in \mathcal{T}$ and it has arity at least 3, and $c_8 \neq 0$, we have $c_7 = 0$. Thus, f has only two nonzero entries that are on the all-0 input and the all-1 input. A contradiction with our assumption that $\mathcal{S}(f) \not\subseteq \{0^n, 1^n\}$.
- Suppose $n = 4$. If $c_2 = 0$, then with the same proof as in the case that $n \geqslant 5$, we have $c_4 = c_6 = 0$, $c_8 \neq 0$ and then $c_3 = c_5 = 0$. Thus, $\mathcal{S}(f) \subseteq \{0000, 1111, 1100\}$. Contradiction. Otherwise, $c_2 \neq 0$. Suppose that $c_2 = kc_1$. Then $c_4 = kc_3$ since $c_1c_4 = c_2c_3$ and $c_6 = kc_5$ since $c_1c_6 = c_2c_5$. If c_3 and c_4 are not zero, then $c_8 = kc_7$ since $c_3c_8 = c_4c_7$. Then, $f = b(x_v, x_w) \otimes (1, 0, 0, k) \in \mathcal{T}$. Contradiction. Thus, $c_3 = c_4 = 0$. Similarly, if c_5 and c_6 are not zero, then we still have $c_8 = kc_7$ since $c_5c_8 = c_6c_7$. Then, we have $f \in \mathcal{T}$. Contradiction. Thus, $c_5 = c_6 = 0$. Then, $\mathcal{S}(f) \subseteq \{0000, 1111, 0011, 1100\}$. Contradiction.

Therefore, there exists some i such that f_i^0 or f_i^1 is not in $\mathcal{T}$. ◀

Our result can be used in the classification of entanglement under SLOCC equivalence. An n-qubit state $|\Psi\rangle$ is equivalent to another n-qubit state $|\Phi\rangle$ under SLOCC if there exist some invertible 2-by-2 matrices M_1, M_2, …, M_n such that $|\Psi\rangle = M_1 \otimes M_2 \otimes \ldots \otimes M_n|\Phi\rangle$. Physicists are interested in the classification of SLOCC equivalence classes. For 2-qubit states there are two SLOCC classes, and for 3-qubit states there are six SLOCC classes [24]. However, for states of 4 or more qubits there are infinitely many SLOCC classes [24]. Then, the goal is to categorize these classes into some finitely many families with common physical or mathematical properties. Depending on which properties are used, there are different approaches. One powerful approach that can possibly handle states of a high number of qubits is by induction [36, 35, 1, 30]. In this approach, the classification of n-qubit states relies on the classification of $(n-1)$-qubit states.

Consider an n-qubit state $|\Psi\rangle$. We can pick some index i and write $|\Psi\rangle$ as $|\Psi\rangle = |0\rangle|\Psi_i^0\rangle + |1\rangle|\Psi_i^1\rangle$. Families of entanglement classes of $|\Psi\rangle$ can be defined according to the types of entanglements found in the linear span$\{|\Psi_i^0\rangle, |\Psi_i^1\rangle\}$ which is related to the entanglement types of $|\Psi_i^0\rangle$ and $|\Psi_i^1\rangle$ themselves. Theorem 7 gives a direct relation between the entanglement types of $|\Psi\rangle$ and $\{|\Psi_i^0\rangle, |\Psi_i^1\rangle\}$. For example, consider a 5-qubit state exhibiting multipartite entanglement. First, by performing SLOCC using the matrix $N_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ on this state, we can always get a state $|\Psi\rangle$ where the coefficient of $|0^5\rangle$ is nonzero. If $|\Psi\rangle$ has the form $a|0^5\rangle + b|1^5\rangle$, then it is equivalent to $|\mathrm{GHZ}_5\rangle = |0^5\rangle + |1^5\rangle$. Otherwise, we can apply Theorem 7. There exists some i such that $|\Psi_i^0\rangle$ or $|\Psi_i^1\rangle$ exhibits multipartite entanglement. Then, in order to classify the state $|\Psi\rangle$, we only need to consider possible entanglement types of $\{|\Psi_i^0\rangle, |\Psi_i^1\rangle\}$ where at least one state exhibits multipartite entanglement. This eliminates many cases compared to considering all entanglement types of $\{|\Psi_i^0\rangle, |\Psi_i^1\rangle\}$.

