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*Original Citation:*

*Availability:*

This version is available at: 11577/3358096 since: 2021-06-13T18:23:08Z

*Publisher:*

American Institute of Mathematical Sciences

*Published version:*

DOI: 10.3934/dcds.2020371

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1                   **STABILIZABILITY IN OPTIMIZATION PROBLEMS**  
2                   **WITH UNBOUNDED DATA**

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(Communicated by the associate editor name)

ABSTRACT. In this paper we extend the notions of sample and Euler stabilizability to a set of a control system to a wide class of systems with unbounded controls, which includes nonlinear control-polynomial systems. In particular, we allow discontinuous stabilizing feedbacks, which are unbounded approaching the target. As a consequence, sampling trajectories may present a chattering behaviour and Euler solutions have in general an *impulsive character*. We also associate to the control system a cost and provide sufficient conditions, based on the existence of a special Lyapunov function, which allow for the existence of a stabilizing feedback that keeps the cost of all sampling and Euler solutions starting from the same point below the same value, in a uniform way.

3 1. **Introduction.** In the last decades, the problem of the feedback stabilization of  
4 a nonlinear control system  $\dot{x} = f(x, u)$  to a point or, more in general, to a set  $\mathcal{C}$ , has  
5 been the subject of intense research and the theory is now well established. In par-  
6 ticular, it is well know that a continuous stabilizing feedback fails to exist in general,  
7 and a smooth Lyapunov function, which guarantees the asymptotic controllability  
8 of the system, may not exist either. For these reasons, nonsmooth Lyapunov func-  
9 tions, discontinuous feedback laws  $K$ , and a “sample and hold” solution concept for  
10  $\dot{x} = f(x, K(x))$ , similar to that used in differential games [13], have been introduced  
11 (see, e.g. [3, 1, 26, 25, 9, 27, 7, 8, 28, 14]). In particular, semiconcave Lyapunov func-  
12 tions have proven to be a powerful tool for the explicit construction of stabilizing  
13 feedback strategies [23, 24]. (For a broader overview of the topic, see review paper  
14 [10]). A key hypothesis in these results is that the vector field  $f(x, K(x))$  associated

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2010 *Mathematics Subject Classification.* Primary: 93D05, 93D20; Secondary: 49J15, 49N25.

*Key words and phrases.* Asymptotic stabilizability, Discontinuous feedback law, Optimal Control, Exit-time problems, Impulsive control systems.

This research is partially supported by the Padua University grant SID 2018 “Controllability, stabilizability and infimum gaps for control systems”, prot. BIRD 187147, and by the INdAM-GNAMP Project 2020 “Extended control problems: gap, higher order conditions and Lyapunov functions”.

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1 to the stabilizing feedback  $K$  is bounded in a neighborhood of the target. In fact,  
 2 it is usually assumed that the feedback  $K$  itself is bounded close to  $\mathcal{C}$ .

3 One of the main goals of this article is, given a (nondifferentiable) Lyapunov  
 4 function, to construct directly a (discontinuous) stabilizing feedback  $K$  and to in-  
 5 troduce a notion of solution to  $\dot{x} = f(x, K(x))$  in the case of *unbounded* dynamics,  
 6 which include, for instance, nonlinear control-polynomial systems (the precise as-  
 7 sumptions are stated in Subsection 1.2). This necessarily leads to feedbacks which  
 8 may be unbounded approaching the target. For example, in [2] the authors exhibit  
 9 some applications to Lagrangian mechanics, where the system is quadratically de-  
 10 pendent on (the derivatives of) the control and stabilization can only be achieved  
 11 by “vibrating controls”, i. e. allowing unbounded inputs.

12 More generally, since control systems are often associated with costs, we aim to  
 13 build up feedback strategies which, besides stabilizing to a target  $\mathcal{C}$  the system

$$\dot{x} = f(x, u), \quad u(t) \in U, \quad (1)$$

14 also provide an upper bound for an integral cost of the form

$$\int_0^{T_x} l(x(\tau), u(\tau)) d\tau. \quad (2)$$

15 Here the control set  $U$  is a closed, possibly unbounded subset of  $\mathbb{R}^m$ , the target  
 16 set  $\mathcal{C} \subseteq \mathbb{R}^n$  is closed with compact boundary, the Lagrangian  $l$  is  $\geq 0$ .  $T_x \leq +\infty$   
 17 denotes the first exit-time of  $x$  from  $\mathbb{R}^n \setminus \mathcal{C}$ . In this case, when the dynamics are  
 18 unbounded and the cost is “cheap” (i. e., there are no coercivity hypotheses, which  
 19 would make the use of unbounded controls “disadvantageous”) it could be necessary  
 20 to implement an unbounded feedback to stabilize the system and keep the cost finite,  
 21 even if there exists a bounded stabilizing feedback. In this regard, see Example 1,  
 22 Section 2.

23 The generalization of the classical stabilizability theory is therefore twofold: be-  
 24 sides an extension of the concepts of sampling and Euler solutions to unbounded  
 25 dynamics, we introduce a suitable notion of associated cost and of *stabilizability*  
 26 *with regulated cost*. Furthermore, we obtain an explicit construction of stabilizing  
 27 feedbacks with regulated cost based upon the existence of a special Lyapunov func-  
 28 tion, known as a Minimum Restraint function. The original notions of Sampling  
 29 and Euler stabilizability associated to a discontinuous feedback and their relation-  
 30 ship with the existence of a Lyapunov function can be found in [7, 8], where the  
 31 target is zero and the dynamics are assumed to be bounded near the origin. Later,  
 32 these results have been extended to more general targets, but always for  $f$  (and  $K$ )  
 33 bounded close to the target (see e.g. [14] and the references therein). Minimum  
 34 Restraint functions were first introduced in [19], where the existence of a function  
 35 of this type was shown to guarantee global asymptotic controllability to a set, with  
 36 regulated cost (see also [18]). The problem of defining a stabilizing feedback law  
 37 with regulated cost through the use of a Minimum Restraint function has been  
 38 addressed only recently in [16], just in the case of bounded data. The extension  
 39 to unbounded dynamics is not achieved by refining the techniques already used.  
 40 Rather, our strategy is to associate an equivalent, *rescaled*, problem with the start-  
 41 ing problem, under assumptions that include and generalize those most used in the  
 42 study of problems with unbounded data.

1 More in detail, we first assume  $f$  and  $l$  merely continuous on  $(\mathbb{R}^n \setminus \mathcal{C}) \times U$ . Hence,  
 2 sampling trajectories can have a finite blow-up time and chattering phenomena may  
 3 occur (see Subsection 2.1). As a consequence, classical Euler solutions –defined as  
 4 uniform limits of sampling solutions– may not exist. This leads us to propose in  
 5 Section 3 a notion of *weak* Euler solutions and costs, given by the pointwise limit  
 6 of a sequence of suitably truncated sampling trajectories and costs. In support of  
 7 the well-posedness of these definitions, we show that: (i) when they exist, classical  
 8 Euler cost-solution pairs are weak Euler cost-solutions pairs; (ii) the sample stabiliz-  
 9 ability with regulated cost implies the weak Euler stabilizability with regulated cost  
 10 (see Theorem 3.4); (iii) when the system is sample stabilizable with regulated cost  
 11 and the data meet some conditions of weak coercivity –quite usual in optimization  
 12 problems with unbounded controls, see e.g. [20, 22, 21] and the references therein–,  
 13 (stabilizing) weak Euler cost-solution pairs do exist (see Proposition 4, Section 3).

Furthermore, we suppose in addition that there exists some continuous *rescaling*  
*function*  $\nu = \nu(x, u) \geq 0$  such that the *rescaled dynamics and Lagrangian*, given by

$$\bar{f} := f/(1 + \nu), \quad \bar{l} := l/(1 + \nu),$$

14 respectively, are bounded and uniformly continuous on  $(B_R(\mathcal{C}) \setminus \mathcal{C}) \times U$ , for some  
 15  $R > 0$  (see hypotheses **(H.1-2)** below). Under this assumption, in Section 2 we  
 16 can prove the equivalence between the sample stabilizability to the target with reg-  
 17 ulated cost of (1)-(2) and that of the *rescaled problem*, where  $\bar{f}$  and  $\bar{l}$  replace  $f$   
 18 and  $l$ , respectively (see Theorem 2.5). This result is crucial to establish in Section  
 19 3 sufficient conditions for sample (and therefore, weak Euler) stabilizability with  
 20 regulated cost and to build explicit feedback strategies, by means of Minimum Re-  
 21 straint functions for the original or for the rescaled problem (see Theorems 4.2, 4.4).  
 22 Finally, in Theorem 4.6 we show how to implement the previous feedback construc-  
 23 tion starting from a Lipschitz continuous (not necessarily semiconcave) Minimum  
 24 Restraint function, when the data are Lipschitz continuous in the state variable. In  
 25 particular, by choosing  $l \equiv 0$  in (2), Theorems 4.4, 4.6 imply that, given a semicon-  
 26 cave or a Lipschitz continuous Lyapunov function for the unbounded control system  
 27 (1), respectively, we build a (possibly unbounded) stabilizing feedback.

28 The introduction of  $\bar{f}$  and  $\bar{l}$  can be seen as a generalization of well-known com-  
 29 pactification techniques usually exploited to deal with unbounded data. For in-  
 30 stance, if  $\nu := |(f, l)|$ , then  $(\bar{f}, \bar{l})$  coincides with the so-called Erdmann transform of  
 31  $(f, l)$ , used e.g. in [18], while in case  $f$  and  $l$  are functions with a maximal  $u$ -growth  
 32  $\tilde{\nu}(|u|)$ , by choosing  $\nu(x, u) := \tilde{\nu}(|u|)$ , we can recover the extended Lagrangian and  
 33 dynamics considered in impulsive control (see e.g. [22, 20, 12]). The assumptions  
 34 considered in this paper allow for a vast class of dynamics and Lagrangians, in-  
 35 cluding those with a polynomial dependence on  $u_1, \dots, u_m, |u_1|, \dots, |u_m|, |u|$ , and  
 36 compositions of polynomials with exponential and Lipschitz continuous functions.

37 The paper is organized as follows. In the rest of the Introduction we provide  
 38 some preliminary definitions and the precise assumptions. In Section 2 we introduce  
 39 the notion of sample stabilizability with regulated cost and Example 1, and prove  
 40 the equivalence Theorem 2.5. Section 3 is devoted to define weak Euler solutions  
 41 and costs, derive their main properties, and discuss the concept of weak Euler  
 42 stabilizability with regulated cost. Section 4 deals with sufficient conditions for  
 43 sample, Euler and weak Euler stabilizability with regulated cost.

1.1. **Notations and preliminaries.** For every  $r \geq 0$  and  $\Omega \subset \mathbb{R}^n$ , we set  $B_r(\Omega) := \{x \in \mathbb{R}^n \mid d(x, \Omega) \leq r\}$ , where  $d$  is the usual Euclidean distance. When  $\Omega = \{z\}$  for some  $z \in \mathbb{R}^n$ , we also make use of the notation  $B(z, r) := B_r(\{z\})$ . We use  $\overline{\Omega}$  to denote the closure of  $\Omega$ . For  $a, b \in \mathbb{R}$ ,  $a \vee b := \max\{a, b\}$ ,  $a \wedge b := \min\{a, b\}$ . For any  $F : \Omega \rightarrow \mathbb{R}^M$  we call *modulus (of continuity) of  $F$*  any increasing, continuous function  $\omega : [0, +\infty) \rightarrow [0, +\infty)$  such that  $\omega(0) = 0$ ,  $\omega(r) > 0$  for every  $r > 0$  and  $|F(x_1) - F(x_2)| \leq \omega(|x_1 - x_2|)$  for all  $x_1, x_2 \in \Omega$ . We say that a map  $F : I \rightarrow J$ ,  $I, J$  real intervals, is increasing (decreasing) when it is monotone nondecreasing (nonincreasing). We use  $\mathcal{KL}$  to denote the set of all continuous functions  $\beta : [0, +\infty) \times [0, +\infty) \rightarrow [0, +\infty)$  such that: (1)  $\beta(0, t) = 0$  and  $\beta(\cdot, t)$  is strictly increasing and unbounded for each  $t \geq 0$ ; (2)  $\beta(r, \cdot)$  is strictly decreasing for each  $r \geq 0$ ; (3)  $\beta(r, t) \rightarrow 0$  as  $t \rightarrow +\infty$  for each  $r \geq 0$ .

Let us summarize some basic notions in nonsmooth analysis – see e.g. [4, 6, 29] for a thorough treatment. Let  $\Omega \subseteq \mathbb{R}^n$  be a nonempty open set. A continuous function  $F : \overline{\Omega} \rightarrow \mathbb{R}$  is said *positive definite on  $\Omega$*  if  $F(x) > 0 \forall x \in \Omega$  and  $F(x) = 0 \forall x \in \partial\Omega$ . The function  $F$  is called *proper on  $\Omega$*  if the pre-image  $F^{-1}(K)$  of any compact set  $K \subset [0, +\infty)$  is compact.

Let  $F : \Omega \rightarrow \mathbb{R}$  be a locally Lipschitz function. For every  $x \in \Omega$ ,  $\partial_P F(x)$  is defined as the *proximal subdifferential* of  $F$  at  $x$ :  $p \in \partial_P F(x)$  if and only if there exist  $\rho, \eta > 0$  such that

$$F(y) - F(x) + \rho|y - x|^2 \geq \langle p, y - x \rangle \quad \forall y \in B(x, \eta) \ (\subset \Omega).$$

The *limiting subdifferential*  $\partial_L F(x)$  at  $x$ , is given by

$$\partial_L F(x) := \left\{ \lim p_i : p_i \in \partial_P F(x_i), \lim x_i = x \right\}.$$

The proximal subdifferential  $\partial_P F(x)$  may be empty at some point; nevertheless, the set of such points has zero measure. The limiting subdifferential  $\partial_L F(x)$  instead, is nonempty at every point. The Clarke generalized gradient can be derived as  $\text{co } \partial_L F(x)$  at any  $x$ .

