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ON THE RELATIVE PROJECTIVE SPACE

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ABSTRACT. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be an abelian symmetric monoidal category satisfying certain exactness conditions. In this paper we define a presheaf $\mathbb{P}^n_{\mathcal{C}}$ on the category of commutative algebras in $\mathcal{C}$ and we prove that this functor is a $\mathcal{C}$-scheme in the sense of B. Toen and M. Vaquié. We give another definition and prove that they give isomorphic $\mathcal{C}$-schemes. This construction gives us a context of non-associative relative algebraic geometry. The most important example of the construction is the octonionic projective space.

1. Introduction

The study of the octonionic projective plane was initiated by R. Moufang in 1933 [Moufang, 1933]. She constructed it by coordinatizing using the octonion algebra, also known as the Cayley-Dickson algebra. Her point was to show an example of a non-Desarguessian plane.

Another way to define the octonionic plane is via Jordan algebras. The idea is to consider the exceptional simple Jordan algebra $H(\mathbb{O}_3)$ of 3×3 matrices with entries in the octonions, which are symmetric with respect to the involution. This attempt was first made by P. Jordan in 1949 [Jordan, 1949]. He considered the real octonion algebra and used the idempotents of $H(\mathbb{O}_3)$ to represent the points and lines in the octonionic projective plane. Later in 1953, H. Freudenthal [Freudenthal, 1953] rediscovered the same construction and used it to study the exceptional Lie groups F_4 and E_6. In the 1960 T. A. Springer generalized the Jordan-Freudenthal definition to the octonion algebra over fields of characteristic not 2 or 3. In this setting, the elements of rank one were used to represent points and lines.

In [Albuquerque-Majid, 1999] the authors showed that the Cayley algebra of the octonions $\mathbb{O}$ can be seen as a commutative algebra object (unital and associative up to the unity and associative constrains) in the monoidal category of G-graded k- vector spaces with G the additive group $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, the monoidal structure of this category is given by a two cocycle on the group G. Once we have a commutative algebra in a symmetric monoidal category, we can define the category of $\mathbb{O}$-modules, with the purpose to imitate

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the algebraic geometry over commutative rings.

Following the results given by Albuquerque and Majid, we give a new approach for the construction of the octonionic projective space, which we shall denote $\mathbb{P}^n_{\mathbb{O}}$. This attempt is via relative algebraic geometry in monoidal categories. Relative algebraic geometry has been widely studied in the literature, see for instance [Deligne, 1989], [Saavedra Rivano, 1972], [Hakim, 1972] and [Toen-Vaquie, 2009].

In this work we use the definition of relative scheme give by B. Toen and M. Vaquie [Toen-Vaquie, 2009]. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a closed monoidal category with limits and colimits and $\mathrm{Comm}(\mathcal{C})$ the category of commutative algebras in $\mathcal{C}$, then the category of affine $\mathcal{C}$-schemes $\mathrm{Aff}_{\mathcal{C}}$ is defined as $\mathrm{Comm}(\mathcal{C})^{op}$. Next, a $\mathcal{C}$-scheme will be a sheaf in $\mathrm{Aff}_{\mathcal{C}}$ which is *covered*, in a precise sense, by finitely many affine schemes. When the monoidal category is the category of abelian groups $\mathbb{Z}$-Mod, the relative algebraic geometry reduces to the usual algebraic geometry over the scheme $\mathrm{Spec}\,\mathbb{Z}$, see [Toen-Vaquie, 2009] and [Demazure-Gabriel, 1970, Ch. 1].

The key ingredient for our definition of the functor of points of the relative scheme $\mathbb{P}^n_{\mathcal{C}}$ is the notion of *line objects* in symmetric monoidal categories. As an example of this construction we define the functor $\mathbb{P}^n_{\mathbb{O}}$ relative the monoidal category $\mathrm{Mod}(\mathbb{O})$ and we prove that $\mathbb{P}^n_{\mathbb{O}}$ is in fact a relative scheme, see Corollary 4.3.

In the noncommutative world there are some approaches for a noncommutative projective scheme. Inspired by a Grothendieck's idea that it is not necessary to have a "noncommutative space": a good category of sheaves on the space should do as well, in [Artin-Zhang, 1994], [Verevkin, 1992] the authors define, independently, a noncommutative projective scheme via a module category, which will play the role of the category of quasi-coherent sheaves on the space. In the classical setting, a commutative graded algebra is associated to a projective scheme and the geometry of the scheme can be described in terms of a quotient category of a category of graded modules. Since this module category is available when the ring is not commutative, this observation provides a way to make the definition in a more general setting. We do not intent this approach here, although there is no apparent reason for not doing it.

An outline of this work is the following: Section two contains a short review of the ideas of relative algebraic geometry developed in [Toen-Vaquie, 2009]. It also deals with Zariski covers of an affine scheme $\mathrm{Spec}\,A$ in terms of a generating family of elements in A. We show that associated to an ideal of a commutative algebra A in $\mathcal{C}$, there is an open subscheme of the affine scheme $\mathrm{Spec}\,A$, see Lemma 2.14. We also give a sufficient condition for a family of Zariski open immersions to be a Zariski cover, Lemma 2.17. This last result will allow us to show that the functor $\mathbb{P}^n_{\mathcal{C}}$ has a finite Zariski cover by affine schemes. Finally we show that for a faithfully flat morphism $A \longrightarrow B$ in $\mathrm{Comm}(\mathcal{C})$ and $M \in \mathrm{Mod}_{\mathcal{C}}(A)$, $L \rightarrowtail M$ is a direct summand whenever $B \otimes L$ is a direct summand of $B \otimes M$. In section 3 we define the functor $\mathbb{P}^n_{\mathcal{C}}$ and we prove that if $\mathcal{C}$ is an *abelian strong relative context* (see Definition 2.6) then $\mathbb{P}^n_{\mathcal{C}}$ is a $\mathcal{C}$-scheme.We also give a definition of this functor of points in terms of quotients Definition 3.9 and we show that these two definitions are equivalent in the sense that they give isomorphic $\mathcal{C}$-schemes. In section 4 we define the category of $\mathbb{O}$-modules denoted by $\mathrm{Mod}(\mathbb{O})$ and we prove that this category is an abelian strong relative context, as a consequence we have defined the relative scheme $\mathbb{P}^n_{\mathbb{O}}$, see Corollary 4.3.

2. Relative algebraic geometry

Let $(\mathcal{C}, \otimes, 1)$ be a symmetric closed monoidal category, with all limits and colimits. Let $\mathrm{Comm}(\mathcal{C})$ be the category of commutative algebras in $\mathcal{C}$. For each object A in $\mathrm{Comm}(\mathcal{C})$, we denote $\mathrm{Mod}_{\mathcal{C}}(A)$ the category of A-modules in $\mathcal{C}$. Then $\mathrm{Mod}_{\mathcal{C}}(A)$ is equally a symmetric closed monoidal category [Osorio, 2017].
The category of affine schemes over $\mathcal{C}$ is defined as $\mathrm{Aff}_{\mathcal{C}} := \mathrm{Comm}(\mathcal{C})^{op}$. For an object A in $\mathrm{Comm}(\mathcal{C})$, we denote $\mathrm{Spec}\, A$ the corresponding object in $\mathrm{Aff}_{\mathcal{C}}$. the pseudo functor M assigns to each affine scheme $X = \mathrm{Spec}\, A$, the category of A-modules $\mathrm{Mod}_{\mathcal{C}}(A)$ and for any morphism $f : \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$, the base change morphism $f^* : \mathrm{Mod}_{\mathcal{C}}(A) \longrightarrow \mathrm{Mod}_{\mathcal{C}}(B)$ is given by $- \otimes_A B$. A morphism $f : \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$ is flat if f^* is an exact functor. A family $(f_i : \mathrm{Spec}\, A_i \longrightarrow \mathrm{Spec}\, A)_{i \in I}$ is M-faithfully flat if there exists a finite set $J \subset I$ such that the family of functors $(f_j^* : \mathrm{Mod}_{\mathcal{C}}(A) \longrightarrow \mathrm{Mod}_{\mathcal{C}}(A_j))_{j \in J}$ is jointly conservative. This is a topology induced by the pseudo functor M and it is called the faithfully flat quasi-compact (fpqc) topology [Toen-Vaquie, 2009]. The Zariski topology in $\mathrm{Aff}_{\mathcal{C}}$ is defined as follows:

2.1. Definition. *$f : \mathrm{Spec}\, B \longrightarrow \mathrm{Spec}\, A$ is said to be a Zariski open if the morphism $f : A \longrightarrow B$ in $\mathrm{Comm}(\mathcal{C})$ is a flat epimorphism of finite presentation.*
The family $(f_i : \mathrm{Spec}\, A_i \longrightarrow \mathrm{Spec}\, A)_{i \in I}$ in $\mathrm{Aff}_{\mathcal{C}}$ is a Zariski cover if it is an M-faithfully flat family such that each morphism $f_i : \mathrm{Spec}\, A_i \longrightarrow \mathrm{Spec}\, A$ is a Zariski open.

We are mostly interested in the Zariski topology, however, the importance of the fpqc-topology lies in the following observation:

2.2. Remark. The fpqc topology in $\mathrm{Aff}_{\mathcal{C}}$ turns out to be subcanonical, that is, for every $X \in \mathrm{Aff}_{\mathcal{C}}$ the presheaf h_X is a sheaf with respect to the fpqc topology [Toen-Vaquie, 2009, Corollary 2.11]. Since every Zariski cover is a fpqc cover, one has the following inclusion

$$\mathrm{Sh}^{fpqc}(\mathrm{Aff}_{\mathcal{C}}) \subset \mathrm{Sh}^{Zar}(\mathrm{Aff}_{\mathcal{C}}) \subset \mathrm{Psh}(\mathrm{Aff}_{\mathcal{C}}).$$

It follows that the Zariski topology is also subcanonical. For each $X \in \mathrm{Aff}_{\mathcal{C}}$, the sheaf h_X will be simply denoted by $X \in \mathrm{Sh}^{Zar}(\mathrm{Aff}_{\mathcal{C}})$ and the category of Zariski sheaves will be denoted just by $\mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}})$.

