

HOCHSCHILD COHOMOLOGY OF FROBENIUS ALGEBRAS

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ABSTRACT. Let k be a field, A a finite dimensional Frobenius k -algebra and $\rho: A \rightarrow A$, the Nakayama automorphism of A with respect to a Frobenius homomorphism $\varphi: A \rightarrow k$. Assume that ρ has finite order m and that k has a primitive m -th root of unity w . Consider the decomposition $A = A_0 \oplus \cdots \oplus A_{m-1}$ of A , obtained defining $A_i = \{a \in A : \rho(a) = w^i a\}$, and the decomposition $\mathrm{HH}^*(A) = \bigoplus_{i=0}^{m-1} \mathrm{HH}_i^*(A)$ of the Hochschild cohomology of A , obtained from the decomposition of A . In this paper we prove that $\mathrm{HH}^*(A) = \mathrm{HH}_0^*(A)$ and that if decomposition of A is strongly $\mathbb{Z}/m\mathbb{Z}$ -graded, then $\mathbb{Z}/m\mathbb{Z}$ acts on $\mathrm{HH}^*(A_0)$ and $\mathrm{HH}^*(A) = \mathrm{HH}_0^*(A) = \mathrm{HH}^*(A_0)^{\mathbb{Z}/m\mathbb{Z}}$.

1. INTRODUCTION

Let k be a field, A a finite dimensional k -algebra and $DA = \mathrm{Hom}_k(A, k)$ endowed with the usual A -bimodule structure. Recall that A is said to be a Frobenius algebra if there exists a linear form $\varphi: A \rightarrow k$, such that the map $A \rightarrow DA$, defined by $x \mapsto x\varphi$ is a left A -module isomorphism. This linear form $\varphi: A \rightarrow k$ is called a Frobenius homomorphism. It is well known that this is equivalent to say that the map $x \mapsto \varphi x$, from A to DA , is an isomorphism of right A -modules. From this it follows easily that there exists an automorphism ρ of A , called the Nakayama automorphism of A with respect to φ , such that $x\varphi = \varphi\rho(x)$, for all $x \in A$. It is easy to check that a linear form $\tilde{\varphi}: A \rightarrow k$ is another Frobenius homomorphism if and only if there exists $x \in A$ invertible, such that $\tilde{\varphi} = x\varphi$. It is also easy to check that the Nakayama automorphism of A with respect to $\tilde{\varphi}$ is the map given by $a \mapsto \rho(x)^{-1}\rho(a)\rho(x)$.

Let A be a Frobenius k -algebra, $\varphi: A \rightarrow k$ a Frobenius homomorphism and $\rho: A \rightarrow A$ the Nakayama automorphism of A with respect to φ .

Definition 1.1. We say that ρ has order $m \in \mathbb{N}$ and we write $\mathrm{ord}_\rho = m$ if $\rho^m = \mathrm{id}_A$ and $\rho^r \neq \mathrm{id}_A$, for all $r < m$.

Assume that ρ has finite order and that k has a primitive ord_ρ -th root of unity w . Since the polynomial $X^{\mathrm{ord}_\rho} - 1$ has distinct roots w^i ($0 \leq i < \mathrm{ord}_\rho$), the algebra A becomes a $\frac{\mathbb{Z}}{\mathrm{ord}_\rho\mathbb{Z}}$ -graded algebra

$$A = A_0 \oplus \cdots \oplus A_{\mathrm{ord}_\rho - 1}, \quad \text{where } A_i = \{a \in A : \rho(a) = w^i a\}.$$

Let $(\mathrm{Hom}_k(A^{\otimes*}, A), b^*)$ be the cochain Hochschild complex of A with coefficients in A . For each $0 \leq i < \mathrm{ord}_\rho$, we let $(\mathrm{Hom}_k(A^{\otimes*}, A)_i, b^*)$ denote the subcomplex of

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$(\mathrm{Hom}_k(A^{\otimes*}, A), b^*)$, defined by

$$\mathrm{Hom}_k(A^{\otimes n}, A)_i = \bigoplus_{\tilde{B}_{i,n}} \mathrm{Hom}_k(A_{u_1} \otimes \cdots \otimes A_{u_n}, A_v),$$

where $\tilde{B}_{i,n} = \{(u_1, \dots, u_n, v) \text{ such that } v - u_1 - \cdots - u_n \equiv i \pmod{\mathrm{ord}_\rho}\}$. The cochain Hochschild complex $(\mathrm{Hom}_k(A^{\otimes*}, A), b^*)$ decomposes as the direct sum

$$(\mathrm{Hom}_k(A^{\otimes*}, A), b^*) = \bigoplus_{i=0}^{\mathrm{ord}_\rho - 1} (\mathrm{Hom}_k(A^{\otimes*}, A)_i, b^*).$$

Thus, the Hochschild cohomology $\mathrm{HH}^n(A)$, of A with coefficients in A , decomposes as the direct sum

$$\mathrm{HH}^n(A) = \bigoplus_{i=0}^{\mathrm{ord}_\rho - 1} \mathrm{HH}_i^n(A),$$

where $\mathrm{HH}_i^n(A) = H^n(\mathrm{Hom}_k(A^{\otimes*}, A)_i, b^*)$.