3 Preliminaries for Holant Problems

3.1 Definitions and Notations

Let f be a complex-valued signature. If $\overline{f(\alpha)} = f(\overline{\alpha})$ for all α where $\overline{f(\alpha)}$ denotes the complex conjugation of $f(\alpha)$ and $\overline{\alpha}$ denotes the bit-wise complement of α, we say f satisfies *arrow reversal symmetry* (ARS). We may use f^α to denote $f(\alpha)$.

We use $=_n$ to denote the EQUALITY signature of arity n, which takes value 1 on the all-0 or all-1 inputs and 0 elsewhere. Note that $=_2$ represents the Bell state $|\phi^+\rangle$. Let $\mathcal{EQ} = \{=_1, =_2, \ldots, =_n, \ldots\}$ denote the set of all EQUALITY signatures. Then $\#\mathrm{CSP}(\mathcal{F})$ is exactly $\mathrm{Holant}(\mathcal{EQ} \mid \mathcal{F})$. Also, let $\mathcal{EQ}_k = \{=_k, =_{2k}, \ldots, =_{nk}, \ldots\}$, and we define $\#\mathrm{CSP}_k(\mathcal{F})$ to be $\mathrm{Holant}(\mathcal{EQ}_k \mid \mathcal{F})$. The following two reductions are known [10]:

$$\#\mathrm{CSP}(\mathcal{F}) \leqslant_T \mathrm{Holant}(=_3, \mathcal{F}) \quad \text{and} \quad \#\mathrm{CSP}_2(\mathcal{F}) \leqslant_T \mathrm{Holant}(=_4, \mathcal{F}).$$

Here, $\leqslant_T$ denotes P-time Turing reduction. We use $\neq_2$ to denote the binary DISEQUALITY signature with truth table $(0, 1, 1, 0)$. It represents the Bell state $|\psi^+\rangle$.

A signature f of arity $n \geqslant 2$ can be expressed as a $2 \times 2^{n-1}$ matrix $M_i(f)$, which lists the 2^n entries of f with variable $x_i \in \{0, 1\}$ as row index and the assignments of the other $n-1$ variables in lexicographic order as column index, i.e. $M_i(f) = \begin{bmatrix} f^{0,00\ldots0} & f^{0,00\ldots1} & \ldots & f^{0,11\ldots1} \\ f^{1,00\ldots0} & f^{1,00\ldots1} & \ldots & f^{1,11\ldots1} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_i^0 \\ \mathbf{f}_i^1 \end{bmatrix}$, where $\mathbf{f}_i^a$ denotes the row vector indexed by $x_i = a$ in $M_i(f)$. For $=_2$, it has the 2-by-2 signature matrix $M(=_2) = I_2 = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. For $\neq_2$, $M(\neq_2) = N_2 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$.

3.2 Holographic Transformation

For an invertible matrix $T \in \mathbf{GL}_2(\mathbb{C})$ and a signature f of arity n, written as a column vector $f \in \mathbb{C}^{2^n}$, we denote by $Tf = T^{\otimes n} f$ the transformed signature. For a signature set $\mathcal{F}$, define $T\mathcal{F} = \{Tf \mid f \in \mathcal{F}\}$ to be the set of transformed signatures. For signatures written as row vectors we define fT^{-1} and $\mathcal{F}T^{-1}$ similarly.

Let $T \in \mathbf{GL}_2(\mathbb{C})$. The holographic transformation defined by T is the following operation: given a signature grid $\Omega = (H, \pi)$ of $\mathrm{Holant}(\mathcal{F} \mid \mathcal{G})$, for the same bipartite graph H, we get a new signature grid $\Omega' = (H, \pi')$ of $\mathrm{Holant}\left(\mathcal{F}T^{-1} \mid T\mathcal{G}\right)$ by replacing each signature in $\mathcal{F}$ or $\mathcal{G}$ with the corresponding signature in $\mathcal{F}T^{-1}$ or $T\mathcal{G}$.

