New Fault Tolerant Subset Preservers

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Abstract

Fault tolerant distance preservers are sparse subgraphs that preserve distances between given pairs of nodes under edge or vertex failures. In this paper, we present the first non-trivial constructions of subset distance preservers, which preserve all distances among a subset of nodes S, that can handle either an edge *or* a vertex fault.

- For an n-vertex undirected weighted graph or weighted DAG $G = (V, E)$ and $S \subseteq V$, we present a construction of a subset preserver with $\widetilde{O}(|S|n)$ edges that is resilient to a single fault. In the single pair case ($|S| = 2$), the bound improves to $O(n)$. We further provide a nearly-matching lower bound of $\Omega(|S|n)$ in either setting, and we show that the same lower bound holds conditionally even if attention is restricted to unweighted graphs.
- For an n-vertex directed unweighted graph $G = (V, E)$ and $r \in V, S \subseteq V$,we present a construction of a preserver of distances in $\{r\} \times S$ with $\widetilde{O}(n^{4/3}|S|^{5/6})$ edges that is resilient to a single fault. In the case $|S| = 1$ the bound improves to $O(n^{4/3})$, and for this case we provide another matching conditional lower bound.
- For an n-vertex directed weighted graph $G = (V, E)$ and $r \in V, S \subseteq V$, we present a construction of a preserver of distances in $\{r\} \times S$ with $\widetilde{O}(n^{3/2}|S|^{3/4})$ edges that is resilient to a single vertex fault. (It was proved in [14] that the bound improves to $O(n^{3/2})$ when $|S| = 1$, and that this is conditionally tight.)

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1 Introduction

Distance Preservers. This paper is about *distance preservers*, a graph-theoretic primitive that appears in work on spanners [16, 32, 23, 17, 2, 1, 24], hopsets [2, 28, 27], shortcutting [27, 2], shortest path algorithms [5, 4, 21, 18], etc.; recently, distance preservers have also been a popular topic in their own right [17, 20, 16, 12, 25, 18].

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▶ **Definition 1** (Distance Preservers). *For a graph $G = (V, E)$ and a set $P \subseteq V \times V$ of "demand pairs", a subgraph $H = (V, E_H \subseteq E)$ is a distance preserver of G, P if*

$$\mathtt{dist}_G(s,t) = \mathtt{dist}_H(s,t) \text{ for all } (s,t) \in P.$$

When $P = S \times S$ for some $S \subseteq V$, we say that H is a subset distance preserver of G, S.

Most often, the goal is to determine the worst-case tradeoff between the number of demand pairs and the number of edges needed for a distance preserver. For example, a classic result in the area is that any p demand pairs in an n-node graph have a distance preserver on $O(np^{1/2})$ edges [20].

The subset distance preserver structure $P = S \times S$ is quite common in the known applications of distance preservers (e.g. [4, 21, 18]), and one of the primary reasons distance preservers were first developed was to address this setting [22]. Despite this, our understanding of subset distance preservers lags far behind our understanding of the general case. There is *no* graph setting in which the structure $P = S \times S$ is known to be useful towards proving sparser distance preservers, and yet our lower bounds get much worse when they are required to have this structure. To illustrate, one of the main questions in the area is whether one can generally have a subset preserver with only a constant number of edges per demand pair:

Is there an absolute constant $c > 0$ such that, for any $|P| = n^{2-c}$ demand pairs in an n-node graph, there is always a distance preserver on $O(|P|)$ edges?

In the *sourcewise* setting $P = S \times V$, the answer is clearly yes (build a shortest path tree rooted at each $s \in S$). In the *pairwise* setting, i.e. P is arbitrary, it was proved by Coppersmith and Elkin [20] that the answer is no (even if attention is restricted to undirected unweighted graphs). But in the intermediate *subset* setting $P = S \times S$, the question is completely open: we can neither prove it for undirected unweighted graphs, nor refute it for directed weighted graphs; all we know is that $c \leq \frac{2}{3}$ [12].

Fault Tolerance. Distance preservers and friends are often applied to networks or distributed systems where parts can spontaneously fail. A natural additional requirement for these applications is *fault-tolerance*, meaning roughly that the preserver is robust to these failures:

▶ **Definition 2** (Fault Tolerant Distance Preservers). *For a graph $G = (V, E)$ and demand pairs P, a distance preserver H of G, P is* fault tolerant (FT) *if for any vertex or edge x we have that $H \setminus \{x\}$ is still a distance preserver of $G \setminus \{x\}$.*[1] *We say H is* vertex (edge) fault tolerant*, abbreviated VFT (EFT), to indicate that x must specifically be a vertex (edge).*

The current literature reflects a world in which vertex faults are harder to analyze than edge faults; many basic questions in the area are closed for EFT but open for (V)FT [19, 9, 8, 13, 15]. To highlight one example:

For a single demand pair (s, t) in an undirected (possibly weighted) graph G, is there always an FT distance preserver on $O(n)$ edges?

In [14], it is proved that the answer is *yes* for the special case of edge failures. The argument leverages a convenient structural fact about edge failures, which may generally explain some of the EFT/VFT discrepancy: a shortest path in $G \setminus \{x\}$ is always the union of two shortest paths in G when x is an edge [3], but nothing like this seems to hold when x is a vertex. Accordingly, the above question is open when the failures can be vertices.

[1] One can also consider a version where several nodes/edges fail at once. However, recent lower bounds have proved that the available preserver quality is quite poor already for $f = 2$ faults [30, 14], so $f = 1$ may be the more applicable setting.

1.1 Our Results

In this work, we show that the $P = S \times S$ structure does actually seem to be useful in the FT setting. For background, it was proved by Bilo et al. [8] that any $|S|$ source nodes in an n-node undirected weighted graph has an EFT preserver on $\widetilde{O}(n|S|)$ edges. Our main results fill in several gaps around this result, extending to the general FT setting and providing corresponding lower bounds.

▶ **Theorem 3** (Undirected Graphs). *For any undirected weighted n-node graph $G = (V, E, w)$ and set of source nodes $S \subseteq V$, there is a general FT subset distance preserver on $\widetilde{O}(n|S|)$ edges. When $|S| = 2$, the bound improves to $O(n)$. Moreover, for any given value of $|S|$ there are examples where $\Omega(n|S|)$ edges are needed.*

We note that Theorem 3 positively resolves the latter open question mentioned above. We think this bound $\widetilde{O}(n|S|)$ is surprising; to explain why, let us compare to the corresponding bounds in the non-faulty setting. Here the current state-of-the-art [20] is that $|S| = \sigma$ source nodes in an n-node graph have a subset distance preserver of size

$$O\left(\min\left\{n + n^{1/2}\sigma^2, n\sigma\right\}\right).$$

In particular, when $\sigma \geq \sqrt{n}$ the bound is $O(n\sigma)$. In other words, following Theorem 3, one can tolerate a fault in this parameter regime essentially *for free*. It is more likely that this reflects the weakness of our current understanding of non-faulty subset distance preservers, rather a world in which fault tolerance is actually free. Still, the non-faulty subset distance preserver bounds have resisted improvement for the last 15 years, so the unintuitive hypothesis of free fault tolerance suggested by Theorem 3 may be hard to refute.

Since Theorem 3 is essentially best possible, for the rest of the paper we investigate to what extent it extends to other graph settings. For example, what if attention is restricted to unweighted graphs? Our lower bound construction fundamentally relies on the use of edge weights, and thus it does not constrain this setting. However, we show that it can be replaced with a *conditional* lower bound: it will not be possible to meaningfully improve Theorem 3 for unweighted graphs without first improving on the non-faulty setting. This result will use a new parameter for distance preservers that we call the *gap*. For a graph G and demand pairs P, the gap is

$$\gamma(P) := \max_{(s,t),(s',t') \in P} \texttt{dist}_G(s,t) - \texttt{dist}_G(s',t').$$

Intuitively, the gap measures how close G, P are to a "layered" instance: except for a few degenerate cases, the gap is 0 if and only if G is a layered graph, P is a subset of the first $\times$ last layer, and the shortest paths for P cross the layers directly without backtracking. We use the following technical hypothesis for our unweighted lower bound:

▶ **Hypothesis 4.** *For any $\sigma = \sigma_n$, there is an n-node undirected unweighted graph G and demand pairs $P = S \times T$ with*

$$|S| \leq \sigma \qquad \text{and} \qquad |T| \leq \sqrt{\frac{n\sigma}{(\gamma(P)+1)}}$$

such that any distance preserver has $\Omega(n\sigma)$ edges.

We remark that this hypothesis is only plausible when $|T| = \Omega(\sigma)$, and hence we need $\gamma(P) = O(n/\sigma)$. But the current understanding of non-faulty distance preservers is compatible with the bounds in Hypothesis 4, even if we were to assume more strongly that $\gamma(P) = 0$.