The aim of this paper is to prove the following results:

Theorem 1.2. *Let A be a Frobenius k -algebra, $\varphi: A \rightarrow k$ a Frobenius homomorphism and $\rho: A \rightarrow A$ the Nakayama automorphism of A with respect to φ . If ρ has finite order and k has a primitive ord_ρ -th root of unity w , then*

$$\mathrm{HH}^n(A) = \mathrm{HH}_0^n(A), \quad \text{for all } n \geq 0.$$

Recall that $A = A_0 \oplus \cdots \oplus A_{\mathrm{ord}_\rho - 1}$ is said to be strongly $\mathbb{Z}/\mathrm{ord}_\rho\mathbb{Z}$ -graded if $A_i A_j = A_{i+j}$, for all $i, j \in \{0, \dots, \mathrm{ord}_\rho - 1\}$, where $i + j$ denotes the sum of i and j in $\mathbb{Z}/\mathrm{ord}_\rho\mathbb{Z}$.

Theorem 1.3. *Let A be a Frobenius k -algebra, $\varphi: A \rightarrow k$ a Frobenius homomorphism and $\rho: A \rightarrow A$ the Nakayama automorphism of A with respect to φ . If ρ has finite order, k has a primitive ord_ρ -th root of unity w and $A = A_0 \oplus \cdots \oplus A_{\mathrm{ord}_\rho - 1}$ is strongly $\mathbb{Z}/\mathrm{ord}_\rho\mathbb{Z}$ -graded, then*

$$\mathrm{HH}^n(A) = \mathrm{HH}^n(A_0)^{\mathbb{Z}/\mathrm{ord}_\rho\mathbb{Z}}, \quad \text{for all } n \geq 0.$$

Corollary 1.4. *Assume that the hypothesis of Theorem 1.3 are verified. If the Hochschild cohomology $\mathrm{HH}^2(A_0) = 0$, then A is rigid.*

Remark 1.5. As it is well known, every finite dimensional Hopf algebra H is Frobenius, being a Frobenius homomorphism any right integral $\varphi \in H^* \setminus \{0\}$. Moreover, by Proposition 3.6 of [S], the composition inverse of the Nakayama map ρ with respect to φ , is given by

$$\rho^{-1}(h) = \alpha(h_{(1)})\overline{S}^2(h_{(2)}),$$

where $\alpha \in H^*$ is the modular element of H^* and \overline{S} is the composition inverse of S (note that the automorphism of Nakayama considered in [S] is the composition inverse of the one considered by us). Using this formula and that $\alpha \circ S^2 = \alpha$, it is easy to check that $\rho(h) = \alpha(S(h_{(1)}))S^2(h_{(2)})$, and more generality, that

$$\rho^l(h) = \alpha^{*l}(S(h_{(1)}))S^{2l}(h_{(2)}),$$

where α^{*l} denotes the l -fold convolution product of α . Since α has finite order respect to the convolution product and, by the Radford formula for S^4 (see Theorem 3.8 of [S]), the antipode S has finite order respect to the composition, we have that ρ has finite order. So, the above theorems apply to finite dimensional Hopf algebras.

We think that the decomposition of H associated with ρ can be useful to study the structure of finite dimensional Hopf algebras. In this paper we exploit it in a cohomological level. Recently has been considered another decomposition of H , similar to this one, but distinct. Namely the one associated to S^2 (see [R-S]).

Example 1.6. Let k a field and N a natural number. Assume that k has a primitive N -th root of unity w . Let H be the Taft algebra of order N . That is, H is the algebra generated over k by two elements g and x subject to the relations $g^N = 1$, $x^N = 0$ and $xg = wx$. The Taft algebra H is a Hopf algebra with comultiplication Δ , counity ϵ and antipode S given by

$$\begin{aligned}\Delta(g) &= g \otimes g, & \Delta(x) &= 1 \otimes x + x \otimes g, \\ \epsilon(g) &= 1, & \epsilon(x) &= 0, \\ S(g) &= g^{-1}, & S(x) &= -xg^{-1}.\end{aligned}$$

Using that $t = \sum_{j=0}^{N-1} w^{-j} g^j x^{N-1}$ is a right integral of H , it is easy to see that the modular element $\alpha \in H^*$ verifies $\alpha(g) = w^{-1}$ and $\alpha(x) = 0$. By the remark above, the Nakayama map $\rho: H \rightarrow H$ is given by $\rho(g) = wg$ and $\rho(x) = w^{-1}x$. Hence, $H = H_0 \oplus \cdots \oplus H_{N-1}$, where

$$\begin{aligned}H_i &= \{a \in H : \rho(a) = w^{-i}a\} \\ &= \langle x^i, x^{i+1}g, \dots, x^{N-1}g^{N-i-1}, g^{N-i}, xg^{N-i+1}, \dots, x^{i-1}g^{N-1} \rangle.\end{aligned}$$