▶ **Theorem 18** (Valiant's Holant Theorem [45]). *For any $T \in \boldsymbol{GL}_2(\mathbb{C})$,*

$$\mathrm{Holant}(\mathcal{F} \mid \mathcal{G}) \equiv_T \mathrm{Holant}(\mathcal{F}T^{-1} \mid T\mathcal{G}).$$

$\mathrm{Holant}(\mathcal{F})$ is equivalent to its bipartite form $\mathrm{Holant}\,(=_2\mid \mathcal{F})$. A particular holographic transformation that will be commonly used is the transformation defined by $Z^{-1} = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & -\mathfrak{i} \\ 1 & \mathfrak{i} \end{smallmatrix}\right]$, with $Z = \frac{1}{\sqrt{2}} \left[\begin{smallmatrix} 1 & 1 \\ \mathfrak{i} & -\mathfrak{i} \end{smallmatrix}\right]$. Since $(=_2)Z = (\neq_2)$, we have $\mathrm{Holant}\,(=_2\mid \mathcal{F}) \equiv_T \mathrm{Holant}\left(\neq_2\mid Z^{-1}\mathcal{F}\right)$. We denote $Z^{-1}\mathcal{F}$ by $\widehat{\mathcal{F}}$ and $Z^{-1}f$ by $\widehat{f}$. The following relation between f and $\widehat{f}$ is known.

▶ **Lemma 19** (Lemma A.2 in [15]). *f is a real valued signature iff $\widehat{f}$ satisfies* ARS.

3.3 Gadget Construction

One basic reduction for Holant problems is gadget construction. We say a signature f is *realizable* from a signature set $\mathcal{F}$ (by gadget construction) if there is a graph $G = (V, E, D)$ with internal edges E and dangling edges D where each vertex $v \in V$ is labeled by a signature f_v from $\mathcal{F}$, and the graph defines the signature f by its sum-of-products with inputs on the dangling edges. If f is realizable from a set $\mathcal{F}$, then $\operatorname{Holant}(f, \mathcal{F}) \equiv_T \operatorname{Holant}(\mathcal{F})$.

A basic gadget construction is *merging*; we connect two variables x_i and x_j of f using $=_2$. We use $\partial_{ij} f = f_{ij}^{00} + f_{ij}^{11}$ to denote a signature realized by merging, where f_{ij}^{ab} denotes the signature obtained by setting $(x_i, x_j) = (a, b) \in \{0,1\}^2$. The merging operation using $=_2$ is synonymous with performing a self-loop by the Bell state $|\phi^+\rangle$.

A gadget construction often used in this paper is *mating*. Given a *real*-valued signature f of arity $n > m \geqslant 1$, we connect two copies of f in the following manner: Fix a set S of $n - m$ variables among all n variables of f. For each $x_k \in S$, connect x_k of one copy of f with x_k of the other copy using $=_2$. The variables that are not in S are called dangling variables. For $m = 1$, there is one dangling variable x_i. Then, the mating construction realizes a binary signature, denoted by $\mathfrak{m}_i f$. It can be represented by matrix multiplication. We have

$$M(\mathfrak{m}_i f) = M_i(f) I_2^{\otimes(n-1)} M_i^{\mathtt{T}}(f) = \begin{bmatrix} \mathbf{f}_i^0 \\ \mathbf{f}_i^1 \end{bmatrix} \begin{bmatrix} \mathbf{f}_i^{0\,\mathtt{T}} & \mathbf{f}_i^{1\,\mathtt{T}} \end{bmatrix} = \begin{bmatrix} |\mathbf{f}_i^0|^2 & \langle \mathbf{f}_i^0, \mathbf{f}_i^1 \rangle \\ \langle \mathbf{f}_i^0, \mathbf{f}_i^1 \rangle & |\mathbf{f}_i^1|^2 \end{bmatrix}. \tag{3.1}$$

