A Profunctorial Scott Semantics

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Abstract

In this paper, we study the bicategory of profunctors with the free finite coproduct pseudo-comonad and show that it constitutes a model of linear logic that generalizes the Scott model. We formalize the connection between the two models as a change of base for enriched categories which induces a pseudo-functor that preserves all the linear logic structure. We prove that morphisms in the co-Kleisli bicategory correspond to the concept of strongly finitary functors (sifted colimits preserving functors) between presheaf categories. We further show that this model provides solutions of recursive type equations which provides 2-dimensional models of the pure lambda calculus and we also exhibit a fixed point operator on terms.

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1 Introduction

1.1 Scott semantics and linear logic

Domain theory provides a mathematical structure to study computability with a notion of approximation of information. The elements of a domain represent partial stages of computation and the order relation represents increasing computational information. Among the desired properties of the interpretation of a program are monotonicity and continuity, i.e. the more a function has information on its input, the more it will provide information on its output and any finite part of the output can be attained through a finite computation. These features form the basis of Scott semantics of λ-calculus whose framework is Scott-continuous functions (monotonous maps preserving directed suprema) between domains. A fundamental property of Scott-continuous functions is that they admit a least fixed point which allows for the study of recursively defined programs.

Linear logic (**LL**) arose from the analysis by Girard of denotational models of system F (second order λ-calculus). It allows the study of how programs or proofs manage their resources by using exponential modalities that distinguish linear arguments that can be used exactly once and non-linear ones that can be used an arbitrary number of times [13]. One of the most basic models of linear logic is the category of sets and relations **Rel** which provides a quantitative semantics of **LL** as it allows to recover the number of times a program or a proof uses its argument to compute a given output. In quantitative models of **LL**, non-linear programs are thought of as analytic maps that are infinitely differentiable and represented by power series which can be approximated by polynomials. Viewing programs as series, a natural question was to understand the logical counterpart of differentiation, which led Ehrhard and Regnier to introduce differential linear logic and the syntactic notion of Taylor expansion which associates a formal sum of resource λ-terms to a given λ-term [7, 8].

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Huth showed that the Scott model of λ-calculus can be extended to a model of **LL** where the objects are prime algebraic complete lattices, the linear maps are functions preserving all suprema and the co-Kleisli maps are Scott-continuous functions [14, 15]. Independently, Winskel gave a simpler presentation based on preorders and ideal relations [22, 23]. In both cases, the co-Kleisli category is equivalent to the category of prime algebraic complete lattices and Scott-continuous functions between them. The obtained linear logic model is qualitative in that it only provides information about which arguments were used to compute a given output but not how many times. The qualitative Scott model is connected to the quantitative differential relational model through an extensional collapse construction discovered by Ehrhard [6]. This construction has been used in the context of intersection types which characterize normalization properties of λ-calculus. The quantitative relational model corresponds to a non-idempotent intersection type system whereas the qualitative Scott model corresponds to an idempotent type system. The extensional collapse construction provides a connection between the two type systems that allows to translate non-idempotent normalization to the indempotent one [5].

1.2 Categorifying Scott semantics

When taking a categorical approach to domain theory, preorders are generalized to categories and a morphism $f : x \to y$ is now an explicit name to represent the fact that y contains more computational information than x. This approach was extensively studied by Winskel among others and has proved in many ways fruitful in the theory of concurrent computation [3, 24]. This analogy can be formalized in the setting of enriched categories. A preorder $A = (|A|, \leq_A)$ corresponds to a category enriched over the two element lattice $\mathbb{2} = (\{\emptyset \leq \mathbb{1}\}, \wedge, \mathbb{1})$ where for every $a, a' \in |A|$, the homset $A(a, a')$ is equal to $\mathbb{1}$ if $a \leq_A a'$ and is empty otherwise. A $\mathbb{2}$-functor between preorders A and B is simply an order-preserving function $f : |A| \to |B|$ and the presheaf category of a preorder $[A^{op}, \mathbb{2}]$ corresponds to the set of down-closed subsets of A ordered with by inclusion. An ideal relation between preorders A and B (a relation up-closed in A and down-closed in B) corresponds to a monotone function $A \to [B^{op}, \mathbb{2}]$. Using the cartesian closed structure, it can be identified with a monotone map $A \times B^{op} \to \mathbb{2}$ which gives the direct correspondence with 2-profunctors.

Following this analogy, Cattani and Winskel showed that the bicategory of profunctors with the finite colimit completion pseudo-comonad $\mathcal{F}$ forms a model of linear logic that generalizes intuitions from the Scott model [3]. In their model, filtered colimits generalize directed suprema and Scott-continuous functions correspond to finitary functors. More recently, Fiore, Gambino, Hyland and Winskel used profunctors with the free symmetric monoidal completion pseudo-comonad $\mathcal{S}$ and showed that it forms a differential model of linear logic that generalizes the theory of combinatorial species of structures [10]. The monoidal structure of the exponential modality $\mathcal{S}$ encodes linear substitution and $\mathcal{S}$-species can be considered as a categorified version of the differential relational model.

In this paper, we study the free coproduct completion pseudo-comonad $\mathcal{C}$ (which corresponds to the finite Fam-construction) which models non-linear operations such as duplication and erasure. In the setting of algebraic theories and operads, symmetric operads are monads in the category of combinatorial species $[\mathcal{S}\,\mathbb{1}, \mathbf{Set}]$ with the Day convolution product and a Lawvere theory is a monad in the category $[\mathbf{FinSet}, \mathbf{Set}] \simeq [\mathcal{C}\,\mathbb{1}, \mathbf{Set}]$ with the substitution product. This analogy extends to the many-sorted case where symmetric many-sorted operads correspond to monads in the bicategory of $\mathcal{S}$-species [10]. Similarly, monads for $\mathcal{C}$-species correspond to many-sorted Lawvere theories. $\mathcal{C}$-species are also related to the cartesian closed bicategory of cartesian profunctors studied by Fiore and Joyal [12] where $\mathcal{C}$-species can be obtained by restricting to free cartesian categories.

Our motivation is two-fold: firstly, when we take $\mathcal{C}$ as a pseudo-comonad to interpret the exponential modality, we obtain a model of linear logic that generalizes the Scott model. There is indeed a monoidal functor from **Set** to the two-element lattice $\mathbb{2}$ that induces a change of base pseudo-functor from $\mathcal{C}$-species to the Scott model which commutes with all the constructions of linear logic. The obtained model of $\mathcal{C}$-species gives a different perspective on how to categorify Scott-continuity: directed suprema now correspond to sifted colimits and Scott-continuous functions correspond to strongly finitary functors. These correspondences are summarized in the table below:

a preorder $A = (\|A\|, \leq_A)$	a small category $\mathbf{A}$
a monotonous function $f : A \to B$	a functor $F : \mathbf{A} \to \mathbf{B}$
a down-closed subset $x \subseteq \|A\|$	a presheaf $X : \mathbf{A}^{op} \to \mathbf{Set}$
an ideal relation $R \subseteq A \times B$	a profunctor $F : \mathbf{A} \nrightarrow \mathbf{B}$
inclusion of relations	a natural transformation
a directed supremum	a sifted colimit
a Scott-continuous function	a strongly finitary functor

Secondly, since $\mathcal{S}$-species categorify the relational model and $\mathcal{C}$-species categorify the Scott-model, our future goal is to connect them using a construction in the spirit of the extensional collapse mentionned above and to explore the intersection type counterpart of this construction in the profunctorial setting.

Contributions

- In Section 3, we show that the model of profunctors with the finite coproduct pseudo-comonad $\mathcal{C}$ is a model of linear logic which is a generalization of the qualitative Scott model with **Rel**.
- The connection is formalized by exhibiting a change of base pseudo-functor that commutes with the linear logic structure (Section 5).
- We prove in Section 4 that morphisms in the associated co-Kleisli bicategory correspond to the notion of functors preserving sifted colimits by providing a biequivalence between the two structures.
- Lastly, we show in Section 6 that every recursive type equation built from linear logic connectives has a least fixed point solution, and we exhibit a fixed point operator on terms which allows for the study of recursively defined terms.

Notation

- For an integer $n \in \mathbb{N}$, we write $\underline{n}$ for the set $\{1, \dots, n\}$.
- The length of a finite sequence of elements $u = \langle a_1, \dots, a_n \rangle$ is denoted by $|u|$.
- Categories will be denoted in boldface whereas simple text will be used for sets. For a small category $\mathbf{A}$, we denote by $\widehat{\mathbf{A}}$ the presheaf category $[\mathbf{A}^{op}, \mathbf{Set}]$ and write $\mathrm{y}_{\mathbf{A}} : \mathbf{A} \to \widehat{\mathbf{A}}$ for the Yoneda embedding.
- We use $\cong$ for natural isomorphisms between functors or category isomorphisms and $\simeq$ for equivalences.

2 The Qualitative Scott Model of Linear Logic

The category of prime algebraic lattices and maps preserving all suprema gives rise to a model of linear logic whose associated co-Kleisli category is equivalent to the Scott model of prime algebraic lattices and Scott-continuous functions between them [14, 15]. It is however more convenient to manipulate linear logic constructions on preorders rather than on lattices and since any prime algebraic lattice can be obtained as the set of downward closed subsets of a preorder, we adopt the viewpoint of taking our objects to be preorders. The Kleisli category of this model is then equivalent to the Scott model of prime algebraic lattices [22, 23].

Define **ScottL** to be the category whose objects are preordered sets $A = (|A|, \leq_A)$ and a morphism from A to B is a relation $R \subseteq |A| \times |B|$ that is up-closed in $(|A|, \leq_A)$ and down-closed in $(|B|, \leq_B)$. Explicitly, it verifies that for all $a, a' \in |A|$ and $b, b' \in |B|$:

$$(a \leq_A a' \ \wedge\ (a,b) \in R \ \wedge\ b' \leq_B b) \ \Rightarrow\ (a', b') \in R$$

The identity is given by $\mathrm{id}_A := \{(a, a') \mid a' \leq_A a\}$ and composition is the usual composition of relations. The dual of a preordered set A is defined to be $A^{\perp} := (|A|, \geq_A)$. Every preordered set A induces a a domain $\mathcal{I}(A)$ of ideals (downward closed subsets of A) ordered by inclusion. Morphisms in the linear category $\mathbf{ScottL}(A, B)$ can then be seen as elements of $\mathcal{I}(A^{\perp} \times B)$; they are also equivalent to functions from $\mathcal{I}(A)$ to $\mathcal{I}(B)$ that commute with all unions.