We will consider also the *set of reachable gradients* of  $F$  at  $x$ :

$$D^*F(x) := \left\{ w \in \mathbb{R}^n : w = \lim_k \nabla F(x_k), x_k \in \text{DIFF}(F) \setminus \{x\}, \lim_k x_k = x \right\}$$

where  $\nabla$  denotes the classical gradient operator and  $\text{DIFF}(F)$  is the set of differentiability points of  $F$ . The set-valued map  $x \rightsquigarrow D^*F(x)$  is upper semicontinuous on  $\Omega$ , with nonempty, compact values, and  $D^*F(x)$  is in general not convex.

A continuous function  $F : \Omega \rightarrow \mathbb{R}$  is said to be *semiconcave on  $\Omega$*  if there exist  $\rho > 0$  such that

$$F(x) + F(\hat{x}) - 2F\left(\frac{x + \hat{x}}{2}\right) \leq \rho|x - \hat{x}|^2,$$

for all  $x, \hat{x} \in \Omega$  such that  $[x, \hat{x}] \subset \Omega$ . The constant  $\rho$  above is called a *semiconcavity constant* for  $F$  in  $\Omega$ .  $F$  is said to be *locally semiconcave on  $\Omega$*  if it semiconcave on every compact subset of  $\Omega$ . We remind that locally semiconcave functions are locally Lipschitz. Actually, they are twice differentiable almost everywhere.

When  $F$  is a locally semiconcave function, then  $D^*F(x) = \partial_L F(x)$  for any  $x \in \Omega$ .

1.2. **Assumptions.** Through the whole paper,  $U \subseteq \mathbb{R}^m$  and  $\mathcal{C} \subset \mathbb{R}^n$  are closed, nonempty sets and the boundary  $\partial\mathcal{C}$  is compact. Given  $\mathbf{f} : (\mathbb{R}^n \setminus \mathcal{C}) \times U \rightarrow \mathbb{R}^n$  and  $\mathbf{I} : (\mathbb{R}^n \setminus \mathcal{C}) \times U \rightarrow [0, +\infty)$ , we will consider the following sets of hypotheses:

1 **(H.1)** the functions  $\mathbf{f}, \mathbf{l}$  are continuous on  $(\mathbb{R}^n \setminus \mathcal{C}) \times U$ ;

2 **(H.2)** there exists a continuous function  $\nu : (\mathbb{R}^n \setminus \mathcal{C}) \times U \rightarrow [0, +\infty)$ , that we call  
3 a rescaling function, such that the rescaled functions  $\bar{\mathbf{f}}, \bar{\mathbf{l}}$ , defined by

$$(\bar{\mathbf{f}}(x, u), \bar{\mathbf{l}}(x, u)) := \left( \frac{\mathbf{f}(x, u)}{1 + \nu(x, u)}, \frac{\mathbf{l}(x, u)}{1 + \nu(x, u)} \right) \quad \forall (x, u) \in (\mathbb{R}^n \setminus \mathcal{C}) \times U, \quad (3)$$

4 are uniformly continuous on  $\mathcal{K} \times U$  for every compact subset  $\mathcal{K} \subset \mathbb{R}^n \setminus \mathcal{C}$ , and  
5 for any  $R > 0$  there is some  $M(R) > 0$  such that

$$|\bar{\mathbf{f}}(x, u)| \leq M(R), \quad \bar{\mathbf{l}}(x, u) \leq M(R) \quad \forall (x, u) \in (B_R(\mathcal{C}) \setminus \mathcal{C}) \times U. \quad (4)$$

6 In the following, we set  $\mathbf{d}(x) := d(x, \mathcal{C})$ .

7 **2. Sample stabilizability with regulated cost.** In this section we extend the  
8 notion of Sample stabilizability with regulated cost firstly introduced in [16, 17], to  
9 more general, unbounded data. Furthermore, in Theorem 2.5 we show that, if  $f$   
10 and  $l$  satisfy **(H.2)**, the original problem is sample stabilizable with regulated cost  
11 if and only if the *rescaled problem* is.

12 **2.1. Sampling processes.** Let  $\mathbf{f}, \mathbf{l}$  verify **(H.1)**.

13 **Definition 2.1** (Admissible process). We say that a triple  $(x^0, x, u)$  is an *admissible*  
14 *process* (for  $\mathbf{f}, \mathbf{l}$ ) if there exists  $T_x \leq +\infty$  such that: the control  $u$  belongs to  
15  $L_{loc}^\infty([0, T_x], U)$ ;  $x : [0, T_x] \rightarrow \mathbb{R}^n \setminus \mathcal{C}$  is a solution of the control system

$$\dot{x}(t) = \mathbf{f}(x(t), u(t)), \quad \text{a.e. } t \in (0, T_x), \quad (5)$$

16 verifying, if  $T_x < +\infty$ ,  $\lim_{t \rightarrow T_x^-} \mathbf{d}(x(t)) = 0$ ; the cost  $x^0$  is given by

$$x^0(t) := \int_0^t \mathbf{l}(x(\tau), u(\tau)) d\tau, \quad \forall t \in [0, T_x]. \quad (6)$$

17 For every  $z \in \mathbb{R}^n \setminus \mathcal{C}$ , we call  $(x^0, x, u)$  as above an *admissible process from  $z$* , when  
18  $x(0) = z$ .

19 A *partition* (of  $[0, +\infty)$ ) is a sequence  $\pi = (t_k)$  such that  $t_0 = 0$ ,  $t_{k-1} < t_k$   
20  $\forall k \geq 1$ , and  $\lim_{k \rightarrow +\infty} t_k = +\infty$ . The value  $\text{diam}(\pi) := \sup_{k \geq 1} (t_k - t_{k-1})$  will be  
21 called the *diameter* or the *sampling time* of the partition  $\pi$ .

22 We will call *feedback* any locally bounded function  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$ . In particular,  
23 when  $U$  is unbounded we allow feedbacks  $\mathbf{K}$  verifying  $\limsup_{x \rightarrow \bar{x} \in \partial \mathcal{C}} |\mathbf{K}(x)| = +\infty$ .

**Definition 2.2** (Sampling process). Given a locally bounded feedback  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$ , a partition  $\pi = (t_k)$ , and a point  $z \in \mathbb{R}^n \setminus \mathcal{C}$ , we call  $\pi$ -*sampling process* (for  $\mathbf{f}, \mathbf{l}$ ) *from  $z$* , a triple  $(x^0, x, u)$ , where  $x$ , called the *sampling trajectory*, is a continuous function defined by recursively solving

$$\dot{x} = \mathbf{f}(x(t), \mathbf{K}(x(t_{k-1}))) \quad \text{a.e. } t \in [t_{k-1}, t_k], \quad (x(t) \in \mathbb{R}^n \setminus \mathcal{C})$$

from the initial time  $t_{k-1}$  up to time

$$\tau_k := t_{k-1} \vee \sup\{\tau \in [t_{k-1}, t_k] : x \text{ is defined on } [t_{k-1}, \tau)\},$$

24 such that  $x(t_0) = x(0) = z$ . In this case, the trajectory  $x$  is defined on the right-  
25 open interval from time zero up to time  $T^- := \inf\{\tau_k : \tau_k < t_k\}$ . Accordingly, for  
26 every  $k \geq 1$  and for all  $t \in [t_{k-1}, t_k] \cap [0, T^-)$ , the *sampling control* is defined as

$$u(t) := \mathbf{K}(x(t_{k-1})) \quad \forall t \in [t_{k-1}, t_k] \cap [0, T^-), \quad k \geq 1. \quad (7)$$

1 The *sampling cost*,  $x^0$ , is given by

$$x^0(t) := \int_0^t \mathbf{l}(x(\tau), u(\tau)) d\tau \quad t \in [0, T^-]. \quad (8)$$

If  $(x^0, x, u)$  is an admissible process and  $T^- = T_x < +\infty$ , we extend  $x$  to  $[0, +\infty)$  by setting  $x(t) := \bar{z} \quad \forall t \geq T_x$ , where  $\bar{z}$  is a point of the set

$$\omega(x) := \left\{ \lim_{j \rightarrow +\infty} x(t_j) : (t_j) \text{ is increasing and } \lim_{j \rightarrow +\infty} t_j = T_x \right\}.$$

2 If  $\lim_{t \rightarrow T_x^-} x^0(t) < +\infty$ , we also extend  $x^0$ , by setting  $x^0(t) := \lim_{t \rightarrow T_x^-} x^0(t) \quad \forall t \geq T_x$ .

3 By the definition of  $T_x$ , the set  $\omega(x)$  is always not empty, since  $\partial\mathcal{C}$  is compact.  
 4 In general,  $\omega(x)$  is not a singleton, unless  $\mathbf{f}$  is bounded on a neighborhood of  $\mathcal{C}$ ,  
 5 uniformly with respect to the control. Notice that for any admissible  $\pi$ -sampling  
 6 process  $(x^0, x, u)$ , the trajectory  $x$ , possibly extended as above, is always defined on  
 7 the whole interval  $[0, +\infty)$ .

8 **Definition 2.3** (Sample stabilizability with regulated cost). A locally bounded  
 9 feedback  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  is said to sample stabilize the control system  $\dot{x} = \mathbf{f}(x, u)$   
 10 to  $\mathcal{C}$  if there is a function  $\beta \in \mathcal{KL}$ , that we call *descent rate*, satisfying the following:  
 11 for each pair  $0 < r < R$  there exists  $\delta = \delta(r, R) > 0$ , such that, for every partition  
 12  $\pi$  with  $\text{diam}(\pi) \leq \delta$  and for any initial state  $z \in \mathbb{R}^n \setminus \mathcal{C}$  such that  $\mathbf{d}(z) \leq R$ , any  
 13  $\pi$ -sampling process  $(x^0, x, u)$  for  $\mathbf{f}, \mathbf{l}$  with  $x(0) = z$  is admissible and verifies:

$$\mathbf{d}(x(t)) \leq \max\{\beta(\mathbf{d}(z), t), r\} \quad \forall t \in [0, +\infty). \quad (9)$$

14 We call  $\dot{x} = \mathbf{f}(x, u)$  *sample stabilizable (to  $\mathcal{C}$ )* if it admits a feedback  $\mathbf{K}$  as above.

15 If moreover there exist  $p_0 > 0$  and a continuous map  $W : \mathbb{R}^n \setminus \mathcal{C} \rightarrow [0, +\infty)$ ,  
 16 positive definite and proper in  $\mathbb{R}^n \setminus \mathcal{C}$ , such that  $(x^0, x, u)$  also verifies

$$x^0(\bar{T}_x^r) = \int_0^{\bar{T}_x^r} \mathbf{l}(x(\tau), u(\tau)) d\tau \leq \frac{W(z)}{p_0} \quad (10)$$

17 where

$$\bar{T}_x^r := \inf\{t > 0 : \mathbf{d}(x(\tau)) \leq r \quad \forall \tau \geq t\}, \quad (11)$$

18 we say that (5)–(6) is *sample stabilizable (to  $\mathcal{C}$ ) with  $(p_0, W)$ -regulated cost*. To unify  
 19 the notation, when  $\dot{x} = \mathbf{f}(x, u)$  is merely sample stabilizable, we say that (5)–(6) is  
 20 *sample stabilizable with  $(p_0, W)$ -regulated cost for  $p_0 = 0$* .

21 Disregarding the cost, if the dynamics  $\mathbf{f}$  is bounded on  $(B_R(\mathcal{C}) \setminus \mathcal{C}) \times U$  for some  
 22  $R > 0$ , the above notion of sample stabilizability is a slight extension of the original  
 23 one in [7, 15], consisting of the fact that our target is not necessarily a point and  
 24 the feedback  $\mathbf{K}$  can be unbounded on it. However, when both feedback controls  
 25 and resulting dynamics are unbounded in a neighborhood of the target we are far  
 26 beyond the classical theory (see e.g. [8, 14]).

27 **Example 1** (A cheap control problem). This example shows how the presence of a  
 28 cost can drastically change the choice of a stabilizing feedback. In particular, in the  
 29 following simple problem there is a continuous and bounded stabilizing feedback,  
 30 but in order to obtain stabilizability with regulated cost it is necessary to choose  
 31 an unbounded feedback. We consider the scalar control system

$$\dot{x} = f(x, u) := x^2 u, \quad u \in \mathbb{R}, \quad (12)$$

- 1 with target  $\mathcal{C} := \{0\}$  and associated cost

$$\int_0^{T_x} l(x) dt, \quad l(x) := |x|, \quad (13)$$

where  $T_x$  is as in Definition 2.1. The bounded feedback  $\bar{K}(x) := -\text{sign}(x)$  sample stabilizes the system. Indeed, given  $r, R, 0 < r < R$ , for every partition  $\pi$  of  $[0, +\infty)$ , any  $\pi$ -sampling solution  $x$  associated to  $\bar{K}$  with  $x(0) = z \neq 0, |z| \leq R$ , satisfies

$$|x(t)| = \frac{|z|}{|z|t + 1} =: \beta(|z|, t) \quad \forall t \in [0, +\infty),$$

and it is immediate to check that  $\beta$  is a  $\mathcal{KL}$  function. However, by straightforward calculations one has

$$\int_0^{\bar{T}_x^r} |x(t)| dt = \ln \left( 1 + \frac{|z| - r}{r} \right) \implies \lim_{r \rightarrow 0^+} \int_0^{\bar{T}_x^r} |x(t)| dt = +\infty$$

- 2 where  $\bar{T}_x^r$  is as in (11). So  $\bar{K}$  does not sample stabilize the system *with regulated*  
 3 *cost*. By similar arguments, it can be shown that there is no bounded stabilizing  
 4 feedback that gives a regulated cost.