▶ **Theorem 5.** *Assuming Hypothesis 4, for any $\sigma = \sigma_n$, there are examples of undirected unweighted n-node graphs and node subsets of size $|S| = \sigma$ where any FT distance preserver needs $\Omega(n\sigma)$ edges.*

Thus Theorem 3 is conditionally tight even for unweighted graphs. Next, we ask if it extends to directed graphs. For the special case of DAGs, we observe that any replacement path avoiding a failure can be represented by concatenation of two original shortest paths linked together by an edge (much like the EFT setting). This allows the above content to extend directly to preservers for DAGs.

▶ **Theorem 6** (DAGs)**.** *The upper and lower bound of Theorem 3 both still hold when the input graph G is a weighted DAG. If G is an unweighted DAG, then the lower bound of Theorem 5 still holds as well, assuming that Hypothesis 4 also holds for unweighted DAGs.*

The general directed setting is trickier. The question of single-pair FT preservers for directed weighted graphs was settled in [14]; these need at least $\Omega(n^{4/3})$ and at most $O(n^{3/2})$ edges,[2] and hence Theorem 3 does not extend to this setting. For directed unweighted graphs, we provide the first improvements on this upper bound:

▶ **Theorem 7** (Directed Unweighted Graphs)**.** *For any directed unweighted n-node graph $G = (V, E)$, and demand pairs $(r, S) \in V \times 2^V$ there is an FT distance preserver on $\widetilde{O}(n^{4/3}|S|^{5/6})$ edges. When $|S| = 1$, the bound improves to $O(n^{4/3})$.*

Like before, we show conditional tightness of this bound, although here only in the single-pair setting. Our lower bound needs the following hypothesis:

▶ **Hypothesis 8.** *There is an n-node directed unweighted graph G and demand pairs $P = S \times T$ with*

$$|S|, |T|, \gamma(P) = O(n^{1/3})$$

such that any distance preserver has $\Omega(n^{4/3})$ edges.

Hypothesis 8 is more tenuous than Hypothesis 4, as it is right on the boundary of current techniques. We will discuss the interpretation of these hypotheses more shortly. Still, the point is that one cannot meaningfully improve our single-fault distance preservers without first gaining an improved understanding of the non-faulty case:

▶ **Theorem 9.** *Assuming Hypothesis 8, there are examples of directed unweighted n-node graphs and single demand pairs where any FT distance preserver needs $\Omega(n^{4/3})$ edges.*

We remark that if the goal is just to refute the extension of Theorem 3 to directed graphs, rather than to show exact tightness of Theorem 7, then one can get by with a weaker assumption than Hypothesis 8 (e.g. the preserver lower bound can be $\Omega(n^{1.01})$, or one can trade this off with a relaxation of $|S|, |T|, \gamma(P)$).

Lastly, we mention that our techniques give a nontrivial extension of the single-pair result in [14] to the multi-target setting.

[2] More specifically, it was proved that the correct size bound is exactly that of a non-faulty distance preserver of n demand pairs in an n-node directed weighted graph. Due to [20], this is at least $\Omega(n^{4/3})$ and at most $O(n^{3/2})$.

▶ **Theorem 10** (Directed Weighted Graphs). *For any directed weighted n-node graph $G = (V, E)$ and demand pairs $(r, S) \in V \times 2^V$, there is an FT distance preserver on $\widetilde{O}(n^{3/2}|S|^{3/4})$ edges.*

We recall from [14] that the bound improves to $O(n^{3/2})$ when $|S| = 1$, and that this single-pair bound is unimprovable under the hypothesis that the current bounds for non-faulty distance preservers of directed weighted graphs are tight.

1.2 Interpretation of Hypotheses

Since our lower bounds rely on Hypotheses 4 and 8, we will give them some more context here and discuss how likely they are to be true. Currently, except for a certain tiny range of parameters [12], the state-of-the-art upper bounds for non-faulty $S \times T$ preservers are based entirely on a property called *consistency*:

▶ **Definition 11** (Consistency). *A set of paths Π in a (possibly directed) graph G are* consistent *if, for any $\pi_1, \pi_2 \in \Pi$ with nodes $u < v \in \pi_1, u < v \in \pi_2$, the subpaths $\pi_1[u \rightsquigarrow v], \pi_2[u \rightsquigarrow v]$ are equal.*

In other words, given any set of consistent paths with endpoints in $S \times T$, one can exploit the consistency property to prove upper bounds on the total number of edges contained in the union of all these paths [20, 12]. Interpreting $S \times T$ as demand pairs for a preserver, it is not hard to break shortest path ties in such a way that the chosen paths are consistent, and so these "consistency bounds" yield preserver upper bounds, which are essentially state-of-the-art.

With this in mind, let us imagine a weaker version of Hypotheses 4, where we hypothesize a *consistent* set of $S \times T$ paths with $\Omega(n\sigma)$ edges in their union, the gap $\gamma(S \times T)$ defined as the maximum difference between any two path lengths, and $|S|, |T|$ are bounded as before. From a construction in [16], this hypothesis is true. The similar "consistency-weakened" version of Hypothesis 8 is also true, from an unpublished construction of Bodwin and Reingold.

So, the truth of Hypotheses 4 and 8 essentially depends on how smoothly one can pass from consistent paths to unique shortest paths without destroying the other important properties of the construction, like the edge density and the gap. It is hard to say for sure whether this will be possible. Our guess is that the consistency bounds can be improved – and thus Hypotheses 4, 8 are false – but that this will require major new technical ideas that are not currently in the literature. The main evidence for this is that the consistency bounds have been polynomially improved for general *pairwise* preservers, which do not require the structure $P = S \times T$ [16]. But in the $P = S \times T$ setting the consistency bounds have resisted improvement for the last 15 years [20], despite significant research effort, so we feel that Hypotheses 4, 8 accurately mark the boundaries of current knowledge. Thus we interpret these hypotheses, and the corresponding lower bounds, as proof that no more progress can really be made in the FT setting until the non-faulty setting is understood first.

1.3 Related Work

For an n-vertex unweighted graph G and a set of sources S, Parter and Peleg [31] showed that there exists an $S \times V$ FTP with $O\left(n^{3/2}|S|^{1/2}\right)$ edges. This bound holds also for the case of a single node failure and when the graph is directed. They also showed that this result is existentially optimal, namely, there exist n-vertex graphs and a set of sources S such that any $S \times V$ FTP has at least $\Omega\left(n^{3/2}|S|^{1/2}\right)$ edges.

For the more general case of $P = S \times T$, [8] showed an FTP of size $\widetilde{O}\left(n^{4/3}|S|^{1/3}|T|^{1/3}\right)$ under edge failure. An FTP for a single pair in undirected (possibly weighted) graph of linear

size was explicitly presented in [14], but was also implied by previous replacement-paths algorithms, e.g., [29].

Considerably much less is known for the multiple fault setting (even for edges faults). Parter [30] showed an upper bound of $O(n^{5/3})$ edges for $\{s\} \times V$ (edge) f-FTP with $f = 2$, in undirected unweighted graphs. A matching lower bound of $\Omega\left(|S|^{1/(f+1)} \cdot n^{2-1/(f+1)}\right)$ for sourcewise ($P = S \times V$) f-FTP was also provided in [30], for any f. Gupta and Khan [26] extended this result, providing a tight upper bound for sourcewise ($S \times V$) f-FTP with $f = 2$. This bound holds also for the case of a two vertex failures and when the graph is directed. For the more general case of f failures, Bodwin et al. [14] showed an upper bound of$\widetilde{O}(f \cdot |S|^{1/2^f} \cdot n^{2-1/2^f})$ edges for a $S \times V$ f-FTP. This result holds under both edge and vertex faults, and in directed graphs. For weighted graphs, [14] showed that even a single pair f-FTP with $f \geq 2$ has $\Theta(n^2)$ edges.

Baswana and Khanna [7] proved the existence of a $(1+\epsilon)$ multiplicative vertex FT spanner for $P = \{s\} \times V$, with $O(n/\epsilon^3 + n \log n)$ edges, for any $\epsilon > 0$. Recently, Bilo et al. [11] improved this result to $O(n \log n/\epsilon^2)$, for both edge and vertex single failure. In [10], Bilò et al. showed construction of approximate FTP to handle multiple edge failures. They showed that for any $f \geq 1$ and for $P = \{s\} \times V$, we can compute an FTP $O(fn)$ size that after failure of f edge preserves distance up to a multiplicative stretch of $(2f + 1)$.