The (complex) inner product $\langle \cdot, \cdot \rangle$ uses complex conjugation. But since f is real-valued, this is the same as the usual dot product. $|\mathbf{f}|$ denotes its 2-norm. In the setting of $\operatorname{Holant}(\neq_2 |\, \widehat{\mathcal{F}})$, the above mating operation is equivalent to connecting variables in S using $\neq_2$. We denote the resulting signature by $\widehat{\mathfrak{m}}_i \widehat{f} = \widehat{\mathfrak{m}_i f}$. Note that $\widehat{f}$ satisfies ARS since f is real. Thus,

$$N_2^{\otimes(n-1)} \widehat{\mathbf{f}}_i^{0\,\mathtt{T}} = (\widehat{f}^{0,11\ldots1}, \widehat{f}^{0,11\ldots0}, \ldots, \widehat{f}^{0,00\ldots0})^{\mathtt{T}} = (\overline{\widehat{f}^{1,00\ldots0}}, \overline{\widehat{f}^{1,00\ldots1}}, \ldots, \overline{\widehat{f}^{1,11\ldots1}})^{\mathtt{T}} = \overline{\widehat{\mathbf{f}}_i^1}^{\mathtt{T}}.$$

Then, we have

$$M(\widehat{\mathfrak{m}}_i \widehat{f}) = \begin{bmatrix} \widehat{\mathbf{f}}_i^0 \\ \widehat{\mathbf{f}}_i^1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{\otimes(n-1)} \begin{bmatrix} \widehat{\mathbf{f}}_i^{0\,\mathtt{T}} & \widehat{\mathbf{f}}_i^{1\,\mathtt{T}} \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{f}}_i^0 \\ \widehat{\mathbf{f}}_i^1 \end{bmatrix} \begin{bmatrix} \overline{\widehat{\mathbf{f}}_i^1}^{\mathtt{T}} & \overline{\widehat{\mathbf{f}}_i^0}^{\mathtt{T}} \end{bmatrix} = \begin{bmatrix} \langle \widehat{\mathbf{f}}_i^0, \widehat{\mathbf{f}}_i^1 \rangle & |\widehat{\mathbf{f}}_i^0|^2 \\ |\widehat{\mathbf{f}}_i^1|^2 & \langle \widehat{\mathbf{f}}_i^1, \widehat{\mathbf{f}}_i^0 \rangle \end{bmatrix}. \tag{3.2}$$

Here, due to ARS, the complex inner product can also be represented by mating using $\neq_2$.

3.4 Known results

We give some known signature sets that define tractable, i.e., polynomial time computable, counting problems. There are three families: product-type signatures, affine signatures and local affine signatures denoted by $\mathscr{P}$, $\mathscr{A}$ and $\mathscr{L}$ respectively. Please see the full paper or [22] for definitions and more details. Problems defined by $\mathscr{T}$ are also tractable.

▶ **Definition 20.** *We say a signature set $\mathcal{F}$ is $\mathscr{C}$-transformable if there exists a $T \in \mathbf{GL}_2(\mathbb{C})$ such that $(=_2)(T^{-1})^{\otimes 2} \in \mathscr{C}$ and $T\mathcal{F} \subseteq \mathscr{C}$.*

By Theorem 18, if $\operatorname{Holant}(\mathscr{C})$ is tractable, then $\operatorname{Holant}(\mathcal{F})$ is tractable for any $\mathscr{C}$-transformable set $\mathcal{F}$. The following tractable results are known [22, 3].

▶ **Theorem 21.** *For any complex-valued signature set $\mathcal{F}$,* $\operatorname{Holant}(\mathcal{F})$ *is P-time computable if*

$$\mathcal{F} \subseteq \mathscr{T}, \quad \mathcal{F} \text{ is } \mathscr{P}\text{-transformable}, \quad \mathcal{F} \text{ is } \mathscr{A}\text{-transformable}, \quad \text{or } \mathcal{F} \text{ is } \mathscr{L}\text{-transformable}. \tag{T}$$

Based on dichotomy results of #CSP and Holant^c [21, 22, 3], the following #P-hardness results are known.