ScottL is a compact closed category where the tensor product $A \otimes B$ is given by $(|A| \times |B|, \leq_A \times \leq_B)$ and has the singleton preordered set $\mathbb{1}$ as a unit. The additive structure is given by the disjoint union of preorders $A \,\&\, B := (|A| + |B|, \leq_A + \leq_B)$ with the empty preordered set $\mathbb{0}$ as zero object.

The exponential modality $! : \mathbf{ScottL} \to \mathbf{ScottL}$ takes a preordered set A to the preordered set whose web $|!A|$ is the set of finite sequences of elements in $|A|$ i.e $|!A| := \{\langle a_1, \ldots, a_n\rangle \mid a_i \in |A|, n \in \mathbb{N}\}$ and the preorder relation is defined as follows:

$$\langle a_1, \ldots, a_n\rangle \leq_{!A} \langle b_1, \ldots, b_m\rangle \quad :\Leftrightarrow \quad \forall i \in \underline{n}, \exists j \in \underline{m}, a_i \leq_A b_j$$

On morphisms, a relation $R \in \mathbf{ScottL}(A, B)$ is mapped to

$$!R := \{(\langle a_1, \ldots a_n\rangle, \langle b_1, \ldots, b_m\rangle) \mid \forall j \in \underline{m}, \exists i \in \underline{n}, (a_i, b_j) \in R\}.$$

The obtained co-Kleisli category $\mathbf{ScottL}_!$ is then equivalent to the category of prime algebraic lattices and Scott-continuous functions between them as every relation in $\mathbf{ScottL}_!(A, B)$ corresponds to a Scott-continuous function $\mathcal{I}(A) \to \mathcal{I}(B)$.

▶ Remark 1. We chose this presentation of the comonad instead of finite subsets [14, 22] or finite multisets [5, 6] since it is more convenient for the profunctorial generalization with the free coproduct pseudo-comonad. Note that for the three presentations, the associated lattices of downward closed subsets are all isomorphic and the associated co-Kleisli categories are all equivalent to the Scott model.

3 The Model of Profunctors

3.1 The bicategory of profunctors

The notion of *profunctor* (or *distributor*) has become increasingly important in theoretical computer science as a tool to model a wide range of bidimensional computational structures. For small categories $\mathbf{A}$ and $\mathbf{B}$, a profunctor $F : \mathbf{A} \nrightarrow \mathbf{B}$ is a functor $F : \mathbf{A} \times \mathbf{B}^{op} \to \mathbf{Set}$ or equivalently a functor $F : \mathbf{A} \to \widehat{\mathbf{B}}$ [2]. Profunctors can be seen as a generalization of **Rel** as a relation $R \subseteq A \times B$ corresponds to a profunctor between discrete categories such that each component is either the empty set or a singleton.

The composite of two profunctors $F : \mathbf{A} \nrightarrow \mathbf{B}$ and $G : \mathbf{B} \nrightarrow \mathbf{C}$ is the profunctor $G \circ F : \mathbf{A} \times \mathbf{C}^{op} \to \mathbf{Set}$ given by the coend formula:

$$(a, c) \mapsto \int^{b \in \mathbf{B}} F(a, b) \times G(b, c).$$

and the identity $\mathrm{id}_{\mathbf{A}} : \mathbf{A} \nrightarrow \mathbf{A}$ is given by the yoneda embedding $\mathcal{y}_{\mathbf{A}} : \mathbf{A} \to \widehat{\mathbf{A}}$. Composition of profunctors is however associative only up to natural isomorphisms which puts us in the setting of a bicategory [17].

▶ **Definition 2.** *The bicategory of profunctors* **Prof** *consists of*

- 0**-cells:** *small categories* $\mathbf{A}, \mathbf{B}$*,*
- 1**-cells:** *profunctors* $F : \mathbf{A} \nrightarrow \mathbf{B}$*,*
- 2**-cells:** *natural transformations between profunctors.*

In [10], Fiore et al. showed that **Prof** is a bicategorical model of **LL** that constitutes a generalization of Joyal's species of structures. **Prof** can be equipped with a symmetric monoidal structure where the unit $\mathbb{1}$ is the category with a unique object and a unique arrow and the tensor product $\otimes : (\mathbf{A}, \mathbf{B}) \mapsto \mathbf{A} \times \mathbf{B}$ is the cartesian product of categories in **Cat**. The dualizer $-^{\perp}$ which takes a small category $\mathbf{A}$ to $\mathbf{A}^{op}$ provides **Prof** with a compact closed structure. The additive structure $\& : (\mathbf{A}, \mathbf{B}) \mapsto \mathbf{A} + \mathbf{B}$ is given by the coproduct in **Cat** which makes **Prof** a cartesian bicategory whose zero object is the empty category $\mathbb{0}$. The exponential modality in their model relies on the free symmetric monoidal completion $\mathcal{S}\mathbf{A}$ for a small category $\mathbf{A}$.

3.2 The free finite coproduct pseudo-comonad

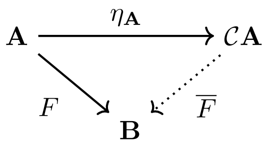
Cattani and Winskel showed that by taking the free finite colimit completion pseudo-comonad $\mathcal{F}$, we obtain a model of **LL** that generalizes the Scott model [3]. The maps obtained in the co-Kleisli bicategory do not however preserve bisimulation which led them to consider the pseudo-comonad of indexed families instead. Among the examples given is the restriction to finite families which corresponds to the free finite coproduct completion $\mathcal{C}$. In this section, we expand this example and exhibit that **Prof** together with the pseudo-comonad $\mathcal{C}$ forms a model of **LL** that gives a different perspective on how to categorify the Scott model. While 1-categorical semantics of linear logic has been extensively studied (see [18] for a complete review of **LL**-models and [7] for differential linear logic), no complete account of what is a bicategorical model of differential linear logic has been given yet. In this section, we take the same compact closed structure for the linear bicategory described in the previous paragraph (see [3] and [10] for more details). The remaining ingredients to obtain a model of **LL** are a pseudo-comonad structure and Seely equivalences satisfying the coherence conditions for a linear exponential pseudo-comonad.

▶ **Definition 3.** *For a small category* $\mathbf{A}$*, define* $\mathcal{C}\mathbf{A}$ *to be the category whose objects are finite sequences* $\langle a_1, \dots, a_n \rangle$ *of objects of* $\mathbf{A}$ *and a morphism between two sequences* $\langle a_1, \dots, a_n \rangle$ *and* $\langle b_1, \dots, b_m \rangle$ *consists of a pair* $(\sigma, (f_i)_{i \in \underline{n}})$ *of a function* $\sigma : \underline{n} \to \underline{m}$ *and a family of morphisms* $f_i : a_i \to b_{\sigma(i)}$ *in* $\mathbf{A}$ *for* $i \in \underline{n}$*. Equivalently, the hom-sets can be described by:*

$$\mathcal{C}\mathbf{A}(\langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_m \rangle) = \prod_{i \in \underline{n}} \sum_{j \in \underline{m}} \mathbf{A}(a_i, b_j).$$

We recall below a classical result:

▶ **Lemma 4.** *For two finite sequences u and v in $\mathcal{C}\mathbf{A}$, the concatenation (denoted by $u \oplus v$) provides a coproduct structure for $\mathcal{C}\mathbf{A}$ and the empty sequence $\langle\rangle$ is initial. $\mathcal{C}\mathbf{A}$ is the free finite coproduct completion of $\mathbf{A}$, i.e. for any functor $F : \mathbf{A} \to \mathbf{B}$ where $\mathbf{B}$ is a category with finite coproducts, there exists a unique (up to natural isomorphism) functor $\overline{F} : \mathcal{C}\mathbf{A} \to \mathbf{B}$ that preserves finite coproducts and makes the following diagram commute:*



▶ Note 5. To obtain the free symmetric monoidal completion $\mathcal{S}\mathbf{A}$, it suffices to take the subcategory of $\mathcal{C}\mathbf{A}$ where we restrict σ in Definition 3 to be a bijection.

The endofunctor $\mathcal{C} : \mathbf{Cat} \to \mathbf{Cat}$ can be equipped with a 2-monad structure. In order to obtain a pseudo-comonad on **Prof**, one needs to start with the dual construction of the free finite product 2-monad $\mathcal{P} : \mathbf{Cat} \to \mathbf{Cat}$ which takes a small category $\mathbf{A}$ to $\mathcal{P}(\mathbf{A}) = (\mathcal{C}(\mathbf{A}^{op}))^{op}$. In [11], Fiore et al. show that the 2-monad $\mathcal{P}$ lifts to a pseudo-monad on **Prof**. Taking its dual, one obtains the pseudo-comonad of finite coproducts on **Prof** which we briefly describe below.

For a profunctor $F : \mathbf{A} \nrightarrow \mathbf{B}$ between small categories $\mathbf{A}$ and $\mathbf{B}$, $\mathcal{C}F : \mathcal{C}\mathbf{A} \nrightarrow \mathcal{C}\mathbf{B}$ is given by:

$$\mathcal{C}F : (u, v) \mapsto \prod_{j \in |v|} \int^{a_j \in \mathbf{A}} F(a_j, v_j) \times \mathcal{C}\mathbf{A}(\langle a_j \rangle, u)$$

The counit and comultiplication pseudo-natural transformations have the following components:

$$\begin{array}{ll} \varepsilon_{\mathbf{A}} : \mathcal{C}\mathbf{A} \nrightarrow \mathbf{A} & \delta_{\mathbf{A}} : \mathcal{C}\mathbf{A} \nrightarrow \mathcal{C}^2\mathbf{A} \\ (u, a) \mapsto \mathcal{C}\mathbf{A}(\langle a \rangle, u) & (u, \langle u_1, \dots, u_n \rangle) \mapsto \mathcal{C}\mathbf{A}(u_1 \oplus \cdots \oplus u_n, u) \end{array}$$

A morphism $F : \mathcal{C}\mathbf{A} \nrightarrow \mathbf{B}$ in the co-Kleisli bicategory $\mathbf{Prof}_{\mathcal{C}}$ is called a $\mathcal{C}$*-species* and its lifting or promotion $F^{\mathcal{C}} : \mathcal{C}\mathbf{A} \nrightarrow \mathcal{C}\mathbf{B}$ is given by:

$$F^{\mathcal{C}}(u, v) = \mathcal{C}F \circ \delta_{\mathbf{A}}(u, v) = \prod_{j \in |v|} F(u, v_j)$$

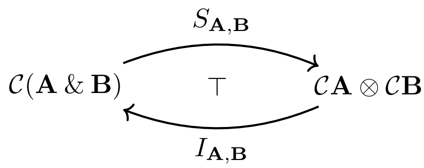
The composite in $\mathbf{Prof}_{\mathcal{C}}$ of two $\mathcal{C}$-species $F : \mathcal{C}\mathbf{A} \nrightarrow \mathbf{B}$ and $G : \mathcal{C}\mathbf{B} \nrightarrow \mathbf{C}$ is then given by the profunctorial composition $G \circ F^{\mathcal{C}} : \mathcal{C}\mathbf{A} \nrightarrow \mathbf{C}$.