As in the classical setting in algebraic geometry, a relative scheme is that of a sheaf which has a Zariski open cover by affine schemes. In order to define $\mathcal{C}$-schemes the Zariski topology has to be extended to $\mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}})$.

2.3. Definition. *[Toen-Vaquie, 2009, Definition 2.12]*

1. *Let $X \in \mathrm{Aff}_{\mathcal{C}}$ and $F \subset X$ a sub sheaf. F is said to be a Zariski open of X if there exists a family of Zariski opens $\{X_i \longrightarrow X\}_{i \in I}$ in $\mathrm{Aff}_{\mathcal{C}}$ such that F is the image of the morphism of sheaves $\coprod_{i \in I} X_i \longrightarrow X$.*

2. *$f : F \longrightarrow G$ in $\mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}})$ is a Zariski open (Zariski open immersion, open sub functor) if for every affine scheme X and every morphism $X \longrightarrow G$ the induced morphism $F \times_G X \longrightarrow X$ is a monomorphism with image a Zariski open of X, i.e., $F \times_G X$ is a Zariski open of X.*

2.4. Remark. Recall that given sheaves F, G, the image of a morphism of sheaves $\eta : F \longrightarrow G$ is the following subsheaf $\mathrm{Im}(\eta) \subset G$, $s \in \mathrm{Im}(\eta)(U) \subset G(U)$ if and only if there is a covering $(f_i : U_i \longrightarrow U)_{i \in I}$ and $t_i \in F(U_i)$ such that $\eta(t_i) = s \cdot f_i = s|_{U_i}$.
Concretely in our case 2.3.1, if $X = \mathrm{Spec}\, A$, and $X_i = \mathrm{Spec}\, A_i$, and $f_i : \mathrm{Spec}\, A_i \longrightarrow \mathrm{Spec}\, A$ is the family of Zariski opens, then $F \subset \mathrm{Spec}\, A$ is the image of the morphism of sheaves $\coprod_{i \in I} X_i \longrightarrow X$ if given $B \in \mathrm{Aff}_{\mathcal{C}}$, then $s \in F(B)$ if and only if there is a Zariski cover $(g_j : \mathrm{Spec}\, B_j \longrightarrow \mathrm{Spec}\, B)_{j \in J}$, an index $i \in I$, and $s_j \in \mathrm{Spec}\, A_i(B_j)$ such that:

$$\begin{tikzcd} A \arrow[r, "f_i"] \arrow[d, "s"'] & A_i \arrow[d, "s_j"] \\ B \arrow[r, "g_j"'] & B_j \end{tikzcd}$$

2.5. Definition. *[Toen-Vaquie, 2009, Definition 2.15] $F \in \mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}})$ is a scheme relative to $\mathcal{C}$ or a $\mathcal{C}$-scheme if there exists a family $\{X_i\}_{i \in I} \in \mathrm{Aff}_{\mathcal{C}}$ such that for all i there exists $X_i \longrightarrow F$ satisfying*

1. *The morphism $X_i \longrightarrow F$ is a Zariski open of F for all i.*

2. *The induced morphism $p : \coprod_{i \in I} X_i \longrightarrow F$ is an epimorphism of sheaves.*

A morphism of $\mathcal{C}$-schemes $f : F \longrightarrow G$ is a morphism of sheaves, i.e. the category of $\mathcal{C}$-schemes $\mathrm{Sch}(\mathcal{C})$ is a full subcategory of $\mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}})$.

2.6. Definition. *$(\mathcal{C}, \otimes, \mathbb{1})$ is called an abelian strong relative context if it is abelian, bicomplete, symmetric closed monoidal category such that $\mathbb{1}$ is a projective finitely presentable generator. This condition on $\mathbb{1}$ means that the forgetful functor $V_0 = \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, -)$ to Set is conservative, preserves and reflects epimorphisms and filtered colimits.*

Because of the adjunction $\mathcal{C} \rightleftarrows \mathrm{Mod}_{\mathcal{C}}(A)$, if $\mathcal{C}$ is an abelian strong relative context then $\mathrm{Mod}_{\mathcal{C}}(A)$ is also an abelian strong relative context.
From now on $(\mathcal{C}, \otimes, \mathbb{1})$ is an abelian strong relative context. In the rest of this section we prove several lemmas needed in order to show that what we define as the functor of points of the projective space is in fact a $\mathcal{C}$-scheme. These lemmas are the relative counterpart

of results in commutative algebra.

For A a commutative algebra in $\mathcal{C}$, we define the operation

$$*: \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) \times \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$$

as follows. Given $f, g \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$, $f * g$ is given by

$$\mathbb{1} \cong \mathbb{1} \otimes \mathbb{1} \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A.$$

2.7. Lemma. *$(\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A), *, \eta)$ is a commutative monoid (in* **Set***). The operation $*$ is the adjoint of the composition operation in the monoid $\mathrm{Hom}_A(A, A)$. Even more, the adjunction $\varphi : \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A) \overset{\cong}{\to} \mathrm{Hom}_A(A, A)$ is an isomorphism of commutative monoids.*

Proof. For details, see [Marty, 2009] ■

From now on we abuse notation and identify $f : \mathbb{1} \longrightarrow A$ with its adjoint $\varphi_f : A \longrightarrow A$, i.e. $\varphi_f = f$.

2.8. Definition. *For $A \in \mathrm{Comm}(\mathcal{C})$, we say that $(f_i : \mathbb{1} \longrightarrow A)_{i \in I}$ is a generating family of A if $\coprod_i f_i : \coprod_i A \longrightarrow A$ is an epimorphism in $\mathrm{Mod}_{\mathcal{C}}(A)$. The family $(f_i)_{i \in I}$ is a partition of unity if there exists $J \subset I$ a finite set and a family $(s_j : A \longrightarrow A)_{j \in J}$ such that $\sum_{j \in J} s_j f_j = 1$.*

2.9. Definition. *Let $A \in \mathrm{Comm}(\mathcal{C})$, an ideal of A is a subobject of A in $\mathrm{Mod}_{\mathcal{C}}(A)$. Given a family of morphisms $(f_i : A \longrightarrow A)_{i \in I}$ in $\mathrm{Mod}_{\mathcal{C}}(A)$, the ideal generated by the family, denoted by $< f_i : i \in I >$ is defined as the image submodule of the morphism $\coprod_i f_i : \coprod_i A \longrightarrow A$.*

2.10. Remark. There is a definition for the sum and product of ideals, prime and maximal ideals, and for A finitely presentable, and $I \subset A$ proper, there is a maximal ideal such that $I \subset \mathfrak{m} \subset A$. For details, see [Brandenburg, 2014, Ch. 4]

2.11. Lemma. *Let $(f_i : A \longrightarrow A)_{i \in I}$ be a generating family, then $(f_i)_{i \in I}$ is a partition of unity on A.*

Proof. Let us see that $(f_i)_{i \in I}$ can be reduced to a finite family. In fact, for each finite subset $J = \{i_1, \cdots, i_k\} \subset I$ consider the generated ideal $< f_{i_1}, \cdots, f_{i_k} >$. Then, these ideals determine a filtered diagram. Since the family $(f_i)_{i \in I}$ is epimorphic in $\mathrm{Mod}_{\mathcal{C}}(A)$ then A is the filtered colimit of these ideals, i.e.,

$$A \cong \operatorname*{colim}_{J \subset I} < f_{i_1}, \cdots, f_{i_k} > .$$

Given that A is finitely presented in $\mathrm{Mod}_{\mathcal{C}}(A)$, we have the isomorphism

$$\mathrm{Hom}_A(A, A) \cong \operatorname*{colim}_{J \subset I} \mathrm{Hom}_A(A, < f_{i_1}, \cdots, f_{i_k} >).$$

Then there exists a subset $J = \{i_1, \cdots, i_k\}$ such that the identity arrow $1 : A \longrightarrow A$ factorizes through $< f_{i_1}, \ldots, f_{i_k} >$, that is to say $A \cong< f_{i_1}, \cdots, f_{i_k} >$.
Now, let us see that the finite family indexed by J is a partition of unity. As we have an epimorphism $\bigoplus_{j\in J} A \xrightarrow{\oplus f_j} A$ and A is projective, there exists a family $(s_j)_{j\in J}$ such that the diagram commutes

$$\begin{tikzcd} & \bigoplus_{j\in J} A \arrow[d, "\sum f_j"] \\ A \arrow[ur, dashed, "(s_j)_j"] \arrow[r, "1"'] & A \end{tikzcd}$$

i.e. such that $\sum_{j\in J} s_j f_j = 1$. ∎

2.12. Lemma. *Let* $(A \xrightarrow{f_i} A)_{i\in I}$ *be a generating family. Then* $(\operatorname{Spec} A_{f_i} \longrightarrow \operatorname{Spec} A)_i$ *is a Zariski cover.*

Proof. Each $A \longrightarrow A_{f_i}$ is a flat epimorphism of finite presentation. By Lemma 2.11 the family $(f_i)_{i\in I}$ is a partition of unity, then there exists a finite subset $J \subset I$ such that the family of functors $\operatorname{Mod}(A) \longrightarrow \prod_{j\in J} \operatorname{Mod}(A_{f_j})$ is jointly conservative (see [Banerjee, 2015, Proposition 2.7]). ∎

Next we define the open subscheme of an affine scheme $\operatorname{Spec} A$ associated to an ideal of A.