▶ **Theorem 22.** *Let $\mathcal{F}$ be a set of real-valued signatures. If $\mathcal{F}$ does not satisfy condition* (T)*, then* $\#\text{CSP}(\mathcal{F})$*,* $\#\text{CSP}_2(\mathcal{F})$ *and* $\text{Holant}^c(\mathcal{F})$ *are #P-hard.*

For reducible signatures, the following reduction was proved by Lin and Wang [37].

▶ **Lemma 23.** *If a nonzero real-valued signature f has a factorization $g \otimes h$ where g and h are also real-valued signatures, then* $\text{Holant}(g, h, \mathcal{F}) \equiv_T \text{Holant}(f, \mathcal{F})$*. In this case, we say that g and h are realizable from f by factorization.*

4 Proof Sketch for Theorem 10

We give a proof sketch for Theorem 10.

By Theorem 21, if $\mathcal{F}$ satisfies condition (T), then $\text{Holant}(\mathcal{F})$ is tractable. We prove #P-hardness when $\mathcal{F}$ does not satisfy condition (T). First, we show that under some holographic transformations, either one can use a signature of odd arity in $\mathcal{F}$ to realize the unary signature $\Delta_0 = (1, 0)$, or one can realize an equality signature $=_k$ for some $k \geqslant 3$.

▶ **Lemma 24.** *Let $\mathcal{F}$ be a set of real-valued signatures containing a signature of odd arity. Then* $\text{Holant}(\neq_2|=_k, \widehat{\mathcal{F}}) \leqslant_T \text{Holant}(\mathcal{F})$ *for some $k \geqslant 3$, or* $\text{Holant}(\Delta_0, Q\mathcal{F}) \leqslant_T \text{Holant}(\mathcal{F})$ *for some real orthogonal 2-by-2 matrix $Q \in \mathbf{O}_2(\mathbb{R})$.*

We first prove the #P-hardness of $\text{Holant}(\neq_2|=_k, \widehat{\mathcal{F}})$ given $k \geqslant 3$ and $\mathcal{F}$ does not satisfy condition (T). We give the following reduction.

▶ **Lemma 25.** *If $k \geqslant 3$, then* $\#\text{CSP}_k(\neq_2, \mathcal{G}) \equiv_T \text{Holant}(\mathcal{EQ}_k \mid \neq_2, \mathcal{G}) \leqslant_T \text{Holant}(\neq_2|=_k, \mathcal{G})$.

Proof. The first equivalence is by definition. For the second reduction, we show that $=_{nk}$ can be realized on the LHS by induction on n. First, we connect one variable of each of k copies of $\neq_2$ on the LHS with the k variables of $=_k$ on the RHS (Figure 1a). This gadget realizes $=_k$ on the LHS.

Then, suppose that $=_{nk}$ is realizable on the LHS. We take one copy of $=_{nk}$ and two copies of $=_k$ on the LHS, and one copy of $=_k$ on the RHS. Remember that $k \geqslant 3$. We connect two variables of $=_k$ on the RHS with one variable of each of the two copies of $=_k$ on the LHS, and connect the other $k-2$ variables of $=_k$ on the RHS with $k-2$ variables of $=_{nk}$ on the LHS (Figure 1b). This gadget realizes $=_{(n+1)k}$ on the LHS.

Also, connecting $k-1$ variables of one copy of $=_k$ on the RHS with $k-1$ variables of another copy of $=_k$ on the RHS using $\neq_2$ on the LHS realizes $\neq_2$ on the RHS. ◀

Then, we give a dichotomy of $\#\text{CSP}_k(\neq_2, \mathcal{G})$ for any *complex*-valued signature set $\mathcal{G}$. This result should be of independent interest. Let $\rho_k = e^{\frac{i\pi}{2k}}$ be a $4k$-th primitive root of unity, $T_k = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \rho_k \end{smallmatrix}\right]$, and $\mathcal{A}_k^d = T_k^d \mathcal{A} = \{T_k^d f \mid f \in \mathcal{A}\}$ where $d \in [k]$.