▶ **Lemma 6.** *There is a Seely adjoint equivalence of categories $\mathcal{C}(\mathbf{A} \,\&\, \mathbf{B}) \simeq \mathcal{C}\mathbf{A} \otimes \mathcal{C}\mathbf{B}$.*

Proof. Define $I_{\mathbf{A},\mathbf{B}} : \mathcal{C}\mathbf{A} \otimes \mathcal{C}\mathbf{B} \to \mathcal{C}(\mathbf{A} \,\&\, \mathbf{B})$ as follows:

$$I_{\mathbf{A},\mathbf{B}} : (u, v) \mapsto \mathcal{C}(i_1)(u) \oplus \mathcal{C}(i_2)(v) \in \mathcal{C}(\mathbf{A} \,\&\, \mathbf{B})$$

where $i_1 : \mathbf{A} \to \mathbf{A} \,\&\, \mathbf{B}$ and $i_2 : \mathbf{B} \to \mathbf{A} \,\&\, \mathbf{B}$ are the coprojections maps. Consider now the functor $p_1 : \mathbf{A} \,\&\, \mathbf{B} \to \mathcal{C}\mathbf{A}$ defined by $p_1(1, a) := \langle a \rangle$ and $p_1(2, b) := \langle\rangle$. This functor induces a functor $\overline{p_1} : \mathcal{C}(\mathbf{A} \,\&\, \mathbf{B}) \to \mathcal{C}\mathbf{A}$ (using the universal property of the free finite coproduct completion) that is a retract of $\mathcal{C}(i_1) : \mathcal{C}\mathbf{A} \to \mathcal{C}(\mathbf{A} \,\&\, \mathbf{B})$. We define similarly a functor $\overline{p_2} : \mathcal{C}(\mathbf{A} \& \mathbf{B}) \to \mathcal{C}\mathbf{B}$ that is a retract of $\mathcal{C}(i_2) : \mathcal{C}\mathbf{B} \to \mathcal{C}(\mathbf{A} \& \mathbf{B})$. For $w \in \mathcal{C}(\mathbf{A} \& \mathbf{B})$, we denote by $w.1 \in \mathcal{C}\mathbf{A}$ its image by $\overline{p_1}$ and by $w.2 \in \mathcal{C}\mathbf{B}$ its image by $\overline{p_2}$. $S_{\mathbf{A},\mathbf{B}} : \mathcal{C}(\mathbf{A} \,\&\, \mathbf{B}) \to \mathcal{C}\mathbf{A} \otimes \mathcal{C}\mathbf{B}$ is then defined to be the functor $w \mapsto (w.1, w.2) \in \mathcal{C}\mathbf{A} \otimes \mathcal{C}\mathbf{B}$.



We now exhibit two natural isomorphisms $\eta : \mathrm{Id}_{\mathcal{C}\mathbf{A}\otimes\mathcal{C}\mathbf{B}} \Rightarrow S_{\mathbf{A},\mathbf{B}} \circ I_{\mathbf{A},\mathbf{B}}$ and $\varepsilon : I_{\mathbf{A},\mathbf{B}} \circ S_{\mathbf{A},\mathbf{B}} \Rightarrow \mathrm{Id}_{\mathcal{C}(\mathbf{A}\&\mathbf{B})}$. For $(u, v) \in \mathcal{C}\mathbf{A} \otimes \mathcal{C}\mathbf{B}$, we have that

$$((\mathcal{C}(i_1)(u) \oplus \mathcal{C}(i_2)(v)).1, (\mathcal{C}(i_1)(u) \oplus \mathcal{C}(i_2)(v)).2) = (u, v)$$

so η is just the identity. Let $w \in \mathcal{C}(\mathbf{A} \,\&\, \mathbf{B})$, ε_w is the reshuffling isomorphism from $\mathcal{C}(i_1)(w.1) \oplus \mathcal{C}(i_2)(w.2)$ to w. The adjunction is obtained by seeing that for $(u, v) \in \mathcal{C}\mathbf{A} \otimes \mathcal{C}\mathbf{B}$ and $w \in \mathcal{C}(\mathbf{A} \,\&\, \mathbf{B})$ there is a natural isomorphism:

$$\mathcal{C}(\mathbf{A} \,\&\, \mathbf{B})(\mathcal{C}(i_1)(u) \oplus \mathcal{C}(i_2)(v), w) \cong \mathcal{C}\mathbf{A}(u, w.1) \times \mathcal{C}\mathbf{B}(v, w.2).$$ ◀

In [10], Fiore et al. show that **Prof** together with the free symmetric monoidal pseudo-comonad $\mathcal{S}$ is a model of differential linear logic which can be seen as a categorification of the differential relational model. We show below that similarly to the Scott model with preorders, $\mathbf{Prof}_{\mathcal{C}}$ is not a model of differential linear logic.

▶ **Lemma 7.** $\mathbf{Prof}_{\mathcal{C}}$ *is not a model of differential linear logic.*

Proof. If $\mathbf{Prof}_{\mathcal{C}}$ were a model of differential linear logic, there would exist a pseudo-natural transformation $\bar{\varepsilon} : \mathrm{Id}_{\mathbf{Prof}} \to \mathcal{C}$ interpreting the codereliction rule. One of the required coherence axioms for the codereliction is $\varepsilon \circ \bar{\varepsilon} = \mathrm{Id}_{\mathbf{Prof}}$. For all $\mathbf{A} \in \mathbf{Cat}$ and $a, a' \in \mathbf{A}$, we then have:

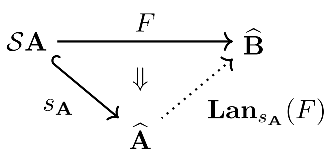
$$\int^{u\in\mathcal{C}\mathbf{A}} \bar{\epsilon}_{\mathbf{A}}(a, u) \times \mathcal{C}\mathbf{A}(\langle a' \rangle, u) \cong \mathbf{A}(a', a)$$

which implies $\bar{\epsilon}(a, \langle a' \rangle) \cong A(a', a)$. Another required coherence diagrams for the codereliction map is that for any object $\mathbf{A}$, $w_{\mathbf{A}} \circ \bar{\varepsilon}_{\mathbf{A}} = \mathbb{0}_{\mathbf{A}}$ where $w_{\mathbf{A}} : \mathcal{C}\mathbf{A} \nrightarrow \mathbb{1}$ is the weakening map given by $u \mapsto \mathcal{C}\mathbf{A}(\langle\rangle, u)$ and $\mathbb{0}_{\mathbf{A}} : \mathbf{A} \nrightarrow \mathbb{1}$ is the empty profunctor. For $a \in \mathbf{A}$, we have:

$$w_{\mathbf{A}} \circ \bar{\varepsilon}_{\mathbf{A}}(a) = \int^{u\in\mathcal{C}\mathbf{A}} \mathcal{C}\mathbf{A}(\langle\rangle, u) \times \bar{\epsilon}_{\mathbf{A}}(a, u) \cong \bar{\epsilon}_{\mathbf{A}}(a, \langle\rangle)$$

Since there is a map $\langle\rangle \to \langle a \rangle$ in $\mathcal{C}\mathbf{A}$, it induces a function from $\bar{\epsilon}_{\mathbf{A}}(a, \langle a \rangle)$ to $\bar{\epsilon}_{\mathbf{A}}(a, \langle\rangle)$. The set $\bar{\epsilon}_{\mathbf{A}}(a, \langle a \rangle) \cong \mathbf{A}(a, a)$ is not empty as it contains id_a so the set $\bar{\epsilon}_{\mathbf{A}}(a, \langle\rangle)$ cannot be empty which contradicts our hypothesis. ◀

The extensional collapse construction between the relational model and the Scott model gives a connection between $\mathbf{Rel}_!$ which is not well-pointed to the well-pointed category $\mathbf{ScottL}_!$. In the categorified setting, the situation is however more subtle. In the case of $\mathcal{S}$-species, Fiore introduced the notion of generalized analytic functor as the Taylor series counterpart of species that generalizes Joyal's original definition for combinatorial species [9]. For small categories $\mathbf{A}$ and $\mathbf{B}$, a functor $P : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ is said to be *analytic* if there exists a generalized species $F : \mathcal{S}\mathbf{A} \nrightarrow \mathbf{B}$ such that P is isomorphic to $\mathbf{Lan}_{s_{\mathbf{A}}} F$ (the left Kan extension of F along $s_{\mathbf{A}}$)



where $s_{\mathbf{A}} : \mathcal{S}\mathbf{A} \to \widehat{\mathbf{A}}$ is the functor that takes a sequence $\langle a_1, \ldots a_n \rangle$ in $\mathcal{S}\mathbf{A}$ to the presheaf $\sum_{i=1}^{n} \text{よ}_{\mathbf{A}}(a_i)$ in $\widehat{\mathbf{A}}$. The functor $s_{\mathbf{A}} : \mathcal{S}\mathbf{A} \to \widehat{\mathbf{A}}$ is not fully faithful which entails that the functor giving the correspondence between $\mathcal{S}$-species and analytic functors:

$$\mathbf{Lan}_{s_{\mathbf{A}}} : \mathbf{Prof}_{\mathcal{S}}(\mathbf{A}, \mathbf{B}) \to [\widehat{\mathbf{A}}, \widehat{\mathbf{B}}]$$

is not fully faithful. Fiore however showed that it is possible to reconstruct an $\mathcal{S}$-species from its analytic functor if we restrict the objects to be groupoids [9]. Formally, he showed that there is a biequivalence between the bicategory of $\mathcal{S}$-species restricted to groupoids and the 2-category of analytic functors (whose 0-cells are small groupoids, 1-cells are analytic functors and 2-cells are weak cartesian natural transformations). If we extend the functor $s_{\mathbf{A}}$ to the category $\mathcal{C}\mathbf{A}$, we obtain a fully faithful functor which entails that $\mathbf{Lan}_{s_{\mathbf{A}}} : \mathbf{Prof}_{\mathcal{C}}(\mathbf{A}, \mathbf{B}) \to [\widehat{\mathbf{A}}, \widehat{\mathbf{B}}]$ is now fully faithful as a corollary of a classical result on Kan extension that we recall below.