1.4 Preliminaries and Tools

Graph Notations. We use the following graph notations and definitions in the context of a given undirected graph $G = (V, E, w)$ with $n = |V|$, $m = |E|$ and a weight function $w : E \to \mathbb{R}^+$. To avoid complications due to shortest-paths of the same length, we assume throughout that all shortest path are computed with a consistent tie-breaking function π that guarantees the uniqueness of the shortest-paths[3]. For every $x, y \in V$, and a subgraph $G' \subseteq G$, let $\pi(x, y, G')$ be the (unique) x-y shortest path in G', when $G' = G$ we may simply write $\pi(x, y)$. For any path P, let $P[x, y]$ be the subpath of P between x and y and let $P(x, y) = P[x, y] \setminus \{x, y\}$. For a node $v \in V$, let T_v be the shortest path tree rooted at v. For $v, x \in V$, let $T_v(x)$ the subtree of T_v rooted at x.

For $s, t \in V$ and a failure $x \in V$, the *replacement path* $P_{s,t,x}$ is the unique s-t shortest path in $G \setminus \{x\}$. To avoid cumbersome notation, when s or t are clear from context, we may omit them from the notation. Let $D_{s,t,x}$ be the *detour* segment of the replacement path defined by $P_{s,t,x} \setminus \pi(s, t)$.

For a path P, let $E(P)$ denote its edges and $V(P)$ denote its vertices. Similarly, for a collection of paths $\mathcal{P}$, we denote their edges and vertices with $E(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} E(P)$ and $V(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} V(P)$ respectively. We denote the first and last vertex of a path P by first(P) and last(P) respectively. Similarly, for a collection of path $\mathcal{P}$, we denote their sources and terminals with first$(\mathcal{P}) = \bigcup_{P \in \mathcal{P}}$ first(P) and last$(\mathcal{P}) = \bigcup_{P \in \mathcal{P}}$ last(P). For a tree T and vertices u, v let LCA(u, v) denote the *least common ancestor* of u and v in T.

Heavy Path Decomposition. For a shortest path tree T_s rooted at s, we use the *heavy-path decomposition* technique devised by Sleator and Tarjan [33] in order to break the tree T_s into vertex-disjoint paths with several desired properties. The following lemma summarizes the main properties of this partitioning scheme (proof can be also found in [7]).

[3] See Def. 11 for a formal definition of a consistent tie-breaking.

▶ **Lemma 12** ([33]). *There exist a linear time algorithm that given an n-vertex tree T computes a path Q in T whose removal breaks T into vertex-disjoint subtrees $T_1, ..., T_\ell$ such that for each $i \leq \ell$:*

- $|V(T_i)| \leq n/2$ *and* $V(Q) \cap V(T_i) = \emptyset$,
- T_i *is connected to* Q *through some edge.*

The desired decomposition is obtained by recursively applying Lemma 12 (on each of the subtrees T_i), getting a partition of T into vertex disjoint paths $\mathcal{HP}(T)$. A useful property of the decomposition is that for every edge $(u, v) \in T$ that does not appear in any of the paths $\mathcal{HP}(T)$, it holds that $|V(T_s(v))| \leq |V(T_s(u))|/2$. This yields the following useful lemma.

▶ **Lemma 13** ([7]). *For any vertex v, its path to the root in T intersects at most $\log n$ paths in $\mathcal{HP}(T)$.*

2 Fault Tolerant Preservers for Undirected Graphs

2.1 Preservers for Undirected Weighted Graphs

▶ **Theorem 14.** *Any undirected (possibly weighted) $G = (V, E, w)$ and a set $S \subseteq V$ of sources has a $S \times S$ subset (vertex) fault tolerant preserver H with $O(n|S| \log n)$ edges.*

The subgraph H simply contains all the $S \times S$ replacement paths, that is,

$$H = \{P_{s,t,x} \mid s, t \in S,\ x \in \pi(s, t)\}.$$

To prove Theorem 14, we will provide a bound on the size of H.

Size Analysis. A replacement path $P_{s,t,x}$ is *short* if it has at most $n/|S|$ edges; otherwise it is *long*. Throughout this section, we mainly focus on bounding the number of edges in the collection of all short replacement paths, denoted throughout by $\mathcal{P}_{short}$:

▶ **Lemma 15.** $|E(\mathcal{P}_{short})| = O(n|S| \log n)$.

We start by observing that to prove Theorem 14 it is indeed sufficient to consider only the short replacement paths.

▷ Claim 16. Lemma 15 implies Theorem 14.

Proof. Let S^* be a sample of nodes obtained by including each node independently with probability $p = |S|/n$, and let $S' = S^* \cup S$. For each edge e in a long replacement path $P = P_{s,t,x}$, by standard Chernoff bounds, with constant probability there are nodes $s_1, s_2 \in S' \cap P$, separated by $\leq n/|S|$ edges in P, such that e comes between s_1 and s_2 in P. We thus have $e \in P_{s_1,s_2,x}$, so e is now part of a short replacement path. Hence e is counted in Lemma 15 with at least constant probability. Since only $O(n|S| \log n)$ edges may be counted in this way, by linearity of expectation it follows that there are $O(n|S| \log n)$ total edges in long replacement paths. ◁

Road-Map for Proving Lemma 15. From now on, we focus on the collection of short replacement paths, $\mathcal{P} = \mathcal{P}_{short}$. We will show the existence of a subgraph H' with $\widetilde{O}(|S|n)$ edges that contains all the edges of $\mathcal{P}$. This subgraph will be the union of three auxiliary sub-graphs each containing a different subset of replacement paths. For any source $s \in S$, we first show the existence of a subgraph H_s of size $\widetilde{O}(n)$ such that $H_s \cup T(S)$ include

most of the replacement paths which originate at s, where $T(S) = \cup_s T_s$. Then, letting $H_2 = \bigcup_{s \in S} H_s$, we have that $|H_2| = \widetilde{O}(n|S|)$. Finally, the third subgraph H_3 will include a collection of left-over $O(|S|^2 \log n)$ (short) replacement paths, and thus its size will be bounded by $O(n|S| \log n)$ as well.

Partitioning of Replacement Paths based on Heavy-Path Decomposition. Throughout, we consider a fixed source node $s \in S$ and the set $\mathcal{P}_s = \{P_{s,t,x} \mid P_{s,t,x} \in \mathcal{P}, t \in S\}$ of all its short replacement paths. We then divide these replacement paths into several subsets based on the heavy-path decomposition $\mathcal{HP}(T_s)$ of the shortest-path tree T_s as follows. For every path $Q \in \mathcal{HP}(T_s)$, let

$$\mathcal{P}_s(Q) := \{P_{s,t,x} \mid x \in Q \text{ and } x \neq \mathrm{LCA}(t, \mathrm{last}(Q))\} \quad \text{and} \quad \mathcal{P}'_s = \bigcup_{Q \in \mathcal{HP}(T_s)} \mathcal{P}_s(Q) \ .$$

We will also define a small subset of the *left-over* replacement paths $\mathcal{L}_s = \mathcal{P}_s \setminus \mathcal{P}'_s$. The analysis shows that $|\mathcal{L}_s| = O(|S| \log n)$, and since all these replacement paths are short, the total number of edges in these left-over replacement paths can be bounded by $O(|S| n \log n)$. We next bound the number of edges in each of the subsets $\mathcal{P}_s(Q)$, enjoying the fact that the failures of all the replacement paths in these sets lie on a *single* path.

Bounding the number of edges in $\mathcal{P}_s(Q)$ (failure on a single path). Let $Q = \langle x_0, ..., x_k \rangle$. Following Baswana and Khanna [7], for every $x_i \in Q$, we define the vertex partition of $T_s \setminus \{x_i\}$ into

$$U_i : V(T_s \setminus T_s(x_i)), \quad D_i := V(T_s(x_{i+1})) \text{ and } O_i := V(T_s) \setminus (U_i \cup D_i \cup \{x_i\}).$$

Note that for any $i \neq j$, O_i and O_j are pairwise disjoint. This property is crucial for our construction.