Let $C_N = \{1, t, \dots, t^{N-1}\}$ be the cyclic group of order N . It is easy to see that C_N acts on H_0 via $t \cdot x^i g^i = w^i x^i g^i$ and that H is isomorphic to the skew product of $H_0 \# C_N$. By Theorem 1.3,

$$\mathrm{HH}^n(H) = \mathrm{HH}^n(H_0)^{C_N} \quad \text{for all } n \geq 0,$$

where the action of C_N on $\mathrm{HH}^n(H_0)$ is induced by the one of C_N on $\mathrm{Hom}_k(H_0^{\otimes n}, H_0)$, given by

$$t \cdot \varphi(x^{i_1} g^{i_1} \otimes \cdots \otimes x^{i_n} g^{i_n}) = g^{N-1} \varphi(t \cdot x^{i_1} g^{i_1} \otimes \cdots \otimes t \cdot x^{i_n} g^{i_n})g.$$

2. PROOF OF THEOREMS 1.2 AND 1.3

Let k be a field, A a k -algebra and V a k -module. To begin, we fix some notations:

- (1) As in the introduction, we let DA denote $\mathrm{Hom}_k(A, k)$, endowed with the usual A -bimodule structure.
- (2) We let $V^{\otimes n}$ denote the n -fold tensor product $V \otimes \cdots \otimes V$.
- (3) Given $x \in A \cup DA$, we write

$$\pi_A(x) = \begin{cases} x & \text{if } x \in A, \\ 0 & \text{if } x \in DA, \end{cases} \quad \text{and} \quad \pi_{DA}(x) = \begin{cases} x & \text{if } x \in DA, \\ 0 & \text{if } x \in A, \end{cases}$$

- (4) For $n \geq 1$, we let $B^n \subseteq (A \oplus DA)^{\otimes n}$ denote the vector subspace spanned by n -tensors $x_1 \otimes \cdots \otimes x_n$ such that exactly 1 of the x_i 's belong to DA , while the other x_i 's belong to A .
- (5) Given $i < j$ and $x_i, x_{i+1}, \dots, x_j \in A \cup DA$, we write $\mathbf{x}_{i,j} = x_i \otimes \cdots \otimes x_j$.
- (6) For each map $f: X \rightarrow Y$ and each element $x \in X$, we let $\langle f, x \rangle$ denote the evaluation of f in x .

2.1. **The complex $X^{*,*}(A)$.** For each k -algebra A , we consider the double complex

$$\begin{array}{ccc}
 & \begin{array}{c} \vdots \\ \uparrow b^{0,4} \\ \text{Hom}_k(A^{\otimes 3}, A) \end{array} & \xrightarrow{\delta^{1,3}} & \begin{array}{c} \vdots \\ \uparrow b^{1,4} \\ \text{Hom}_k(B^4, DA) \end{array} \\
 & \uparrow b^{0,3} & & \uparrow b^{1,3} \\
 X^{*,*}(A) := & \text{Hom}_k(A^{\otimes 2}, A) & \xrightarrow{\delta^{1,2}} & \text{Hom}_k(B^3, DA) \\
 & \uparrow b^{0,2} & & \uparrow b^{1,2} \\
 & \text{Hom}_k(A, A) & \xrightarrow{\delta^{1,1}} & \text{Hom}_k(B^2, DA) \\
 & \uparrow b^{0,1} & & \uparrow b^{1,1} \\
 & \text{Hom}_k(k, A) & \xrightarrow{\delta^{1,0}} & \text{Hom}_k(B^1, DA),
 \end{array}$$

where

$$\begin{aligned}
 \langle \langle b^{0,n+1}, f \rangle, \mathbf{x}_{1,n+1} \rangle &= x_1 \langle f, \mathbf{x}_{2,n+1} \rangle + \sum_{i=1}^n (-1)^i \langle f, \mathbf{x}_{1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+1,n+1} \rangle \\
 &\quad + (-1)^{n+1} \langle f, \mathbf{x}_{1,n} \rangle x_{n+1}, \\
 \langle \langle b^{1,n}, g \rangle, \mathbf{y}_{1,n+1} \rangle &= \langle \pi_A, y_1 \rangle \langle g, \mathbf{y}_{2,n+1} \rangle + \sum_{i=1}^n (-1)^i \langle g, \mathbf{y}_{1,i-1} \otimes y_i y_{i+1} \otimes \mathbf{y}_{i+1,n+1} \rangle \\
 &\quad + (-1)^{n+1} \langle g, \mathbf{y}_{1,n} \rangle \langle \pi_A, y_{n+1} \rangle, \\
 \langle \langle \delta^{1,n}, f \rangle, \mathbf{y}_{1,n+1} \rangle &= \langle \pi_{DA}, y_1 \rangle \langle f, \mathbf{y}_{2,n+1} \rangle + (-1)^{n+1} \langle f, \mathbf{y}_{1,n} \rangle \langle \pi_{DA}, y_{n+1} \rangle,
 \end{aligned}$$

for $f \in \text{Hom}_k(A^{\otimes n}, A)$, $g \in \text{Hom}_k(B^n, DA)$, $\mathbf{x}_{1,n+1} = x_1 \otimes \cdots \otimes x_{n+1} \in A^{\otimes n+1}$ and $\mathbf{y}_{1,n+1} = y_1 \otimes \cdots \otimes y_{n+1} \in B^n$.