▶ **Proposition 8** ([16]). *Let $S : \mathbf{A} \to \mathbf{B}$ be a fully faithful functor from a small category $\mathbf{A}$. Then, for every functor $F : \mathbf{A} \to \mathbf{D}$ into a cocomplete category $\mathbf{D}$, the natural transformation $F \Rightarrow \mathbf{Lan}_S(F) \circ S$ is an isomorphism and the functor $\mathbf{Lan}_S : [\mathbf{A}, \mathbf{D}] \to [\mathbf{B}, \mathbf{D}]$ is fully faithful.*

3.3 The cartesian closed structure

▶ **Definition 9.** *A cartesian bicategory $\mathcal{B}$ is* closed *if for every pair of objects $A, B \in \mathcal{B}$, we have:*

1. *an exponential object $A \Rightarrow B$ together with an evaluation map $Ev_{A,B} \in \mathcal{B}((A \Rightarrow B) \& A, B)$ and*
2. *for every $X \in \mathcal{B}$, an adjoint equivalence*

$$\mathcal{B}(X, B^A) \underset{\lambda}{\overset{Ev_{A,B} \circ ((-) \,\&\, A)}{\rightleftarrows}} \mathcal{B}(X \,\&\, A, B) \qquad (\bot)$$

▶ **Proposition 10.** $\mathbf{Prof}_{\mathcal{C}}$ *is cartesian closed.*

Proof.

1. For small categories $\mathbf{A}$ and $\mathbf{B}$, the exponential object $\mathbf{A} \Rightarrow \mathbf{B}$ is defined as $(\mathcal{C}\mathbf{A})^{op} \times \mathbf{B}$ and the evaluation map $\mathrm{Ev}_{\mathbf{A},\mathbf{B}} : \mathcal{C}\,((\mathbf{A} \Rightarrow \mathbf{B}) \,\&\, \mathbf{A}) \to\!\!\!\!\!\!\;\mapsto \mathbf{B}$ takes $(W, b) \in \mathcal{C}\,((\mathbf{A} \Rightarrow \mathbf{B}) \,\&\, \mathbf{A}) \times \mathbf{B}^{op}$ to the set:
$$\int^{u_1 \in \mathcal{C}(\mathbf{A} \Rightarrow \mathbf{B}), u_2 \in \mathcal{C}\mathbf{A}} \mathcal{C}(\mathbf{A} \Rightarrow \mathbf{B})(u_1, W.1) \times \mathcal{C}\mathbf{A}(u_2, W.2) \times \mathcal{C}(\mathbf{A} \Rightarrow \mathbf{B})(\langle (u_2, b) \rangle, u_1)$$
$$\cong \mathcal{C}(\mathbf{A} \Rightarrow \mathbf{B})(\langle (W.2, b) \rangle, W.1)$$
2. For $G : \mathcal{C}(\mathbf{X} \,\&\, \mathbf{A}) \to\!\!\!\!\!\!\;\mapsto \mathbf{B}$, $\lambda(G) : \mathcal{C}\mathbf{X} \to\!\!\!\!\!\!\;\mapsto (\mathcal{C}\mathbf{A} \multimap \mathbf{B})$ is defined by
$$\lambda(G) : (z, (u, b)) \mapsto F(\mathcal{C}(i_1)(z) \oplus \mathcal{C}(i_2)(u), b).$$
Let $F : \mathcal{C}\mathbf{X} \to\!\!\!\!\!\!\;\mapsto (\mathbf{A} \Rightarrow \mathbf{B})$, $F \,\&\, \mathbf{A} : \mathcal{C}(\mathbf{X} \,\&\, \mathbf{A}) \to\!\!\!\!\!\!\;\mapsto (\mathbf{A} \Rightarrow \mathbf{B}) \,\&\, \mathbf{A}$ is the profunctor that takes $(w, (1, (u, b)))$ in $\mathcal{C}(\mathbf{X} \,\&\, \mathbf{A}) \times ((\mathbf{A} \Rightarrow \mathbf{B}) \,\&\, \mathbf{A})^{op}$ to:
$$F \circ \Pi_1(w, (u, b)) = \int^{z \in \mathcal{C}\mathbf{X}} F(z, (u, b)) \times \mathcal{C}(\mathbf{X} \,\&\, \mathbf{A})(\mathcal{C}(i_1)z, w)$$
$$\cong \int^{z \in \mathcal{C}\mathbf{X}} F(z, (u, b)) \times \mathcal{C}\mathbf{X}(z, w.1) \times \mathcal{C}\mathbf{A}(\langle\rangle, w.2) \cong F(w.1, (u, b))$$

and the image of an element $(w,(2,a)) \in \mathcal{C}(\mathbf{X} \,\&\, \mathbf{A}) \times ((\mathbf{A} \Rightarrow \mathbf{B}) \,\&\, \mathbf{A})^{op}$ is given by

$$\Pi_2(w,a) = \mathcal{C}(\mathbf{X} \,\&\, \mathbf{A})(\langle(2,a)\rangle, w) \cong \mathcal{C}\mathbf{A}(\langle a\rangle, w.2).$$

Hence, its lifting $(F \,\&\, \mathbf{A})^{\mathcal{C}} : \mathcal{C}(\mathbf{X} \,\&\, \mathbf{A}) \nrightarrow \mathcal{C}((\mathbf{A} \Rightarrow \mathbf{B}) \,\&\, \mathbf{A})$ is given by:

$$(w,W) \mapsto \cong F^{\mathcal{C}}(w.1, W.1) \times \mathcal{C}\mathbf{A}(W.2, w.2)$$

We can now compute $\mathrm{Ev}_{\mathbf{A},\mathbf{B}} \circ (F \,\&\, \mathbf{A}) : \mathcal{C}(\mathbf{X} \,\&\, \mathbf{A}) \nrightarrow \mathbf{B}$:

$$\begin{aligned}(w,b) \mapsto \int^{W \in \mathcal{C}(\mathbf{A}\Rightarrow\mathbf{B})\&\mathbf{A}} & \mathrm{Ev}_{\mathbf{A},\mathbf{B}}(W,b) \times (F \,\&\, \mathbf{A})^{\mathcal{C}}(w,W) \\ \cong \int^{W} & \mathcal{C}(\mathbf{A} \Rightarrow \mathbf{B})(\langle(W.2,b)\rangle, W.1) \times F^{\mathcal{C}}(w.1,W.1) \times \mathcal{C}\mathbf{A}(W.2,w.2) \\ \cong & \, F^{\mathcal{C}}(w.1, \langle(w.1,b)\rangle) \cong F(w.1,(w.2,b))\end{aligned}$$

Consider now two profunctors $F : \mathcal{C}\mathbf{X} \nrightarrow (\mathbf{A} \Rightarrow \mathbf{B})$ and $G : \mathcal{C}(\mathbf{X} \,\&\, \mathbf{A}) \nrightarrow \mathbf{B}$, we exhibit the following natural ismorphisms:

$$\eta_F : F \overset{\sim}{\Longrightarrow} \lambda(\mathrm{Ev}_{\mathbf{A},\mathbf{B}} \circ (F \,\&\, \mathbf{A})) \qquad \beta_G : \mathrm{Ev}_{\mathbf{A},\mathbf{B}} \circ (\lambda(G) \,\&\, \mathbf{A}) \overset{\sim}{\Longrightarrow} G$$

For $(z,(u,b)) \in \mathcal{C}\mathbf{X} \times (\mathbf{A} \Rightarrow \mathbf{B})^{op}$, we have:

$$\begin{aligned}\lambda(\mathrm{Ev}_{\mathbf{A},\mathbf{B}} \circ (F \,\&\, \mathbf{A}))(z,(u,b)) &\cong (\mathrm{Ev}_{\mathbf{A},\mathbf{B}} \circ (F \,\&\, \mathbf{A}))(\mathcal{C}i_1 z \oplus \mathcal{C}i_2 u, b) \\ &\cong F((\mathcal{C}i_1 z \oplus \mathcal{C}i_2 u).1, (\mathcal{C}i_1 z \oplus \mathcal{C}i_2 u).2, b)) \cong F(z,(u,b))\end{aligned}$$

and for $(w,b) \in \mathcal{C}(\mathbf{X} \,\&\, \mathbf{A}) \times \mathbf{B}^{op}$, we obtain:

$$\begin{aligned}&\mathrm{Ev}_{\mathbf{A},\mathbf{B}}(\lambda(G) \,\&\, \mathbf{A})(w,b) = \lambda(G)(w.1,(w.2,b)) \\ &\cong G((\mathcal{C}(i_1)(w.1) \oplus \mathcal{C}(i_2)(w.2)), b) \cong G(w,b)\end{aligned}$$

◀

4 Strongly finitary functors

In the case of analytic functors for $\mathcal{S}$-species (restricted to groupoids), one can characterize them as functors preserving filtered colimits and weak wide pullbacks [9]. Cattani and Winskel showed that $\mathcal{F}$-species correspond to the notion of finitary functors, i.e. functors preserving filtered colimits [3]. Filtered colimits are the classical way of generalizing directed suprema in Scott's topology, and they are characterized as colimits which commute with finite limits in **Set**. In this section, we focus on a larger class of colimits, called *sifted colimits* which are colimits which commute with finite products in **Set**. A large part of the theory of locally finitely presentable categories and finitely presentable objects has analogues for sifted colimits. An object a in a category $\mathbf{A}$ is said to be *strongly finitely presentable* if $\mathbf{A}(a,-) : \mathbf{A} \to \mathbf{Set}$ preserves sifted colimits. The full subcategory of these objects in $\mathbf{A}$ is denoted by $\mathbf{A}_{\mathrm{sfp}}$. For a preorder, finitely and strongly presentable objects coincide with the compact elements and in the category **Set**, the two notions coincide with finite sets [1]. A category $\mathbf{A}$ is *strongly locally finitely presentable* if it is cocomplete, $\mathbf{A}_{\mathrm{sfp}}$ is a small category and every object of $\mathbf{A}$ is a sifted colimit of a diagram in $\mathbf{A}_{\mathrm{sfp}}$.