▶ **Observation 17.** *Fix $x_i \in Q$. If $P_{s,t,x_i} \in \mathcal{P}_s(Q)$ and $P_{s,t,x_i} \neq \pi(s,t)$ then $t \in D_i$.*

Proof. Since $U_i \cup O_i \cup D_i = V \setminus \{x_i\}$, we need to rule out two cases. (i) Assume that $t \in U_i$. In this case $x_i \notin \pi(s,t)$, and thus $P_{s,t,x_i} = \pi(s,t)$. (ii) Assume that, $t \in O_i$. In this case, we must also have that $x_i = \mathrm{LCA}(t, \mathrm{last}(Q))$ and thus $P_{s,t,x_i} \notin \mathcal{P}_s(Q)$ (this path will be included in the left-over set $\mathcal{L}_s$). As $U_i \cup O_i \cup D_i = V$, we conclude that $t \in D_i$. ◀

▶ **Lemma 18.** *Fix $x_i \in Q$ and $t \in S \cap D_i$. The replacement path $P = P_{s,t,x_i}$ is of the form $p_1(P) \circ e_1(P) \circ p_2(P) \circ e_2(P) \circ p_3(P)$ where $e_1(P), e_2(P)$ are edges, $p_1(P) \subseteq T_s$, $p_3(P) \subseteq T_t$ and $V(p_2(P)) \subseteq O_i$. Each of the edges $e_1(P), e_2(P)$ (but not both) and some of the paths $p_i(P)$, $i \in \{1,2,3\}$ might be empty.*

Proof. Let $P_i = p_i(P)$ for $i \in \{1,2,3\}$, and $e_j = e_j(P)$ for $j \in \{1,2\}$. Let z be the *first* vertex of the path P (i.e., closest to s) that belongs to D_i. Let u be the *last* vertex of the path P (i.e., closest to t) that belongs to U_i. Let $P_1 = P[s,u]$ and $P_3 = P[z,t]$. We first claim that $P_1 \subset T_s$. To see this, observe that since $u \in U_i$, $x_i \notin \pi(s,u)$ and thus $P_1 = \pi(s,u) \subseteq T_s$. Since $P_1 = \pi(s,u)$, it also implies that u appears strictly before z on the path P.

Next, we show that $P_3 \subset T_t$ by proving that $x_i \notin \pi(z,t)$. Assume towards contradiction otherwise, i.e., that $x_i \in \pi(z,t)$. We have that $\pi(z,t) = \pi(z,x_i) \circ \pi(x_i,t)$. Since (x_i, x_{i+1}) appears on both of the paths $\pi(z,x_i)$ and $\pi(x_i,t)$, we get an alternative z-t path $\pi(z,x_{i+1}) \circ \pi(x_{i+1},t)$ that is strictly shorter, leading to a contradiction. As $x_i \notin \pi(z,t)$, we get that $P_3 = \pi(z,t) \subset T_t$ as required. It remains to consider the path $P(u,z)$. If $P \cap O_i = \emptyset$ then

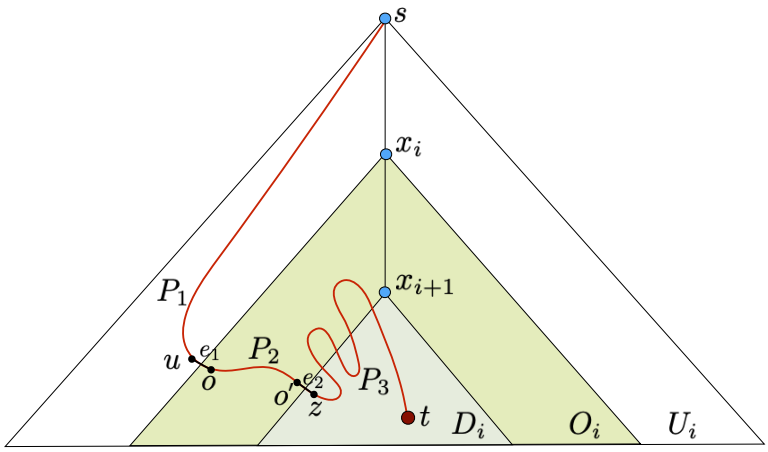

Figure 1 The partition of the path $P = P_{s,t,x_i}$ suggested in Lemma 18 is presented. $P_1 = p_1(P)$ is the portion of P that appears before o, $P_3 = p_3(P)$ is the portion of P from z, and the remaining portion is $P_2 = p_2(P)$. In the figure, $e_1 = e_1(P)$ and $e_2 = e_2(P)$.

we are done. Otherwise, Let o, o' be the first (resp. last) vertices in O_i on the path P. Note that it might be the case that $o = o'$. Letting $e_1 = (u, o)$, $e_2 = (o', z)$, and $P_2 = P[o, o']$, we get that $P = P_1 \circ e_1 \circ P_2 \circ e_2 \circ P_3$ and by definition $V(P_2) \subseteq O_i$. The claim follows. ◀

From now on, for any path $P \in \mathcal{P}_s(Q)$, let $p_i(P), e_j(P)$ denote the partition defined in Lemma 18 (for $i \in [3], j \in [2]$). As $p_1(P), p_3(P) \subseteq T(S)$, it remains to mainly bound the edges contributed by $p_2(P)$ for every $P \in \mathcal{P}_s(Q)$. To do that, we focus on a fixed failure $x_i \in Q$ and define a special graph G_i (which is not necessarily a sub-graph of G) in which $p_2(P)$ is a shortest path. A similar approach has been taken in [7]. For every $x_i \in Q$, define the graph $G_i = (V_i, E_i)$ such that $V_i = O_i \cup \{s\}$ and E_i includes the edges of $G[O_i]$ and the following additional edges. For each $o \in O_i$ with a neighbor in U_i, E_i contains the edge (s, o) with weight $\min_{(u,o)\in E, u\in U_i}\{w(\pi(s, u, G)) + w((u, o))\}$. Letting τ_i denote the shortest paths tree rooted at s in G_i, define

$$H(s, Q) := \Big(\bigcup_{i<k}(\tau_i \cap G)\Big) \cup \Big(\bigcup_{P\in\mathcal{P}_s(Q), j\in\{1,2\}} e_j(P)\Big).$$

We show the following:

▶ **Lemma 19.**
(i) *For every $P_{s,t,x_i} \in \mathcal{P}_s(Q)$, it holds that $P_{s,t,x_i} \subseteq T(S) \cup H(s, Q)$.*
(ii) $|E(H(s, Q))| = O(|T_s(\text{first}(Q))| + |\mathcal{P}_s(Q)|)$.

Proof. (i) Since $p_1(P_{s,t,x_i}), p_3(P_{s,t,x_i}) \subseteq T(S)$ and $e_1(P_{s,t,x_i}), e_2(P_{s,t,x_i}) \in H(s, Q)$, it is sufficient to show that $p_2(P_{s,t,x_i}) \subseteq \tau_i \cap G$. Specifically, we show that $p_2(P_{s,t,x_i})$ is a suffix of a shortest path rooted in s in the graph G_i.

Let o, o' be the first (resp., last) nodes in $P_{s,t,x_i} \cap O_i$. First, observe that the existence of an edge $e = (s, o) \in G_i$ with weight $w(e)$ implies that there exists a path in $G[U_i \cup O_i]$ of weight $w(e)$ from s to o, such that all of its vertices are in U_i except the last one. Thus, if there exists a path of weight W from s to $o \in O$ in G_i, it implies that there exists a path in $G[U_i \cup O_i]$ of weight W from s to o. We now claim $p_2(P_{s,t,x_i}) \subseteq \tau_i \cap G$. Let $u \in U_i$ be the vertex preceding o on the path P_{s,t,x_i} such that $e_1(P_{s,t,x_i}) = (u, o)$ (i.e., alternatively, u is the last vertex in U_i on the path P_{s,t,x_i}). By the optimal subpath property,

$P_{s,t,x_i}[s,o']$ is a shortest path in $G \setminus \{x_i\}$, and by Lemma 18 it is in the subgraph $G[U_i \cup O_i]$. Recall that $V(P_{s,t,x_i}[s,u]) \subseteq U_i$ and $V(P_{s,t,x_i}[o,o']) \subseteq O_i$. Thus $e_1(P_{s,t,x_i})$ is the edge that minimizes $w(\pi(s,u,G)) + w((u,o))$, namely, $w((s,o)) = w(\pi(s,u,G)) + w((u,o))$. Therefore $(s,o) \circ P_{s,t,x_i}[o,o']$ is a shortest path from s to o' in G_i, and $p_2(P_{s,t,x_i}) \subseteq \tau_i \cap G$.

We proceed with (ii). As $V(\tau_i \cap G) \subseteq O_i$, by the mutual disjointness of O_i's, it follows that

$$\sum_{i<k} |\tau_i \cap G| \leq \sum_{i<k} |O_i| \leq |T_s(\text{first}(Q))| \ .$$

The claim follows by noting that contribution of $e_1(P)$ and $e_2(P)$ for every $P \in \mathcal{P}_s(Q)$ is at most $2|\mathcal{P}_s(Q)|$. ◀

Interestingly, Lemma 19(ii) yields an immediate linear upper bound for single pair preservers. That is, in the case where $S = \{s,t\}$ and $Q = \pi(s,t)$, the set $\mathcal{P}_s(Q)$ contains all the replacement paths. As $|\mathcal{P}_s(Q)|$ includes at most one path for each failure, its size is bounded by $|\pi(s,t)| = O(n)$.