▶ **Theorem 26.** *Let $\mathcal{G}$ be a set of complex-valued signatures. If $\mathcal{G} \subseteq \mathcal{P}$ or $\mathcal{G} \subseteq \mathcal{A}_k^d$ for some $d \in [k]$. then* $\#\text{CSP}_k(\neq_2, \mathcal{G})$ *is tractable; otherwise,* $\#\text{CSP}_k(\neq_2, \mathcal{G})$ *is #P-hard.*

When $\mathcal{F}$ does not satisfy condition (T), we can show that $\widehat{\mathcal{F}} \notin \mathcal{P}$ and $\widehat{\mathcal{F}} \notin \mathcal{A}_k^d$ for any $d \in [k]$. Combining with Lemma 25 and Theorem 26, we have the following result.

▶ **Lemma 27.** *Let $\mathcal{F}$ be a set of complex-valued signatures. If $\mathcal{F}$ does not satisfy condition* (T) *and $k \geqslant 3$, then* $\text{Holant}(\neq_2|=_k, \widehat{\mathcal{F}})$ *is #P-hard.*

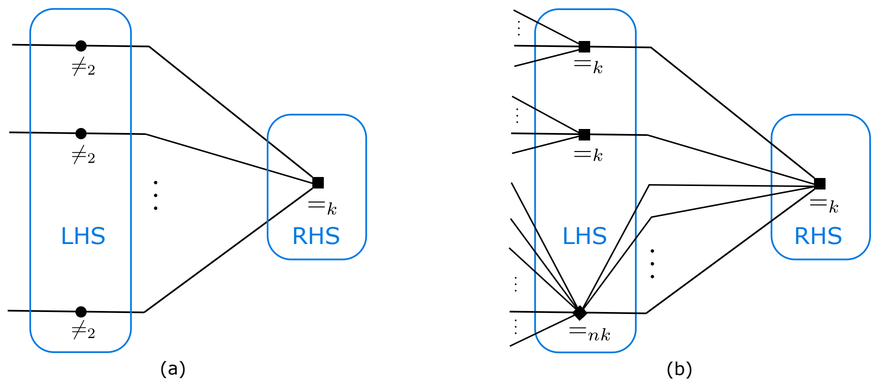

Figure 1 Gadgets realizing $=_k$ and $=_{(n+1)k}$ on the LHS.

Next, we prove the #P-hardness of Holant$(\Delta_0, Q\mathcal{F})$ given $\mathcal{F}$ is a real-valued signature set not satisfying condition (T) and $Q \in \mathbf{O}_2(\mathbb{R})$. Under this condition, we can show that $Q\mathcal{F}$ is also a real-valued signature set not satisfying condition (T). Thus, in order to prove Holant$(\Delta_0, Q\mathcal{F})$ is #P-hard, it suffices to show that Holant$(\Delta_0, \mathcal{F})$ is #P-hard for any real-valued signature set $\mathcal{F}$ not satisfying condition (T). Here, we will apply our entanglement result (Lemma 9) to the proof of #P-hardness. If Δ_1 is realizable from Holant$(\Delta_0, \mathcal{F})$, then we reduce Holant$(\Delta_0, \mathcal{F})$ from Holant$^c(\mathcal{F})$ and we are done by the dichotomy of Holant$^c(\mathcal{F})$. By using Δ_0, we first give two conditions that Δ_1 can be easily realized by pinning or interpolation. We show that either Holant$^c(\mathcal{F}) \leqslant_T$ Holant$(\Delta_0, \mathcal{F})$, or every irreducible $f \in \mathcal{F}$ satisfies the following important *first order orthogonality* condition.