2.13. Definition. *Let* $X = \operatorname{Spec} A$ *in* $\operatorname{Aff}_{\mathcal{C}}$, $I \overset{j_I}{\rightarrowtail} A$ *an ideal. There is a subfunctor of* X *associated to the ideal* I *defined by:* $U_I(B) = \{u : A \longrightarrow B,\ BI \cong B\}$ *where* $BI \subset B$ *is the* B*-submodule defined as the image of the morphism* $B \otimes I \xrightarrow{u\circ j_I \otimes B} B \otimes B \xrightarrow{m_B} B$.

2.14. Lemma. U_I *is a* $\mathcal{C}$*-scheme, it is called the complementary open subscheme.*

Proof. First we prove that U_I is a sub sheaf. Let $(B \longrightarrow B_i)_{i\in J}$ be a Zariski cover and let $(f_i)_i$ be a compatible family in $\prod_i U(B_i) \rightarrowtail \prod_i \mathrm{h}_A(B_i)$. Since h_A is a sheaf, there exists a unique $f \in \mathrm{h}_A(B)$ whose restrictions to every open $\operatorname{Spec} B_i$ is f_i. Let us check that this f is in fact a section in $U(B)$, i.e., $f : A \longrightarrow B$ induces an isomorphism $BI \cong B$. Since the B_i form an open cover for B we have that family of functors

$$- \otimes_B B_i : \operatorname{Mod}(B) \longrightarrow \operatorname{Mod}(B_i)$$

is jointly conservative, so if we consider the inclusion $BI \rightarrowtail B$, we know that for every $i \in J$, $BI \otimes_B B_i \cong B_i I \xrightarrow{\sim} B_i$, therefore $BI \cong B$. Now we show that if $(f_i)_i \subset \operatorname{Hom}_A(A, A)$ is a generating family of the ideal I then $U_i = \operatorname{Spec} A_{f_i} \longrightarrow U$ is a Zariski open immersion and $\{U_i \longrightarrow U\}_{i\in J}$ is a Zariski cover. First, note that by the universal property of localizations

$$U_i(B) = \operatorname{Hom}_{\operatorname{Comm}(\mathcal{C})}(A_{f_i}, B) \cong \{f : A \longrightarrow B \ : \ B < f_i > \cong B\}. \tag{1}$$

Moreover, the inclusion $U_i \longrightarrow \operatorname{Spec} A$ induces a morphism $U_i \longrightarrow U$, by (1) this morphism is a monomorphism. We will check that this morphism is in fact a Zariski open immersion. Let $\operatorname{Spec} B$ in $\mathrm{Aff}_{\mathcal{C}}$ and $u : \operatorname{Spec} B \longrightarrow U$ and consider the pullback diagram

$$\begin{tikzcd}
U_i \times_A \operatorname{Spec} B \arrow[r] \arrow[d, tail] & U_i \arrow[d, tail] \arrow[ddr, bend left] & \\
\operatorname{Spec} B \arrow[r, "u"'] \arrow[drr, bend right] & U \arrow[dr] & \\
 & & \operatorname{Spec} A
\end{tikzcd}$$

we have to prove that $U_i \times_A \operatorname{Spec} B \longrightarrow \operatorname{Spec} B$ is a Zariski open immersion. To give the morphism $u : \operatorname{Spec} B \longrightarrow U$ is the same as giving an element in $U(B)$, that is to say, a morphism $u : A \longrightarrow B$ such that $IB \cong B$, then the result follows by the isomorphism $U_i \times_A \operatorname{Spec} B \cong \operatorname{Spec} B_{u(f_i)}$, where $u(f_i) : A \xrightarrow{f_i} A \xrightarrow{u} B$ and $\operatorname{Spec} B_{u(f_i)} \longrightarrow \operatorname{Spec} B$ is a Zariski open, therefore $U_i \longrightarrow U$ is a Zariski open.
On the other hand, in view of $BI \cong B$, $(u(f_i))_i$ is a generating family of B as an A-algebra. This family can be reduced to a finite family $(u(f_j))_{j\in J}$, thus by Lemma 2.11, $\coprod_{j\in J} \operatorname{Spec} B_{uf_j} \longrightarrow \operatorname{Spec} B$ is an epimorphism of sheaves so is $\coprod_{j\in J} U_j \longrightarrow U$ ■

We will give a sufficient condition for a morphism of sheaves to be an epimorphism, for this we need the definition of field objects in a symmetric monoidal category. This condition is very useful in order to prove that the functor of the projective space is covered by affine Zariski open immersions.

2.15. Definition. [Field objects in $\mathcal{C}$] $K \in \mathrm{Comm}(\mathcal{C})$, *is called a field object, or just a field, if* $K \neq 0$ *and it has no proper non trivial ideals.*

2.16. Proposition. $K \in \mathrm{Comm}(\mathcal{C})$ *is a field if and only if for all* $0 \neq f \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, K)$ *there exists* $g \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, K)$ *such that* $f * g = \eta$ *with* η *the unity in* K.

Proof. $\Leftarrow$): Let I be a proper ideal of K, then since V_0 is conservative, there exists $0 \neq f \in I$, this means that $f : \mathbb{1} \longrightarrow K$ factorizes through I. On the other hand, the ideal generated by f is K, for f is invertible, therefore $< f >= K \subset I$, a contradiction.
$\Rightarrow$): Let $f : \mathbb{1} \longrightarrow K$ a non zero arrow and consider its adjoint morphism $\varphi_f : K \longrightarrow K$, then the image $\mathrm{Im}(\varphi_f)$, being a subobject of K is in fact K itself. On the other hand, $\ker\varphi_f = 0$, since the kernel is a submodule, therefore, $\varphi_f : K \longrightarrow K$ being an epimorphism and a monomorphism is an isomorphism with inverse $(\varphi_f)^{-1}$. Now, via the adjunction there exists the arrow $f^{-1} : \mathbb{1} \longrightarrow K$ such that $(\varphi_f)^{-1} = \varphi_{f^{-1}}$, then $f * f^{-1} = \eta$. ■

2.17. Lemma. *Let* $\{U_i \longrightarrow F\}$ *be a finite family of affine Zariski open immersions in* $\mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}})$. *If for every field object* $K \in \mathrm{Comm}(\mathcal{C})$, $\coprod_i U_i(K) \longrightarrow F(K)$ *is surjective then* $\coprod_i U_i \longrightarrow F$ *is an epimorphism of sheaves.*

Proof. It is enough to prove the lemma for $F = \operatorname{Spec} A$ since a necessary and sufficient condition for $G \longrightarrow F$ to be a sheaf epimorphism is that for every affine scheme $\operatorname{Spec} A$, $\operatorname{Spec} A \times_F G \longrightarrow \operatorname{Spec} A$ is an epimorphism. In this case, we have to check that for each $\operatorname{Spec} A_j \xrightarrow{u_j} \operatorname{Spec} A$, the family of functors $\operatorname{Mod}(A) \longrightarrow \operatorname{Mod}(A_j)$ is jointly conservative. Let $0 \neq M \in \operatorname{Mod}(A)$, let us prove that $M_j := A_j \otimes_A M \neq 0$ for all j. As $M \neq 0$, then M contains a submodule of the form A/I. In fact, there is a non zero $f : A \longrightarrow M$, so we take $I = ker f$, then we have the factorization

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr] & & M \\ & A/I \arrow[ur, tail] & \end{tikzcd}$$

Let $\mathfrak{m}$ be a maximal ideal containing I, then the morphism φ from A to the field object $K = A/\mathfrak{m}$ represents an element in $F(K)$. Denote $U_i = \operatorname{Spec} A_i$, then we have a surjective function $\coprod_i U_i(K) \longrightarrow F(K)$, the element φ seen as an arrow factorizes through some $u_j : A \longrightarrow A_j$, this means that there exists φ_j such that the diagram commutes

$$\begin{tikzcd} A \arrow[rr, "\varphi"] \arrow[dr, "u_j"'] & & K \\ & A_j \arrow[ur, "\varphi_j"'] & \end{tikzcd}$$

Now, by the universal property of $\ker\varphi_j$, there exist a unique morphism $\mathfrak{m} \longrightarrow \ker\varphi_j$, then we have the pullback diagram

$$\begin{tikzcd} \mathfrak{m} \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & u_j^{-1}(\ker\varphi_j) \arrow[r] \arrow[d, tail] & \ker\varphi_j \arrow[d, tail] \\ & A \arrow[r, "u"'] \arrow[d, "\varphi"'] & A_j \arrow[d, "\varphi_j"] \\ & K \arrow[r, equal] & K \end{tikzcd}$$

with the morphism $\mathfrak{m} \longrightarrow u_j^{-1}(\ker\varphi_j)$ being a monomorphism.