▶ **Corollary 20** (Single-Pair FT Preservers). *Any undirected (possibly weighted) graph $G = (V,E)$, and a pair of nodes $s,t \in V$ has a single-pair FT Preserver of linear size.*

Bounding the set $\mathcal{P}'_s$. So far, we assume that the failing vertex appears on a fixed path $Q \in \mathcal{HP}(T_s)$. We next bound the total number of edges in the union of all paths $\mathcal{P}'_s = \bigcup_{Q \in \mathcal{HP}(T_s)} \mathcal{P}_s(Q)$. By Lemma 19, every $P_{s,t,x_i} \in \mathcal{P}_s(Q)$ is contained in $T(S) \cup H(s,Q)$. Therefore, it remains to bound the number of edges in the subgraph $H_s = \cup_{Q \in \mathcal{HP}(T_s)} H(s,Q)$.

▶ **Lemma 21.** $|E(H_s)| = O(n \log n)$.

Proof. Since $H_s = \cup_{Q \in \mathcal{HP}(T_s)} H(s,Q)$, by Lemma 19 it is sufficient to show that:

- (i) $\Sigma_{Q \in \mathcal{HP}(T_s)} |T_s(\text{first}(Q))| = O(n \log n)$ and
- (ii) $\Sigma_{Q \in \mathcal{HP}(T_s)} |\mathcal{P}_s(Q)| = O(n \log n)$.

Begin with (i). By Lemma 13, for every $t \in S$, the s-t path in T_s intersects with at most $\log n$ paths $Q \in \mathcal{HP}(T_s)$. Since a vertex t appears in $T_s(\text{first}(Q))$ only if $\text{first}(Q) \in \pi(s,t)$, it holds that each vertex belongs to at most $\log n$ such subtrees.

We proceed with (ii) and first show that $\Sigma_{Q \in \mathcal{HP}(T_s)} |\mathcal{P}_s(Q)| \leq |\mathcal{P}'_s|$. Recall that $\mathcal{P}_s(Q)$ includes the replacement paths of $\mathcal{P}'_s$ whose failure appears in Q. Since $\mathcal{HP}(T_s)$ is a partition of T_s, every two paths $Q \neq Q'$ in this partitioning are vertex disjoint. Thus for $Q \neq Q' \in \mathcal{HP}(T_s)$, $\mathcal{P}'_s(Q) \cap \mathcal{P}'_s(Q') = \emptyset$, implying that $\{\mathcal{P}'_s(Q)\}_{Q \in \mathcal{HP}(T_s)}$ is a partition of $\mathcal{P}'_s$ and the claim follows.

Next, we show that $|\mathcal{P}'_s| = O(n \log n)$ by claiming that for every $t \in S$, there are at most $O(n \log n / |S|)$ replacement paths between s and t in $\mathcal{P}'_s$. Fix $t \in S$ and let $\mathcal{P}'_s(t) = \{P_{s,t,x} \mid P_{s,t,x} \in \mathcal{P}'_s\}$. Our goal is to show that $|\mathcal{P}'_s(t)| \leq O(n \log n / |S|)$. Note that in the case where G is unweighted, for every short replacement path $P_{s,t,x}$ with hop-length of $O(n \log n / |S|)$, it holds that the original shortest path $\pi(s,t)$ is short as well, and thus it has at most $O(n \log n / |S|)$ vertex failures that require a replacement path. Consider a shortest path $\pi(s,t)$ in a weighted graph and let $\ell := c \cdot n \log n / |S|$. In the case that $\pi(s,t)$ has hop-length less than 2ℓ, the lemma trivially holds. Otherwise, we assume $|\pi(s,t)| > 2\ell$. Let s' be the ℓ'th vertex on the path from s, namely $\pi(s,s')$ has hop-length of ℓ. Similarly, let t' be the ℓ'th vertex on the path from t, namely $\pi(t',t)$ has hop-length of ℓ. Observe that any short replacement path $P \in \mathcal{P}'_s(t)$ avoids both s', t', otherwise its length is larger than ℓ. This implies that the starting point of any detour of a path in $\mathcal{P}'_s(t)$ is before s' and its ending point is after t'. We show that there is at most one replacement path $P \in \mathcal{P}'_s(t)$ for all the

failures in $\pi(s', t')$. Assume towards contradiction that there are two replacement paths $P_v, P_u \in \mathcal{P}'_s(t)$, and $u, v \in \pi(s', t')$. As both P_v, P_u avoid $\pi(s', t')$, both avoid u, v. Thus, we get that both paths are shortest paths in $G \setminus \{u, v\}$, which is a contradiction to the uniqueness of the shortest paths. As any other shortest path must have a failure on $\pi(s, s')$ or $\pi(t', t)$, we have that $|\mathcal{P}'_s(t)| \leq 1 + 2\ell$. Thus, for any $t \in S$, we have that $|\mathcal{P}'_s(t)| = O(n \log n/|S|)$, and $|\mathcal{P}'_s| = O(n \log n)$, as desired. ◀

Bounding the number of edges in the left-over subset $\mathcal{L}_s$. It remains to bound the number of edges contributed by the left-over replacement paths.

▷ Claim 22. $|\mathcal{L}_s| = O(|S| \log n)$ and thus $|E(\mathcal{L}_s)| = O(n \log n)$.

Proof. We first claim that for every fixed $t \in S$, $\mathcal{L}_s$ contains $O(\log n)$ replacement paths between s and t. Thus, $|\mathcal{L}_s| = O(|S| \log n)$. Recall that $P_{s,t,x} \in \mathcal{L}_s$ if and only if there exists a path $Q \in \mathcal{HP}(T_s)$ such that $x \in Q$ and $x = \mathrm{LCA}(t, \mathrm{last}(Q))$. Thus, we have that $x \in \pi(s, t) \cap Q$. According to Lemma 13, there are at most $\log n$ paths from $\mathcal{HP}(T_s)$ intersecting with $\pi(s, t)$ in T_s. Since an *LCA* is unique for every pair of nodes, for each such path Q there is only one failure x such that $x = \mathrm{LCA}(t, \mathrm{last}(Q))$. Since each path in $\mathcal{L}_s$ is short, i.e., has at most $O(n/|S|)$ edges, the total number of edges in the paths of $\mathcal{L}_s$ is bounded by $O(n|S| \log n)$ as desired. ◁

We are now ready to complete the proof of Lemma 15.

Proof of Lemma 15. The collection of all short replacement paths $\mathcal{P}$ is divided into $\bigcup_s \mathcal{P}'_s$ and the left-over sets $\bigcup_s \mathcal{L}_s$. By Lemma 19(i), $E(\mathcal{P}'_s) \subseteq H_s \cup T(S)$. By Lemma 21, $|E(H_s)| = O(n \log n)$ and thus the total number of edges in $\bigcup_s \mathcal{P}'_s$ is bounded by $O(n|S| \log n)$. By Claim 22, $|E(\mathcal{L}_s)| = O(n \log n)$ and thus $E(\bigcup_s \mathcal{L}_s) = O(n|S| \log n)$. The lemma follows. ◀

2.2 Preservers for Undirected Unweighted Graphs

We next extend our constructions to the $S \times T$ setting. In [8], such an extension has been provided for the case of the single edge failure. We obtain the following theorem.

▶ **Theorem 23.** *For any undirected unweighted $G = (V, E)$ and subsets $S, T \subseteq V$, one can compute a (vertex) fault tolerant $S \times T$ preserver H with $\widetilde{O}(n^{4/3}|S|^{1/3}|T|^{1/3})$ edges.*

Proof. The subgraph H simply contains all the $S \times T$ replacement paths, i.e., $H = \{P_{s,t,x} \mid s, t \in S,\ x \in \pi(s, t)\}$. We will bound size the size of this subgraph, by bounding the size of subgraph H' that contains H. Let R be a random subset of $O(n/L)$ nodes where L is a parameter to be optimized later. Let $\ell = \lceil L \log n \rceil$. The subgraph $H' = H_1 \cup H_2$ is defined as follows.

1. Let H_1 be an $W \times W$ FPT for $W = R \cup S$ obtained by Theorem 14.
2. Let $H_2 = \{\ell\text{-length suffix of } P_{s,t,x} \mid s, t \in S \times T, x \in \pi(s, t), \mathtt{dist}_G(x, t) \leq \ell\}$.

Correctness. Fix $\{s, t\} \in S \times T$ and $x \in \pi(s, t)$. First assume that $|P_{s,t,x}| \leq \ell$. It then implies that also $|\pi(s, t)| \leq \ell$ and thus also that $\mathtt{dist}(x, t) \leq \ell$. Concluding that $P_{s,t,x} \subseteq H_2$. Next, assume that $|P_{s,t,x}| > \ell$. By Chernoff bound, w.h.p. it holds that $|P_{s,t,x} \cap R| \neq \emptyset$. Let $r \in R$ be the closest vertex to t from R on $\pi(s, t)$, we then have that $P_{s,t,x} = P_{s,r,x} \circ P_{r,t,x}$, and $|P_{r,t,x}| \leq \ell$ (as r is the closest vertex to t from R). Since H_1 is an $W \times W$ FPT, we have that $P_{s,r,x} \subset H_1$. It is left to show that $P_{r,t,x}$ is included

in either H_1 or H_2. First, assume that $\texttt{dist}(x,t) > \ell$. As $\ell \geq |P_{r,t,x}| \geq |\pi(r,t)|$, it holds that $x \notin \pi(r,t)$ and thus $P_{r,t,x} = \pi(r,x)$. Since the $W \times W$ preservers of Theorem 14 contains all BFS trees of the W nodes, it includes the BFS tree T_r, and $P_{r,t,x} \subset H_1$. Next, assume that $\texttt{dist}(x,t) \leq \ell$. By construction, as $|P_{r,t,x}| \leq \ell$, it holds that $P_{r,t,x} \subset H_2$.