▶ **Lemma 11.** *For a small category $\mathbf{A}$, the presheaf category $\widehat{\mathbf{A}}$ is strongly finitely presentable and every presheaf is a sifted colimit of finite coproducts of representables.*

Proof. Let $\mathbf{A}$ be a small category, then $\widehat{\mathbf{A}}$ is strongly finitely presentable and the strongly finitely presentable objects are the regular projective presheaves [1]. In presheaf categories, the full subcategory of coproducts of representables is a regular projective cover [20]. Hence every presheaf is a sifted colimit of coproducts of representables. Since every coproduct is a filtered colimit of finite coproducts, we obtain the desired result. ◀

Functors preserving sifted colimits are called *strongly finitary functors*. On **Set**, finitary and strongly finitary functors coincide [1].

▶ **Definition 12.** *The 2-category* **Sift** *has small categories as objects and a morphism between two categories* $\mathbf{A}$ *and* $\mathbf{B}$ *is a strongly finitary functor* $\mathcal{P} : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$*. The 2-cells between two such functors are natural transformations.*

The main result of this section is to show that there is a biequivalence between the bicategory $\mathbf{Prof}_{\mathcal{C}}$ and the 2-category **Sift**.

▶ **Lemma 13.** *For a $\mathcal{C}$-species $F : \mathcal{C}\mathbf{A} \nrightarrow \mathbf{B}$, $\mathbf{Lan}_{s_{\mathbf{A}}}(F) : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ preserves sifted colimits.*

Proof. Let $\mathcal{D} : I \to \widehat{\mathbf{A}}$ be a sifted diagram, we have:

$$\begin{aligned}
&\mathbf{Lan}_{s_{\mathbf{A}}} F(\varinjlim_{i\in I} \mathcal{D}(i))(b) = \int^{u=\langle a_1,\ldots,a_n\rangle} F(u,b) \times \widehat{\mathbf{A}}(s_{\mathbf{A}}(u), \varinjlim_{i\in I} \mathcal{D}(i)) \\
&\cong \int^{u} F(u,b) \times \prod_{j=1}^{n} \widehat{\mathbf{A}}(\mathsf{y}(a_j), \varinjlim_{i\in I} \mathcal{D}(i)) \cong \int^{u} F(u,b) \times \prod_{i=j}^{n} \varinjlim_{i\in I} \mathcal{D}(i)(a_j) \\
&\cong \int^{u} F(u,b) \times \varinjlim_{i\in I} \prod_{j=1}^{n} \mathcal{D}(i)(a_j) \cong \int^{u} F(u,b) \times \varinjlim_{i\in I} \left(\widehat{\mathbf{A}}(s_{\mathbf{A}}(u), \mathcal{D}(i))\right) \\
&\cong \int^{u} \varinjlim_{i\in I} \left(F(u,b) \times \widehat{\mathbf{A}}(s_{\mathbf{A}}(u), \mathcal{D}(i))\right) = \varinjlim_{i\in I} \left(\int^{u} F(u,b) \times \widehat{\mathbf{A}}(s_{\mathbf{A}}(u), \mathcal{D}(i))\right)
\end{aligned}$$

Since sifted colimits commute with finite products, it allows us to obtain the third isomorphism. We then make use of the facts that $(F(u,b) \times -)$ is a left adjoint, and hence colimit-preserving, and that the coend is a colimit and hence commutes with colimits. ◀

▶ **Lemma 14.** *For small categories* $\mathbf{A}$ *and* $\mathbf{B}$*, there is an adjoint equivalence between the categories:*

$$\mathbf{Prof}_{\mathcal{C}}(\mathbf{A},\mathbf{B}) \; \underset{-\circ s_{\mathbf{A}}}{\overset{\mathbf{Lan}_{s_{\mathbf{A}}}(-)}{\rightleftarrows}} \; \mathbf{Sift}(\mathbf{A},\mathbf{B}) \qquad (\bot)$$

Proof. Since $s_{\mathbf{A}}$ is fully faithful, for any $\mathcal{C}$-species F in $\mathbf{Prof}_{\mathcal{C}}(\mathbf{A},\mathbf{B})$ there is a natural isomorphism $\alpha_F : F \Rightarrow (\mathbf{Lan}_{s_{\mathbf{A}}}(F)) \circ s_{\mathbf{A}}$. Hence, for a natural transformation $\beta : F_1 \Rightarrow F_2$ in $\mathbf{Prof}_{\mathcal{C}}(\mathbf{A},\mathbf{B})$, its image by $\mathbf{Lan}_{s_{\mathbf{A}}}(-)$ is the unique natural transformation $\gamma : \mathbf{Lan}_{s_{\mathbf{A}}}(F_1) \Rightarrow \mathbf{Lan}_{s_{\mathbf{A}}}(F_2)$ such that $\gamma s_{\mathbf{A}} \alpha_{F_1} = \beta \alpha_{F_2}$ which provides us with a natural isomorphism $\eta : \mathrm{Id}_{\mathbf{Prof}_{\mathcal{C}}(\mathbf{A},\mathbf{B})} \Rightarrow (\mathbf{Lan}_{s_{\mathbf{A}}}(-)) \circ s_{\mathbf{A}}$ by Proposition 8.

Let $P : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}$ be a functor that preserves sifted colimits. We want to exhibit a natural isomorphism

$$\mathbf{Lan}_{s_{\mathbf{A}}}(P \circ s_{\mathbf{A}})(X) \cong P(X)$$

By Lemma 11, X is a sifted colimit of finite coproducts of representables, i.e. there exists a sifted diagram $D : I \to \mathcal{C}\mathbf{A}$ such that $X \cong \varinjlim_{i\in I} s_{\mathbf{A}}(D(i))$:

$$\begin{aligned}
&\mathbf{Lan}_{s_{\mathbf{A}}}(P \circ s_{\mathbf{A}})(X) = \int^{u=\langle a_1,\dots,a_n\rangle} P(s_{\mathbf{A}}(u)) \times \widehat{\mathbf{A}}(s_{\mathbf{A}}(u), X) \\
&\cong \int^{u} P(s_{\mathbf{A}}(u)) \times \prod_{j=1}^{n} \widehat{\mathbf{A}}(\mathsf{y}(a_j), \varinjlim_{i\in I} s_{\mathbf{A}} D(i)) \cong \int^{u} P(s_{\mathbf{A}}(u)) \times \varinjlim_{i\in I} \prod_{j=1}^{n} \widehat{\mathbf{A}}(\mathsf{y}(a_j), s_{\mathbf{A}} D(i)) \\
&\cong \int^{u} P(s_{\mathbf{A}}(u)) \times \varinjlim_{i\in I} \widehat{\mathbf{A}}(s_{\mathbf{A}}(u), s_{\mathbf{A}} D(i)) \cong \varinjlim_{i\in I} \int^{u} P(s_{\mathbf{A}}(u)) \times \mathcal{C}\mathbf{A}(s_{\mathbf{A}}(u), s_{\mathbf{A}} D(i)) \\
&\cong \varinjlim_{i\in I} P(s_{\mathbf{A}}(D(i))) \cong P(X)
\end{aligned}$$

which entails the existence of a natural isomorphism $\varepsilon : \mathbf{Lan}_{s_{\mathbf{A}}}(- \circ s_{\mathbf{A}}) \Rightarrow \mathrm{Id}_{\mathbf{Sift}(\mathbf{A},\mathbf{B})}$ as desired. The adjunction

$$[\mathcal{C}\mathbf{A}, \widehat{\mathbf{B}}](F, P \circ s_{\mathbf{A}}) \cong [\widehat{\mathbf{A}}, \widehat{\mathbf{B}}](\mathbf{Lan}_{s_{\mathbf{A}}} F, P).$$

is a direct consequence of the universal property of left Kan extensions (see Theorem 4.38 in [16] for example). ◀

▶ **Proposition 15.** *The bicategory* $\mathbf{Prof}_{\mathcal{C}}$ *is biequivalent to the* 2*-category* $\mathbf{Sift}$.

Proof. We prove that the pseudofunctor $\mathcal{F} : \mathbf{Prof}_{\mathcal{C}} \to \mathbf{Sift}$ defined below is a biequivalence. For $\mathbf{A}$ and $\mathbf{B}$ small categories, we define $\mathcal{F}(\mathbf{A}) := \mathbf{A}$ and

$$\begin{aligned}
\mathcal{F}_{\mathbf{A},\mathbf{B}} &: \mathbf{Prof}_{\mathcal{C}}(\mathbf{A}, \mathbf{B}) \to \mathbf{Sift}(\mathbf{A}, \mathbf{B}) \\
&F : \mathcal{C}\mathbf{A} \nrightarrow \mathbf{B} \mapsto \mathbf{Lan}_{s_{\mathbf{A}}}(F) : \widehat{\mathbf{A}} \to \widehat{\mathbf{B}}
\end{aligned}$$

Since $\mathbf{Prof}_{\mathcal{C}}$ and $\mathbf{Sift}$ have the same objects, it follows immediately that $\mathcal{F} : \mathbf{Prof}_{\mathcal{C}} \to \mathbf{Sift}$ is essentially surjective. Lemma 14 entails that $\mathcal{F}_{\mathbf{A},\mathbf{B}}$ is an adjoint equivalence of categories. ◀

5 From Prof to ScottL

In this section, we formalize the connection between the categorical approach and the preorder model as a change of base for enriched categories. A category enriched over $\mathbb{2} = (\{\emptyset \leq \mathbb{1}\}, \wedge, \mathbb{1})$ is a preorder and a $\mathbb{2}$-profunctor between two preorders $A = (|A|, \leq_A)$ and $B = (|B|, \leq_B)$ corresponds to a relation in $\mathbf{ScottL}(A, B)$. The functor $M : \mathbf{Set} \to \mathbb{2}$ defined by

$$X \mapsto \begin{cases} \emptyset & \text{if } X = \emptyset \\ \mathbb{1} & \text{otherwise} \end{cases}$$

is monoidal and therefore induces a lax pseudo-functor Ψ from $\mathbf{Prof}_{\mathbf{Set}}$ (just denoted by $\mathbf{Prof}$) to $\mathbf{Prof}_{\mathbb{2}} = \mathbf{ScottL}$ [4]. In this section, we give an explicit description of this change of base pseudo-functor $\Psi : \mathbf{Prof} \to \mathbf{ScottL}$ and show that it is in fact a strong pseudo-functor that preserves all the structure of linear logic. The viewpoint of enriched categories enables us to work in a unified setting where both models coexist and the change of base becomes a pseudo-functor that connects the preorder world and the categorified world in a way that preserves the structure of linear logic.