Let $\mathfrak{m}_j$ be a proper maximal ideal containing $\ker(\varphi_j)$, since u_j is flat we have that $u_j^{-1}(\ker\varphi_j) \rightarrowtail u_j^{-1}(\mathfrak{m}_j)$, then $\mathfrak{m} \rightarrowtail u_j^{-1}(\mathfrak{m}_j)$. We claim that $u^{-1}(\mathfrak{m}_j)$ is a proper ideal of A. In fact, if $u^{-1}(\mathfrak{m}_j) = A$, then the morphism $u_j : A \longrightarrow A_j$ factorizes through $A \twoheadrightarrow \mathfrak{m}_j$, we get a contradiction. By maximality $\mathfrak{m} = u_j^{-1}(\mathfrak{m}_j)$. Then we have the commutative diagram

$$\begin{tikzcd} \mathfrak{m} \arrow[r] \arrow[d, tail] & \mathfrak{m}_j \arrow[d, tail] \\ A \arrow[r, "u_j"'] & A_j \end{tikzcd}$$

tensoring with the A-algebra A_j we have a morphism $A_j \otimes_A \mathfrak{m}_j \cong A_j\mathfrak{m} \longrightarrow \mathfrak{m}_j$ commuting with the inclusion to A_j. Then this morphism must be a monomorphism. On the other hand, we have a monomorphism $A_jI \rightarrowtail A_j\mathfrak{m}$, it follows that $\mathfrak{m}_j$ contains the ideal A_jI then $A_j/A_jI \neq 0$ and we have a monomorphism

$$A/I \otimes_A A_j \cong A_j/A_jI \rightarrowtail A_j \otimes_A M = M_j\ ,$$

this means that $M_j \neq 0$ for all j, therefore $\{U_i \longrightarrow \operatorname{Spec} A\}_i$ is a Zariski cover. ∎

Let $A \longrightarrow B$ be a morphism in $\mathrm{Comm}(\mathcal{C})$ and let M, N objects in $\mathrm{Mod}_{\mathcal{C}}(A)$, we would like to define a morphism

$$B \otimes_A \underline{\mathrm{hom}}_A(M,N) \xrightarrow{\ \zeta\ } \underline{\mathrm{hom}}_B(B \otimes_A M, B \otimes_A N)\ \cdot$$

Where $\underline{\mathrm{hom}}_A(,)$ denotes the internal hom of the closed category $\mathrm{Mod}_{\mathcal{C}}(A)$. We have the morphism $1 \otimes \varepsilon : B \otimes_A M \otimes_A \underline{\mathrm{hom}}_A(M,N) \longrightarrow B \otimes_A N$ which by adjunction corresponds to a morphism

$$\underline{\mathrm{hom}}_A(M,N) \xrightarrow{\ \chi\ } \underline{\mathrm{hom}}_A(B \otimes_A M, B \otimes_A N)\ \cdot$$

On the other hand, as $B \otimes_A M$ and $B \otimes_A N$ are B-modules, the object $\underline{\mathrm{hom}}_A(B \otimes_A M, B \otimes_A N)$ is also a B-module, with action

$$B \otimes_A \underline{\mathrm{hom}}_A(B \otimes_A M, B \otimes_A N) \xrightarrow{\ \mu\ } \underline{\mathrm{hom}}_A(B \otimes_A M, B \otimes_A N)$$

by composing these two morphism, we have the morphism

$$B \otimes_A \underline{\mathrm{hom}}_A(M,N) \xrightarrow{\mu(1\otimes\chi)} \underline{\mathrm{hom}}_A(B \otimes_A M, B \otimes_A N)\ \cdot$$

It's not hard to see that $\mu(1 \otimes \chi)$ equalizes the following two morphisms [Osorio, 2017, Ch. 4]

$$\underline{\mathrm{hom}}_A(B \otimes_A M, B \otimes_A N) \rightrightarrows \underline{\mathrm{hom}}_B(B, \underline{\mathrm{hom}}_A(B \otimes_A M, B \otimes_A N))\ .$$

Since $\underline{\mathrm{hom}}_B(B \otimes_A M, B \otimes_A N)$ is, by definition the equalizer of these two morphisms, there exists an arrow

$$B \otimes_A \underline{\mathrm{hom}}_A(M,N) \xrightarrow{\ \zeta\ } \underline{\mathrm{hom}}_B(B \otimes_A M, B \otimes_A N)\ \cdot$$

With notations as above we have the following results:

2.18. Lemma. *If $A \longrightarrow B \in \mathrm{Comm}(\mathcal{C})$ is faithfully flat and M is finitely presentable, then the induced morphism*

$$B \underset{A}{\otimes} \underline{\hom}_A(M, N) \xrightarrow{\zeta} \underline{\hom}_B(B \underset{A}{\otimes} M, B \underset{A}{\otimes} N)$$

is an isomorphism.

Proof. The proof of this result is verbatim of the classical result given in [Eisenbud, 1995, Proposition 2.10]. The key is that under our hypothesis on $\mathcal{C}$, an object $M \in \mathcal{C}$ is finitely presentable in the sense that the functor $\mathrm{Hom}_{\mathcal{C}}(M, -)$ preserves filtered colimits if and only if it has a finite presentation, that is to say, there exist integers n, m such that the following diagram is exact

$$\mathbb{1}^m \longrightarrow \mathbb{1}^n \longrightarrow M$$

■

2.19. Lemma. *Let $A \longrightarrow B \in \mathrm{Comm}(\mathcal{C})$ faithfully flat, $M \in \mathrm{Mod}(A)$ of finite presentation and $L \rightarrowtail M$ an A-submodule. If $B \otimes_A L$ is a direct summand of $B \otimes_A M$ then L is a direct summand of M.*

Proof. L is a direct summand of M if and only if there exists a morphism $r : M \longrightarrow L$ such that $r \circ i = 1_L$. This is equivalent to prove that the function between the homs is surjective, i.e., $\mathrm{Hom}_A(M, L) \twoheadrightarrow \mathrm{Hom}_A(L, L)$.
Since the forgetful functor $\mathrm{Hom}_A(A, -)$ preserves epimorphisms, it is enough to prove that $\varphi : \underline{\hom}_A(M, L) \longrightarrow \underline{\hom}_A(L, L)$ is an epimorphism. The result comes from the following commutative diagram

$$\begin{tikzcd}
B \underset{A}{\otimes} \underline{\hom}_A(M, L) \arrow[r, "B\otimes\varphi"] \arrow[d, "\zeta_1\cong"'] & B \underset{A}{\otimes} \underline{\hom}_A(L, L) \arrow[d, "\cong\zeta_2"] \\
\underline{\hom}_B(B \underset{A}{\otimes} M, B \underset{A}{\otimes} L) \arrow[r, two heads, "\psi"'] & \underline{\hom}_B(B \underset{A}{\otimes} L, B \underset{A}{\otimes} L)
\end{tikzcd}$$

ψ is an epimorphisms since $B \otimes_A I$ is a direct summand of $B \otimes_A M$ and the forgetful functor $\mathrm{Hom}_B(B, -)$ reflects epimorphisms. Lemma 2.18 shows that ζ_1 and ζ_2 are isomorphisms, therefore $B \otimes \psi$ is an epimorphism and we get the result. ■

Line Objects. Next, following [Saavedra Rivano, 1972, Brandenburg, 2014] we review the definition and properties of *line objects* in monoidal categories. These objects are the key ingredient in the definition of the projective space. Line objects are the categorification of rank one invertible sheaves over a scheme.

2.20. Definition. [Invertible object] *If $\mathcal{C}$ is a symmetric monoidal category, $L \in \mathcal{C}$ is called invertible if there exists an object $L^\vee$ and an isomorphism $\delta : \mathbb{1} \longrightarrow L \otimes L^\vee$.*

Note that if L is invertible then $L \otimes - : \mathcal{C} \longrightarrow \mathcal{C}$ is an equivalence with inverse $L^\vee \otimes -$.

2.21. Remark.

1. $\mathbb{1}$ is invertible and invertible objects are closed under tensor products. Isomorphisms classes of invertible objects form a group denoted $Pic(\mathcal{C})$. For more details on $Pic(\mathcal{C})$, see [May,2001].

2. If L is invertible, then for every isomorphism $\delta : \mathbb{1} \longrightarrow L \otimes L^\vee$ there exists an isomorphism $\epsilon : L^\vee \otimes L \longrightarrow \mathbb{1}$ satisfying the triangle axioms

$$\begin{tikzcd} \mathbb{1} \otimes L \arrow[r, "\delta\otimes L"] \arrow[d, "\cong"'] & L \otimes L^\vee \otimes L \arrow[dl, "L\otimes\epsilon"] \\ L & \end{tikzcd} \qquad \begin{tikzcd} L^\vee \otimes \mathbb{1} \arrow[r, "L^\vee\otimes\delta"] \arrow[d, "\cong"'] & L^\vee \otimes L \otimes L^\vee \arrow[dl, "\epsilon\otimes L^\vee"] \\ L^\vee & \end{tikzcd}$$

   therefore $(L, L^\vee, \epsilon, \delta)$ is a duality in $\mathcal{C}$.

3. If L is invertible and $\mathbb{1}$ is projective, then L is a projective object in $\mathcal{C}$. In fact, since L is invertible, $\underline{\hom}_\mathcal{C}(L, -)$ is left adjoint to $\underline{\hom}_\mathcal{C}(L^\vee, -)$, therefore it preserves colimits. As $\mathrm{Hom}_\mathcal{C}(L, -) \cong \mathrm{Hom}_\mathcal{C}(\mathbb{1}, \underline{\hom}_\mathcal{C}(L, -))$ and $\mathrm{Hom}_\mathcal{C}(\mathbb{1}, -)$ preserves epimorphisms we have that $\mathrm{Hom}_\mathcal{C}(L, -)$ preserves epimorphisms.

Now, for invertible objects there is a well defined signature

2.22. Definition. [Signature] *Since $L\otimes -$ is an equivalence we have bijections $End_\mathcal{C}(\mathbb{1}) \cong End_\mathcal{C}(L) \cong End_\mathcal{C}(L \otimes L)$, then the signature is the endomorphism of $\mathbb{1}$ corresponding to the symmetry $\sigma_{L,L} : L \otimes L \longrightarrow L \otimes L$ via that bijection.*

2.23. Definition. [Line object.] *$L \in \mathcal{C}$ is called a line object if it is invertible and its signature is the identity morphism.*

2.24. Remark. An object M in $\mathcal{C}$ is said to be *symtrivial* if $\sigma_{M,M} : M \otimes M \longrightarrow M \otimes M$ is the identity arrow, where σ denotes the symmetry in $\mathcal{C}$. Since the signature of an invertible object L in $\mathcal{C}$ is the endomorphism associated to the symmetry of $L \otimes L$, then a line object is simply an invertible symtrivial object.