Let us now consider the unbounded feedback  $K(x) := -1/x$ . Given  $r, R, 0 < r < R$ , for every partition  $\pi := (t_i)$  of  $[0, +\infty)$  and any  $\pi$ -sampling trajectory of  $K$  with  $x(0) = z, 0 < |z| \leq R$ , we have

$$x(t) = \frac{x(t_i)}{1 + (t - t_i)} \quad \forall t \in [t_i, t_{i+1}].$$

From this, observing that, fixed  $\varepsilon \in (0, 1)$ , there exists some  $\bar{\delta} > 0$  such that  $e^{\varepsilon\delta} \leq 1 + \delta$  for any  $\delta \in (0, \bar{\delta}]$ , we deduce that,

$$|x(t)| = \frac{|z|}{(1 + t - t_i) \prod_{k=0}^{i-1} (1 + t_{k+1} - t_k)} \leq \frac{|z|}{e^{\varepsilon t}} =: \beta(|z|, t) \quad \forall t \in [t_i, t_{i+1}], \forall i \in \mathbb{N},$$

as soon as  $\text{diam}(\pi) \leq \bar{\delta}$ . Since the right-hand side of the above inequality is a  $\mathcal{KL}$  function of  $|z|$  and  $t$ , then  $K$  is a sample stabilizing feedback to the origin. For the associated cost, we get

$$\int_0^{\bar{T}_x^r} |x(t)| dt \leq \int_0^{+\infty} |x(t)| dt \leq \int_0^{+\infty} |z| e^{-\varepsilon t} dt = \frac{|z|}{\varepsilon}.$$

- 5 Therefore, the feedback  $K(x) = -1/x$  stabilizes (12)-(13) to the origin with  $(\varepsilon, W)$ -  
 6 regulated cost, where  $W(x) := |x|$  for all  $x \in \mathbb{R}$ .

Incidentally,  $W(z) = |z|$  does not coincide with the *value function* of the problem, defined for any  $z \neq 0$  as

$$V(z) := \inf_{\{(x,u) \text{ admissible}, x(0)=z\}} \int_0^{T_x} l(x) dt,$$

which is identically zero, as it is not difficult to show. However, for any  $n \in \mathbb{N}, n \geq 1$ , there exists a locally bounded sample stabilizing feedback  $K_n : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}$  with  $(1, W_n)$ -regulated cost, such that  $\lim_{n \rightarrow +\infty} W_n(z) = 0$ , for each  $z$ . This fact could be easily proved directly, by constructing a suitable sequence of feedbacks with this property. On the other hand, it also follows from Theorem 4.4 (see also Theorem 4.2) below, observing that, for each  $n \geq 1$ , the function

$$W_n(z) := n^{-1}(|z| \wedge |z|^n) \quad \forall z \in \mathbb{R},$$

- 7 is a 1-MRF (see Definition 4.1 below) for  $f, l$  as above.



1 **2.2. Sample stabilizability with regulated cost: a rescaled problem.** Con-  
 2 sider the original data  $f$  and  $l$  verifying hypotheses **(H.1-2)** for some rescaling  
 3 function  $\nu$  and let  $\bar{f}, \bar{l}$  be the associated rescaled functions.

4 In the following, we denote by  $(x^0, x, u)$  any admissible process for  $f, l$  (see  
 5 Definition 2.1). Precisely,  $u \in L_{loc}^\infty([0, T_x], U)$ ,  $x : [0, T_x) \rightarrow \mathbb{R}^n \setminus \mathcal{C}$  solves the  
 6 control system

$$\dot{x}(t) = f(x(t), u(t)), \quad \text{a.e. } t \in (0, T_x), \quad (14)$$

7 and verifies  $\lim_{t \rightarrow T_x^-} \mathbf{d}(x(t)) = 0$  as soon as  $T_x < +\infty$ , while the cost  $x^0$  is given by  
 8

$$x^0(t) = \int_0^t l(x(\tau), u(\tau)) d\tau \quad \forall t \in [0, T_x]. \quad (15)$$

9 Moreover, we introduce the *rescaled control system*

$$y'(s) = \bar{f}(y(s), v(s)), \quad \text{a.e. } s \in (0, S_y), \quad (16)$$

10 and the *rescaled cost*

$$y^0(s) = \int_0^s \bar{l}(y(\sigma), v(\sigma)) d\sigma, \quad \forall s \in [0, S_y], \quad (17)$$

11 whose admissible processes will be denoted by  $(y^0, y, v)$ , with domain  $[0, S_y)$ . In par-  
 12 ticular, we will call  $(y^0, y, v)$  an (admissible) *rescaled process*,  $y$  a *rescaled trajectory*,  
 13 and  $v$  a *rescaled control*.

14 In (16) we use the apex “ $'$ ” to denote differentiation with respect to the new  
 15 parameter  $s$ , in order to stress that it does not coincide, in general, with the time  
 16 variable  $t$ , of (14). Indeed, any rescaled process is composition of a process of the  
 17 original problem with a suitable time-scale and vice-versa, as stated in the following  
 18 lemma. Since every  $L^1$  equivalence class contains Borel measurable representatives,  
 19 from now on we assume without loss of generality that  $u$  and  $v$  are Borel measurable.

20 **Lemma 2.4.** *Fix  $z \in \mathbb{R}^n \setminus \mathcal{C}$ .*

(i) *Given an admissible process  $(x^0, x, u)$  from  $z$ , set*

$$s(t) := \int_0^t (1 + \nu(x(\tau), u(\tau))) d\tau \quad \forall t \in [0, T_x), \quad S_y := \lim_{t \rightarrow T_x^-} s(t), \quad t(\cdot) := s^{-1}(\cdot).$$

21 *Then  $(y_0, y, v)(s) := (x^0, x, u) \circ t(s)$ ,  $s \in [0, S_y)$ , is an admissible rescaled process  
 22 from  $z$ .*

(ii) *Vice-versa, let  $(y^0, y, v)$  be an admissible rescaled process from  $z$  and set*

$$t(s) := \int_0^s (1 + \nu(y(\sigma), v(\sigma)))^{-1} d\sigma \quad \forall s \in [0, S_y), \quad T_x := \lim_{s \rightarrow S_y^-} t(s), \quad s(\cdot) := t^{-1}(\cdot).$$

23 *Then  $(x^0, x, u)(t) := (y_0, y, v) \circ s(t)$ ,  $t \in [0, T_x)$ , is an admissible process from  $z$ .*

24 *Proof.* Claims (i), (ii) can be derived by a standard application of the chain rule,  
 25 once observed that the inverse of an absolutely continuous real map with derivative  
 26  $> 0$  almost everywhere, is absolutely continuous (see e.g. [11, Theorem 2.10.13]).  
 27  $\square$

28 In Theorem 2.5 below we establish the equivalence between the sample stabiliz-  
 29 ability with regulated cost of the original problem and that of the rescaled problem.

1 **Theorem 2.5.** *Assume that  $f, l$  satisfy (H.1-2). Then a locally bounded feedback*  
 2  *$K : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  sample stabilizes the original problem (14)–(15) to  $\mathcal{C}$  with  $(p_0, W)$ -*  
 3 *regulated cost for some  $p_0 \geq 0$ , if and only if it sample stabilizes the rescaled problem*  
 4 *(16)–(17) to  $\mathcal{C}$  with  $(p_0, W)$ -regulated cost.*

5 *Proof.* Let  $K$  be a sample stabilizing feedback with  $(p_0, W)$ -regulated cost for the  
 6 rescaled problem. Then there exists a function  $\beta \in \mathcal{KL}$  such that for any  $r, R > 0$ ,  
 7  $r < R$ , there is some  $\tilde{\delta} = \tilde{\delta}(r, R) > 0$  such that for every  $z \in \mathbb{R}^n \setminus \mathcal{C}$  with  $\mathbf{d}(z) \leq R$   
 8 and every partition  $\tilde{\pi} = (s_k)$  of  $\text{diam}(\tilde{\pi}) \leq \tilde{\delta}$ , any  $\tilde{\pi}$ -sampling rescaled process  
 9  $(y^0, y, v)$ , with initial datum  $z$ , is admissible and verifies

$$\begin{aligned} \mathbf{d}(y(s)) &\leq \max\{\beta(\mathbf{d}(z), s), r\} \quad \forall s \geq 0, \\ y^0(s) &\leq \frac{W(z)}{p_0} \quad \forall s \in [0, \bar{S}_y^r] \quad (\text{if } p_0 > 0), \end{aligned} \quad (18)$$

10 where  $\bar{S}_y^r = \inf\{s \geq 0 : \mathbf{d}(y(\sigma)) \leq r \ \forall \sigma \geq s\}$ . Let  $r' = r'(r) > 0$  verify the relation

$$\beta(2r', 0) = r, \quad (19)$$

11 and let us define  $\tilde{N}(r, R) \geq 0$  and  $\tilde{M}(r, R) \geq 1$ , as

$$\begin{aligned} \tilde{N}(r, R) &:= \sup\{\nu(x, K(x)) : r'(r) \leq \mathbf{d}(x) \leq \beta(R, 0) + 2R\}, \\ \tilde{M}(r, R) &:= \sup\{|(f, l)(x, K(x))| : r'(r) \leq \mathbf{d}(x) \leq \beta(R, 0) + 2R\} \vee 1. \end{aligned} \quad (20)$$

12 Notice that, by the very definition of  $\beta$ ,  $\beta(2r', 0) \geq 2r'$ , so that  $r' \leq \frac{r}{2}$ . Hence  
 13  $\tilde{N}(r, R)$  and  $\tilde{M}(r, R)$  are well-defined. Clearly, for every fixed  $R > 0$ , on  $(0, R)$   
 14 we can suppose  $r \mapsto r'(r)$  strictly increasing and continuous, so that  $r \mapsto \tilde{N}(r, R)$ ,  
 15  $\tilde{M}(r, R)$ , are locally bounded and decreasing (possibly diverging to  $+\infty$  as  $r$  tends  
 16 to  $0^+$ ). Hence they can be dominated by some  $r$ -continuous and strictly decreasing  
 17 maps  $N(r, R) \geq \tilde{N}(r, R)$ ,  $M(r, R) \geq \tilde{M}(r, R)$ . We set

$$\delta = \delta(r, R) := \frac{\tilde{\delta}(r, R)}{1 + N(r, R)} \wedge \frac{1}{M(r, R)}. \quad (21)$$

For every  $z \in \mathbb{R}^n \setminus \mathcal{C}$  with  $\mathbf{d}(z) \leq R$  and every partition  $\pi = (t_k)$  of  $[0, +\infty)$  with  
 $\text{diam}(\pi) \leq \delta$ , let us consider an arbitrary  $\pi$ -sampling process  $(x^0, x, u)$  from  $z$  for  
 the original problem (14)–(15). Let  $[0, T^-)$  be the maximal definition interval of  $x$   
 and let us define

$$\hat{T} = \hat{T}_x(r, R) := \sup\{t \in [0, T^-) : r'(r) \leq \mathbf{d}(x(t)) \leq \beta(R, 0) + 2R\}.$$

18 Since  $\mathbf{d}(x(0)) = \mathbf{d}(z) \leq R$  and  $\mathbf{d}$  is positive definite and proper on  $\mathbb{R}^n \setminus \mathcal{C}$ , one has  
 19  $0 < \hat{T} < T^-$ . Set

$$s(t) := \int_0^t (1 + \nu(x(\tau), u(\tau))) d\tau \quad \forall t \in [0, T^-), \quad \hat{S} := s(\hat{T}), \quad S^- := \lim_{t \rightarrow T^-} s(t), \quad (22)$$

20 and let  $t = t(s)$  be the inverse map of  $s : [0, T^-) \rightarrow [0, S^-)$ . By Lemma 2.4, the  
 21 process  $(y^0, y, v) := (x^0, x, u) \circ t$  is (the restriction to  $[0, \hat{S}]$  of) a  $\tilde{\pi}$ -sampling rescaled  
 22 process with  $y(0) = z$  and  $\text{diam}(\tilde{\pi}) \leq \tilde{\delta}$ . Indeed, setting  $\hat{n} := \sup\{i \in \mathbb{N} : t_i \leq \hat{T}\}$ ,  
 23  $s_k := s(t_k) \ \forall k = 0, \dots, \hat{n}$ , and  $s_k = s_{k-1} + \tilde{\delta}$  for all  $k > \hat{n}$ , then for every  $k = 1, \dots, \hat{n}$ ,  
 24 one has

$$s_k - s_{k-1} = \int_{t_{k-1}}^{t_k} (1 + \nu(x(\tau), u(\tau))) d\tau \leq (1 + N(r, R))\delta(r, R) \leq \tilde{\delta}(r, R). \quad (23)$$

- 1 Therefore, any  $\tilde{\pi}$ -sampling extension of  $(y, v)$  (associated to  $K$ ) to  $[0, +\infty)$  satisfies  
 2 (18), so that, for all  $t \in [0, \hat{T}]$ , we get

$$\begin{aligned} \mathbf{d}(x(t)) &= \mathbf{d}(y(s(t))) \leq \max\{\beta(\mathbf{d}(z), s(t)), r\} \leq \max\{\beta(\mathbf{d}(z), t), r\}, \\ x^0(t) &= y^0(s(t)) \leq \frac{W(z)}{p_0} \quad (\text{if } p_0 > 0), \end{aligned} \quad (24)$$

- 3 where the last inequality in the first expression holds true since  $s(t) \geq t$  and the  
 4 map  $t \mapsto \beta(\mathbf{d}(z), t)$  is decreasing. It remains only to show that  $x$  is a sampling  
 5 trajectory extendable to the whole interval  $[0, +\infty)$  as described in Definition 2.2  
 6 and such that

$$\mathbf{d}(x(t)) \leq r \quad \forall t \geq \hat{T}. \quad (25)$$

- 7 In fact, proven (25), we get (9) by the first relation in (24). In addition, (25) also  
 8 implies that  $\bar{T}_x^r = \inf\{t \geq 0 : \mathbf{d}(x(\tau)) \leq r \forall \tau \geq t\} \leq \hat{T}$  and this, together with the  
 9 last relation in (24), yields the cost estimate (10). By the arbitrariness of  $(x^0, x, u)$ ,  
 10 this concludes the proof.