On objects, Ψ sends a small category $\mathbf{A}$ to the following preorder:

$$(\mathrm{Ob}(\mathbf{A}), \leq_A) \quad \text{where} \quad a \leq_A a' \quad :\Leftrightarrow \quad \mathbf{Hom}_{\mathbf{A}}(a, a') \neq \emptyset$$

For a profunctor $F : \mathbf{A} \nrightarrow \mathbf{B}$, $\Psi_{\mathbf{A},\mathbf{B}}(F)$ is given by $\Psi_{\mathbf{A},\mathbf{B}}(F) := \{(a, b) \mid F(a, b) \neq \emptyset\}$.

▶ **Lemma 16.** *For every* $\mathbf{A}, \mathbf{B}$, $\Psi_{\mathbf{A},\mathbf{B}} : \mathbf{Prof}(\mathbf{A}, \mathbf{B}) \to \mathbf{ScottL}(\Psi(\mathbf{A}), \Psi(\mathbf{B}))$ *is functorial.*

Proof. We first need to check that $\Psi_{\mathbf{A},\mathbf{B}}(F)$ is indeed an element of $\mathbf{ScottL}(\Psi(\mathbf{A}), \Psi(\mathbf{B}))$, i.e. that for all $(a, b) \in \Psi_{\mathbf{A},\mathbf{B}}(F)$, $(a', b') \leq_{\mathbf{A}^{op}\times\mathbf{B}} (a, b)$ implies $(a', b') \in \Psi_{\mathbf{A},\mathbf{B}}(F)$. If $(a, b) \in \Psi_{\mathbf{A},\mathbf{B}}(F)$, then $F(a, b) \neq \emptyset$ so there exists an element $s \in F(a, b)$. The inequality $(a', b') \leq_{\mathbf{A}^{op}\times\mathbf{B}} (a, b)$ implies that there exist morphisms $f : a \to a'$ in $\mathbf{A}$ and $g : b' \to b$ in $\mathbf{B}$. Hence, $F(f, g)(s) \in F(a', b')$ which is not empty as desired. When we consider $\mathbf{ScottL}$ as a bicategory, morphisms in $\mathbf{ScottL}(\Psi(\mathbf{A}), \Psi(\mathbf{B}))$ are just inclusions of relations so we only need to show that if there exists a natural transformation $\alpha : F \Rightarrow G$ in $\mathbf{Prof}(\mathbf{A}, \mathbf{B})$, then $\Psi_{\mathbf{A},\mathbf{B}}(F) \subseteq \Psi_{\mathbf{A},\mathbf{B}}(G)$. For $(a, b) \in \Psi_{\mathbf{A},\mathbf{B}}(F)$, if there exists an element $s \in F(a, b)$ then $\alpha_{(a,b)}(s) \in G(a, b)$ which implies that $(a, b) \in \Psi_{\mathbf{A},\mathbf{B}}(G)$ as desired. ◀

▶ **Proposition 17.** Ψ *is a strong pseudo-functor that preserves the linear logic structure.*

Proof.

- For profunctors $F : \mathbf{A} \nrightarrow \mathbf{B}$ and $G : \mathbf{B} \nrightarrow \mathbf{C}$, the following equalities hold:

$$\begin{aligned}\Psi_{\mathbf{A},\mathbf{C}}(G \circ_{\mathbf{Prof}} F) &= \{(a, c) \mid \int^{b\in\mathbf{B}} F(a, b) \times G(b, c) \neq \emptyset\} \\ &= \{(a, c) \mid \exists b \in \mathrm{Ob}(\mathbf{B}), F(a, b) \neq \emptyset \text{ and } G(b, c) \neq \emptyset\} \\ &= \{(a, c) \mid \exists b \in \mathrm{Ob}(\mathbf{B}), (a, b) \in \Psi_{\mathbf{A},\mathbf{B}}(F) \text{ and } (b, c) \in \Psi_{\mathbf{B},\mathbf{C}}(G)\} \\ &= \Psi_{\mathbf{B},\mathbf{C}}(G) \circ_{\mathbf{ScottL}} \Psi_{\mathbf{A},\mathbf{B}}(F)\end{aligned}$$

- We only show that Ψ commutes with the pseudo-comonad structure, the other cases being similar. For a small category $\mathbf{A}$, $!\Psi(\mathbf{A})$ is the preorder whose underlying set is equal to the object set of $\mathcal{C}\mathbf{A}$ so $!\Psi$ and $\Psi\mathcal{C}$ coincide on objects. For a profunctor $F : \mathbf{A} \nrightarrow \mathbf{B}$, we have:

$$\begin{aligned}!\Psi_{\mathbf{A},\mathbf{B}}(F) &= \{(\langle a_1, \ldots, a_n\rangle, \langle b_1, \ldots, b_m\rangle) \mid \forall j \in \underline{m}, \exists i \in \underline{n}, (a_i, b_j) \in \Psi(F)\} \\ &= \{(\langle a_1, \ldots, a_n\rangle, \langle b_1, \ldots, b_m\rangle) \mid \forall j \in \underline{m}, \exists i \in \underline{n}, F(a_i, b_j) \neq \emptyset\} \\ &= \{(\langle a_1, \ldots, a_n\rangle, \langle b_1, \ldots, b_m\rangle) \mid \prod_{j\in\underline{m}} \sum_{i\in\underline{n}} F(a_i, b_j) \neq \emptyset\} \\ &= \{(\langle a_1, \ldots, a_n\rangle, \langle b_1, \ldots, b_m\rangle) \mid \mathcal{C}F(\langle a_1, \ldots, a_n\rangle, \langle b_1, \ldots, b_m\rangle) \neq \emptyset\} = \Psi(\mathcal{C}F)\end{aligned}$$

The following equalities also hold for the dereliction and the digging pseudo-natural transformations:

$$\begin{aligned}\Psi(\varepsilon_{\mathbf{A}}) &= \{(u, a) \mid \varepsilon_{\mathbf{A}}(u, a) \neq \emptyset\} = \{(u, a) \mid \sum_{i\in|u|} \mathbf{A}(a, u_i) \neq \emptyset\} \\ &= \{(u, a) \mid \forall i \in |u|, a \leq_{\Psi(\mathbf{A})} u_i\} = \varepsilon_{\Psi(\mathbf{A})} \\ \Psi(\delta_{\mathbf{A}}) &= \{(u, \langle u_1, \ldots, u_n\rangle) \mid \mathcal{C}\mathbf{A}(u_1 \oplus \cdots \oplus u_n, u) \neq \emptyset\} \\ &= \{(u, \langle u_1, \ldots, u_n\rangle) \mid u_1 \oplus \cdots \oplus u_n \leq_{\Psi(\mathcal{C}\mathbf{A})} u\} = \delta_{\Psi\mathcal{C}\mathbf{A}}\end{aligned}$$

◀

6 Recursive Type and Term Equations

6.1 Fixed points of Types

Recursive domain equations play a central role in denotational semantics. A classical example is Scott's D_∞ construction providing an extensional model of the untyped λ-calculus. In $\mathbf{Prof}_{\mathcal{C}}$, we show that full subcategory inclusion is a partial order relation on objects such that all linear logic constructions define Scott-continuous maps on this partially ordered class. It entails that we can give solutions to any recursive type equation constituted of linear logic operators and we exhibit in this section an example of a 2-dimensional model of pure λ-calculus in $\mathbf{Prof}_{\mathcal{C}}$.

▶ **Definition 18.** *For small categories* $\mathbf{A}$ *and* $\mathbf{B}$*, we write* $\mathbf{A} \sqsubseteq \mathbf{B}$ *if* $\mathbf{A}$ *is a full subcategory of* $\mathbf{B}$*, i.e.* $Ob(\mathbf{A}) \subseteq Ob(\mathbf{B})$ *and for all* a *and* a' *in* $Ob(\mathbf{A})$*,* $\mathbf{A}(a,a') = \mathbf{B}(a,a')$*.*

One can easily check that $\sqsubseteq$ defines a partial order relation on the class of small categories. We denote by $\mathbf{Cat}_{\sqsubseteq}$ the obtained partially ordered class and show the following lemma:

▶ **Lemma 19.** $\mathbf{Cat}_{\sqsubseteq}$ *is closed under directed colimits.*

Proof. Let $D : I \to \mathbf{Cat}_{\sqsubseteq}$ be a directed diagram. We denote by $\bigvee_{i\in I} D_i$ the category whose set of objects is $\bigcup_{i\in I} \mathrm{Ob}(D_i)$ so that for any $a, b \in \mathrm{Ob}(\bigvee_{i\in I} D_i)$, there exist $i, j \in I$ such that $a \in \mathrm{Ob}(D_i)$ and $b \in \mathrm{Ob}(D_j)$. Since I is directed, there exists $k \in I$ such that $a, b \in \mathrm{Ob}(D_k)$ so we define $\bigvee_{i\in I} D_i(a,b)$ to be $D_k(a,b)$. ◀

▶ **Lemma 20.** *All the linear logic constructions are Scott-continuous with respect to the order* $\sqsubseteq$*.*

Proof. The proof is routine, we only exhibit the dual and exponential cases:

- **Dual:** It is noteworthy to observe that the dual is monotonous with respect to this order. For $\mathbf{A} \sqsubseteq \mathbf{B}$, we have that $\mathrm{Ob}(\mathbf{A}^{op}) = \mathrm{Ob}(\mathbf{A}) \subseteq \mathrm{Ob}(\mathbf{B}) = \mathrm{Ob}(\mathbf{B}^{op})$ and for any $a, a' \in \mathbf{A}^{op}$, $\mathbf{A}^{op}(a,a') = \mathbf{A}(a',a) = \mathbf{B}(a',a) = \mathbf{B}^{op}(a,a')$ which entails that $\mathbf{A}^{op} \sqsubseteq \mathbf{B}^{op}$. Let $D : I \to \mathbf{Cat}_{\sqsubseteq}$ be a directed diagram, we want to show that $\left(\bigvee_{i\in I} D_i\right)^{op} = \bigvee_{i\in I} D_i^{op}$. It is immediate to show that these two categories have the same objects and for $a, a' \in \bigvee_{i\in I} D_i^{op}$, there exists $k \in I$ such that $a, a' \in \mathrm{Ob}(D_k)$ so that:
$$\bigvee_{i\in I} D_i^{op}(a,a') = D_k^{op}(a,a') = D_k(a',a) = (\bigvee_{i\in I} D_i)(a',a) = (\bigvee_{i\in I} D_i)^{op}(a,a').$$
- **Exponential:** For $\mathbf{A} \sqsubseteq \mathbf{B}$, $\mathrm{Ob}(\mathcal{C}\mathbf{A}) = \{\langle a_1, \ldots, a_n\rangle \mid a_i \in \mathrm{Ob}(\mathbf{A})\} \subseteq \{\langle b_1, \ldots, b_n\rangle \mid b_i \in \mathrm{Ob}(\mathbf{B})\} = \mathrm{Ob}(\mathcal{C}\mathbf{B})$ and for u, v in $\mathrm{Ob}(\mathcal{C}\mathbf{A})$:
$$\mathcal{C}\mathbf{A}(u,v) = \prod_{i\in|u|} \sum_{j\in|v|} \mathbf{A}(u_i, v_j) = \prod_{i\in|u|} \sum_{j\in|v|} \mathbf{B}(u_i, v_j) = \mathcal{C}\mathbf{B}(u,v)$$
which entails that $\mathcal{C}\mathbf{A} \sqsubseteq \mathcal{C}\mathbf{B}$ as desired. Let $D : I \to \mathbf{Cat}_{\sqsubseteq}$ be a directed diagram, we want to show that $\mathcal{C}(\bigvee_{i\in I} D_i) = \bigvee_{i\in I} \mathcal{C}D_i$. For the object sets, we haveL
$$\mathrm{Ob}(\mathcal{C}(\bigvee_{i\in I} D(i))) = \bigcup_{n\in\mathbb{N}} \mathrm{Ob}(\bigvee_{i\in I} D(i))^n = \bigcup_{n\in\mathbb{N}} (\bigcup_{i\in I} \mathrm{Ob}(D_i))^n = \bigcup_{n\in\mathbb{N}} \bigcup_{i\in I} (\mathrm{Ob}(D(i)))^n$$
$$= \bigcup_{i\in I} \bigcup_{n\in\mathbb{N}} (\mathrm{Ob}(D_i))^n = \mathrm{Ob}\left(\bigvee_{i\in I} \mathcal{C}D_i\right)$$
The third equality follows from the fact that directed unions commute with finite products. Consider now two elements $u := \langle x_1, \ldots, x_n\rangle$ and $v := \langle y_1, \ldots, y_m\rangle$ in $\bigvee_{i\in I} \mathcal{C}(D_i)$. Since I is directed, there exists $k \in I$ such that $u, v \in \mathrm{Ob}(\mathcal{C}(D_k))$, we therefore obtain:
$$(\bigvee_{i\in I} \mathcal{C}(D_i)(u,v) = \mathcal{C}(D_k)(u,v) = \prod_{l\in\underline{n}} \sum_{r\in\underline{m}} D_k(x_l, y_r) = \prod_{l\in\underline{n}} \sum_{r\in\underline{m}} \bigvee_{i\in I} D_k(x_l, y_r) = \mathcal{C}(\bigvee_{i\in I} D(i))(u,v)$$
The last equality follows from the fact that $D_k \sqsubseteq \bigvee_{i\in I} D_i$. ◀

▶ **Example 21.** By the previous lemma, any recursive type equation on $\mathbf{Cat}_{\sqsubseteq}$ built from linear logic connectives has a least fixed point. Let $\mathbf{N}$ be the least fixed point solution of $\mathbf{N} = \mathbb{1} \oplus \mathbf{N}$, it can be explicitly described as the category $\mathbf{N} = \bigoplus_{i\in\mathbb{N}} \mathbb{1}$. Consider now $\mathbf{D}$ to be the least fixed point solution of $\mathbf{D} = (\mathcal{C}(\mathbf{N} \multimap \mathbf{D}))^{op}$. Using the Seely equivalence in Lemma 6, we can first note that $\mathbf{D}$ verifies the following equivalence:

$$\begin{aligned}\mathbf{D} &= (\mathcal{C}(\mathbf{N} \multimap \mathbf{D}))^{op} \simeq (\mathcal{C}((\mathbb{1} \oplus \mathbf{N}) \multimap \mathbf{D}))^{op} \simeq (\mathcal{C}((\mathbb{1} \multimap \mathbf{D}) \,\&\, (\mathbf{N} \multimap \mathbf{D})))^{op} \\ &\simeq ((\mathcal{C}(\mathbf{D})) \otimes \mathcal{C}(\mathbf{N} \multimap \mathbf{D}))^{op} = (\mathcal{C}\mathbf{D})^{op} \parr \mathbf{D} = (\mathbf{D} \Rightarrow \mathbf{D})\end{aligned}$$

The category $\mathbf{D}$ provides an extensional reflexive object for the pure λ-calculus in the cartesian closed bicategory $\mathbf{Prof}_{\mathcal{C}}$. We make explicit its structure below by first giving the application and lambda profunctors:

$$Ap : \mathcal{C}(\mathbf{D} \Rightarrow \mathbf{D}) \nrightarrow \mathbf{D} \qquad \lambda : \mathcal{C}\mathbf{D} \nrightarrow (\mathbf{D} \Rightarrow \mathbf{D})$$

as follows: for $W \in \mathcal{C}(\mathbf{D} \Rightarrow \mathbf{D})$ and $d \in \mathbf{D}^{op}$, let $k \in \mathbb{N}$ be the smallest index such that $W \in \mathcal{C}(\mathbf{D}_k \Rightarrow \mathbf{D}_k)$ and $d \in \mathbf{D}_k^{op}$. Since $\mathbf{D}_k^{op} = (\mathcal{C}((\mathbb{1} \oplus \mathbf{N}) \multimap \mathbf{D}_{k-1})) \cong (\mathcal{C}(\mathbf{D}_{k-1}) \& (\mathbf{N} \multimap \mathbf{D}_{k-1})))$, we use the Seely equivalence and obtain $d.1 \in \mathcal{C}(\mathbf{D}_{k-1}) \sqsubseteq \mathcal{C}(\mathbf{D}_k)$ and $d.2 \in \mathcal{C}(\mathbf{N} \multimap \mathbf{D}_{k-1}) = \mathbf{D}_k^{op}$. We now define Ap as the profunctor taking (W, d) to $\mathcal{C}(\mathbf{D}_k \Rightarrow \mathbf{D}_k)(\langle(d.1, d.2)\rangle, W)$.

To define $\lambda(u, (v, d))$ for $u \in \mathcal{C}\mathbf{D}$ and $(v, d) \in (\mathbf{D} \Rightarrow \mathbf{D})^{op}$, we first let l to be the smallest index such that $u \in \mathcal{C}(\mathbf{D})_l$, $v \in \mathcal{C}(\mathbf{D}_l)$ and $d \in \mathbf{D}_l^{op} \sqsubseteq \mathbf{D}_{l+1}^{op} = \mathcal{C}((\mathbb{1} \oplus \mathbf{N}) \multimap \mathbf{D}_l) \cong \mathcal{C}(\mathbf{D}_l \,\&\, (\mathbf{N} \multimap \mathbf{D}_l))$. Considering the diagram below,

$$\begin{array}{ccccc}\mathcal{C}(\mathbf{D})_l & \xrightarrow[\mathcal{C}(i_1)]{} & \mathcal{C}(\mathbf{D}_l \,\&\, (\mathbf{N} \multimap \mathbf{D}_l)) & \xleftarrow[\mathcal{C}(i_2)]{} & \mathcal{C}(\mathbf{N} \multimap \mathbf{D}_l) \\ & & \parallel & & \\ & & \mathbf{D}_{l+1}^{op} & & \end{array}$$

we obtain that $\mathcal{C}(i_1)(u) \oplus \mathcal{C}(i_2)(d)$ is an element of $\mathbf{D}_{l+1}^{op}$, so we define $\lambda(u, (v, d))$ to be $\mathcal{C}(\mathbf{D}_{l+1})(\mathcal{C}(i_1)(v) \oplus \mathcal{C}(i_2)(d), u)$. We then obtain:

$$\begin{aligned}\lambda \circ Ap(W, (v, d)) &= \int^{u\in\mathcal{C}\mathbf{D}} \lambda(u, (v, d)) \times Ap^{\mathcal{C}}(W, u) = \int^{u} \mathcal{C}\mathbf{D}(\mathcal{C}(i_1)(v) \oplus \mathcal{C}(i_2)(d), u) \times Ap^{\mathcal{C}}(W, u) \\ &\cong Ap(W, \mathcal{C}(i_1)(v) \oplus \mathcal{C}(i_2)(d)) = \mathcal{C}(\mathbf{D} \Rightarrow \mathbf{D})(\langle(v, d)\rangle, W) = \mathrm{Id}_{\mathbf{D}\Rightarrow\mathbf{D}}(W, (v, d))\end{aligned}$$

The second to last equality follows from the fact that $(\mathcal{C}(i_1)(v) \oplus \mathcal{C}(i_2)(d)).1 = v$ and $(\mathcal{C}(i_1)(v) \oplus \mathcal{C}(i_2)(d)).2 = d$. We also obtain the following isomorphism:

$$\begin{aligned}Ap \circ \lambda(u, d) &= \int^{W\in\mathcal{C}(\mathbf{D}\Rightarrow\mathbf{D})} Ap(W, d) \times \lambda^{\mathcal{C}}(u, W) \\ &= \int^{W} \mathcal{C}(\mathbf{D} \Rightarrow \mathbf{D})(\langle(d.1, d.2)\rangle, W) \times \lambda^{\mathcal{C}}(u, W) \cong \lambda(u, (d.1, d.2)) \\ &= \mathcal{C}\mathbf{D}(\mathcal{C}(i_1)(d.1) \oplus \mathcal{C}(i_2)(d.2), u) \cong \mathcal{C}\mathbf{D}(\langle d\rangle, u) = \mathrm{Id}_{\mathbf{D}}(u, d)\end{aligned}$$

The second to last equality follows from the fact that d is isomorphic to $\mathcal{C}(i_1)(d.1) \oplus \mathcal{C}(i_2)(d.2)$ in $\mathcal{C}(\mathbf{D})$.