Proposition 2.1. *Let $X^*(A)$ be the total complex of $X^{*,*}(A)$. It is true that*

$$H^n(X^*(A)) = \begin{cases} H^0(X^{0,*}(A)) & \text{if } n = 0, \\ H^n(X^{0,*}(A)) \oplus H^{n-1}(X^{0,*}(A)) & \text{if } n \geq 1. \end{cases}$$

Proof. Let

$$\delta^{1,*}: (\text{Hom}_k(A^{\otimes *}, A), -b^{0,*+1}) \rightarrow (\text{Hom}_k(B^{*+1}, DA), b^{1,*+1})$$

be the map defined by

$$\langle \langle \delta^{1,n}, f \rangle, \mathbf{x}_{1,n+1} \rangle = \langle \pi_{DA}, x_1 \rangle \langle f, \mathbf{x}_{2,n+1} \rangle + (-1)^{n+1} \langle f, \mathbf{x}_{1,n} \rangle \langle \pi_{DA}, x_{n+1} \rangle.$$

Since $X^*(A)$ is the mapping cone of $\delta^{1,*}$, in order to obtain the result it suffices to check that $\delta^{1,*}$ is null homotopic. Let $\sigma_*: \text{Hom}_k(A^{\otimes*}, A) \rightarrow \text{Hom}_k(B^*, DA)$ be the family of maps defined by

$$\langle\langle\sigma_n, f\rangle, \mathbf{x}_{1,n}\rangle, a\rangle = (-1)^{jn+1}\langle x_j, \langle f, \mathbf{x}_{j+1,n} \otimes a \otimes \mathbf{x}_{1,j-1}\rangle\rangle \quad \text{if } x_j \in DA.$$

We assert that σ_* is an homotopy from $\delta^{1,*}$ to 0. By definition,

$$\begin{aligned} \langle\langle b^{1,n}, \langle\sigma_n, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle &= \langle\pi_A, x_1\rangle\langle\langle\sigma_n, f\rangle, \mathbf{x}_{2,n+1}\rangle \\ &+ \sum_{i=1}^n (-1)^i \langle\langle\sigma_n, f\rangle, \mathbf{x}_{1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+1}\rangle \\ &+ (-1)^{n+1} \langle\langle\sigma_n, f\rangle, \mathbf{x}_{1,n}\rangle\langle\pi_A, x_{n+1}\rangle. \end{aligned}$$

Hence, if $x_1 \in DA$, then

$$\begin{aligned} \langle\langle\langle b^{1,n}, \langle\sigma_n, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle, x_{n+2}\rangle &= (-1)^{n+2} \langle x_1, x_2 \langle f, \mathbf{x}_{3,n+2}\rangle\rangle \\ &- \sum_{i=2}^{n+1} (-1)^{n+i} \langle x_1, \langle f, \mathbf{x}_{2,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+2}\rangle\rangle; \end{aligned}$$

if $x_j \in DA$ for $1 < j \leq n$, then

$$\begin{aligned} \langle\langle\langle b^{1,n}, \langle\sigma_n, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle, x_0\rangle &= (-1)^{(j-1)n+j} \langle x_j, \langle f, \mathbf{x}_{j+1,n+1} \otimes \mathbf{x}_{0,j-2}\rangle x_{j-1}\rangle \\ &- \sum_{i=0}^{j-2} (-1)^{(j-1)n+i} \langle x_j, \langle f, \mathbf{x}_{j+1,n+1} \otimes \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,j-1}\rangle\rangle \\ &- (-1)^{jn+j} \langle x_j, x_{j+1} \langle f, \mathbf{x}_{j+2,n+1} \otimes \mathbf{x}_{0,j-1}\rangle\rangle \\ &- \sum_{i=j+1}^n (-1)^{jn+i} \langle x_j, \langle f, \mathbf{x}_{j+1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+1} \otimes \mathbf{x}_{0,j-1}\rangle\rangle \\ &+ (-1)^{jn+n} \langle x_j, \langle f, \mathbf{x}_{j+1,n} \otimes x_{n+1} x_0 \otimes \mathbf{x}_{1,j-1}\rangle\rangle; \end{aligned}$$

and if $x_{n+1} \in DA$, then

$$\begin{aligned} \langle\langle\langle b^{1,n}, \langle\sigma_n, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle, x_0\rangle &= \sum_{i=0}^{n-1} (-1)^{n+i+1} \langle x_{n+1}, \langle f, \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n}\rangle\rangle \\ &- \langle x_{n+1}, \langle f, \mathbf{x}_{0,n-1}\rangle x_n\rangle. \end{aligned}$$