So, let us check (25). By the definition of  $\hat{T}$  and by (24), one has  $\mathbf{d}(x(\hat{T})) = r'$   
 and  $T^- > \hat{T}$ , where either  $T^- = T_x$  or  $\lim_{t \rightarrow T^-} |x(t)| = +\infty$ . Suppose first that  
 $\mathbf{d}(x(t)) \leq r$  for all  $t \in [0, T^-)$ . Then either  $T^- = +\infty$  and (25) holds true, or  
 $T^- = T_x$ . In this case,  $x$  can be extended to  $[0, +\infty)$  in such a way that  $\mathbf{d}(x(t)) = 0$   
 for all  $t \geq T_x$  as in Definition 2.2 and (25) is proven. Suppose now, by contradiction,  
 that  $\mathbf{d}(x(t)) > r$  for some  $t \in (0, T^-)$ . By the continuity of  $x$  and  $\mathbf{d}$ , there exist  
 some  $\hat{t}^0, \hat{t}^1$  such that  $\hat{T} < \hat{t}^0 < \hat{t}^1 < T^-$ , and, for every  $t \in [\hat{t}^0, \hat{t}^1]$ ,

$$r' < \mathbf{d}(x(\hat{t}^0)) = 2r' \leq r < \mathbf{d}(x(\hat{t}^1)) < R, \quad \mathbf{d}(x(\hat{t}^0)) \leq \mathbf{d}(x(t)) \leq \mathbf{d}(x(\hat{t}^1)).$$

Hence by (22), for all  $\underline{t}, \bar{t} \in [\hat{t}^0, \hat{t}^1]$ ,  $\underline{t} < \bar{t}$ , such that  $\bar{t} - \underline{t} \leq \delta$ , one has

$$s(\bar{t}) - s(\underline{t}) \leq (1 + N(r, R))(\bar{t} - \underline{t}) \leq \tilde{\delta}.$$

So, setting  $\hat{s}^0 := s(\hat{t}^0)$ ,  $\hat{s}^1 := s(\hat{t}^1)$ , we can see the process

$$(y^0, y, v)(s) := (x^0, x, u)(t(s + \hat{s}^0)) \quad \text{for } s \in [0, \hat{s}^1 - \hat{s}^0],$$

as the restriction of a  $\tilde{\pi}$ -sampling rescaled process with  $y(0) = x(\hat{t}^0)$  and  $\text{diam}(\tilde{\pi}) \leq \tilde{\delta}$ .  
 But then (18) yields

$$\mathbf{d}(x(t)) = \mathbf{d}(y(s(t) - \hat{s}^0)) \leq \beta(\mathbf{d}(y(0)), 0) = \beta(2r', 0) = r \quad \forall t \in [\hat{t}^0, \hat{t}^1],$$

- 11 which contradicts the hypothesis that  $\mathbf{d}(x(\hat{t}^1)) > r$ .

Suppose now that  $K$  is a sample stabilizing feedback to  $\mathcal{C}$  with  $(p_0, W)$ -regulated  
 cost for the original problem. Let  $\beta, \delta = \delta(r, R)$  for any pair  $r, R$  with  $0 < r < R$  be  
 as in Definition 2.3, so that every  $\pi$ -sampling process  $(x^0, x, u)$  with initial datum  
 $z \in \mathbb{R}^n \setminus \mathcal{C}$ ,  $\mathbf{d}(z) \leq R$ , and  $\text{diam}(\pi) \leq \delta$ , verifies

$$\mathbf{d}(x(t)) \leq \max\{\beta(\mathbf{d}(z), t), r\} \quad \forall t \geq 0, \quad x^0(t) \leq \frac{W(z)}{p_0} \quad \forall t \in [0, \bar{T}_x^r] \quad (\text{if } p_0 > 0),$$

- 12 where  $\bar{T}_x^r = \inf\{t \geq 0 : \mathbf{d}(x(\tau)) \leq r \forall \tau \geq t\}$ . Let  $r' > 0$ ,  $N(r, R)$ , and  $M(r, R)$  be  
 13 defined as in (19), (20).

For any  $\tilde{\pi}$ -sampling rescaled process  $(y^0, y, v)$  with  $y(0) = z$  and  $\text{diam}(\tilde{\pi}) \leq \delta$ ,  
 consider the time-scaling

$$t(s) := \int_0^s (1 + \nu(y(\sigma), v(\sigma)))^{-1} d\sigma \quad \forall s \in [0, S^-), \quad \hat{T} := t(\hat{S}), \quad s(\cdot) := t^{-1}(\cdot),$$

where  $[0, S^-)$  is the maximal definition interval of  $y$  and

$$\hat{S} = \hat{S}_y(r, R) := \sup\{s \in [0, S^-) : r'(r) \leq \mathbf{d}(y(s)) \leq \beta(R, 0) + 2R\}.$$

- 1 Then  $t(s) \geq \frac{s}{1+N(r, R)}$  for all  $s \in [0, \hat{S}]$  and, arguing as in the previous step, one can  
 2 easily conclude that  $y$  is a sampling trajectory of the rescaled system extendable to  
 3  $[0, +\infty)$ , and that  $(y^0, y, v)$  verifies

$$\begin{aligned} \mathbf{d}(y(s)) &\leq \max\{\beta(\mathbf{d}(z), t(s)), r\} \leq \max\left\{\beta\left(\mathbf{d}(z), \frac{s}{1+N(r, R)}\right), r\right\}, \\ y^0(s) &\leq \frac{W(z)}{p_0} \quad \forall s \in [0, \bar{S}_y^r] \quad (\text{if } p_0 > 0), \end{aligned} \quad (26)$$

where  $\bar{S}_y^r = \inf\{s \geq 0 : \mathbf{d}(y(\sigma)) \leq r \ \forall \sigma \geq s\}$ . Let  $S(r, R) > 0$  be the value of  $s$  implicitly defined by the equation

$$\beta\left(R, \frac{s}{1+N(r, R)}\right) = r.$$

By the monotonicity and continuity properties of  $\beta$  and  $N$ , it follows that  $S(\cdot, \cdot)$  is a continuous function on  $\{(r, R) : 0 < r < R\}$ , such that  $r \mapsto S(r, R)$  is strictly decreasing and  $R \mapsto S(r, R)$  is strictly increasing. As a consequence, if we denote by  $\rho = \rho(R, s)$  the inverse of the map  $\rho \mapsto S(\rho, R)$ , it is easy to see that  $\rho$  is a  $\mathcal{KL}$  function. At this point, observe that, for every  $\rho \in [r, R)$ , since  $\text{diam}(\tilde{\pi}) \leq \delta(r, R) \leq \delta(\rho, R)$ , the first relation in (26) implies

$$\mathbf{d}(y(s)) \leq \beta\left(R, \frac{s}{1+N(\rho, R)}\right) \quad \forall s \in [0, S(\rho, R)], \quad \mathbf{d}(y(s)) \leq r \quad \forall s > S(r, R),$$

which, substituting  $\rho = \rho(R, s)$ , yields

$$\begin{aligned} \mathbf{d}(y(s)) &\leq \beta\left(R, \frac{s}{1+N(\rho(R, s), R)}\right) = \rho(R, s) \quad \forall s \leq S(r, R), \\ \mathbf{d}(y(s)) &\leq r \quad \forall s > S(r, R). \end{aligned}$$

Since  $\mathbf{d}(z) \leq R$  implies that  $\delta(r, \mathbf{d}(z)) \geq \delta(r, R)$ , we finally obtain that, for every  $\tilde{\pi}$ -process with  $\text{diam}(\tilde{\pi}) \leq \delta(r, R)$ , one has

$$\mathbf{d}(y(s)) \leq \rho(\mathbf{d}(z), s) \quad \forall s \leq S(r, \mathbf{d}(z)), \quad \mathbf{d}(y(s)) \leq r \quad \forall s > S(r, \mathbf{d}(z)),$$

which is trivially equivalent to

$$\mathbf{d}(y(s)) \leq \max\{\rho(\mathbf{d}(z), s), r\} \quad \forall s \geq 0.$$

- 4 Together with the second relation in (26), this concludes the proof of the sample  
 5 stabilizability to  $\mathcal{C}$  with  $(p_0, W)$ -regulated cost of the rescaled problem.  $\square$

- 6 **Remark 1.** In view of the above proof, when there is stabilizability, a descent rate  
 7  $\beta$  for the rescaled problem is a descent rate also for the original problem. Instead,  
 8 given a descent rate  $\beta$  for the original problem, a descent rate  $\rho$  for the rescaled  
 9 problem is in general larger. In this case, we get an explicit construction of  $\rho$ .

1 **3. Weak Euler stabilizability with regulated cost.** In our previous results,  
 2 the controllers are taken to be discontinuous feedbacks, so, in principle, the dynam-  
 3 ics is discontinuous in the state variable. However, these results are stated in terms  
 4 of sampling trajectories, which are classical solutions corresponding to piecewise  
 5 constant controls. Therefore, the issue of defining a solution concept for discontin-  
 6 uous differential equations has so far been neglected. In this section we address this  
 7 question and define Euler solutions, weak Euler solutions, and the associated costs.  
 8 Furthermore, we introduce the notions of Euler and of weak Euler stabilizability  
 9 with regulated cost, and prove that the sample stabilizability with regulated cost  
 10 implies both of them.

11 **3.1. Weak Euler solutions and costs.** Let the data  $\mathbf{f}, \mathbf{l}$  verify assumption **(H.1)**.

12 Following [16], we define Euler trajectories and costs as locally uniform limits of  
 13 sampling trajectories and costs.

**Definition 3.1** (Euler trajectory and cost). Given a locally bounded feedback  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$ , fix  $z \in \mathbb{R}^n \setminus \mathcal{C}$  and let  $(\pi_i)$  be a sequence of partitions such that  $\delta_i := \text{diam}(\pi_i) \rightarrow 0$  as  $i \rightarrow \infty$ . For every  $i$ , let  $(x_i^0, x_i, u_i)$  be an admissible  $\pi_i$ -sampling process for the data  $\mathbf{f}, \mathbf{l}$  with initial condition  $x_i(0) = z$ , such that  $x_i$  is defined on  $[0, +\infty)$ . If there exists a map  $\mathcal{X} : [0, +\infty) \rightarrow \mathbb{R}^n$  verifying

$$x_i \rightarrow \mathcal{X} \quad \text{locally uniformly in } [0, +\infty), \quad (27)$$

we call  $\mathcal{X}$  an *Euler trajectory from  $z$*  of (5). If moreover, every  $x_i^0$  is defined on  $[0, +\infty)$  and there is a map  $\mathcal{X}^0 : [0, +\infty) \rightarrow [0, +\infty)$  verifying

$$x_i^0 \rightarrow \mathcal{X}^0 \quad \text{locally uniformly in } [0, +\infty), \quad (28)$$

14 we call  $\mathcal{X}^0$  an *Euler cost from  $z$*  associated to  $\mathcal{X}$ .

The above notion of solution is well suited for situations where the discontinuous dynamics associated to the feedback are bounded around the target, while it seems too strong in the general case. Indeed, suppose that (5)–(6) is sample stabilizable to  $\mathcal{C}$  with  $(p_0, W)$ -regulated cost for some  $p_0 > 0$ . Then, fixed  $z$ , any sequence of  $\pi_i$ -sampling cost-trajectory pairs  $(x_i^0, x_i)$  with  $\text{diam}(\pi_i) \rightarrow 0$  is equibounded. If for any  $R > 0$  there is some  $M(R) > 0$  such that

$$|\mathbf{f}(x, K(x))| \leq M(R), \quad \mathbf{l}(x, K(x)) \leq M(R) \quad \forall x \in B_R(\mathcal{C}) \setminus \mathcal{C},$$

15 the sequence  $(x_i^0, x_i)$  is also equi-Lipschitz continuous. Therefore, passing eventually  
 16 to a subsequence, it converges locally uniformly by Ascoli-Arzelá Theorem and the  
 17 existence of an Euler solution to (5) and of an associated Euler cost is guaranteed.

18 When instead the data are truly unbounded, sampling trajectories may approach  
 19 the target faster and faster and even converge to discontinuous functions. Hence  
 20 Euler solutions defined as locally uniform limits of sampling solutions as above,  
 21 may not exist. An analogous remark holds for the associated Euler costs. These  
 22 considerations lead us to consider the following notions of *weak Euler solution* and  
 23 *weak Euler cost*, inspired by the impulsive control theory (see also [17]), for which  
 24 we are able to provide existence under weak coercivity conditions which are satisfied  
 25 in several applications.

**Definition 3.2** (Weak Euler trajectory and cost). Let  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  be a locally bounded feedback, fix  $z \in \mathbb{R}^n \setminus \mathcal{C}$ , and let  $(\pi_i)$  be a sequence of partitions such

that  $\delta_i := \text{diam}(\pi_i) \rightarrow 0$  as  $i \rightarrow \infty$ . For every  $i$ , let  $(x_i^0, x_i, u_i)$  be an admissible  $\pi_i$ -sampling process for  $\mathbf{f}, \mathbf{l}$  with  $x_i(0) = z$ . When there exists a map  $\mathcal{X} : [0, +\infty) \rightarrow \mathbb{R}^n$ , verifying, for some sequence  $(r_i) \subset (0, \mathbf{d}(z))$  converging to 0:

$$\tilde{x}_i \rightarrow \mathcal{X} \quad \text{pointwise in } [0, +\infty) \quad (29)$$

1 where, for each  $i$ ,

$$\begin{aligned} T_i &:= \bar{T}_{x_i}^{r_i} = \inf\{t > 0 : \mathbf{d}(x_i(\tau)) \leq r_i \quad \forall \tau \geq t\} \leq +\infty, \\ \tilde{x}_i(t) &:= x_i(t \wedge T_i) \quad \forall t \geq 0, \end{aligned} \quad (30)$$

2 we call  $\mathcal{X}$  a *weak Euler trajectory from  $z$*  of (5). For each  $i$ , let us set

$$\tilde{x}_i^0(t) := x_i^0(t \wedge T_i) \quad \forall t \geq 0. \quad (31)$$

When it exists, we call *weak Euler cost* associated to  $\mathcal{X}$  a map  $\mathcal{X}^0 : [0, +\infty) \rightarrow [0, +\infty)$ , verifying

$$\tilde{x}_i^0 \rightarrow \mathcal{X}^0 \quad \text{pointwise in } [0, +\infty). \quad (32)$$

3 In short, we will say that  $(\mathcal{X}^0, \mathcal{X})$  a *weak Euler cost-trajectory pair from  $z$* .

4 Clearly, Euler solutions and costs are also weak Euler solutions and costs, re-  
5 spectively. Some relevant properties of weak Euler solutions and costs are stated in  
6 Propositions 1, 2, and 3 below.