Size Analysis. By Theorem 14, $|E(H_1)| = O\left(n(|S| + n/\ell)\log^2 n\right)$. In addition, $|E(H_2)| = O(|S||T|\ell^2)$ as it includes ℓ edges of ℓ s-t replacement paths for every s,t pair. Letting $\ell = n^{2/3}(|S||T|)^{-1/3}$ gives $O(n^{4/3}(|S||T|)^{1/3}\log^2 n)$. This bound matches the state of the art bound known for one edge failure by [8] (Theorem 11). ◀

3 Fault Tolerant Preservers for General Directed Graphs

We start by defining a couple of notations needed for our constructions. Recall the for each pair of vertices $s,t \in V$, and a node fault $x \in \pi(s,t)$, the detour of replacement path $P_{s,t,x}$ is represented as $D_{s,t,x}$. We denote $\partial D_{s,t,x}$ to denote the partial subpath of $D_{s,t,x}$ obtained by removing the first and last vertex from $D_{s,t,x}$.

Any partial detour $\partial D_{s,t,x}$ is a shortest path in the graph $G \setminus \pi(s,t)$, so we have following lemma.

▶ **Lemma 24.** *For any $s,t \in V$, and $x \in \pi(s,t)$, the path $\partial D_{s,t,x}$ is a shortest path in $G \setminus \pi(s,t)$.*

We now describe the fault-tolerant distance preservers with respect to pair-set $\{r\} \times S$, for a given choice of a root node $r \in V$ and a set $S \subseteq V$. Let T represent the shortest path tree rooted at r in G. We initialize H_0 to T. Further for each $s \in S$ and failure $x \in \pi(r,s)$, we add the two edge present in $E(D_{s,t,x}) \setminus E(\partial D_{s,t,x})$ to H_0. In the process, we add at most $O(n|S|)$ edges to H.

For a directed path Q, we use "H_Q" to represent the minimal sub-graph such that $H_0 + H_Q$ is a 1-FT distance preserver for pairs in $(r \times S)$ when vertex-faults in G are *restricted* to path Q. One of our main-contributions in achieving sparseness for (non-acyclic graphs) is in obtaining tight bound over the size of H_Q. This is captured in the Proposition 25 and Proposition 29.

3.1 Preservers for Directed Weighted Graphs

We obtain the following bound on H_Q for directed weighted graphs.

▶ **Proposition 25.** *For any directed path $Q = (y \rightsquigarrow z)$ in T (between two vertices y, z with y being ancestor of z in T), the graph H_Q, for a directed possibly weighted graph G, requires at most $O(|T(y)|\ |S|^{3/4}\ \sqrt{|Q|})$ edges, where $|Q|$ denotes the number of vertices in path Q.*

Proof. Let Q be a directed $y \rightsquigarrow z$ tree-path in T, for some $y,z \in V$ with y being ancestor of z. Let $k = |S \cap T(y)|$, and α be an integer parameter to be decided later on. For simplicity we assume $K = k/\alpha$ is integer. Let $S_1, \ldots, S_\alpha$ be an arbitrary partition of $S \cap T(y)$, each of size K, satisfying the constraint that for each $(s,s') \in S_i \times S_{i+1}$, $\text{LCA}(s,z)$ is either equal to or an ancestor of $\text{LCA}(s',z)$. Further let $w_0 = y$, w_i be $\text{LCA}(S_i \cup \{z\})$ for $1 \leq i \leq \alpha$, and $w_{\alpha+1} = z$. Observe that for $i \in [1,\alpha]$, w_i is either equal to, or a ancestor of w_{i+1}. Partition Q in consecutive segments (blocks): $B_0 = Q[w_0, w_1]$, $B_1 = Q[w_1, w_2]$, $\ldots$, $B_\alpha = Q[w_\alpha, w_{\alpha+1}]$. Further, let L_i be the number of vertices in B_i, for $0 \leq i \leq \alpha$. (See Figure 2).

We will analyze the failures in each block separately. Fix an index $1 \leq i \leq K$. We distinguish two cases as below.

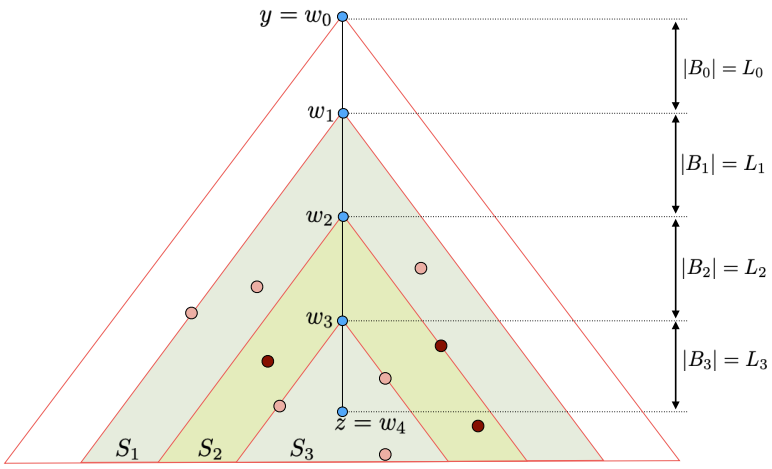

Figure 2 Depiction of block-partitioning of $y \rightsquigarrow z$ tree-path, for $K = \alpha = 3$. Observe that the sets $S_1, S_2, \ldots, S_\alpha$ are each of size $K = 3$.

Analysis of replacement path to vertices in S_i on vertex failure in B_i. Consider a pair $(f, s) \in B_i \times S_i$. Observe that on failure of f, the partial detour $\partial D_{r,s,f}$ (of replacement path $P_{r,s,f}$) is a shortest path in $G \setminus B_i[w_i, \mathrm{LCA}(s, z)]$. (Recall $B_i[w_i, \mathrm{LCA}(s, z)]$ is the sub-path of B_i comprising of those vertices that are ancestor of s in T). Furthermore, it follows from our tie-breaking scheme that none of the vertices of $\partial D_{r,s,f}$ can lie outside $T(y)$. So, for a fixed $s \in S_i$, to compute an (r, s) FT-preserver containing partial-detours corresponding to $f \in B_i$, we can simply use Coppersmith-Elkin's [20] pairwise-distance preserver over graph $G[T(y)] \setminus B_i[w_i, \mathrm{LCA}(s, z)]$, where, $G[T(y)]$ is the graph induced by vertices in subtree $T(y)$.

Since fault f has L_i choices, this gives a bound of $O(|T(y)| \cdot \sqrt{L_i})$ edges, for a single $s \in S_i$. On summing over $|S_i| = K = k/\alpha$ nodes in S_i, and each of the α blocks, we get a bound (say X_1):

$$X_1 = O\Big(|T(y)| \cdot \frac{k}{\alpha} \cdot \sum_{i \in [1,\alpha]} \big(\sqrt{L_i}\big)\Big) \tag{1}$$

Analysis of replacement path to vertices in $\cup_{j>i} S_j$ on vertex failure in B_i. Consider a pair $(f, s) \in B_i \times \cup_{j>i} S_j$. Observe that on failure of f, the partial detour $\partial D_{r,s,f}$ is a shortest path in $G \setminus B_i$, since all the vertices in B_i are ancestor of s. So here we will use a distance preserver over graph $G[T(y)] \setminus B_i$, and the number of pairs (as well as partial-detours) is at most $|S \cap T(y)| \cdot L_i$. Hence, handling failures on B_i incurs us $O(|T(y)| \cdot \sqrt{k\ L_i})$ cost. On summing over all blocks, we get a bound (say X_2):

$$X_2 = O\Big(|T(y)| \cdot \sum_{i \in [0,\alpha]} \big(\sqrt{k \cdot L_i}\big)\Big) \tag{2}$$

For a given α, we have $X_1 \leq O(|T(y)| \cdot (k^2/\alpha)^{1/2} \cdot \sqrt{|Q|})$ and $X_2 \leq O(|T(y)| \cdot (k\alpha)^{1/2} \cdot \sqrt{|Q|})$, where $|Q|$ denotes the number of vertices on path Q. Optimizing over α, we get α must be $\Theta(\sqrt{k})$. This provides a bound of at most $O(|T(y)| k^{3/4} \sqrt{|Q|})$ edges on the size of H_Q. ◀

We are now ready to prove our results for directed weighted graphs. Previously it was know by Bodwin et al. [14] that for a single-pair $(r, s) \in V \times V$, we can compute a FTP with at most $O(n^{1.5})$ edges. A direct implementation of this result over pairs in $\{r\} \times S$ would result in a bound of $O(n^{1.5}|S|)$ edges. However using Proposition 25 and heavy-path decomposition, we are able to obtain a better bound of $o(n^{1.5}|S|)$ size for the $\{r\} \times S$ setting.