On the other hand, if $x_1 \in DA$, then

$$\begin{aligned} \langle\langle\sigma_{n+1}, -\langle b^{0,n+1}, f\rangle\rangle, \mathbf{x}_{1,n+1}\rangle, x_{n+2}\rangle &= (-1)^{n+1} x_1 (b^{0,n+1}(f)(\mathbf{x}_{2,n+2})) \\ &= (-1)^{n+1} \langle x_1, x_2 \langle f, \mathbf{x}_{3,n+2}\rangle\rangle + \langle x_1, \langle f, \mathbf{x}_{2,n+1}\rangle x_{n+2}\rangle \\ &+ \sum_{i=2}^{n+1} (-1)^{n+i} \langle x_1, \langle f, \mathbf{x}_{2,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+2}\rangle\rangle; \end{aligned}$$

if $x_j \in DA$ for $1 < j \leq n$, then

$$\begin{aligned}
& \langle \langle \sigma_{n+1}, -\langle b^{0,n+1}, f \rangle, \mathbf{x}_{1,n+1} \rangle, x_0 \rangle \\
&= (-1)^{j(n+1)} \langle x_j, \langle \langle b^{0,n+1}, f \rangle, \mathbf{x}_{j+1,n+1} \otimes \mathbf{x}_{0,j-1} \rangle \rangle \\
&= (-1)^{j(n+1)} \langle x_j, x_{j+1} \langle f, \mathbf{x}_{j+2,n+1} \otimes \mathbf{x}_{0,j-1} \rangle \rangle \\
&+ \sum_{i=j+1}^n (-1)^{j(n+1)+i-j} \langle x_j, \langle f, \mathbf{x}_{j+1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+1} \otimes \mathbf{x}_{0,j-1} \rangle \rangle \\
&+ (-1)^{j(n+1)+n-j+1} \langle x_j, \langle f, \mathbf{x}_{j+1,n} \otimes x_{n+1} x_0 \otimes \mathbf{x}_{1,j-1} \rangle \rangle \\
&+ \sum_{i=0}^{j-2} (-1)^{j(n+1)+i+n-j} \langle x_j, \langle f, \mathbf{x}_{j+1,n+1} \otimes \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,j-1} \rangle \rangle \\
&+ (-1)^{j(n+1)+n+1} \langle x_j, \langle f, \mathbf{x}_{j+1,n+1} \otimes \mathbf{x}_{0,j-2} \rangle x_{j-1} \rangle;
\end{aligned}$$

and if $x_{n+1} \in DA$, then

$$\begin{aligned}
\langle \langle \sigma_{n+1}, -\langle b^{0,n+1}, f \rangle, \mathbf{x}_{1,n+1} \rangle, x_0 \rangle &= (-1)^{n+1} \langle x_{n+1}, \langle \langle b^{0,n+1}, f \rangle, \mathbf{x}_{0,n} \rangle \rangle \\
&= (-1)^{n+1} \langle x_{n+1}, x_0 \langle f, \mathbf{x}_{1,n} \rangle \rangle + \langle x_{n+1}, \langle f, \mathbf{x}_{0,n-1} \rangle x_n \rangle \\
&+ \sum_{i=0}^{n-1} (-1)^{n+i} \langle x_{n+1}, \langle f, \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n} \rangle \rangle.
\end{aligned}$$

The assertion follows immediately from these equalities. \square

2.2. The complex $Y^{*,*}(A)$. From now on we fix a Frobenius algebra A , a Frobenius homomorphism $\varphi: A \rightarrow k$ of A and we let denote ρ the Nakayama automorphism of A with respect to φ . Let A_ρ be A , endowed with the A -bimodule structure given by $a \cdot x \cdot b := \rho(a)xb$. Let $\Theta: DA \rightarrow A_\rho$ be the A -bimodule isomorphism given by $\Theta(\varphi x) = x$ and let

$$A_\rho \xleftarrow{\mu} A \otimes A_\rho \xleftarrow{b'_1} A^{\otimes 2} \otimes A_\rho \xleftarrow{b'_2} A^{\otimes 3} \otimes A_\rho \xleftarrow{b'_3} A^{\otimes 4} \otimes A_\rho \xleftarrow{b'_4} \cdots,$$

be the bar resolution of A_ρ .

Proposition 2.2. *The following assertions hold:*

(1) *The complex*

$$DA \xleftarrow{\mu'} A \otimes B^1 \otimes A \xleftarrow{b''_1} A \otimes B^2 \otimes A \xleftarrow{b''_2} A \otimes B^3 \otimes A \xleftarrow{b''_3} \cdots,$$

where $\langle \mu', x_0 \otimes x_1 \otimes x_2 \rangle = x_0 x_1 x_2$ and

$$\begin{aligned}
\langle b''_n, \mathbf{x}_{0,n+2} \rangle &= x_0 \langle \pi_A, x_1 \rangle \otimes \mathbf{x}_{2,n+2} + \sum_{i=1}^n (-1)^i \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+2} \\
&+ (-1)^{n+1} \mathbf{x}_{0,n} \otimes \langle \pi_A, x_{n+1} \rangle x_{n+2},
\end{aligned}$$

is a projective resolution of DA .