7 Given a weak Euler solution  $\mathcal{X}$ , let us define the exit-time  $T_{\mathcal{X}} \leq +\infty$  as

$$T_{\mathcal{X}} := \inf\{t > 0 : \mathcal{X}([0, t]) \subset \mathbb{R}^n \setminus \mathcal{C}, \quad \lim_{\tau \rightarrow t^-} \mathbf{d}(\mathcal{X}(\tau)) = 0\} \leq +\infty, \quad (33)$$

8 ( $T_{\mathcal{X}} := +\infty$  if the set is empty). Notice that the function  $\mathcal{X}$  is in general discontin-  
9 uous and it may happen that  $\lim_{t \rightarrow +\infty} \mathbf{d}(\mathcal{X}(t)) \neq 0$  or  $T_{\mathcal{X}} < +\infty$  and  $\mathbf{d}(\mathcal{X}(T + \varepsilon)) = 0$   
10 for some  $\varepsilon > 0$ , but  $\lim_{t \rightarrow T_{\mathcal{X}}^-} \mathbf{d}(\mathcal{X}(t)) \neq 0$ , despite each sampling process  $(x_i^0, x_i, u_i)$   
11 in the definition of  $\mathcal{X}$  is admissible and verifies  $\lim_{t \rightarrow +\infty} \mathbf{d}(x_i(t)) = 0$ .

12 Next result provides a uniform lower bound for the exit time  $T_{\mathcal{X}}$ . In particular,  
13 the local boundedness of the feedbacks prevents the existence of purely impulsive  
14 weak Euler trajectories, that jump from the initial state to the target in zero time.

15 **Proposition 1.** *Given a locally bounded feedback  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$ , let  $x$  be an*  
16 *admissible  $\pi$ -sampling trajectory of  $\dot{x} = \mathbf{f}(x, u)$  from  $z \in \mathbb{R}^n \setminus \mathcal{C}$  associated to  $\mathbf{K}$ .*  
17 *Then for each  $\varepsilon \in (0, \mathbf{d}(z))$ , one has*

$$0 < T^\varepsilon := \frac{\mathbf{d}(z) - \varepsilon}{M(\varepsilon, \mathbf{d}(z))} \leq T_x^\varepsilon \quad (34)$$

18 where

$$T_x^\varepsilon := \inf\{t > 0 : \mathbf{d}(x(t)) \leq \varepsilon\} \quad (35)$$

19 and  $M$  is the function mapping the pairs  $(\varepsilon, R)$  with  $0 < \varepsilon < R$  to

$$M(\varepsilon, R) := \sup\{|\mathbf{f}(x, u)| \mid \varepsilon \leq \mathbf{d}(x) \leq R, \quad u \in \mathbf{K}(x)\}. \quad (36)$$

20 Moreover, if  $\mathcal{X}$  is a weak Euler solution from  $z$ , then

$$0 < T^\varepsilon \leq T_{\mathcal{X}}. \quad (37)$$

---

<sup>1</sup> When  $T_x \leq +\infty$  and  $\lim_{t \rightarrow T_x^-} \mathbf{d}(x(t)) = 0$ ,  $T_i$  is obviously finite. It may happen that  $T_i = +\infty$  only in case  $T_x = +\infty$  and  $\lim_{t \rightarrow +\infty} \mathbf{d}(x(t)) \neq 0$ . Obviously,  $t \wedge (+\infty) = t$ .

1 *Proof.* Fix  $z \in \mathbb{R}^n \setminus \mathcal{C}$  and let  $x$  be an admissible  $\pi$ -sampling trajectory of  $\dot{x} = \mathbf{f}(x, u)$   
 2 associated to the feedback  $\mathbf{K}$  and verifying  $x(0) = z$ . Given  $\varepsilon \in (0, \mathbf{d}(z))$ , let  
 3  $T_x^\varepsilon := \inf\{t \geq 0 : \mathbf{d}(x(t)) \leq \varepsilon\}$ .

When  $T_x^\varepsilon = +\infty$ , the lower bound (34) is trivially verified. Let  $T_x^\varepsilon$  be finite. Then  
 $\mathbf{d}(x(T_x^\varepsilon)) = \varepsilon$  and there is some  $z^\varepsilon \in \partial\mathcal{C}$  such that  $\varepsilon = \mathbf{d}(x(T_x^\varepsilon)) = |x(T_x^\varepsilon) - z^\varepsilon|$ .  
 Moreover, by continuity, there exists some time  $t_x^\varepsilon \in [0, T_x^\varepsilon]$ , such that

$$\mathbf{d}(z) = \mathbf{d}(x(t_x^\varepsilon)) \geq \mathbf{d}(x(t)) \geq \varepsilon \quad \forall t \in [t_x^\varepsilon, T_x^\varepsilon].$$

Hence  $T_x^\varepsilon$  verifies (34), since

$$\begin{aligned} \mathbf{d}(z) = \mathbf{d}(x(t_x^\varepsilon)) &\leq |x(t_x^\varepsilon) - z^\varepsilon| \leq |x(t_x^\varepsilon) - x(T_x^\varepsilon)| + |x(T_x^\varepsilon) - z^\varepsilon| \\ &\leq M(\varepsilon, \mathbf{d}(z)) (T_x^\varepsilon - t_x^\varepsilon) + \varepsilon \leq M(\varepsilon, \mathbf{d}(z)) T_x^\varepsilon + \varepsilon, \end{aligned}$$

4 where  $M(\varepsilon, \mathbf{d}(z))$  is as in (36). Incidentally,  $M(\varepsilon, \mathbf{d}(z)) < +\infty$  by the properties of  
 5  $\mathbf{f}$ ,  $\mathbf{d}$  and  $\mathbf{K}$ .

Let  $\mathcal{X}$  be a weak Euler solution for  $\mathbf{K}$  with initial condition  $z \in \mathbb{R}^n \setminus \mathcal{C}$ , determined  
 by a sequence  $(x_i^0, x_i, u_i)$  of admissible  $\pi_i$ -sampling processes from  $z$  with  $\text{diam}(\pi_i) =$   
 $\delta_i$  and by  $(r_i)$ , as in Definition 3.2. In particular,  $\mathcal{X}$  is the pointwise limit of  $(\tilde{x}_i)$ ,  
 where  $\tilde{x}_i(t) = x_i(t)$  for any  $t \leq \bar{T}_{x_i}^{r_i} := \inf\{t \geq 0 : \mathbf{d}(x_i(\tau)) \leq r_i \forall \tau \geq t\}$ . Given  
 $\varepsilon \in (0, \mathbf{d}(z))$ , for any  $i \in \mathbb{N}$ , set  $T_i^\varepsilon := \inf\{t > 0 : \mathbf{d}(x_i(t)) \leq \varepsilon\}$ . We can assume  
 without loss of generality  $r_i < \varepsilon$  for all  $i \in \mathbb{N}$ , since  $r_i \rightarrow 0$ . Hence,  $\bar{T}_{x_i}^{r_i} \geq T_i^\varepsilon$  and  
 by the previous step it follows that

$$\bar{T}_{x_i}^{r_i} \geq T_i^\varepsilon \geq T^\varepsilon = \frac{\mathbf{d}(z) - \varepsilon}{M(\varepsilon, \mathbf{d}(z))}.$$

6 Hence, for every  $t \in [0, T_\varepsilon]$ ,<sup>2</sup>  $\mathbf{d}(\tilde{x}_i(t)) = \mathbf{d}(x_i(t)) \geq \varepsilon$  and passing to the limit as  
 7  $i \rightarrow +\infty$  we get  $\mathbf{d}(\mathcal{X}(t)) \geq \varepsilon$ . As a consequence, we can conclude that  $T_\mathcal{X} \geq T^\varepsilon$ .  
 8 □

9 The sequence  $(r_i)$  plays a key role in Definition 3.2. In particular, when, for  
 10 some  $i$ , the time  $T_i$  is finite, the truncated functions  $\tilde{x}_i^0, \tilde{x}_i$ , differently from  $x_i^0$   
 11 and  $x_i$ , cannot have a chattering behaviour. However, the restriction to the interval  
 12  $[0, T_\mathcal{X}]$  of a weak Euler cost-trajectory pair  $(\mathcal{X}^0, \mathcal{X})$  associated to a sampling sequence  
 13  $(x_i^0, x_i, u_i)$  does not depend on the choice of the sequence  $(r_i)$ . Precisely, we have:

14 **Proposition 2.** *Given a locally bounded feedback  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$ , let  $(\mathcal{X}^0, \mathcal{X})$  be*  
 15 *a weak Euler cost-trajectory pair with initial condition  $z \in \mathbb{R}^n \setminus \mathcal{C}$ . Let  $(x_i^0, x_i, u_i)$ ,*  
 16  *$(\delta_i)$ , and  $(r_i)$  determine  $(\mathcal{X}^0, \mathcal{X})$ , as in Definition 3.2. Then the following properties*  
 17 *hold true.*

18 (i) *Setting*

$$\bar{T} := \liminf_{i \rightarrow +\infty} T_i \quad (T_i \text{ as in (30)}), \quad (38)$$

19 *we have that  $0 < T_\mathcal{X} \leq \bar{T} \leq +\infty$ , and*

$$(\mathcal{X}^0, \mathcal{X})(t) = \lim_{i \rightarrow +\infty} (\tilde{x}_i^0, \tilde{x}_i)(t) = \lim_{i \rightarrow +\infty} (x_i^0, x_i)(t) \quad \forall t \in [0, \bar{T}]. \quad (39)$$

20 (ii) *Let  $(\hat{r}_i) \subset (0, \mathbf{d}(z))$  be any sequence converging to zero and such that  $\hat{r}_i \geq r_i$*   
 21 *for all  $i \in \mathbb{N}$ . Define, for each  $i \in \mathbb{N}$ ,*

$$\hat{T}_i := \bar{T}_{x_i}^{\hat{r}_i} = \inf\{t > 0 : \mathbf{d}(x_i(\tau)) \leq \hat{r}_i \quad \forall \tau \geq t\}, \quad \hat{T} := \liminf_{i \rightarrow +\infty} \hat{T}_i. \quad (40)$$

<sup>2</sup>If  $T_\varepsilon = +\infty$ , we obviously mean  $t \in [0, +\infty)$ .

Then  $T_{\mathcal{X}} \leq \hat{T} (\leq \bar{T})$  and the sequence  $(\hat{x}_i^0, \hat{x}_i)$ , where  $(\hat{x}_i^0, \hat{x}_i)(t) := (x_i^0, x_i)(t \wedge \hat{T}_i)$  for all  $t \geq 0$  and  $i \in \mathbb{N}$ , verifies

$$\lim_{i \rightarrow +\infty} (\hat{x}_i^0, \hat{x}_i)(t) = (\mathcal{X}^0, \mathcal{X})(t) \quad \forall t \in [0, \hat{T}).$$

*Proof of Proposition 2.* (i) By Proposition 1 it follows that, given an arbitrary  $\varepsilon \in (0, \mathbf{d}(z))$  one has  $\bar{T} \geq T^\varepsilon > 0$ . For every  $t \in [0, \bar{T})$ , it is clear that  $t < T_i$  for all  $i$  large enough. Hence  $(\tilde{x}_i^0, \tilde{x}_i)(t) = (x_i^0, x_i)(t)$  for such  $i$  and the definitions (29), (32) imply that

$$\lim_{i \rightarrow +\infty} (x_i^0, x_i)(t) = (\mathcal{X}^0, \mathcal{X})(t) \quad \forall t \in [0, \bar{T}).$$

If  $\bar{T} = +\infty$ , the proof is concluded. If instead  $\bar{T} < +\infty$ , it remains to show that  $T_{\mathcal{X}} \leq \bar{T}$ . To this aim, let us consider a subsequence  $k \mapsto i_k$  such that  $\bar{T} = \lim_{k \rightarrow +\infty} T_{i_k}$ . For every  $\varepsilon_1 > 0$ ,  $T_{i_k} < \bar{T} + \varepsilon_1$  for all  $k$  large enough, so that  $\mathbf{d}(\tilde{x}_{i_k}(\bar{T} + \varepsilon_1)) = \mathbf{d}(x_{i_k}(T_{i_k})) = r_{i_k}$  for such  $k$ . Hence, we get

$$\mathbf{d}(\mathcal{X}(\bar{T} + \varepsilon_1)) \leq |\mathcal{X}(\bar{T} + \varepsilon_1) - \tilde{x}_{i_k}(\bar{T} + \varepsilon_1)| + \mathbf{d}(\tilde{x}_{i_k}(\bar{T} + \varepsilon_1)) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

1 By the arbitrariness of  $\varepsilon_1 > 0$ , this implies that  $T_{\mathcal{X}} \leq \bar{T}$ , so concluding the proof.