▶ **Definition 28** (First order orthogonality). *Let f be a complex-valued signature of arity $n \geqslant 2$, we say that it satisfies* first order orthogonality (1ST-ORTH) *if there exists some $\mu \neq 0$ such that for all indices $i \in [n]$, the entries of f satisfy the following equations*

$$|\boldsymbol{f}_i^0|^2 = |\boldsymbol{f}_i^1|^2 = \mu, \text{ and } \langle \boldsymbol{f}_i^0, \boldsymbol{f}_i^1 \rangle = 0.$$

To restate it in the quantum terminology, let $|\Psi\rangle$ be a normalized n-qubit ($n \geqslant 2$) state, i.e., $\langle \Psi|\Psi\rangle = 1$. Then it satisfies first order orthogonality if for every i-th qubit of $|\Psi\rangle$, $\langle \Psi_i^0|\Psi_i^0\rangle = \langle \Psi_i^1|\Psi_i^1\rangle = 1/2$ and $\langle \Psi_i^0|\Psi_i^1\rangle = 0$.

▶ Remark 29. When f is a real-valued signature, the inner product is just the ordinary dot product which can be represented by mating using $=_2$. Thus, f satisfies 1ST-ORTH iff there is some real $\mu \neq 0$ such that for all indices i, $M(\mathfrak{m}_i f) = \mu I_2$. On the other hand, when $\widehat{f}$ is a signature with ARS, by equation (3.2), the complex inner product can also be represented by mating using $\neq_2$. Thus, $\widehat{f}$ satisfies 1ST-ORTH iff there is some real $\mu \neq 0$ such that for all i, $M(\widehat{\mathfrak{m}}_i \widehat{f}) = \mu N_2$. Moreover, f satisfies 1ST-ORTH iff $\widehat{f}$ satisfies it. Although 1ST-ORTH is well-defined for any complex-valued signature, the properties of $\mathfrak{m}_i f$ and $\widehat{\mathfrak{m}}_i \widehat{f}$ crucially depend on f being real (equivalently $\widehat{f}$ satisfying ARS).

Back to the proof of the #P-hardness of Holant$(\Delta_0, \mathcal{F})$. Since $\mathcal{F}$ does not satisfy condition (T), $\mathcal{F} \nsubseteq \mathcal{T}$. Hence, there is a signature $f \in \mathcal{F}$ of arity $n \geqslant 3$ such that $f \notin \mathcal{T}$. In other words, $\mathcal{F}$ contains an n-qubit state exhibiting multipartite entanglement. We will prove #P-hardness by induction on n. We first consider the base case that $n = 3$. We show that an irreducible ternary signature (a genuinely entangled 3-qubit state) satisfying first order orthogonality has some special forms, from which one can realize $=_3$ or $=_4$ after some holographic transformations. Then, we can reduce the problem from #CSP$(\mathcal{F})$, or

$\#\mathrm{CSP}_2(\mathcal{F})$, or $\mathrm{Holant}(\neq_2|=_3, \widehat{\mathcal{F}})$, to $\mathrm{Holant}(\Delta_0, \mathcal{F})$. This allows us to finish the proof by invoking existing dichotomy results for $\#\mathrm{CSP}(\mathcal{F})$, or $\#\mathrm{CSP}_2(\mathcal{F})$, or the #P-hardness result we showed above for $\mathrm{Holant}(\neq_2|=_k, \widehat{\mathcal{F}})$ where $k \geqslant 3$.