(2) *There is a chain map $\Psi'_*: (A^{\otimes *+1} \otimes A_\rho, b'_*) \rightarrow (A \otimes B^{*+1} \otimes A, b''_*)$, given by*

$$\langle \Psi'_n, \mathbf{x}_{0,n+1} \rangle = \sum_{i=0}^n (-1)^{i+n} \mathbf{x}_{0,i} \otimes \varphi \otimes \langle \rho, x_{i+1} \rangle \otimes \cdots \otimes \langle \rho, x_n \rangle \otimes x_{n+1}.$$

$$(3) \Theta \circ \mu' \circ \Psi'_0 = \mu.$$

Proof. Items (2) and (3) follow by a direct computation and item (1) is well known. For instance, the family of maps

$$\sigma_0: DA \rightarrow A \otimes B^1 \otimes A \quad \text{and} \quad \sigma_n: A \otimes B^n \otimes A \rightarrow A \otimes B^{n+1} \otimes A \quad (n \geq 1),$$

given by

$$\langle \sigma_0, x \rangle = 1 \otimes x \otimes 1,$$

$$\langle \sigma_{n+1}, \mathbf{x}_{0,n+1} \rangle = \begin{cases} 1 \otimes \mathbf{x}_{0,n+1} + (-1)^{n+1} \otimes x_0 x_1 \otimes \mathbf{x}_{2,n+1} \otimes 1 & \text{if } x_1 \in DA, \\ 1 \otimes \mathbf{x}_{0,n+1} & \text{if } x_1 \notin DA, \end{cases}$$

where $\mathbf{x}_{0,n+1} = x_0 \otimes \cdots \otimes x_{n+1} \in A \otimes B^n \otimes A$, is a contracting homotopy of the complex of item (1) as a k -module complex. \square

Let $Y^{*,*}(A)$ be the double complex

$$Y^{*,*}(A) := \begin{array}{ccc} & \begin{array}{c} \vdots \\ \uparrow \tilde{b}^{0,4} \\ \text{Hom}_k(A^{\otimes 3}, A) \end{array} & \begin{array}{c} \vdots \\ \uparrow \tilde{b}^{1,4} \\ \text{Hom}_k(A^{\otimes 3}, A) \end{array} \\ & \begin{array}{c} \uparrow \tilde{b}^{0,3} \\ \text{Hom}_k(A^{\otimes 2}, A) \end{array} & \xrightarrow{\tilde{\delta}^{1,3}} & \begin{array}{c} \uparrow \tilde{b}^{1,3} \\ \text{Hom}_k(A^{\otimes 2}, A) \end{array} \\ & \begin{array}{c} \uparrow \tilde{b}^{0,2} \\ \text{Hom}_k(A, A) \end{array} & \xrightarrow{\tilde{\delta}^{1,2}} & \begin{array}{c} \uparrow \tilde{b}^{1,2} \\ \text{Hom}_k(A, A) \end{array} \\ & \begin{array}{c} \uparrow \tilde{b}^{0,1} \\ \text{Hom}_k(k, A) \end{array} & \xrightarrow{\tilde{\delta}^{1,1}} & \begin{array}{c} \uparrow \tilde{b}^{1,1} \\ \text{Hom}_k(k, A) \end{array} \\ & & \xrightarrow{\tilde{\delta}^{1,0}} & \text{Hom}_k(k, A), \end{array}$$

with boundary maps

$$\begin{aligned} \langle \langle \tilde{b}^{u,n}, f \rangle, \mathbf{x}_{1,n} \rangle &= x_1 f(\mathbf{x}_{2,n}) + \sum_{i=1}^{n-1} (-1)^i \langle f, \mathbf{x}_{1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n} \rangle \\ &\quad + (-1)^n \langle f, \mathbf{x}_{1,n-1} \rangle x_n, \\ \langle \langle \tilde{\delta}^{1,n-1}, f \rangle, \mathbf{x}_{1,n-1} \rangle &= (-1)^n \langle f, \mathbf{x}_{1,n-1} \rangle \\ &\quad + (-1)^{n-1} \langle \rho^{-1}, \langle f, \langle \rho, x_1 \rangle \otimes \cdots \otimes \langle \rho, x_{n-1} \rangle \rangle \rangle, \end{aligned}$$

where $u = 0, 1$, $f \in \text{Hom}_k(A^{\otimes n-1}, A)$ and $\mathbf{x}_{1,n} = x_1 \otimes \cdots \otimes x_n \in A^{\otimes n}$.