(ii) By the hypothesis that  $\hat{r}_i \geq r_i$ , it follows that  $\hat{T}_i \leq T_i \quad \forall i \in \mathbb{N}$ . Hence  $\hat{T} \leq \bar{T}$ . The facts that  $\hat{T} > 0$  and

$$\lim_{i \rightarrow +\infty} (\hat{x}_i^0, \hat{x}_i)(t) = \lim_{i \rightarrow +\infty} (x_i^0, x_i)(t) = (\mathcal{X}^0, \mathcal{X})(t) \quad \forall t \in [0, \hat{T}),$$

can be proved arguing as in the previous step. It remains to show that  $T_{\mathcal{X}} \leq \hat{T}$ . If  $\hat{T} = \bar{T}$ , the thesis follows by (i). Suppose now  $\hat{T} < \bar{T}$ . Let  $k \mapsto i_k$  be a subsequence such that  $\lim_{k \rightarrow +\infty} \hat{T}_{i_k} = \hat{T}$ . For every  $\varepsilon_1 > 0$  such that  $\hat{T} < \hat{T} + \varepsilon_1 < \bar{T}$ , by the definition of  $\bar{T}$  one has  $\hat{T} + \varepsilon_1 \leq T_{i_k}$  for all  $k$  large enough. Therefore,  $(\tilde{x}_{i_k}^0, \tilde{x}_{i_k})(\hat{T} + \varepsilon_1) = (x_{i_k}^0, x_{i_k})(\hat{T} + \varepsilon_1)$  for such  $k$ , and

$$\begin{aligned} \mathbf{d}(\mathcal{X}(\hat{T} + \varepsilon_1)) &\leq |\mathcal{X}(\hat{T} + \varepsilon_1) - \tilde{x}_{i_k}(\hat{T} + \varepsilon_1)| + \mathbf{d}(\tilde{x}_{i_k}(\hat{T} + \varepsilon_1)) \\ &= |\mathcal{X}(\hat{T} + \varepsilon_1) - \tilde{x}_{i_k}(\hat{T} + \varepsilon_1)| + \mathbf{d}(x_{i_k}(\hat{T} + \varepsilon_1)). \end{aligned}$$

2 Since  $\hat{T}_{i_k} \leq \hat{T} + \varepsilon_1$  as soon as  $k$  is large enough,  $\mathbf{d}(x_{i_k}(\hat{T} + \varepsilon_1)) \leq \hat{r}_{i_k}$  for such  $k$ .

3 Taking the limit as  $k \rightarrow +\infty$  one derives that  $\mathbf{d}(\mathcal{X}(\hat{T} + \varepsilon_1)) = 0$  for every  $\varepsilon_1 > 0$ .

4 Therefore,  $T_{\mathcal{X}} \leq \hat{T}$  and the proof is concluded.  $\square$

5 If the target is a singleton, the definition of weak Euler solution  $\mathcal{X}$  does not  
6 depend at all on the sequence  $(r_i)$  and can be simplified as follows.

**Proposition 3.** *Assume that the target  $\mathcal{C}$  is reduced to a point. Let  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  be a locally bounded feedback and let  $z \in \mathbb{R}^n \setminus \mathcal{C}$  be given. Then a function  $\mathcal{X}$  is a weak Euler solution of (5) from  $z$  if and only if there exists a sequence  $(x_i, u_i)$  of admissible  $\pi_i$ -sampling processes of (5) from  $z$  such that  $\text{diam}(\pi_i) \rightarrow 0$  as  $i \rightarrow +\infty$ , and verifying*

$$x_i \rightarrow \mathcal{X} \quad \text{pointwise in } [0, +\infty). \quad (41)$$

*Proof.* Let  $\mathcal{C} = \{\bar{z}\}$ . The proof consists in showing that, given a sequence  $(x_i, u_i)$  of admissible  $\pi_i$ -sampling processes of (5) from  $z$  such that  $\delta_i := \text{diam}(\pi_i) \rightarrow 0$  as  $i \rightarrow +\infty$ , and any sequence  $(r_i) \subset (0, \mathbf{d}(z))$  converging to zero, there exists the limit

$$\mathcal{X}(t) := \lim_{i \rightarrow +\infty} x_i(t) \quad \forall t \in [0, +\infty),$$



if and only if there exists the limit

$$\mathcal{X}_1(t) := \lim_{i \rightarrow +\infty} \tilde{x}_i(t) \quad \forall t \in [0, +\infty),$$

1 where  $\tilde{x}_i(t) := x_i(t \wedge T_i)$  and  $T_i$  is as in (30). Moreover,  $\mathcal{X} \equiv \mathcal{X}_1$ .

If  $T_i = +\infty$ , one has  $\tilde{x}_i = x_i$  trivially. For each  $i \in \mathbb{N}$  with  $T_i < +\infty$ , by definition,  $\tilde{x}_i(t) = x_i(t)$  for all  $t \in [0, T_i]$ ,  $\tilde{x}_i(t) = x_i(T_i)$  for all  $t \geq T_i$ , and  $\mathbf{d}(x_i(T_i)) = |x_i(T_i) - \bar{z}| = r_i$ . Moreover,  $\mathbf{d}(x_i(t)) = |x_i(t) - \bar{z}| \leq r_i$  for all  $t \geq T_i$ . Then, for every  $t > T_i$ , one has

$$|x_i(t) - \tilde{x}_i(t)| = |x_i(t) - x_i(T_i)| \leq |x_i(t) - \bar{z}| + |\bar{z} - x_i(T_i)| \leq 2r_i.$$

2 Therefore, for every  $t \geq 0$ ,  $(x_i(t))$  converges if and only if  $(\tilde{x}_i(t))$  converges and

3  $\lim_{i \rightarrow +\infty} \tilde{x}_i(t) = \lim_{i \rightarrow +\infty} x_i(t)$ . □

4 **Definition 3.3** (Euler and weak Euler stabilizability with regulated cost). A locally  
5 bounded feedback  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  is said to Euler [resp., weak Euler] stabilize  
6  $\dot{x} = \mathbf{f}(x, u)$  to  $\mathcal{C}$  if there is a function  $\beta \in \mathcal{KL}$  such that, for each  $z \in \mathbb{R}^n \setminus \mathcal{C}$ , every  
7 Euler [resp., weak Euler] trajectory  $\mathcal{X}$  of (5) from  $z$  verifies

$$\mathbf{d}(\mathcal{X}(t)) \leq \beta(\mathbf{d}(z), t) \quad \forall t \in [0, +\infty). \quad (42)$$

8 If moreover there exist  $p_0 > 0$  and a continuous map  $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \rightarrow [0, +\infty)$   
9 which is positive definite and proper in  $\mathbb{R}^n \setminus \mathcal{C}$ , such that, for any  $\mathcal{X}$  as above, every  
10 Euler [resp. weak Euler] cost  $\mathcal{X}^0$  associated to  $\mathcal{X}$  verifies

$$\mathcal{X}^0(t) \leq \frac{W(z)}{p_0} \quad \forall t \in [0, T_{\mathcal{X}}), \quad (43)$$

11 where  $T_{\mathcal{X}}$  is as in (33), we call (5)–(6) Euler [resp., weak Euler] stabilizable to  $\mathcal{C}$   
12 with  $(p_0, W)$ -regulated cost. When  $\dot{x} = \mathbf{f}(x, u)$  is merely Euler [resp., weak Euler]  
13 stabilizable to  $\mathcal{C}$ , we also say that (5)–(6) is Euler [resp., weak Euler] stabilizable to  
14  $\mathcal{C}$  with  $(p_0, W)$ -regulated cost for  $p_0 = 0$ .

15 **Remark 2.** Since any Euler cost-solution pair for (5)–(6) is a weak Euler cost-  
16 solution pair, the weak Euler stabilizability with regulated cost implies the Euler  
17 stabilizability with regulated cost.

18 **3.2. Sample and weak Euler stabilizability with regulated cost.** Sample  
19 stabilizability to  $\mathcal{C}$  with  $(p_0, W)$ -regulated cost implies Euler and weak Euler stabi-  
20 lizability to  $\mathcal{C}$  with  $(p_0, W)$ -regulated cost.

21 **Theorem 3.4.** Assume that  $\mathbf{f}, \mathbf{l}$  verify assumption (H.1). If a locally bounded  
22 feedback  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  sample stabilizes (5)–(6) to  $\mathcal{C}$  with  $(p_0, W)$ -regulated  
23 cost for some  $p_0 \geq 0$ , then  $\mathbf{K}$  Euler and weak Euler stabilizes (5)–(6) to  $\mathcal{C}$  with  
24  $(p_0, W)$ -regulated cost and with the same descent rate.

25 *Proof.* In view of Remark 2, it is sufficient to show that the sample stabilizability  
26 with regulated cost implies the weak Euler stabilizability with regulated cost.

Preliminarily, let us reformulate the notion of sample stabilizability with regu-  
lated cost. Let  $\beta, \delta = \delta(r, R)$  for  $0 < r < R$ , be as in Definition 2.3. Since we can  
assume without loss of generality that

$$r \mapsto \delta(r, R)$$

1 is a continuous, strictly increasing map, that verifies  $\lim_{r \rightarrow 0^+} \delta(r, R) = 0$  and  
 2  $\delta(R) := \lim_{r \rightarrow R^-} \delta(r, R) < +\infty$ , we can define the inverse map

$$\delta \mapsto r(\delta) \quad \forall \delta \in [0, \delta(R)]. \quad (44)$$

3 This function is continuous, strictly increasing and verifies  $r(0) = 0$ . As a conse-  
 4 quence, by the sample stabilizability to  $\mathcal{C}$  of (5)-(6) with  $(p_0, W)$ -regulated cost,  
 5 for each  $\delta \in (0, \delta(R))$ , for every partition  $\pi$  with  $\text{diam}(\pi) = \delta$ , and for every  $\pi$ -  
 6 sampling process  $(x^0, x, u)$  with  $x(0) = z$ ,  $\mathbf{d}(z) = R$ , one has  $x$  defined in  $[0, +\infty)$   
 7 and verifying:

$$\mathbf{d}(x(t)) \leq \max\{\beta(\mathbf{d}(z), t), r(\delta)\} \quad \forall t \geq 0, \quad (45)$$

8 and, if  $p_0 > 0$ ,

$$x^0(\bar{T}_x^{r(\delta)}) = \int_0^{\bar{T}_x^{r(\delta)}} l(x(\tau), u(\tau)) d\tau \leq \frac{W(z)}{p_0}, \quad (46)$$

9 where  $\bar{T}_x^{r(\delta)} = \inf\{t > 0 : \mathbf{d}(x(\tau)) \leq r(\delta) \quad \forall \tau \geq t\}$ , as in (11).

Given  $z \in \mathbb{R}^n \setminus \mathcal{C}$ , let  $(\mathcal{X}^0, \mathcal{X})$  be an arbitrary weak Euler cost-solution pair associated to the sampling stabilizing feedback  $\mathbf{K}$  and with initial condition  $\mathcal{X}(0) = z$ . By definition, there are a sequence of partitions  $(\pi_i)$  such that  $\delta_i := \text{diam}(\pi_i) \rightarrow 0$  as  $i \rightarrow +\infty$ , a sequence of admissible  $\pi_i$ -sampling processes  $(x_i^0, x_i, u_i)$  for (5)-(6) with  $x_i(0) = z$  for each  $i$ , and a vanishing sequence  $(r_i) \subset (0, \mathbf{d}(z))$ , such that

$$\lim_{i \rightarrow +\infty} (\tilde{x}_i^0, \tilde{x}_i)(t) = (\mathcal{X}^0, \mathcal{X})(t) \quad \forall t \in [0, +\infty),$$

where  $(\tilde{x}_i^0, \tilde{x}_i)$  is the sequence of truncated functions introduced in Definition 3.2, namely,

$$(\tilde{x}_i^0, \tilde{x}_i)(t) = (x_i^0, x_i)(t \wedge \bar{T}_{x_i}^{r_i}), \quad \bar{T}_{x_i}^{r_i} = \inf\{t > 0 : \mathbf{d}(x_i(\tau)) \leq r_i \quad \forall \tau \geq t\}.$$

10 Since  $\delta_i \rightarrow 0$ , we can assume without loss of generality that  $\delta_i < \delta(\mathbf{d}(z))$  for all  $i$ .  
 11 Hence, by the previous step it follows that, for every  $i$ ,

$$\mathbf{d}(x_i(t)) \leq \max\{\beta(\mathbf{d}(z), t), r(\delta_i)\} \quad \forall t \geq 0 \quad (47)$$

12 and, if  $p_0 > 0$ ,

$$x_i^0(t) \leq \frac{W(z)}{p_0} \quad \forall t \in [0, \bar{T}_{x_i}^{r(\delta_i)}]. \quad (48)$$

Since  $\tilde{x}_i(t) = x_i(t)$  for all  $t \in [0, \bar{T}_{x_i}^{r_i}]$  and  $\mathbf{d}(\tilde{x}_i(t)) = r_i$  for all  $t \geq \bar{T}_{x_i}^{r_i}$ , (47) implies

$$\mathbf{d}(\tilde{x}_i(t)) \leq \max\{\beta(\mathbf{d}(z), t), r(\delta_i), r_i\} \quad \forall t \geq 0,$$

13 and, passing to the limit as  $i \rightarrow +\infty$ , we get

$$\mathbf{d}(\mathcal{X}(t)) \leq \beta(\mathbf{d}(z), t) \quad \forall t \in [0, +\infty), \quad (49)$$

14 because  $r_i$  and  $r(\delta_i)$  tend to 0. By the arbitrariness of the Euler solution  $\mathcal{X}$ , this  
 15 proves that the feedback  $\mathbf{K}$  weak Euler stabilizes (5) to  $\mathcal{C}$ , with the same descent  
 16 rate  $\beta$  of the sample stabilizability.

17 Suppose  $p_0 > 0$ . To conclude the proof, it remains to show that

$$\lim_{t \rightarrow T_{\mathcal{X}}^-} \mathcal{X}^0(t) \leq \frac{W(z)}{p_0}, \quad (50)$$

- 1 where  $T_{\mathcal{X}} = \inf\{t \geq 0 : \lim_{\tau \rightarrow t^-} \mathbf{d}(\mathcal{X}(\tau)) = 0\} > 0$  by Proposition 1. To this aim,  
 2 for each  $i$ , let us define  $\hat{r}_i := \max\{r_i, r(\delta_i)\}$ , so that  $\hat{r}_i \geq r_i$  and  $\lim_{i \rightarrow +\infty} \hat{r}_i = 0$ . In  
 3 view of Lemma 2, (ii) we have that

$$T_{\mathcal{X}} \leq \hat{T} := \liminf_{i \rightarrow +\infty} \hat{T}_i, \quad \hat{T}_i := \inf\{t > 0 : \mathbf{d}(x_i(\tau)) \leq \hat{r}_i \ \forall \tau \geq t\}. \quad (51)$$

- 4 Fix  $t \in [0, T_{\mathcal{X}})$ . By (51),  $\hat{T}_i > t$  for all  $i$  sufficiently large. Moreover, one has  
 5  $\hat{T}_i \leq \bar{T}_{x_i}^{\delta(r_i)}$ , since  $\hat{r}_i \geq r(\delta_i)$ . This together with (48) implies that

$$\mathcal{X}^0(t) = \lim_{i \rightarrow +\infty} x_i^0(t) \leq \frac{W(z)}{p_0} \quad \forall t \in [0, T_{\mathcal{X}}). \quad (52)$$

- 6 Taking the limit as  $t \rightarrow T_{\mathcal{X}}^-$  we get finally (50).  $\square$

- 7 The weak coercivity hypothesis **(HC)** below, is sufficient to guarantee that,  
 8 when (5)–(6) is sample stabilizable to  $\mathcal{C}$  with  $(p_0, W)$ -regulated cost, the set of the  
 9 corresponding weak Euler cost-solution pairs is nonempty.