Let $Q_1 = (y_1 \rightsquigarrow z_1), \ldots, Q_\gamma = (y_\gamma \rightsquigarrow z_\gamma)$ be the paths in the heavy-path decomposition of T (i.e. $\mathcal{HP}(T)$). Then $\sum_{i=1}^{\gamma} |T(y_i)|$ is $O(n \log n)$ (see Lemma 13). Since $\cup_{i=1}^{\gamma} V(Q_i) = V(T)$, it follows that to handle all failures in T we incur at most $O(n|S|^{3/4}\sqrt{n}\log n)$ cost.

Thus our $\{r\}\times S$ 1-FT distance-preserver for weighted directed graphs require $\widetilde{O}(n^{3/2}|S|^{3/4})$ edges.

▶ **Theorem 26.** *For any n-node directed weighted graph $G = (V, E)$, $(r, S) \in V \times 2^V$, there is a 1-VFT $(r \times S)$ distance preserver H on $\widetilde{O}(n^{3/2}|S|^{3/4})$ edges.*

3.2 Preservers for Directed Unweighted Graphs

Using Proposition 25, together with a deeper insight into shortest-path's structure in unweighted graphs, we provide an alternative and better bound on H_Q.

Let Q be a directed $y \rightsquigarrow z$ tree-path in T, with y being ancestor of z. Let ℓ (a function of Q) be a parameter to be chosen later. For a triplet (r, s, x) we say that $D_{r,s,x}$ is *long* if the partial detour $\partial D_{r,s,x}$ has at least ℓ nodes, and *short* otherwise. We separately analyse the short and long detours.

Long Detours. Let us fix a node $s \in S \cap T(y)$. Let $W_s \subseteq V(T(y))$ be a random sample of nodes obtained by including each node in $T(y) \setminus \pi(r, s)$ independently with probability ℓ^{-1}. For each long detour $D_{r,s,x}$, for $x \in \pi(r, s)$, with constant probability or higher we sample a node $w \in W_s$. Edges of corresponding partial-detour intersecting a node $w \in W_s$ is contained in the union of an in- and out-BFS tree rooted at w. Hence they contain $O(|T(y)|)$ edges. Unioning over all $w \in W_s$, all long detours that intersect W_s contain $O(|T(y)|^2/\ell)$ edges in total. Finally, we note that since each long detour is counted with at least constant probability, there are at most $O(|T(y)|^2/\ell)$ total edges contained in *all* long detours, for a single node s. Summing this over $s \in S$ gives a bound of $O(|S| \cdot |T(y)|^2/\ell)$.

Short Detours. Let $w_0 = y$, and for $i = 1$ to $\alpha = \lfloor |Q|/\ell \rfloor$, let w_i be the descendant of w_{i-1} on Q at a distance ℓ from it. We partition Q into blocks: $B_0, B_1, \ldots, B_\alpha$ such that B_i includes w_i but excludes w_{i+1}, for $i \in [0, \alpha]$. The following lemma presents the disjointness relation for short detours corresponding to non-consecutive blocks.

▶ **Lemma 27.** *For each $s \in S \cap T(y)$ and $x \in B_i$, a short partial detour $\partial\mathcal{D}_{r,s,x}$ lies in $T(w_i) \setminus T(w_{i+2})$.*

Proof. Consider a fault $x \in B_i$ and a node $s \in S \cap T(y)$. Let a, b be respectively the first and last vertices on $\mathcal{D}_{r,s,x}$. As $x \in B_i = Q[w_i, w_{i+1}] \setminus \{w_{i+1}\}$, node a must be at least the grand parent of w_{i+1}. Thus $\texttt{dist}(a, w_{i+2}) \geq \ell + 2$, and moreover, distance from a to all nodes in $T(w_{i+2})$ is also at least $\ell + 2$. Since distance from a to all vertices in $\partial\mathcal{D}_{r,s,x}$ is at most $\ell + 1$, this completes the proof that $\partial\mathcal{D}_{r,s,x}$ is disjoint with $T(w_{i+2})$. ◀

From a more careful implementation of Proposition 29, it follows that the $T(y)$ term in its bound can in-fact be replaced by $T(w_i) \setminus T(w_{i+2})$ term when faults are restricted to B_i (see Lemma 27). Thus the next lemma follows.

▶ **Lemma 28.** *To handle short detours for faults on B_i, for $i \in [0, \alpha]$, we require at most*

$$O\big(|T(w_i) \setminus T(w_{i+2})|\ |S|^{3/4}\ \sqrt{\ell}\big)$$

edges to be added to H_0.

Since $\sum_{i=0}^{\alpha} |T(w_i) \setminus T(w_{i+2})| = O(T(y))$, the total cost of the construction of H_Q is

$$O\big(|S| \cdot |T(y)|^2/\ell + |T(y)|\ |S|^{3/4}\ \sqrt{\ell}\big)\ .$$

Setting $\ell := |S|^{1/6}|T(y)|^{2/3}$ thus proves the following.

▶ **Proposition 29.** *For any directed path $Q = (y \rightsquigarrow z)$ in T (between two vertices y, z with y being ancestor of z in T), the graph H_Q, for a directed unweighted graph G, requires at most $O(|T(y)|^{4/3}\ |S|^{5/6})$ edges.*

As a direct corollary of Proposition 29, we obtain an $O(n^{4/3})$ upper bound for single node-pair preserver. If the input pair is $(r, s) \in V \times V$, then taking $Q = \pi(r, s)$, provides us the following.

▶ **Corollary 30.** *For any n-node directed unweighted graph $G = (V, E)$, $r, s \in V$, there is a 1-VFT (r, s) distance preserver H on $O(n^{4/3})$ edges.*

We now use the technique of heavy path decomposition. Let $Q_i = (y_i \rightsquigarrow z_i)$, $1 \le i \le \gamma$ be the paths in the heavy-path decomposition of T (i.e. $\mathcal{HP}(T)$). Then $\sum_{i=1}^{\gamma} |T(y_i)|$ is $O(n \log n)$. So Proposition 29 directly proves the following.

▶ **Theorem 31.** *For any n-node directed unweighted graph $G = (V, E)$, $(r, S) \in V \times 2^V$, there is a 1-VFT $(r \times S)$ distance preserver H on $\widetilde{O}(n^{4/3}|S|^{5/6})$ edges.*

4 Fault Tolerant Preservers for Directed Acyclic Graphs

For DAGs, we first present our result for single node-pair case.

▶ **Theorem 32.** *For every n-node directed graph $G = (V, E)$ and a pair $p \in V \times V$, there exists a FT-vertex preserver $H \subseteq G$ for p with $O(n)$ edges.*

Proof. Let $p = (s, t)$, and $Q = \pi(s, t)$ denote the s-t shortest path. Let T_1 (T_2) be an outgoing (incoming) shortest path tree rooted at s (t) in G such that the s-t path in it overlaps with Q; and let H be initialized to $T_1 \cup T_2$. Consider a vertex failure x on Q. Let y_x be the last vertex on the replacement path $P_{s,t,x}$ lying outside subtree $T_1(x)$, and let z_x be its successor. As G is acyclic the z_x to t shortest path in G cannot pass through x. Thus we may assume $Q[s, y_x]$ is contained in T_1, and $Q[z_x, t]$ is contained in T_2. So, for each $x \in Q$, it only remains to add edge (y_x, z_x) to H, thereby proving the linear size bound. ◀

In above theorem, we showed that for each $x \in \pi(s, t)$, there is exists an edge $e_{x,s,t} = (y_x, z_x)$ such that

1. $\textsc{treepath}_{T_s}(s, y_x)$ doesn't contains x,
2. no shortest-path from z_x to t can contain x, and
3. the concatenated path $\textsc{treepath}_{T_s}(s, y_x) \cdot (y_x, z_x) \cdot \pi_G(z_x, t)$ is an s-t shortest path in $G \setminus x$,

where T_s, for $s \in S$, denotes the shortest path tree rooted at s.