Proposition 2.3. *The double complexes $X^{*,*}(A)$ and $Y^{*,*}(A)$ are quasiisomorphic.*

Proof. It is immediate that $X^{1,*}(A) \simeq \text{Hom}_{A^e}((A \otimes B^{*+1} \otimes A, b'_*), DA)$. Moreover, by Proposition 2.2, the map $\Psi^* := \text{Hom}_{A^e}(\Psi'_*, DA)$ is a quasiisomorphism from $\text{Hom}_{A^e}((A \otimes B^{*+1} \otimes A, b'_*), DA)$ to $\text{Hom}_{A^e}((A^{\otimes *+1} \otimes A_\rho, b'_*), DA)$. On the other hand the family of bijective maps

$$\Upsilon^n: Y^{1,n}(A) \rightarrow \text{Hom}_{A^e}(A^{\otimes n+1} \otimes A_\rho, DA) \quad (n \geq 0),$$

defined by $\langle \langle \Upsilon^n, f \rangle, \mathbf{x}_{0,n+1} \rangle = x_0 \langle f, \mathbf{x}_{1,n} \rangle \varphi x_{n+1}$, is an isomorphism of complexes from $Y^{1,*}(A)$ to $\text{Hom}_{A^e}((A^{\otimes *+1} \otimes A_\rho, b'_*), DA)$. In fact, we have

$$\begin{aligned}
\langle \langle \Upsilon^{n+1}, \langle \tilde{b}^{1,n+1}, f \rangle \rangle, \mathbf{x}_{0,n+2} \rangle &= x_0 \langle \langle \tilde{b}^{1,n+1}, f \rangle, \mathbf{x}_{1,n+1} \rangle \varphi x_{n+2} \\
&= x_0 x_1 \langle f, \mathbf{x}_{2,n+1} \rangle \varphi x_{n+2} \\
&\quad + \sum_{i=1}^n (-1)^i x_0 \langle f, \mathbf{x}_{1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+1} \rangle \varphi x_{n+2} \\
&\quad + (-1)^{n+1} x_0 \langle f, \mathbf{x}_{1,n} \rangle x_{n+1} \varphi x_{n+2} \\
&= x_0 x_1 \langle f, \mathbf{x}_{2,n+1} \rangle \varphi x_{n+2} \\
&\quad + \sum_{i=1}^n (-1)^i x_0 \langle f, \mathbf{x}_{1,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+1} \rangle \varphi x_{n+2} \\
&\quad + (-1)^{n+1} x_0 \langle f, \mathbf{x}_{1,n} \rangle \varphi \langle \rho, x_{n+1} \rangle x_{n+2} \\
&= \sum_{i=0}^n (-1)^i \langle \langle \Upsilon^n, f \rangle, \mathbf{x}_{0,i-1} \otimes x_i x_{i+1} \otimes \mathbf{x}_{i+2,n+2} \rangle \\
&\quad + (-1)^{n+1} \langle \langle \Upsilon^n, f \rangle, \mathbf{x}_{0,n} \otimes \langle \rho, x_{n+1} \rangle x_{n+2} \rangle \\
&= \langle \langle \Upsilon^n, f \rangle, \langle b'_{n+1}, \mathbf{x}_{0,n+2} \rangle \rangle.
\end{aligned}$$

Hence, to finish the proof it suffices to check that $\Upsilon^* \circ \tilde{\delta}^{1,*} = \Psi^* \circ \delta^{1,*}$. But,

$$\begin{aligned}
&\langle \langle \Psi^n, \langle \delta^{1,n}, f \rangle \rangle, \mathbf{x}_{0,n+1} \rangle \\
&= \sum_{i=0}^n (-1)^{i+n} x_0 \langle \langle \delta^{1,n}, f \rangle, \mathbf{x}_{1,i} \otimes \varphi \otimes \langle \rho, x_{i+1} \rangle \otimes \cdots \otimes \langle \rho, x_n \rangle \rangle x_{n+1} \\
&= (-1)^n x_0 \varphi \langle f, \langle \rho, x_1 \rangle \otimes \cdots \otimes \langle \rho, x_n \rangle \rangle x_{n+1} + (-1)^{n+1} x_0 \langle f, \mathbf{x}_{1,n} \rangle \varphi x_{n+1} \\
&= (-1)^n x_0 \langle \rho^{-1}, \langle f, \langle \rho, x_1 \rangle \otimes \cdots \otimes \langle \rho, x_n \rangle \rangle \rangle \varphi x_{n+1} + (-1)^{n+1} x_0 \langle f, \mathbf{x}_{1,n} \rangle \varphi x_{n+1} \\
&= x_0 \langle \langle \tilde{\delta}^{1,n}, f \rangle, \mathbf{x}_{1,n} \rangle \varphi x_{n+1} \\
&= \langle \langle \Upsilon^n, \langle \tilde{\delta}^{1,n}, f \rangle \rangle, \mathbf{x}_{0,n+1} \rangle,
\end{aligned}$$

as desired. \square

Proposition 2.4. *Let $Y^*(A)$ denote the total complex of $Y^{*,*}(A)$. If the Nakayama automorphism ρ has finite order and k has a primitive ord_ρ -th root of unity w , then*

$$H^n(Y^*(A)) = \begin{cases} \text{HH}_0^0(A) & \text{if } n = 0, \\ \text{HH}_0^n(A) \oplus \text{HH}_0^{n-1}(A) & \text{if } n \geq 1. \end{cases}$$