**(HC)** For some  $\bar{R} > 0$ , there exist  $C_1 \geq 0$  and  $C_2 > 0$  such that  $\mathbf{f}, \mathbf{l}$  verify

$$\mathbf{l}(x, u) \geq C_2 |\mathbf{f}(x, u)| - C_1 \quad \forall (x, u) \in (B_{\bar{R}}(\mathcal{C}) \setminus \mathcal{C}) \times U.$$

Notice that control-polynomial dynamics and running costs of the form

$$\begin{aligned} \mathbf{f}(x, u) &:= f_0(x) + \sum_{i=1}^d \left( \sum_{\alpha \in \mathbb{N}^M, \alpha_1 + \dots + \alpha_M = i} u_1^{\alpha_1} \cdots u_M^{\alpha_M} f_{\alpha_1, \dots, \alpha_M}(x) \right), \\ \mathbf{l}(x, u) &\geq l_0(x) + l_1(x)|u| + \dots + l_{\bar{d}}(x)|u|^{\bar{d}}, \end{aligned}$$

- 10 where the maps  $f_0, f_{\alpha_1, \dots, \alpha_M}, l_i$  are continuous in  $\mathbb{R}^n$  and  $l_i \geq 0$ , verify hypothesis  
 11 **(HC)** as soon as  $\bar{d} \geq d$  and  $l_{\bar{d}}(x) \geq C > 0$  in  $B_{\bar{R}}(\mathcal{C}) \setminus \mathcal{C}$ , for some  $C > 0$ .

- 12 **Proposition 4.** Let  $\mathbf{f}, \mathbf{l}$  verify hypotheses **(H.1)**, **(HC)**, and let  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  be  
 13 a locally bounded feedback that sample stabilizes (5)–(6) to  $\mathcal{C}$  with  $(p_0, W)$ -regulated  
 14 cost, for some  $p_0 > 0$ . Then for any  $z \in \mathbb{R}^n \setminus \mathcal{C}$  there exists at least one weak Euler  
 15 cost-solution pair  $(\mathcal{X}^0, \mathcal{X})$  of (5)–(6) from  $z$  associated to the feedback  $\mathbf{K}$ .

- 16 *Proof.* As already observed in the proof of Theorem 3.4, by the sample stabilizability  
 17 to  $\mathcal{C}$  of (5)–(6) with  $(p_0, W)$ -regulated cost, it follows that there exist a  $\mathcal{KL}$ -function  
 18  $\beta$  and, for every  $R > 0$ , a continuous, strictly increasing map  $\delta \mapsto r(\delta) \in [0, R]$   
 19 for all  $\delta \in [0, \delta(R)]$  with  $r(0) = 0$  (see (44)), such that, for each  $\delta \in (0, \delta(R))$ , for  
 20 every partition  $\pi$  with  $\text{diam}(\pi) = \delta$ , and for every  $\pi$ -sampling process  $(x^0, x, u)$  with  
 21  $x(0) = z$ ,  $\mathbf{d}(z) = R$ ,  $x$  is defined in  $[0, +\infty)$  and  $(x^0, x, u)$  verifies (45), (46). In  
 22 particular, given  $z \in \mathbb{R}^n \setminus \mathcal{C}$ , for any sequence  $(\delta_i)$  converging to 0 small enough,  
 23 for each  $i$ , there exists at least one admissible  $\pi_i$ -sampling process  $(x_i^0, x_i, u_i)$  with  
 24  $x_i(0) = z$ ,  $\text{diam}(\pi_i) = \delta_i$ ,  $x$  defined on  $[0, +\infty)$ , and it verifies:

$$\mathbf{d}(x_i(t)) \leq \max\{\beta(\mathbf{d}(z), t), r(\delta_i)\} \quad \forall t \geq 0, \quad (53)$$

25

$$x_i^0(\bar{T}_{x_i}^{r(\delta_i)}) = \int_0^{\bar{T}_{x_i}^{r(\delta_i)}} l(x_i(t), u_i(t)) dt \leq \frac{W(z)}{p_0}. \quad (54)$$

- 26 By Proposition 1, given an arbitrary  $\varepsilon \in (0, \mathbf{d}(z))$ , for each  $i$ , one has

$$\bar{T}_{x_i}^{r(\delta_i)} \geq T^\varepsilon > 0, \quad (55)$$

where  $T^\varepsilon$  is as in (34). Let us set  $r_i := r(\delta_i)$  and  $T_i := \bar{T}_{x_i}^{r(\delta_i)}$ . Taking a subsequence if necessary, we can assume that there exists

$$\bar{T} := \lim_{i \rightarrow +\infty} T_i \leq +\infty,$$

1 where  $\bar{T} \geq T^\varepsilon > 0$  by (55). We set

$$(\tilde{x}_i^0, \tilde{x}_i)(t) := (x_i^0, x_i)(t \wedge T_i) \quad \forall t \geq 0. \quad (56)$$

2 Assume first  $\mathbf{d}(z) \leq \bar{R}_1$ , where  $\beta(\bar{R}_1, 0) = \bar{R}$  and  $\bar{R}$  is as in hypothesis (HC).  
3 Hence,  $\bar{R}_1 \leq \bar{R}$  by the properties of  $\beta$ , and (53), (54), and (HC) imply that, for  
4 every  $i \in \mathbb{N}$ , one has

$$\mathbf{d}(x_i(t)) \leq \bar{R} \quad \forall t \geq 0, \quad (57)$$

and

$$\begin{aligned} \tilde{x}_i^0(t) &= \int_0^t |\dot{\tilde{x}}_i^0(\tau)| d\tau = \int_0^{t \wedge T_i} l(x_i(\tau), u_i(\tau)) d\tau \leq \frac{W(z)}{p_0} \quad \forall t \geq 0, \\ \int_0^t |\dot{\tilde{x}}_i(\tau)| dt &= \int_0^{t \wedge T_i} |f(x_i(\tau), u_i(\tau))| d\tau \leq \frac{W(z)}{C_2 p_0} + \frac{C_1}{C_2} (t \wedge T_i) \quad \forall t \geq 0. \end{aligned}$$

5 Moreover, by (57) and the compactness of  $\partial\mathcal{C}$ , it follows that there is some  $M > 0$   
6 such that  $|\tilde{x}_i(t)| \leq M$  for all  $t \geq 0$  and for every  $i$ . Hence the sequence  $(\tilde{x}_i^0, \tilde{x}_i)$   
7 is equibounded and has equibounded total variation on  $[0, t]$  for every  $t > 0$ , so  
8 that Helly's Selection Theorem (see [5, Theorem 15.1]) implies that there exist a  
9 subsequence, which we still denote  $(\tilde{x}_i^0, \tilde{x}_i)$ , and a bounded map  $(\mathcal{X}^0, \mathcal{X}) : [0, +\infty) \rightarrow$   
10  $[0, +\infty) \times \mathbb{R}^n$  with locally bounded total variation, such that

$$\lim_{i \rightarrow +\infty} (\tilde{x}_i^0, \tilde{x}_i)(t) = (\mathcal{X}^0, \mathcal{X})(t) \quad \forall t \geq 0. \quad (58)$$

11 In view of Definition 3.2,  $(\mathcal{X}^0, \mathcal{X})$  is weak Euler cost-solution pair.

If instead  $\mathbf{d}(z) > \bar{R}_1$ , for every  $i \in \mathbb{N}$ , we set

$$\hat{T}_i^{\bar{R}_1} := \inf\{t \geq 0 : \mathbf{d}(x_i(t)) \leq \bar{R}_1\}, \quad \hat{T}^{\bar{R}_1} := \inf\{t \geq 0 : \beta(\mathbf{d}(z), t) \leq \bar{R}_1\},$$

where  $0 < \hat{T}_i^{\bar{R}_1} \leq \hat{T}^{\bar{R}_1}$ . Since  $r(\delta_i) \rightarrow 0$  as  $i \rightarrow +\infty$ , one has  $\hat{T}_i^{\bar{R}_1} \leq T_i$  for all  $i$  large enough. For such  $i$ ,  $\mathbf{d}(\tilde{x}_i(t)) = \mathbf{d}(x_i(t)) \geq \bar{R}_1$  for all  $t \leq \hat{T}_i^{\bar{R}_1}$ , and

$$\mathbf{d}(\tilde{x}_i(t)) = \mathbf{d}(x_i(t)) \leq \beta(\bar{R}_1, t - \hat{T}_i^{\bar{R}_1}) \leq \beta(\bar{R}_1, 0) = \bar{R} \quad \forall t \in [\hat{T}_i^{\bar{R}_1}, T_i],$$

by the definition of  $\bar{R}_1$  and the properties of  $\beta$ . Hence, (HC) yields that

$$\begin{aligned} \int_0^t |\dot{\tilde{x}}_i(t)| dt &= \int_0^{t \wedge \hat{T}_i^{\bar{R}_1}} |f(x_i(t), u_i(t))| dt + \int_{t \wedge \hat{T}_i^{\bar{R}_1}}^{t \wedge T_i} |f(x_i(t), u_i(t))| dt \\ &\leq \bar{M} \hat{T}_i^{\bar{R}_1} + \frac{W(z)}{C_2 p_0} + \frac{C_1}{C_2} (t \wedge T_i) \quad \forall t \geq 0, \end{aligned}$$

12 where  $\bar{M} := \sup\{|\mathbf{f}(x, u)| \mid \bar{R}_1 \leq \mathbf{d}(x) \leq \beta(\mathbf{d}(z), 0), u \in \mathbf{K}(x)\}$ . From now on, the  
13 proof proceeds as in the case  $\mathbf{d}(z) \leq \bar{R}_1$  and we omit it.  $\square$

14 **Remark 3.** From the previous proof we can deduce that the statement of Propo-  
15 sition 4 remains valid even if (HC) is replaced by any condition that implies the  
16 equiboundedness of the total variation of the sequence  $(\tilde{x}_i)$  of the stabilizing sam-  
17 pling trajectories, on any interval  $[0, t]$ ,  $t > 0$ .

1 **4. Sufficient stabilizability conditions in optimal control.** In this section we  
 2 provide sufficient conditions for the sample, Euler, and weak Euler stabilizability  
 3 with regulated cost of the original problem (14)-(15). Such conditions rely on the  
 4 existence of a  $p_0$ -Minimum Restraint function.

5 **4.1. Main results.** Given arbitrary functions  $\mathbf{f}, \mathbf{l}$  verifying (H.1), we introduce  
 6 the *Hamiltonian*  $H_{\mathbf{f}, \mathbf{l}} : (\mathbb{R}^n \setminus \mathcal{C}) \times \mathbb{R} \times \mathbb{R}^n \rightarrow [-\infty, +\infty)$ , given by

$$H(x, p_0, p) := \inf_{u \in U} \{ \langle p, \mathbf{f}(x, u) \rangle + p_0 \mathbf{l}(x, u) \}. \quad (59)$$

7 Notice that, because of the unboundedness of the data,  $H$  may be discontinuous  
 8 and also equal to  $-\infty$  at some points. Following [16, 17], we define a  $p_0$ -Minimum  
 9 *Restraint function* as follows.

10 **Definition 4.1** ( $p_0$ -Minimum Restraint Function). Let  $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \rightarrow [0, +\infty)$  be  
 11 a continuous function, and let us assume that  $W$  is locally semiconcave, positive  
 12 definite, and proper on  $\mathbb{R}^n \setminus \mathcal{C}$ . We say that  $W$  is a  $p_0$ -Minimum Restraint function  
 13 – in short,  $p_0$ -MRF – for some  $p_0 \geq 0$  for  $\mathbf{f}, \mathbf{l}$ , if there exists some continuous,  
 14 strictly increasing function  $\gamma : (0, +\infty) \rightarrow (0, +\infty)$ , that we call a *decrease rate*,  
 15 verifying the following *decrease condition*:

$$H_{\mathbf{f}, \mathbf{l}}(x, p_0, D^*W(x)) \leq -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}.^3 \quad (60)$$

16 **Remark 4.** Given  $\mathbf{f}, \mathbf{l}$ , a  $p_0$ -MRF  $W$  with  $p_0 = 0$  is simply a Control Lyapunov  
 17 function, in short CLF, for the control system  $\dot{x} = \mathbf{f}(x, u)$ . If  $p_0 > 0$ ,  $W$  is still a  
 18 CLF, since  $\mathbf{l} \geq 0$ , but condition (60) now includes, for instance, also Petrov-type  
 19 controllability conditions for the minimum time problem, where  $\mathbf{l} = 1$ . However,  
 20 on the one hand, unlike the existence of a CLF, the existence of a  $p_0$ -MRF gives  
 21 cost information. On the other hand, since  $\mathbf{l}$  may be zero on an arbitrary set, it  
 22 is not possible to reformulate the present problem as a minimum time problem for  
 23 the rescaled dynamics  $\mathbf{f}/\mathbf{l}$ . For more details on the notion of  $p_0$ -MRF and examples  
 24 we refer to [19, 18, 16].