▶ **Lemma 30.** *Let $\mathcal{F}$ be a set of real-valued signatures containing a ternary signature $f \notin \mathfrak{T}$. If $\mathcal{F}$ does not satisfy condition* (T)*, then* $\mathrm{Holant}(\Delta_0, \mathcal{F})$ *is #P-hard.*

Then, we consider the inductive step. The general strategy is that we start with a signature $f \in \mathcal{F}$ of arity $n \geqslant 4$ that is not in $\mathfrak{T}$, and realize a signature g of arity $n-1$ or $n-2$ also not in $\mathfrak{T}$, by pinning or merging. (By the definition of $\mathfrak{T}$, when $n = 4$, this g must have arity 3.) By a sequence of reductions (that is constant in length independent of the problem instance size), we can realize a signature h of arity 3 that is not in $\mathfrak{T}$. Then we are done. In other words, given an n-qubit state with multipartite entanglement, we want to show that multipartite entanglement is preserved under projections onto $|0\rangle$ and self-loops by $|\phi^+\rangle$. Lemma 9 says that the preservation holds, or $|1\rangle$ or $|\mathrm{GHZ}_4\rangle$ is realizable. We give an inductive restatement of Lemma 9 in the Holant framework.

▶ **Lemma 31.** *Let $f \in \mathcal{F}$ be a signature of arity $n \geqslant 4$ and $f \notin \mathfrak{T}$. Then one of the following alternatives must hold:*

- Δ_1 *is realizable:* $\mathrm{Holant}(\Delta_0, \Delta_1, \mathcal{F}) \leqslant_T \mathrm{Holant}(\Delta_0, \mathcal{F})$*, or*
- $=_4$ *is realizable:* $\mathrm{Holant}(=_4, \mathcal{F}) \leqslant_T \mathrm{Holant}(\Delta_0, \mathcal{F})$*, or*
- *a signature $g \notin \mathfrak{T}$ of arity $n-1$ or $n-2$ is realizable:* $\mathrm{Holant}(\Delta_0, g, \mathcal{F}) \leqslant_T \mathrm{Holant}(\Delta_0, \mathcal{F})$.

Proof Sketch. For all indices i and all pairs of indices $\{j, k\}$, consider f_i^0 and $\partial_{jk} f$. If there exists i or $\{j, k\}$ such that f_i^0 or $\partial_{jk} f \notin \mathfrak{T}$, then we can realize $g = f_i^0$ or $\partial_{jk} f$ which has arity $n-1$ or $n-2$, and we are done. Otherwise, f_i^0 and $\partial_{jk} f \in \mathfrak{T}$ for all i and all $\{j, k\}$. Under this assumption, our goal is to show that we can realize Δ_1, or there is a unary signature $a(x_u)$ or a binary signature $b(x_v, x_w)$ such that $a(x_u) \mid f$ or $b(x_v, x_w) \mid f$. Then, we have $f = a(x_u) \otimes g$ or $f = b(x_v, x_w) \otimes g$ for some g of arity $n-1$ or $n-2$. We know g can be realized from f by factorization. By the definition of $\mathfrak{T}$, we have $g \notin \mathfrak{T}$ since $f \notin \mathfrak{T}$, and we are done. When $n \geqslant 5$, the above induction proof can be achieved by the interplay of the unique prime factorization, and the commutivity of f_i^0 (pinning) and $\partial_{jk} f$ (merging) gadgets on disjoint indices. For $n = 4$, there is the additional case that $=_4$ can be realized. Thus for $n = 4$, it requires more work; we need to combine the induction proof and first order orthogonality to handle it. ◀

Remember that $\mathrm{Holant}(\Delta_0, \Delta_1, \mathcal{F})$ is just $\mathrm{Holant}^c(\mathcal{F})$ and $\#\mathrm{CSP}_2(\mathcal{F}) \leqslant_T \mathrm{Holant}(=_4, \mathcal{F})$. By Theorem 22, $\#\mathrm{CSP}_2(\mathcal{F})$ and $\mathrm{Holant}^c(\mathcal{F})$ are both #P-hard when $\mathcal{F}$ does not satisfy condition (T). Combining with Lemmas 30 and 31, we have the following result.

▶ **Lemma 32.** *Let $\mathcal{F}$ be a set of real-valued signatures. If $\mathcal{F}$ does not satisfy condition* (T)*, then* $\mathrm{Holant}(\Delta_0, \mathcal{F})$ *is #P-hard.*

Finally, combining Theorem 21 and Lemmas 24, 27 and 32, we finished the proof of Theorem 10.

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