2.25. Proposition.

1. *Symtrivial objects are preserved by strong monoidal functors.*

2. *$M \oplus N$ is symtrivial if and only if $M \otimes N = 0$ and M, N are symtrivial.*

3. *Let A be a faithfully flat commutative algebra in $\mathcal{C}$, $L \in \mathcal{C}$. If $A \otimes L$ is a line object in $\mathrm{Mod}_\mathcal{C}(A)$ then L is a line object in $\mathcal{C}$.*

4. *If L is a line object in $\mathcal{C}$, then every epimorphism $\mathbb{1} \longrightarrow L$ is an isomorphism.*

Proof. For details see [Brandenburg, 2014]. ∎

2.26. Remark. Given R a commutative ring, then an R-module M is invertible if and only if it is finitely generated projective of rank one. In this case every invertible module is automatically symtrivial (see [Brandenburg, 2014, Lemma 4.3.3.]), hence it follows that line objects are the invertible modules. The same holds for the category of quasicoherent sheaves over a scheme X. However, in general invertible objects are not the same as line objects, there are examples of invertible objects for which the signature is $-id$, called anti-line objects, see [Brandenburg, 2014, Prop. 5.4.8.]. The notion of line object turns out to be crucial for the proof of our main results.

2.27. Lemma. *Let* $(\operatorname{Spec} A_i \longrightarrow \operatorname{Spec} A)_{i\in I}$ *be a finite Zariski open cover. Let the A-algebra* $B = \prod_i A_i$*. If for every* $i \in I$*,* L_i *is a line object in* $\operatorname{Mod}_{\mathcal{C}}(A_i)$ *then* $J = \prod_i L_i$ *is a line object in* $\operatorname{Mod}_{\mathcal{C}}(B)$*.*

Proof. We claim that J has an inverse in $\operatorname{Mod}_{\mathcal{C}}(B)$ given by $J^\vee = \prod_i L_i^\vee$ with $L_i^\vee$ is the inverse of L_i in $\operatorname{Mod}_{\mathcal{C}}(A_i)$ for all $i \in I$. If $m_i, m_i^\vee$ denote the actions of A_i on L_i and $L_i^\vee$ respectively, we will prove the following two things:

i. For every $i \in I$, $L_i \otimes_{A_i} L_i^\vee \cong L_i \otimes_B L_i^\vee$.

ii. For every $i \neq j$, $L_i \otimes_B L_j^\vee = 0$.

For item i. let us consider the diagram with exact rows:

$$\begin{tikzcd}
L_i \otimes_A B \otimes_A L_i^\vee \arrow[r, "\bar{r}"] \arrow[d, "1\otimes p_i \otimes 1"'] & L_i \otimes_A L_i^\vee \arrow[r, "\pi"] \arrow[d, equal] & L_i \otimes_B L_i^\vee \arrow[d, dashed, "\varphi"] \\
L_i \otimes_A A_i \otimes_A L_i^\vee \arrow[r, "r"'] \arrow[u, bend right, "1\otimes \lambda_i \otimes 1"'] & L_i \otimes_A L_i^\vee \arrow[r, "\pi'"'] & L_i \otimes_{A_i} L_i^\vee \arrow[u, dashed, bend right, "\psi"']
\end{tikzcd} \tag{2}$$

where $\bar{r} = \bar{m_i} \otimes 1 - 1 \otimes \bar{m_i}^\vee$, $r = m_i \otimes 1 - 1 \otimes m_i^\vee$ and π, π' the cokernel maps. The existence of arrows φ and ψ are due to universal property of π' and π respectively. It is easy to check that they are inverse to each other.
For item ii. we will prove that for $i \neq j$

$$L_i \otimes_A B \otimes_A L_j^\vee \xrightarrow{\quad r_{ij} \quad} L_i \otimes_A L_j^\vee$$

wth $r_{ij} = m_i \otimes 1 - 1 \otimes m_j^\vee$ an epimorphism, thence its cokernel $L_i \otimes_B L_j^\vee$ would be the zero object. Consider for every $i \in I$, the morphism $\lambda^{(i)} : A \longrightarrow B$ given by $(0, \cdots, \eta_i, 0 \cdots,)$ with $\eta_i : A \longrightarrow A_i$ the unit of A_i as an A-algebra in the i-th position, then

$$L_i \otimes_A A \otimes_A L_j^\vee \xrightarrow{r_{ij}(1\otimes\lambda_i\otimes 1)} L_i \otimes_A B \otimes_A L_j^\vee$$

is the identity arrow for $i \neq j$, thus r_{ij} is an epimorphism. Combining i. and ii. we have that

$$\begin{aligned}
J \otimes_B J^{\vee} &\cong \prod_{i,j} coKer\left(L_i \otimes_A B \otimes_A L_j^{\vee} \longrightarrow L_i \otimes_A L_j^{\vee}\right)\\
&\cong \prod_{i} coKer\left(L_i \otimes_A B \otimes_A L_i^{\vee} \longrightarrow L_i \otimes_A L_i^{\vee}\right)\\
&\cong \prod_{i} L_i \otimes_B L_i^{\vee} \cong \prod_{i} L_i \otimes_{A_i} L_i^{\vee} \cong \prod_{i} A_i = B
\end{aligned}$$

Now we show that J is a symtrivial object in $\mathrm{Mod}_{\mathcal{C}}(B)$ provided that each L_i is symtrivial in $\mathrm{Mod}_{\mathcal{C}}(A_i)$ for all i. Consider the following diagram:

$$\begin{tikzcd}
L_i \otimes_A A_i \otimes_A L_i \arrow[r] \arrow[d, "1\otimes\lambda_i\otimes 1"'] & L_i \otimes_A L_i \arrow[r] \arrow[d, equal] & L_i \otimes_{A_i} L_i \arrow[d, "\psi"] \arrow[ddd, bend left=60, dashed, "\sigma^i_{L_i,L_i}"] \\
L_i \otimes_A B \otimes_A L_i \arrow[r] \arrow[d, "(\sigma_{B,L_i}\otimes 1)(1\otimes\sigma_{L_i,L_i})(\sigma_{L_i,B}\otimes 1)"'] & L_i \otimes_A L_i \arrow[r] \arrow[d, "\sigma_{L_i,L_i}"] & L_i \otimes_B L_i \arrow[d, "\sigma^B_{L_i,L_i}"] \\
L_i \otimes_A B \otimes_A L_i \arrow[r] \arrow[d, "1\otimes p_i\otimes 1"'] & L_i \otimes_A L_i \arrow[r] \arrow[d, equal] & L_i \otimes_B L_i \arrow[d, "\varphi"] \\
L_i \otimes_A A_i \otimes_A L_i \arrow[r] & L_i \otimes_A L_i \arrow[r] & L_i \otimes_{A_i} L_i
\end{tikzcd}$$

where $\sigma, \sigma^i, \sigma^B$ denote the symmetries in $\mathrm{Mod}_{\mathcal{C}}(A)$, $\mathrm{Mod}_{\mathcal{C}}(A_i)$, $\mathrm{Mod}_{\mathcal{C}}(B)$ respectively and φ, ψ defined as in (2) with $L_i^{\vee} = L_i$. It follows that $\sigma^B_{L_i,L_i} = 1$, that is, each L_i is a symtrivial object in $\mathrm{Mod}_{\mathcal{C}}(B)$. Finally by Proposition 2.25, $\prod_i L_i$ is symtrivial in $\mathrm{Mod}_{\mathcal{C}}(B)$ ∎

3. The scheme $\mathbb{P}^n_{\mathcal{C}}$

In this section we give two definitions of the Relative Projective Space, one in terms of subobjects and the other in terms of quotients and by the end of the section we prove these two definitions are equivalent.

3.1. Definition. *Let $n \geq 1$ a fixed integer. The projective space relative to a monoidal symmetric category $\mathcal{C}$ is a presheaf $\mathbb{P}^n_{\mathcal{C}}$ in $\mathrm{Aff}_{\mathcal{C}}$ such that for every affine scheme* $\mathrm{Spec}\, A$, *$\mathbb{P}^n_{\mathcal{C}}(A)$ is the set of A-submodules $L \rightarrowtail A^{n+1}$ satisfying*

- *L is a line object in* $\mathrm{Mod}_{\mathcal{C}}(A)$
- *For the monomorphism $\mathbf{x} : L \rightarrowtail A^{n+1}$, there exists a retraction $A^{n+1} \longrightarrow L$. In other words, L is a direct summand of A^{n+1}.*

For every morphism $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$ *in* $\mathrm{Aff}_{\mathcal{C}}$*, the function* $\mathbb{P}^n_{\mathcal{C}}(A) \longrightarrow \mathbb{P}^n_{\mathcal{C}}(B)$ *assigns to* $L \in \mathbb{P}^n_{\mathcal{C}}(A)$ *the corresponding direct summand* $B \otimes_A L \rightarrowtail B^{n+1}$.

Note that $B \otimes_A L$ is a line object in $\mathrm{Mod}_{\mathcal{C}}(B)$ since line objects are preserved by strong monoidal functors.