Proof. For each $0 \leq i < \text{ord}_\rho$, let $Y_i^{*,*}(A)$ be the subcomplex of $Y^{*,*}(A)$ defined by

$$Y_i^{u,n} = \bigoplus_{B_{i,n}} \text{Hom}(A_{u_1} \otimes \cdots \otimes A_{u_n}, A_v),$$

where $B_{i,n} = \{(u_1, \dots, u_n, v) \text{ such that } v - u_1 - \cdots - u_n \equiv i \pmod{\text{ord}_\rho}\}$. It is clear that $Y^{*,*}(A) = \bigoplus_{i=0}^{\text{ord}_\rho} Y_i^{*,*}(A)$. Let $f \in Y_i^{0,n}(A)$. A direct computation shows that

$$\langle \langle \tilde{\delta}_{1,n}^{1,n}, f \rangle, \mathbf{x}_{1,n} \rangle = (-1)^{n+1} (1 - w^{-i}) \langle f, \mathbf{x}_{1,n} \rangle.$$

Hence the horizontal boundary maps of $Y_i^{*,*}(A)$ are isomorphisms if $i \neq 0$, and they are zero maps if $i = 0$. So,

$$H^n(Y_i^*(A)) = \begin{cases} 0 & \text{if } i \neq 0, \\ H^0(Y_i^{0,*}(A)) & \text{if } i = 0 \text{ and } n = 0, \\ H^n(Y_i^{0,*}(A)) \oplus H^{n-1}(Y_i^{1,*}(A)) & \text{if } i = 0 \text{ and } n > 0, \end{cases}$$

where $Y_i^*(A)$ is the total complex of $Y_i^{*,*}(A)$. The result follows easily from this fact, since $Y_0^{0,*}(A) = Y_0^{1,*}(A) \simeq (\text{Hom}_k(A^{\otimes *}, A)_{0, b^*})$. \square

Proof of Theorem 1.2. By Proposition 2.3,

$$H^n(Y^*(A)) = H^n(X^*(A)) \quad \text{and} \quad H^n(Y^{u,*}(A)) = H^n(X^{u,*}(A)), \quad \text{for } u = 0, 1.$$

Hence, by Propositions 2.1 and 2.4,

$$\begin{aligned} \text{HH}_0^0(A) &= H^0(Y^*(A)) = H^0(X^*(A)) \\ &= H^0(X^{0,*}(A)) = H^0(Y^{0,*}(A)) = \text{HH}^0(A) \end{aligned}$$

and

$$\begin{aligned} \text{HH}_0^n(A) \oplus \text{HH}_0^{n-1}(A) &= H^n(Y^*(A)) = H^n(X^*(A)) \\ &= H^n(X^{0,*}(A)) \oplus H^{n-1}(X^{1,*}(A)) \\ &= H^n(Y^{0,*}(A)) \oplus H^{n-1}(Y^{1,*}(A)) \\ &= \text{HH}^n(A) \oplus \text{HH}^{n-1}(A), \end{aligned}$$

for all $n \geq 1$. From this it follows easily that $\text{HH}^n(A) = \text{HH}_0^n(A)$, for all $n \geq 0$, as desired. \square

Proof of Theorem 1.3. By [St] or the cohomological version of [L], $\mathbb{Z}/\text{ord}_\rho \mathbb{Z}$ acts on $H^*(A_0, A)$ and there is a converging spectral sequence

$$E_2^{p,q} = H^p(\mathbb{Z}/\text{ord}_\rho \mathbb{Z}, H^q(A_0, A)) \Rightarrow \text{HH}^{p+q}(A).$$

Since k has a primitive ord_ρ -th root of unity, ord_ρ is invertible in k . Hence, the above spectral sequence gives isomorphisms

$$\text{HH}^n(A) = H^n(A_0, A)^{\mathbb{Z}/\text{ord}_\rho \mathbb{Z}} \quad (n \geq 0).$$

These maps are induced by the canonical inclusion of A_0 in A , and the action of $i \in \mathbb{Z}/\text{ord}_\rho \mathbb{Z}$ on $H^n(A_0, A)$ is induced by the map of complexes

$$\theta_i^* : (\text{Hom}_k(A_0^{\otimes *}, A), b^*) \rightarrow (\text{Hom}_k(A_0^{\otimes *}, A), b^*),$$

defined by

$$\begin{aligned} &\theta_i^n(\varphi)(a_1 \otimes \cdots \otimes a_n) \\ &= \sum_{j_1, \dots, j_{n+1} \in J_i} s'_{i,j_1} \varphi(s_{i,j_1} a_1 s'_{i,j_2} \otimes s_{i,j_2} a_2 s'_{i,j_3} \otimes \cdots \otimes s_{i,j_n} a_n s'_{i,j_{n+1}}) s_{i,j_{n+1}}, \end{aligned}$$

where $(s_{i,j})_{j \in J_i}$ and $(s'_{i,j})_{j \in J_i}$ are families of elements of A_i and A_{n-i} respectively, that satisfy $\sum_{j \in J_i} s'_{i,j} s_{i,j} = 1$. From this it follows easily that we have isomorphisms

$$\text{HH}_i^n(A) = H^n(A_0, A_i)^{\mathbb{Z}/\text{ord}_\rho \mathbb{Z}} \quad (n \geq 0, 0 \leq i < \text{ord}_\rho).$$

By combining this result with Theorem 1.2, we obtain the desired result. \square

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