6.2 Fixed point operator for terms

▶ **Theorem 22** (e.g. [21]). *Let $\mathbf{C}$ be a category with ω-colimits together with an initial object 0 and let $F : \mathbf{C} \to \mathbf{C}$ be an endofunctor that preserves ω-chains. Then F has an initial algebra obtained by taking the colimit of the following diagram:*

$$0 \xrightarrow[i]{} F(0) \xrightarrow[F(i)]{} F^2(0) \xrightarrow[F^2(i)]{} \cdots$$

where i is the unique map from the initial object to $F(0)$.

▶ **Lemma 23** (e.g. [21]). *Let $F : \mathbf{C} \to \mathbf{C}$ be an endofunctor and $a : F(c) \to c$ an initial algebra. Then a is an isomorphism.*

▶ **Definition 24.** *Let $\mathcal{B}$ be a cartesian closed bicategory and A an object of $\mathcal{B}$. A* fixpoint operator *for an object A in $\mathcal{B}$ is a 1-cell $\mathbf{fix}_A \in \mathcal{B}(A \Rightarrow A, A)$ together with an invertible 2-cell α:*

$$\begin{array}{ccc} A \Rightarrow A & & \\ {\scriptstyle \langle Id_{A\Rightarrow A}, \mathbf{fix}_A\rangle}\downarrow & \overset{\sim}{\underset{\alpha}{\Longrightarrow}} & \searrow^{\mathbf{fix}_A} \\ (A \Rightarrow A) \,\&\, A & \xrightarrow[Ev_{A,A}]{} & A \end{array}$$

For $f \in A \Rightarrow A$, we obtain that $Ev_{A,A}\langle f, \mathbf{fix}_A(f)\rangle \overset{\sim}{\Longrightarrow} \mathbf{fix}_A(f)$.

For a small category $\mathbf{A}$, $\mathbf{fix}_{\mathbf{A}} \in \mathbf{Prof}_{\mathcal{C}}(\mathbf{A} \Rightarrow \mathbf{A}, \mathbf{A})$ is obtained as the initial algebra of the following functor:

$$\begin{aligned} \mathcal{Y}_{\mathbf{A}} : \mathbf{Prof}_{\mathcal{C}}(\mathbf{A} \Rightarrow \mathbf{A}, \mathbf{A}) &\to \mathbf{Prof}_{\mathcal{C}}(\mathbf{A} \Rightarrow \mathbf{A}, \mathbf{A}) \\ F &\mapsto \mathrm{Ev} \circ \langle Id, F\rangle \end{aligned}$$

We identify $\mathbf{Prof}_{\mathcal{C}}(\mathbf{A} \Rightarrow \mathbf{A}, \mathbf{A})$ with the presheaf category of $(\mathbf{A} \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A}$ whose initial object is the empty presheaf. Since for any morphism $H : \mathcal{C}\mathbf{X} \nrightarrow \mathbf{Y}$ in $\mathbf{Prof}_{\mathcal{C}}$, $\mathbf{Lan}_{s_{\mathbf{X}}}(H) : \widehat{X} \to \widehat{Y}$ preserves ω-colimits (as a particular case of sifted colimits), we show that $\mathcal{Y}_{\mathbf{A}}$ can be obtained as the left Kan extension of a $\mathcal{C}$-species in $\mathbf{Prof}_{\mathcal{C}}((\mathbf{A} \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A}, (\mathbf{A} \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A})$ which entails the existence of $\mathbf{fix}_{\mathbf{A}}$ by Theorem 22.

Consider the profunctor $\mathcal{Z}_{\mathbf{A}} \in \mathbf{Prof}_{\mathcal{C}}(((\mathbf{A} \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A}) \,\&\, (\mathbf{A} \Rightarrow \mathbf{A}), \mathbf{A})$ defined by the following composition:

$$\begin{array}{l} ((\mathbf{A} \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A}) \,\&\, (\mathbf{A} \Rightarrow \mathbf{A}) \\ \quad \downarrow {\scriptstyle Id \,\&\, \langle Id, Id\rangle} \\ ((\mathbf{A} \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A}) \,\&\, (\mathbf{A} \Rightarrow \mathbf{A}) \,\&\, (\mathbf{A} \Rightarrow \mathbf{A}) \\ \quad \downarrow {\scriptstyle \mathrm{Ev}_{\mathbf{A}\Rightarrow\mathbf{A},\mathbf{A}} \,\&\, Id} \\ \mathbf{A} \,\&\, (\mathbf{A} \Rightarrow \mathbf{A}) \xrightarrow{\langle \pi_2, \pi_1\rangle} (\mathbf{A} \Rightarrow \mathbf{A}) \,\&\, \mathbf{A} \xrightarrow{\mathrm{Ev}_{\mathbf{A},\mathbf{A}}} \mathbf{A} \end{array}$$

By currying, we obtain a profunctor $\lambda(\mathcal{Z}_{\mathbf{A}})$ in $\mathbf{Prof}_{\mathcal{C}}((\mathbf{A} \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A}, (\mathbf{A} \Rightarrow \mathbf{A}) \Rightarrow \mathbf{A})$ whose left Kan extension along $s_{(\mathbf{A}\Rightarrow\mathbf{A})\Rightarrow\mathbf{A}}$ is isomorphic to $\mathcal{Y}_{\mathbf{A}}$ as desired. Explicitely, $\mathcal{Y}_{\mathbf{A}}$ is given by:

$$\mathcal{Y}_{\mathbf{A}} : (F, (U, a)) = \int^{u \in \mathcal{C}\mathbf{A}} F^{\mathcal{C}}(U, u) \times \mathcal{C}(\mathbf{A} \Rightarrow \mathbf{A})(\langle (u, a)\rangle, U)$$

We can now obtain $\mathbf{fix}_{\mathbf{A}} : \mathcal{C}(\mathbf{A} \Rightarrow \mathbf{A}) \nrightarrow \mathbf{A}$ by computing $\varinjlim_{n \in \omega} \mathcal{Y}^n_{\mathbf{A}}(\mathbb{0})$.

▶ **Example 25.**

- In the theory of combinatorial species, the species of lists is a solution of the equation $L = 1 + X \cdot L$ where 1 is the species whose analytic functor $\mathbf{Set} \to \mathbf{Set}$ is given by $S \mapsto \{\star\}$ and X is the singleton species whose analytic functor is the identity endofunctor on **Set**. It follows the intuition that a list is either empty or an element followed by a list. In the case of $\mathbf{Prof}_{\mathcal{C}}$, we can define for every small category $\mathbf{A}$ a $\mathcal{C}$-species of lists $L_{\mathbf{A}} : \mathcal{C}\mathbf{A} \nrightarrow \mathbf{A}$. $L_{\mathbf{A}}$ is obtained as the least fixpoint of the operator:

$$\begin{aligned} E_{\mathbf{A}} &: \mathbf{Prof}_{\mathcal{C}}(\mathbf{A},\mathbf{A}) \to \mathbf{Prof}_{\mathcal{C}}(\mathbf{A},\mathbf{A}) \\ &(F,(u,a)) \mapsto \mathbf{1}_{\mathbf{A}}(u,a) + \mathbb{X}_{\mathbf{A}}(u,a) \times F(u,a) = \mathcal{C}\mathbf{A}(\langle\rangle,u) + \mathcal{C}\mathbf{A}(\langle a\rangle,u) \times F(u,a) \end{aligned}$$

  where $\mathbf{1}_{\mathbf{A}}(u,a)$ is the constant species $(u,a) \mapsto \mathcal{C}\mathbf{A}(\langle\rangle,u) \simeq \{\star\}$ and $\mathbb{X}_{\mathbf{A}}$ is the singleton species $(u,a) \mapsto \mathcal{C}\mathbf{A}(\langle a\rangle,u)$. Note that if we take $\mathbf{A}$ to be the category $\mathbb{1}$, we obtain the species 1 and X mentionned above. Explicitly, the $\mathcal{C}$-species of lists $L_{\mathbf{A}} : \mathcal{C}\mathbf{A} \nrightarrow \mathbf{A}$ maps (u,a) to $\sum_{n\in\mathbb{N}} \mathcal{C}\mathbf{A}(\langle a\rangle,u)^n$ which entails that $\mathbf{Lan}_{s_{\mathbf{A}}}(L_{\mathbf{A}}) : \widehat{\mathbf{A}} \to \widehat{\mathbf{A}}$ is given by

$$(X,a) \mapsto \sum_{n\in\mathbb{N}} (X(a))^n.$$

- Using a similar reasoning, we can obtain a $\mathcal{C}$-species of binary trees, which is a solution of the equation $B = 1 + X \cdot B^2$. For a small category $\mathbf{A}$, if we compute the least fixpoint of the operator:

$$\begin{aligned} H_{\mathbf{A}} &: \mathbf{Prof}_{\mathcal{C}}(\mathbf{A},\mathbf{A}) \to \mathbf{Prof}_{\mathcal{C}}(\mathbf{A},\mathbf{A}) \\ &(F,(u,a)) \mapsto \mathcal{C}\mathbf{A}(\langle\rangle,u) + \mathcal{C}\mathbf{A}(\langle a\rangle,u) \times F(u,a) \times F(u,a) \end{aligned}$$

  we obtain the $\mathcal{C}$-species $B_{\mathbf{A}} : \mathcal{C}\mathbf{A} \nrightarrow \mathbf{A}$ that maps (u,a) to $\sum_{n\in\mathbb{N}} C_n \times \mathcal{C}\mathbf{A}(\langle a\rangle,u)^n$, where C_n is the nth Catalan number.

Conclusion and Perspectives

We have seen that the bicategory of profunctors with the free finite coproduct pseudo-comonad $\mathcal{C}$ provides a different perspective on how to categorify Scott continuity. This construction enables us to work in the unified framework of enriched profunctors where the change of base allows us to go from the categorified model to the preorder model while preserving the linear logic structure. An important construction in domain theory is the ideal completion which associates an algebraic domain to a preorder by completing with all directed joins. In the preorder model, the morphisms in the Eilenberg-Moore category can characterized as Scott-continuous functions between ideal completions of preorders. We aim to obtain in future work a 2-categorical analogue of this result with strongly finitary functors between sifted colimit completions of small categories. Another future direction is to connect the differential model of $\mathcal{S}$-species with the Scott model of $\mathcal{C}$-species by using a categorified version of the extensional collapse established by Ehrhard. The relationship between profunctors and intersection types has also recently been explored by Olimpieri where the non-idempotent intersection type system corresponds to the free symmetric monoidal pseudo-monad and the idempotent case corresponds to the cartesian pseudo-monad [19]. Our future goal is to connect the two type systems with the categorified extensional collapse construction.

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