3.2. Remark. Note that for every $A \in \mathrm{Comm}(\mathcal{C})$ a pair $(L, \mathbf{x})$ in $\mathbb{P}^n_{\mathcal{C}}(A)$ is a subobject, that is, a class of monomorphisms of A^{n+1}, where $(L_1, \mathbf{x}_1)$, $(L_2, \mathbf{x}_2)$ represent the same element subobject, if there exists an isomorphism $\lambda : L_1 \longrightarrow L_2$ such that the diagram commutes

$$\begin{tikzcd} L_1 \arrow[rr, tail, "\mathbf{x}_1"] \arrow[dr, "\lambda"'] & & A^{n+1} \\ & L_2 \arrow[ur, tail, "\mathbf{x}_2"'] & \end{tikzcd}$$

Since L is an invertible object we have that $Aut(L) \cong Aut(A)$, therefore the equivalence relation is given by scalar multiplication by invertible elements in A. So if we think of the pair $(L, \mathbf{x})$ as a "vector" in A^{n+1}, its class in $\mathbb{P}^n_{\mathcal{C}}(A)$ represents the "line" in A^{n+1}. This is kind of the intuition one has of the classical projective space.

3.3. Theorem. *Let* $\mathcal{C}$ *be an abelian strong relative context. Then the presheaf* $\mathbb{P}^n_{\mathcal{C}}$ *is a* $\mathcal{C}$*-scheme.*

Proof. We have to show that $\mathbb{P}^n_{\mathcal{C}}$ is a sheaf with respect to the Zariski topology and that there exist a finite family of Zariski open immersions which covers $\mathbb{P}^n_{\mathcal{C}}$. This will be done in the following lemmas. ∎

3.4. Lemma. $\mathbb{P}^n_{\mathcal{C}}$ *is a sheaf with respect to the Zariski topology.*

Proof. Let $\{\operatorname{Spec} A_i \longrightarrow \operatorname{Spec} A\}_i$ be a Zariski cover, we have to prove the exactness of the sequence

$$\mathbb{P}^n_{\mathcal{C}}(A) \longrightarrow \prod_i \mathbb{P}^n_{\mathcal{C}}(A_i) \rightrightarrows \prod_{i,j} \mathbb{P}^n_{\mathcal{C}}(A_{ij}). \tag{3}$$

Let $L \in \mathbb{P}^n_{\mathcal{C}}(A)$, by the equivalence given in [Toen-Vaquie, 2009, Théorème 2.5] the following sequence is exact

$$L \longrightarrow \prod_i L_i \rightrightarrows \prod_{i,j} L_{ij}$$

then L is determined by $L_i \in \mathbb{P}^n_{\mathcal{C}}(A_i)$ therefore $\mathbb{P}^n_{\mathcal{C}}(A)$ is a cone of the diagram.
Now we have to check that $\mathbb{P}^n_{\mathcal{C}}(A)$ is universal. To see this, consider the compatible family $(L_i)_i \in \prod_i \mathbb{P}^n_{\mathcal{C}}(A_i)$. The compatibility says that we have a family of isomorphisms

$$\begin{tikzcd} L_i \otimes_A A_j \arrow[rr, "\theta_{ij}"] \arrow[dr, tail] & & L_j \otimes_A A_i \arrow[dl, tail] \\ & A_{ij}^{n+1} & \end{tikzcd}$$

$(L_i, \theta_{i,j})_{i,j}$ is a descent datum, that is, θ_{ij} satisfies the cocycle condition $\theta_{j,k} \circ \theta_{i,j} = \theta_{i,k}$ in $\mathrm{Mod}(A_{i,j,k})$ follows from the fact between two subobjects there is at most one arrow. Therefore, the descent datum $(L_i, \theta_{i,j})_{i,j}$ defines an A-module L as the limit of the diagram

$$\begin{tikzcd}
 & L_i \arrow[r] & L_i \otimes A_j \arrow[d, "\theta_{ij}"] \\
L \arrow[ur] \arrow[dr] & & \\
 & L_j \arrow[r] & L_j \otimes A_i.
\end{tikzcd}$$

To prove that $L \in \mathbb{P}^n_{\mathcal{C}}(A)$, consider the product algebra $B = \prod_i A_i$, note that B is a faithfully flat A-algebra as $\mathrm{Mod}_{\mathcal{C}}(B) \cong \prod_i \mathrm{Mod}_{\mathcal{C}}(A_i)$ and the functor $- \otimes_A B$ is naturally isomorphic to $\bigtimes_i (- \otimes_A A_i)$. Now take the B-module $L \otimes B$, then we have that

$$L \otimes B \cong \prod_i L \otimes A_i \cong \prod_i L_i.$$

By Lemma 2.27, $L \otimes B$ is a line object in $\mathrm{Mod}(B)$ therefore by proposition 2.25 we have that L is a line object in $\mathrm{Mod}(A)$. Finally by Lemma 2.19, L is a direct summand of A^{n+1}. ∎

3.5. Lemma. *Let*

$$U_i(A) = \{(L, \mathbf{x}) : L \overset{\mathbf{x}}{\rightarrowtail} A^{n+1} \xrightarrow{\pi_i} A,\ \pi_i \circ \mathbf{x} \textit{ is an isomorphism in } \mathrm{Mod}_{\mathcal{C}}(A)\}.$$

for $i = 1, \dots, n+1$. Then each U_i is representable by an affine scheme.

Proof. Let us fix the index i. Given any element $(L, \mathbf{x}) \in U_i(A)$, we identify L with A as submodules of A^{n+1} via the isomorphism $\pi_i \mathbf{x} : L \cong A$, then we obtain $\tilde{\mathbf{x}} = \mathbf{x}(\pi_i \mathbf{x})^{-1} : A \longrightarrow A^{n+1}$, this means that $(L, \mathbf{x}) = (A, \tilde{\mathbf{x}})$ as subobjects of A^{n+1}. Since $\pi_i \tilde{\mathbf{x}} = 1$, $\tilde{\mathbf{x}}$ is completely determined by specifying the morphisms $\pi_j \tilde{\mathbf{x}} : A \longrightarrow A$ for $j = 1, \cdots n+1$ and $j \neq i$, i.e., the functor U_i is isomorphic to the functor

$$A \longmapsto \prod_{\substack{j=1 \\ j \neq i}}^{n+1} \mathrm{Hom}_A(A, A)$$

As we have the following natural isomorphisms

$$\begin{aligned}
&\prod_{\substack{j=1 \\ j \neq i}}^{n+1} \mathrm{Hom}_A(A, A) \cong \prod_{j=1}^{n} \mathrm{Hom}_{\mathcal{C}}(1, A) \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}^n, A) \\
&\cong \mathrm{Hom}_{\mathrm{Comm}(\mathcal{C})}(\mathbb{1}[x_1, \cdots x_n], A) =: \mathbb{A}^n_{\mathcal{C}}(A),
\end{aligned}$$

it follows that the U_i is representable by an affine scheme. ∎

3.6. LEMMA. *U_i is a Zariski open immersion.*

PROOF. Let us see that for affine scheme h_A and any morphism $\mathrm{h}_A \longrightarrow \mathbb{P}^n_{\mathcal{C}}$, the pullback $\mathrm{h}_A \times_{\mathbb{P}^n_{\mathcal{C}}} U_i$ is a Zariski immersion of h_A.
By Yoneda's Lemma the morphism $\mathrm{h}_A \longrightarrow \mathbb{P}^n_{\mathcal{C}}$ corresponds to $(L, \mathbf{x})$ in $\mathbb{P}^n_{\mathcal{C}}(A)$. Consider the pullback diagram

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U_i \arrow[d] \\ \mathrm{h}_A \arrow[r] & \mathbb{P}^n_{\mathcal{C}} \end{tikzcd}$$

Now, an element in $V_i(B)$ is the same as a morphism $f : A \longrightarrow B \in \mathrm{Comm}(\mathcal{C})$ such that $B \otimes_A L \longrightarrow B^{n+1} \longrightarrow B$ is an isomorphism. If I_i denotes the ideal in A defined by the image of $\pi_i \circ \mathbf{x}$,

$$\begin{tikzcd} L \arrow[rr, "\pi_i \circ \mathbf{x}"] \arrow[dr, "p"'] & & A \\ & I_i \arrow[ur, tail, "j"'] & \end{tikzcd}$$

then tensoring the factorization of this arrow with B, we get the diagram

$$\begin{tikzcd} B \otimes_A L \arrow[r, "B\otimes(\pi_i\circ\mathbf{x})"] \arrow[dr, "B\otimes p"'] & B \otimes_A A \cong B \arrow[d, "m_B(B\otimes f\circ j)"] \\ & B \otimes_A I_i \end{tikzcd} \tag{4}$$

We have that all the arrows in diagram (4) are isomorphisms. On the other hand, consider the ideal BI_j, which by definition is the image of $m_B(B \otimes f \circ j)$ then we have $B \otimes_A I_i \cong BI_i$. This means that V_i is contained in the complementary open subscheme associated to the ideal I_i. Let us see that the complementary open U_{I_i} defined by the ideal I_i is contained in V_i . Let $f \in U_{I_i}(B)$, i.e., $f : A \longrightarrow B$ satisfies that the induced ideal BI_i is isomorphic with B. Then we have that $B \otimes_A L \longrightarrow B$ is an epimorphism and $B \otimes_A L$ is a line object in $\mathrm{Mod}(B)$, so tensoring this epimorphism with the inverse of $B \otimes_A L$, we have again an epi $B \twoheadrightarrow B \otimes_A L^\vee$ which by Proposition 2.25(4) is an isomorphism in $\mathrm{Mod}(B)$, then tensoring again with the inverse we get $B \otimes_A L \xrightarrow{\cong} B$. This means that $f \in V_i(B)$. Finally by Lemma 2.14, $V_i \subset \mathrm{Spec}\, A$ is a Zariski open, so is $U_i \subset \mathbb{P}^n_{\mathcal{C}}$. ∎