To extend our construction to $S \times S$ setting we proceed as follows. We choose a uniformly random set $\widetilde{S}$ of $\Theta(|S|)$ vertices, and let $R = S \cup \widetilde{S}$ and $L = \frac{n \log n}{|S|}$. Initialize H to $\cup_{r \in R}(T_r)$. Next, for each $s, t \in R$ satisfying $\texttt{dist}_G(s, t) \le L$ and each $x \in \pi_G(s, t)$, add the edge $e_{x,s,t}$ to H. Observe that in this process, we include at most $O(n|S| \log n)$ edges to H. Thus the size of H is at most $O(n|S| \log n)$.

It remains to prove the correctness of H. Consider a pair $(s,t) \in S \times S$, and a vertex $x \in \pi_G(s,t)$. Let $(s = r_0, r_1, r_2, \ldots, r_\ell = t)$ be the vertices in R lying on $P_{s,t,x}$, in the order they appear. (Recall $P_{s,t,x}$ denote the s to t replacement path in $G \setminus x$). With high probability, length between consecutive r_i's in $P_{s,t,x}$ is at most L. This shows that with high probability each segment $P_{s,t,x}[r_i, r_{i+1}]$, for $i < \ell$, is present in $G \setminus x$.

Therefore, with high probability, H is a $S \times S$ fault-tolerant preserver for the input DAG G, and it comprises of at most $O(n|S| \log n)$ edges.

▶ **Theorem 33.** *Any DAG (possibly weighted) $G = (V, E, w)$ and a set $S \subseteq V$ of sources has a $S \times S$ sourcewise (vertex) fault tolerant preserver H with $O(n|S| \log n)$ edges.*

Using the same analysis as in Theorem 23, the above result can be extended to obtain a $S \times T$ preserver for unweighted DAGs with $O(n^{4/3}(|S||T|)^{1/3} \log^2 n)$ edges.

▶ **Theorem 34.** *For any unweighted DAG $G = (V, E)$ and subsets $S, T \subseteq V$, one can compute a (vertex) fault tolerant $S \times T$ preserver H with $\widetilde{O}(n^{4/3}|S|^{1/3}|T|^{1/3})$ edges.*

5 Lower Bounds for FT Preservers

5.1 Unconditional Lower-Bounds for Weighted Graphs

We show here a construction of a graph G on $O(n)$ vertices with integral edge weights in range $[1, n^c]$ (for some constant c) such that its $(\{s\} \times S)$-distance preserver requires at least $\Omega(n|S|)$ edges. Our lower bound construction is an adaption of $\{s\} \times V$ $(1+\epsilon)$-FTP by [6] to the $\{s\} \times S$ exact-FTP setting.

The vertex set of $V(G)$ constitutes $n + \sigma$ vertices, and is union of disjoint sets $U = \{u_1, u_2, \ldots, u_n\}$ and $W = \{w_1, w_2, \ldots, w_\sigma\}$. The edge set of $E(G)$ (in both directed as well as undirected scenario) is the union of the following two sets.

- $E_U = \{(u_n, u_{n-1}), \ldots, (u_i, u_{i-1}), \ldots, (u_2, u_1)\}$, with $wt(u_i, u_{i-1}) = 1$.
- $E_{U,W} = \{(u_i, w_j) \mid i \in [1, n], j \in [1, \sigma]\}$, with $wt(u_i, w_j) = (i \cdot n^4)$.

Say that $S = W$, and let $s = u_n$ be the designated source vertex. Let T_s be the shortest path tree rooted at s in G. It is easy to verify the set $E_U \cup (\{u_1\} \times W)$ constitute the edges of T_s. Now for any $i \in [1, n]$ and $j \in [1, \sigma]$, let $P_{i,j}$ denote the path $(u_n, u_{n-1}, \ldots, u_i) \circ (u_i, w_j)$. Then for any i, j, $wt(P_{i,j}) = i \cdot n^4 + (n - i)$.

If vertex u_{i-1} fails then the shortest path from s to vertex w_j $(j \in [1, \sigma])$ in G is path $P_{i,j}$. Hence each w_j must keep all its incoming edges in the FT-preserver. This shows there exists graphs whose $\{s\} \times S$ FT-preserver must contain $\Omega(n|S|)$ edges, thereby, implying the following result.

▶ **Theorem 35.** *For any positive integers n and σ $(\leq n)$, there exists an n-vertex (un)directed weighted graph $G = (V, E)$ with pair $(s, S) \in V \times 2^V$ satisfying $|S| = \sigma$ whose $\{s\} \times S$ 1-FT-distance-preserver must contain $\Omega(n|S|)$ edges.*

5.2 Conditional Lower-Bounds for Undirected Unweighted Graphs

▶ **Hypothesis 36** (Gap $S \times T$ Distance Preserver Lower Bounds). *For any $\sigma = \sigma_n$, there is an n-node undirected unweighted graph $G = (V, E)$ and demand pairs $P = S \times T$ with $|S| \leq \sigma, |T| \leq \sqrt{n\sigma/\gamma(P)}$ such that any distance preserver of P has $\Omega(n\sigma)$ edges.*

▶ **Theorem 37.** *Assuming Hypothesis 36, for any $\sigma = \sigma_n$, there are examples of n-node undirected unweighted graphs $G = (V, E)$ and node subsets of size $|S| = \sigma$ where $\Omega(n\sigma)$ edges are needed for an $S \times S$ FT preserver of an undirected unweighted graph.*

We first say the construction, which uses ideas from [31]. Let $\gamma = \gamma(P)$ be the gap of an instance from Hypothesis 36. For $i \in [\sigma]$, we create identical disjoint trees T_i, each on $O(n/\sigma)$ nodes, computed as follows. Let

$$\beta := \sqrt{\frac{n}{\sigma\gamma}},$$

and start with β disjoint paths $L_{i,1}, \ldots, L_{i,\beta}$, where $L_{i,j}$ is path on $2j\gamma$ nodes with endpoints $(q_{i,j}, s_{i,j})$. For $j \in [\beta - 1]$, the parent of $q_{i,j}$ is set to $q_{i,j+1}$ by adding an edge; thus the node $q_{i,\beta} =: r_i$ is naturally viewed as the root of T_i, and $\{s_{i,1}, \ldots, s_{i,\beta}\}$ are then the leaves. We set

$$\widetilde{T} = \{s_{i,j} \mid i \in [\alpha],\ j \in [\beta]\}$$

and take $\widetilde{S}$ to be a set of $|\widetilde{S}| = \sigma$ new nodes. Noting that $\left|\widetilde{T}\right| = \sigma\beta = \sqrt{n\sigma/\gamma}$, between $\widetilde{S}$ and $\widetilde{T}$ we may plug in an n-node distance preserver lower bound graph H from Hypothesis 36. We then let

$$S := \widetilde{S} \cup \{r_1, \ldots, r_\alpha\}.$$

This completes the construction. One immediately counts that the number of vertices is

$$n + \left|\bigcup_{i=1}^{\alpha} T_i\right| = O(n),$$

and $|S| = 2\sigma$, so it remains to argue that an FT subset distance preserver must contain a non-faulty $\widetilde{S} \times \widetilde{T}$ preserver in the copy of H, which thus has $\Omega(n\sigma)$ edges. Consider nodes $t_{i,j} \in \widetilde{T}, s \in \widetilde{S}$. Following the argument in [31], one can verify that on failure of $q_{i,j-1}$, every shortest $r_i \rightsquigarrow s$ path has the form

$$(q_{i,\beta}, \ldots, q_{i,j}) \circ L_{i,j} \circ \pi(t_{i,j}, s)$$

where $\pi(t_{i,j}, s)$ is a shortest $t_{i,j} \rightsquigarrow s$ path in H.

5.3 Conditional Lower-Bounds for Directed Unweighted Graphs

For directed unweighted graphs, we prove:

▶ **Hypothesis 38** (Layered $S \times T$ Directed Distance Preserver Lower Bounds)**.** *There is an n-node directed unweighted graph $G = (V, E)$ and demand pairs $S \times T$ with $|S|, |T|, \gamma(S \times T) \leq O(n^{1/3})$ such that any distance preserver of P has $\Omega(n^{4/3})$ edges.*

▶ **Theorem 39.** *Assuming Hypothesis 38, there are examples where $\Omega(n^{4/3})$ edges are needed for a single-pair 1-VFT preserver of a directed unweighted graph.*

To prove Theorem 39, we start with an $s \rightsquigarrow t$ shortest path π of length $\Theta(n^{2/3})$ and an n-node distance preserver lower bound H from Hypothesis 38, and then we add some additional nodes and edges to the graph to carefully connect these two parts of the construction. Then, similar in spirit to our previous lower bounds, we prove that for each $s' \in S, t' \in T$ pair one can fault a particular node on π so that every shortest $s \rightsquigarrow t$ path passes through s', t'. This means one must implicitly keep a (non-faulty) distance preserver of H, which thus has $\Omega(n^{4/3})$ nodes.

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