3.7. LEMMA. *The family $(U_i)_{i=1,\ldots,n+1}$ is an affine Zariski open cover.*

PROOF. We have to prove that there is an epimorphism of sheaves:

$$\coprod_i U_i \longrightarrow \mathbb{P}^n_{\mathcal{C}}.$$

By Lemma 2.17 it is enough to prove that $\coprod_i U_i(\mathbb{K}) \longrightarrow \mathbb{P}^n_{\mathcal{C}}(\mathbb{K})$ is surjective for every field $\mathbb{K}$ in $\mathrm{Comm}(\mathcal{C})$. Let $L \in \mathbb{P}^n_{\mathcal{C}}(\mathbb{K})$, i.e., $\mathbf{x} : L \rightarrowtail \mathbb{K}^{n+1}$, then there exists an index j such that the arrow $\pi_j \circ \mathbf{x} : L \longrightarrow \mathbb{K}$ is non zero but then the image ideal I_j in $\mathbb{K}$ must be exactly $\mathbb{K}$ thus we have an epi $L \twoheadrightarrow \mathbb{K}$. Since L is a line object we have $L \otimes_{\mathbb{K}} L^{\vee} \cong \mathbb{K}$ therefore $\mathbb{K} \twoheadrightarrow L^{\vee}$, thus by Proposition 2.25, $\mathbb{K} \cong L^{\vee}$ so $\mathbb{K} \cong L$. ∎

3.8. Proposition. *The fiber product $U_{ij} = U_i \times_{\mathbb{P}^n_{\mathcal{C}}} U_j$ is representable by an affine scheme.*

Proof. For any A in $Comm(\mathcal{C})$, an element in $U_{ij}(A)$ is an isomorphism class of pairs $(L, \mathbf{x})$ where $L \overset{\mathbf{x}}{\rightarrowtail} A^{n+1}$ satisfies that $\pi_i\mathbf{x}, \pi_j\mathbf{x} : L \longrightarrow A$ are isomorphisms. We denote these isomorphisms by $x_i,\ x_j$ respectively. Using these isomorphisms, we identify the pair $(L, \mathbf{x})$ with a family of arrows

$$\left(\tfrac{x_k}{x_i} : A \longrightarrow A\right) \quad \text{for} \quad k = 1, \dots \hat{i}, \dots n+1,$$

with the property that $\frac{x_j}{x_i}$ is an isomorphism. By the universal property of the localization and the polynomial algebra [Marty, 2009], we have that

$$U_{ij}(A) \cong \mathrm{Hom}_{\mathrm{Comm}(\mathcal{C})}(\mathbb{1}[\tfrac{x_1}{x_i}, \dots, \tfrac{x_{n+1}}{x_i}][\tfrac{x_j}{x_i}]^{-1}, A)$$

where $\mathbb{1}[\frac{x_1}{x_i}, \dots, \frac{x_{n+1}}{x_i}][\frac{x_j}{x_i}]^{-1}$ is the localization algebra at the invertible arrow $\frac{x_j}{x_i}$ ∎

Now we give a definition of the relative projective space in terms of quotients. This definition is somehow dual to the one given in Definition 3.1.

3.9. Definition. *Let $n \geq 1$ a fixed integer. For every affine scheme* $\operatorname{Spec} A$ *we define $\overline{\mathbb{P}^n_{\mathcal{C}}}(A)$ to be the set of quotients L of A^{n+1} with L a line object in* $\mathrm{Mod}_{\mathcal{C}}(A)$*. For every morphism* $\operatorname{Spec} B \longrightarrow \operatorname{Spec} A$*, the function $\overline{\mathbb{P}^n_{\mathcal{C}}}(A) \longrightarrow \overline{\mathbb{P}^n_{\mathcal{C}}}(B)$ assigns to $L \in \overline{\mathbb{P}^n_{\mathcal{C}}}(A)$ the corresponding epimorphism $B^{n+1} \twoheadrightarrow B \otimes_A L$.*

As before, $B \otimes_A L$ is a line object in $\mathrm{Mod}_{\mathcal{C}}(B)$ since line objects are preserved by strong monoidal functors.

3.10. Theorem. *If $\mathcal{C}$ is an abelian strong relative context then $\overline{\mathbb{P}^n_{\mathcal{C}}}$ is a $\mathcal{C}$-scheme.*

Proof. The proof of this theorem is quite similar to its analogous result Theorem 3.3, however by the very definition we will not need Lemmas 2.18 and 2.19. The open cover is given by

$$\overline{U_i}(A) = \{(L, \mathbf{x}) : A \xrightarrow{\lambda_i} A^{n+1} \overset{\mathbf{x}}{\twoheadrightarrow} L,\ \mathbf{x} \circ \lambda_i \text{ is an isomorphism in } \mathrm{Mod}_{\mathcal{C}}(A)\}.$$

For an outline on the proof see [Osorio, 2017] ∎

Now we prove that these two definitions are equivalent, in the sense that they give isomorphic $\mathcal{C}$-schemes, compare with [Eisenbud-Harris 1995][Ch. III, Th. III-37] for the case $\mathcal{C} = \mathbb{Z}$-Mod the category of abelian groups.

3.11. Theorem. *$\mathbb{P}^n_{\mathcal{C}}$ and $\overline{\mathbb{P}^n_{\mathcal{C}}}$ are isomorphic as $\mathcal{C}$-schemes.*

Proof. Since the category of $\mathcal{C}$-schemes is a full subcategory of $\mathrm{Sh}(\mathrm{Aff}_{\mathcal{C}})$, we will prove the isomorphism as presheaves, hence as sheaves. We define the natural transformation as follows: for every $A \in \mathrm{Comm}(\mathcal{C})$

$$\begin{tikzcd} \overline{\mathbb{P}^n_{\mathcal{C}}}(A) \arrow[rr, "\Psi"] & & \mathbb{P}^n_{\mathcal{C}}(A) \end{tikzcd}$$

$$A^{n+1} \xrightarrow{\mathbf{x}} L \longmapsto L^{\vee} \xrightarrow{\mathbf{x}^{\vee}} A^{n+1}$$

Since $\mathbf{x}$ is an epimorphism, $\mathbf{x}^{\vee}$ is a monomorphism. As L is invertible, hence projective, there exists a section s for $\mathbf{x}$, then $r = s^{\vee}$ is a retraction for $\mathbf{x}^{\vee}$ thence $(L^{\vee}, \mathbf{x}^{\vee})$ is an element in $\mathbb{P}^n_{\mathcal{C}}(A)$. The injectivity of Ψ follows from the fact that if $L_1^{\vee}$ and $L_2^{\vee}$ are isomorphic as subobjects of A^{n+1} then by dualizing we obtain that L_1 and L_2 are isomorphic as quotients of A^{n+1}, therefore they represent the same element in $\overline{\mathbb{P}^n_{\mathcal{C}}}(A)$.
To see that Ψ is an epimorphism, we check that for every i, the following diagram commutes:

$$\begin{tikzcd} \overline{U_i} \arrow[r, "\psi"] \arrow[d, tail] & U_i \arrow[d, tail] \\ \overline{\mathbb{P}^n_{\mathcal{C}}} \arrow[r, "\Psi"'] & \mathbb{P}^n_{\mathcal{C}} \end{tikzcd} \tag{5}$$

with $\psi : \overline{U_i}(A) \longrightarrow U_i(A)$ defined as follows. First let us make a simplification: if $(L, \mathbf{x})$ belongs to $U_i(A)$ we can make the identification $L \cong A$ as subobjects of A^{n+1}, we will denote the pair $(A, \mathbf{x})$ in $U_i(A)$. The same goes for a pair $(L, \mathbf{y})$ in $\overline{U_i}(A)$. If $(A, \mathbf{x}) \in U_i(A)$ then for each $j = 1, \ldots, n+1$ the diagram

$$\begin{tikzcd} A \arrow[d, "\lambda_j"'] \arrow[dr, "\pi_j \mathbf{x}"] & \\ A^{n+1} \arrow[r, dashed, "\exists ! \mathbf{y}"'] & A \end{tikzcd}$$

commutes. Since $\mathbf{y}\lambda_i = \pi_i \mathbf{x}$ is an isomorphism we have that $\mathbf{y}$ is an epimorphism, even more that $(A, \mathbf{y})$ is in $\overline{U_i}(A)$. This gives an isomorphism with inverse defined analogously. Now, take $(A, \mathbf{x})$ in $\overline{U_i}(A)$, then $\Psi(A, \mathbf{x}) = (A, \mathbf{x}^{\vee})$, since $\mathbf{x}\lambda_i$ is an isomorphism and for all $j = 1, \ldots n+1$, $\lambda_j^{\vee} = \pi_j$, then $\lambda_i^{\vee}\mathbf{x}^{\vee} = \pi_i \mathbf{x}^{\vee}$ is an isomorphism. This says that the pair $(A, \mathbf{x}^{\vee})$ is in $U_i(A)$. On the other hand, $\psi(A, \mathbf{x}) = (A, \mathbf{y})$ with $\mathbf{y} : A \longrightarrow A^{n+1}$ satisfying that $\pi_j \mathbf{y} = \mathbf{x}\lambda_j$. To prove the commutativity of the diagram, that is, the compatibility between Ψ and ψ, it is enough to show that both pairs $(A, \mathbf{x}^{\vee})$ and $(A, \mathbf{y})$ are the same subobject in A^{n+1}. The result follows from the fact that the dual of the morphism $\mathbf{x}\lambda_j : A \longrightarrow A$ is itself in $\mathrm{Mod}_{\mathcal{C}}(A)$, therefore:

$$\pi_j \mathbf{x}^{\vee} = \lambda_j^{\vee} \mathbf{x}^{\vee} = (\mathbf{x}\lambda_j)^{\vee} = \mathbf{x}\lambda_j = \pi_j \mathbf{y}$$

for every $j = 1, \dots n+1$, then $\mathbf{x}^\vee = \mathbf{y}$.
To finish the proof, consider the commutative diagram in $\mathrm{Sh}(\mathrm{Aff}_\mathcal{C})$

$$\begin{tikzcd}
\coprod_i \overline{U_i} \arrow[r, "\psi"] \arrow[d] & \coprod_i U_i \arrow[d] \\
\overline{\mathbb{P}^n_\mathcal{C}} \arrow[r, "\Psi"'] & \mathbb{P}^n_\mathcal{C}
\end{tikzcd}$$

where ψ is an isomorphism, therefore Ψ is a sheaf epimorphism. ∎

3.12. Remark. In the case $\mathcal{C} = \mathbb{Z}$-Mod, then it is well known that the functor of points of the projective scheme $\mathbb{P}^n_\mathbb{Z}$ evaluated at the commutative ring R is the set of quotients of R^{n+1} that are invertible R-modules, see [Eisenbud-Harris 1995, Cor. III-42]. By Remark 2.26, this is equivalent to our definition of $\overline{\mathbb{P}^n_\mathbb{Z}}$.

4. The octonionic projective space.

In [Albuquerque-Majid, 1999] the authors considered the symmetric monoidal category of real G-graded vector spaces $\mathcal{U} = (Vect^G_\mathbb{R}, \otimes_G, \mathbb{R}, \Phi_F, \sigma_F)$ with $G = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $F(x,y) = (-1)^{f(x,y)}$ with

$$\begin{aligned}
f(x,y) =& \sum_{i \le j} x_i y_j + y_1 x_2 x_3 + x_1 y_2 x_3 + x_1 y_2 y_3, \\
\phi_F(x,y,z) =& \partial F = \frac{F(x,y)F(xy,z)}{F(y,z)F(x,yz)}, \\
\Phi_F((x \otimes y) \otimes z) =& \phi_F(x,y,z) x \otimes (y \otimes z) \quad \text{associativity constraint} \\
\sigma_F(x,y) =& \frac{F(x,y)}{F(y,x)} y \otimes x \qquad\qquad \text{symmetry}
\end{aligned}$$

where by abuse of notation the degree of an homogeneous element is denoted by $|x| = x$. Then they proved that the Cayley algebra of the octonions $\mathbb{O}$ can be obtained as the commutative algebra $(\mathbb{R}\mathbb{Z}_2^3, m_F, \eta)$ in $\mathcal{U}$, with multiplication and unity given by $m_F(x,y) = F(x,y)x \cdot_G y, \ \eta(1) = 1_G$. Once we have a commutative algebra in a symmetric monoidal category, we can construct its category of modules and make some other constructions similar to those, one has in commutative algebra with the purpose to imitate the algebraic geometry over commutative rings.
In this section we will work on the properties of the category $\mathrm{Mod}_\mathcal{U}(\mathbb{O})$, concerning to projective and free objects (respect to a left adjoint functor called the free functor). We will prove that in fact $\mathbb{O}$ is a projective, finitely presented generator for the category $\mathrm{Mod}_\mathcal{U}(\mathbb{O})$, this will say that $\mathrm{Mod}_\mathcal{U}(\mathbb{O})$ is in fact equivalent (not in the monoidal sense) to a category of modules over a certain ring. Another proof of this result can be seen in [Panaite-Van Oystaeyen, 2004]. Although we are not interested in using this equivalence,

it is worth to mention it.
Let us start by characterizing the objects in $\mathrm{Mod}_{\mathcal{U}}(\mathbb{O})$. They consist of a pair (X,ρ) with X a $\mathbb{Z}_2^3$-graded real vector space with a graded morphism $\rho : \mathbb{O} \otimes_{\mathcal{U}} X \longrightarrow X$ satisfying the pentagon and triangle axioms for the action. Since ρ is a degree preserving morphism, then to give ρ is to give an 8-tuple of real vector space morphisms $\rho_i : (\mathbb{O} \otimes_{\mathcal{U}} X)_i \longrightarrow X_i$, where the index denotes the i-th degree component.
If we denote $\{e_i, \quad i = 0 \dots 7\}$ a basis for $\mathbb{O}$, then the associativity of the action says that $\rho(e_i, \rho(e_i, x_k)) = -x_k$, this means that for every $i = 0, \dots 7$, the multiplication by e_i induces an isomorphism $X_k \cong X_l$ with k, l such that $e_l = m_F(e_i, e_k)$.
In summary, an $\mathbb{O}$-module is just a graded vector space with distinguished isomorphisms between the homogenous components, given by the multiplication of the basis elements of $\mathbb{O}$. Thus, the datum of being an $\mathbb{O}$-module is in the 0-th degree component and one obtains the rest of the components by multiplication of the $e_i's$.
Next, a morphism between objects in $\mathrm{Mod}_{\mathcal{U}}(\mathbb{O})$ will be a preserving degree morphism between the graded vector spaces compatible with the actions of $\mathbb{O}$, i.e., a morphism between the degree zero components commuting with the respective isomorphisms. More explicitly, if X, Y are objects in $\mathrm{Mod}_{\mathcal{U}}(\mathbb{O})$, then $f : X \longrightarrow Y$ is characterized by the morphism $f_0 : X_0 \longrightarrow Y_0$, since the rest of the morphisms are just conjugations of f_0 by the e_i's as is depicted in the following commutative diagram:

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[d, "\cdot e_i"'] & Y_0 \arrow[d, "\cdot e_i"] \\ X_i \arrow[r, "f_i"'] & Y_i \end{tikzcd} \tag{6}$$

All this implies the following two lemmas:

4.1. Lemma. *Let* $V_0 = \mathrm{Hom}_{\mathbb{O}}(\mathbb{O}, -) : \mathrm{Mod}_{\mathcal{U}}(\mathbb{O}) \longrightarrow \mathrm{Set}$ *be the "canonical" forgetful functor in monoidal categories. Then* V_0 *is a conservative functor.*

Proof. Let $f : (X,\rho) \longrightarrow (Y,\rho')$ be a morphism in $\mathrm{Mod}_{\mathcal{U}}(\mathbb{O})$, such that the induced morphism $V_0(f) : \mathrm{Hom}_{\mathbb{O}}(\mathbb{O}, X) \longrightarrow \mathrm{Hom}_{\mathbb{O}}(\mathbb{O}, Y)$ is an isomorphism. If we denote by $|\text{-}| : \mathrm{Mod}_{\mathcal{U}}(\mathbb{O}) \longrightarrow \mathcal{U}$ the forgetful functor, then the adjunction $- \underset{\mathcal{U}}{\otimes} \mathbb{O} \dashv |\text{-}|$ says that we have an isomorphism $\mathrm{Hom}_{\mathcal{U}}(\mathbb{R}, |X|) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{U}}(\mathbb{R}, |Y|)$. Since $X_0 \cong \mathrm{Hom}_{\mathcal{U}}(\mathbb{R}, |X|)$ in the category $\mathcal{U}$, then we have the isomorphism $f_0 : X_0 \longrightarrow Y_0$. Finally, by the diagram (6), it follows that $f : (X,\rho) \longrightarrow (Y,\rho')$ is in fact an isomorphism. ■

4.2. Lemma. $\mathbb{O}$ *is a projective finitely presented generator in* $\mathrm{Mod}_{\mathcal{U}}(\mathbb{O})$.

Proof. Limits and colimits in $\mathrm{Mod}_{\mathcal{U}}(\mathbb{O})$ are computed in $\mathcal{U}$, this means in particular that $|\text{-}| : \mathrm{Mod}_{\mathcal{U}}(\mathbb{O}) \longrightarrow \mathcal{U}$ preserves them. Now, since $\mathbb{R}$ is a projective object in $\mathcal{U}$, then $\mathbb{O} \cong \mathbb{R} \otimes_{\mathcal{U}} \mathbb{O}$ is projective in $\mathrm{Mod}_{\mathcal{U}}(\mathbb{O})$.

Now, to show that $\mathbb{O}$ is finitely presented, observe that $\mathrm{Hom}_{\mathcal{U}}(\mathbb{R}, -)$ preserves them, hence by the isomorphism

$$\mathrm{Hom}_{\mathbb{O}}(\mathbb{O}, -) \cong \mathrm{Hom}_{\mathcal{U}}(\mathbb{R}, |-|)$$

we get that $\mathrm{Hom}_{\mathbb{O}}(\mathbb{O}, -)$ preserves filtered colimits, that is $\mathbb{O}$ is finitely presented.
Finally, to see that $\mathbb{O}$ is a generator, we have to prove that $\mathrm{Hom}_{\mathbb{O}}(\mathbb{O}, -)$ is faithful. The result follows by "abstract nonsense": In any category $\mathcal{C}$ with equalizers, a conservative functor $F : \mathcal{C} \longrightarrow \mathrm{Set}$ preserving them is faithful. ∎

In summary we have proved the following result:

4.3. Corollary. $\mathrm{Mod}_{\mathcal{U}}(\mathbb{O})$ *is an abelian strong relative context and* $\mathbb{P}^n_{\mathbb{O}}$ *is relative scheme, where we define the functor* $\mathbb{P}^n_{\mathbb{O}}$ *relative to the category* $\mathrm{Mod}_{\mathcal{U}}(\mathbb{O})$ *as in Definition 3.1.*

Motivated by Corollary 4.3 and Remark 3.12 we call $\mathbb{P}^n_{\mathbb{O}}$ the octonionic relative projective space.

4.4. Remark. An interesting problem would be to characterize the line objects in the category $\mathrm{Mod}_{\mathcal{U}}(\mathbb{O})$.

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