25 The existence of a  $p_0$ -MRF  $\bar{W}$  for the rescaled data  $\bar{f}, \bar{l}$ , guarantees the sample  
 26 stabilizability to  $\mathcal{C}$  with  $(p_0, \bar{W})$ -regulated cost of the original problem:

27 **Theorem 4.2.** *Given  $f, l$  verifying (H.1-2), let  $\bar{W}$  be a  $p_0$ -MRF with  $p_0 \geq 0$  for the*  
 28 *rescaled functions  $\bar{f}, \bar{l}$ . Then there exists a locally bounded feedback  $K : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$*   
 29 *that sample stabilizes the original problem (14)-(15) to  $\mathcal{C}$  with  $(p_0, \bar{W})$ -regulated*  
 30 *cost.*

31 *Proof.* In view of hypotheses (H.1-2), the rescaled functions  $\bar{f}, \bar{l}$ , satisfy the regu-  
 32 larity and boundedness assumptions that make [16, Theorem 1.1] applicable. Hence  
 33 there exists a locally bounded feedback strategy  $K : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  that sample stabi-  
 34 lizes to  $\mathcal{C}$  the rescaled problem (16)-(17) with  $(p_0, \bar{W})$ -regulated cost. The claim on  
 35 the sample stabilizability to  $\mathcal{C}$  with  $(p_0, \bar{W})$ -regulated cost of (14)-(15) now follows  
 36 straightforwardly from the equivalence Theorem 2.5.  $\square$

37 It is not difficult to show that any  $p_0$ -MRF for the rescaled problem is a  $p_0$ -MRF  
 38 for the original problem. Instead, a  $p_0$ -MRF  $W$  for  $f, l$  may not be a  $p_0$ -MRF for  
 39  $\bar{f}, \bar{l}$ , but in view of Theorem 4.3 below we can always build an associated  $p_0$ -MRF  
 40  $\bar{W} \geq W$  for the rescaled problem.

<sup>3</sup>This means that  $H_{\mathbf{f}, \mathbf{l}}(x, p_0, p) \leq -\gamma(W(x))$  for every  $p \in D^*W(x)$ .

1 Preliminarily, let us show that a  $p_0$ -MRF for  $f, l$  provides a locally bounded  
 2 feedback satisfying the decrease condition. This is a direct consequence of the  
 3 following, more general result.

**Proposition 5.** *Assume that  $\mathbf{f}, \mathbf{l}$  satisfy **(H.1)**. Let  $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \rightarrow [0, +\infty)$  be a continuous function, which is locally Lipschitz continuous, proper and positive definite on  $\mathbb{R}^n \setminus \mathcal{C}$  and verifies the decrease condition*

$$H_{\mathbf{f}, \mathbf{l}}(x, p_0, \partial_L W(x)) < -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C},$$

4 for some  $p_0 \geq 0$  and some continuous, strictly increasing function  $\gamma : (0, +\infty) \rightarrow$   
 5  $(0, +\infty)$ . Then there exist a strictly increasing continuous map  $\tilde{\gamma} : (0, +\infty) \rightarrow$   
 6  $(0, +\infty)$ ,  $\tilde{\gamma} \leq \gamma$ , and a continuous function  $N : (0, +\infty) \rightarrow (0, +\infty)$  such that

$$\min_{U \cap B(0, N(W(x)))} \{\langle \partial_L W(x), \mathbf{f}(x, u) \rangle + p_0 \mathbf{l}(x, u)\} < -\tilde{\gamma}(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}. \quad (61)$$

Furthermore, for any selection  $p(x) \in \partial_L W(x)$ ,  $x \in \mathbb{R}^n \setminus \mathcal{C}$ , there exists a locally bounded feedback  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  verifying for all  $x \in \mathbb{R}^n \setminus \mathcal{C}$ ,

$$|\mathbf{K}(x)| \leq N(W(x))$$

7 and

$$\langle p(x), \mathbf{f}(x, \mathbf{K}(x)) \rangle + p_0 \mathbf{l}(x, \mathbf{K}(x)) < -\tilde{\gamma}(W(x)). \quad (62)$$

8 The main results of this section rely on:

**Theorem 4.3.** *Assume that  $f, l$  satisfy **(H.1-2)**. Let  $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \rightarrow [0, +\infty)$  be a continuous function, which is locally Lipschitz continuous, proper and positive definite on  $\mathbb{R}^n \setminus \mathcal{C}$  and verifies for some  $p_0 \geq 0$  the decrease condition*

$$H_{f, l}(x, p_0, \partial_L W(x)) \leq -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C},$$

9 where  $\gamma : (0, +\infty) \rightarrow (0, +\infty)$  is a continuous, strictly increasing function.

10 Then for any  $R > 0$  there exist a continuous function  $\bar{W}_R : \overline{\mathbb{R}^n \setminus \mathcal{C}} \rightarrow [0, +\infty)$   
 11 and a continuous, strictly increasing function  $\gamma_R : (0, +\infty) \rightarrow (0, +\infty)$  enjoying the  
 12 following properties.

- 13 (i) The function  $\bar{W}_R : \overline{\mathbb{R}^n \setminus \mathcal{C}} \rightarrow [0, +\infty)$  is locally Lipschitz continuous, proper  
 14 and positive definite on  $\mathbb{R}^n \setminus \mathcal{C}$ ,  $\bar{W}_R \geq W$ , and  $\bar{W}_R(x) = W(x)$  for all  $x \in$   
 15  $B_R(\mathcal{C}) \setminus \mathcal{C}$ . In addition, when  $W$  is locally semiconcave on  $\mathbb{R}^n \setminus \mathcal{C}$  or locally  
 16 Lipschitz continuous on  $\overline{\mathbb{R}^n \setminus \mathcal{C}}$ , so is  $\bar{W}_R$ . One has  $\gamma_R \leq \gamma$ .  
 17 (ii)  $\bar{W}_R$  and  $\gamma_R$  verify the decrease condition

$$H_{\bar{f}, \bar{l}}(x, p_0, \partial_L \bar{W}_R(x)) \leq -\gamma_R(\bar{W}_R(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}. \quad (63)$$

- 18 (iii) Given a selection  $p(x) \in \partial_L W(x)$  for any  $x \in \mathbb{R}^n \setminus \mathcal{C}$  and a locally bounded  
 19 feedback  $K : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  as in Proposition 5, the (unique) selection  $\bar{p}(x) \in$   
 20  $\partial_L \bar{W}_R(x)$  associated to  $p(x)$  verifies

$$\langle \bar{p}(x), \bar{f}(x, K(x)) \rangle + p_0 \bar{l}(x, K(x)) \leq -\gamma_R(\bar{W}_R(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}. \quad (64)$$

21 As a consequence of Proposition 5 and Theorem 4.3, that will be proved in  
 22 Subsection 4.2, the existence of a  $p_0$ -MRF  $W$  for the original problem still implies  
 23 sample stabilizability to  $\mathcal{C}$  with  $(p_0, W)$ -regulated cost. Precisely, we have:

24 **Theorem 4.4.** *Assume that  $f, l$  verify hypotheses **(H.1-2)** and let  $W$  be a  $p_0$ -MRF  
 25 with  $p_0 \geq 0$  for such  $f, l$ . Then there exists a locally bounded feedback  $K : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$   
 26 that sample, Euler and weak Euler stabilizes the original problem (14)–(15) to  $\mathcal{C}$  with  
 27  $(p_0, W)$ -regulated cost.*

1 *Proof.* We only need to prove that, given  $W$  as above, (14)–(15) is sample stabiliz-  
 2 able to  $\mathcal{C}$  with  $(p_0, W)$ -regulated cost, because then the rest of the statement follows  
 3 from Theorem 3.4.

4 To this end, fix an arbitrary  $R_1 > 0$  and consider  $\bar{W} := \bar{W}_{R_1}, \gamma_{R_1}$  and a feedback  
 5  $K$  as in Theorem 4.3. In particular,  $\bar{W}$  is locally semiconcave as  $W$ , so that for every  
 6  $x \in \mathbb{R}^n \setminus \mathcal{C}$ , the limiting subdifferential  $\partial_L \bar{W}(x)$  coincides with the set of reachable  
 7 gradients  $D^* \bar{W}(x)$  at  $x$ . Therefore,  $\bar{W}$  is a  $p_0$ -MRF for the rescaled problem (16)–  
 8 (17), with dynamics  $\bar{f}$  and lagrangian  $\bar{l}$ , and by Theorem 4.2 it follows that  $K$  is  
 9 a locally bounded feedback which sample stabilizes (14)–(15) to  $\mathcal{C}$ , with  $(p_0, \bar{W})$ -  
 10 regulated cost. If  $p_0 = 0$ , this concludes the proof. Otherwise, observe that until now  
 11 we have shown that there exists some function  $\beta \in \mathcal{KL}$  such that, given  $0 < r < R$ ,  
 12 there is some  $\delta = \delta(r, R) > 0$  such that for any  $z \in \mathbb{R}^n \setminus \mathcal{C}$  with  $\mathbf{d}(z) \leq R$ , any  
 13  $\pi$ -sampling process  $(x^0, x, u)$  with  $\text{diam}(\pi) \leq \delta$  and  $x(0) = z$  verifies

$$\mathbf{d}(x(t)) \leq \max\{\beta(\mathbf{d}(z), t), r\} \quad \forall t > 0, \quad (65)$$

14 and  $x^0(t) \leq \bar{W}(z)/p_0$  for all  $t \in [0, \bar{T}_x^r]$  ( $\bar{T}_x^r$  as in (11)). Since  $\bar{W}$  is in general larger  
 15 than  $W$ , it remains to show that we have in fact

$$x^0(t) \leq \frac{W(z)}{p_0} \quad \forall t \in [0, \bar{T}_x^r]. \quad (66)$$

16 By Theorem 4.3, there is a map  $\bar{W}_{2R}$  which is a  $p_0$ -MRF for  $\bar{f}, \bar{l}$  by the previous  
 17 arguments, and verifies  $\bar{W}_{2R} \equiv W$  on  $B_{2R}(\mathcal{C})$ . Hence there exist some  $(\beta_{2R} \in \mathcal{KL}$   
 18 and)  $\delta_{2R} = \delta_{2R}(r, R) > 0$  such that all  $\pi$ -sampling process  $(x^0, x, u)$  with  $\text{diam}(\pi) \leq$   
 19  $\tilde{\delta}(r, R) := \delta_{2R}(r, R) \wedge \delta(r, R)$  and  $x(0) = z$  verify in particular (65), but also have  
 20  $x^0(t) \leq \bar{W}_{2R}(z)/p_0$  for all  $t \in [0, \bar{T}_x^r]$ . The last inequality yields (66), because  
 21  $\bar{W}_{2R}(z) = W(z)$  for every  $z \in \mathbb{R}^n \setminus \mathcal{C}$  with  $\mathbf{d}(z) \leq R$ .  $\square$

22 Whenever the rescaled functions  $\bar{f}(\cdot, u), \bar{l}(\cdot, u)$  are locally Lipschitz continuous in  
 23  $\overline{\mathbb{R}^n \setminus \mathcal{C}}$  uniformly w.r.t.  $u$ , sample stabilizability can be achieved under milder regu-  
 24 larity assumptions on the  $p_0$ -MRFs. In particular, the semiconcavity requirement  
 25 in the definition of a  $p_0$ -MRF can be replaced by local Lipschitz continuity.

**Definition 4.5** (Lipschitz continuous  $p_0$ -Minimum Restraint Function). We call  
*Lipschitz continuous  $p_0$ -Minimum Restraint Function*,  $p_0 \geq 0$ , for  $\mathbf{f}, \mathbf{l}$  satisfying  
 hypothesis (H.1), any function  $W : \overline{\mathbb{R}^n \setminus \mathcal{C}} \rightarrow [0, +\infty)$  which is locally Lipschitz  
 continuous on  $\overline{\mathbb{R}^n \setminus \mathcal{C}}$ , positive definite, and proper on  $\mathbb{R}^n \setminus \mathcal{C}$ , and verifies the  
 decrease condition

$$H(x, p_0, \partial_L W(x)) < -\gamma(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C},$$

26 for some continuous, strictly increasing function  $\gamma : (0, +\infty) \rightarrow (0, +\infty)$ .

27 We consider the following strengthened version of hypotheses (H.1-2):

28 **(HL)** *The data  $\mathbf{f}, \mathbf{l}$  satisfy (H.1-2). Moreover, the rescaled functions  $\bar{\mathbf{f}}, \bar{\mathbf{l}}$  can be*  
 29 *continuously extended to  $\partial\mathcal{C} \times U$  and for every compact set  $\mathcal{K} \subset \overline{\mathbb{R}^n \setminus \mathcal{C}}$  there exists*  
 30  *$L > 0$  such that*

$$|\bar{\mathbf{f}}(x_1, u) - \bar{\mathbf{f}}(x_2, u)| + |\bar{\mathbf{l}}(x_1, u) - \bar{\mathbf{l}}(x_2, u)| \leq L|x_1 - x_2| \quad \forall (x_1, u), (x_2, u) \in \mathcal{K} \times U. \quad (67)$$

31

32 In this setting, the existence of a Lipschitz continuous  $p_0$ -MRF  $W$  for the rescaled  
 33 problem or for the original problem, still guarantees sample stabilizability to  $\mathcal{C}$  with  
 34  $(p_0/2, W)$ -regulated cost.

1 **Theorem 4.6.** *Assume that  $f, l$  satisfy **(HL)** and let  $W$  be a Lipschitz continuous*  
 2  *$p_0$ -MRF with  $p_0 \geq 0$ , either for the rescaled data  $\bar{f}, \bar{l}$ , or for  $f, l$ . Then there exists*  
 3 *a locally bounded feedback  $K : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  that sample, Euler, and weak Euler*  
 4 *stabilizes the original problem (14)–(15) to  $\mathcal{C}$  with  $(p_0/2, W)$ -regulated cost.*

5 *Proof.* Suppose first that  $W$  is a Lipschitz continuous  $p_0$ -MRF for  $\bar{f}, \bar{l}$ . In view of  
 6 hypothesis **(HL)**, the rescaled problem satisfies the assumptions of [16, Theorem  
 7 4.3]. This implies the existence of a (locally semiconcave)  $\frac{p_0}{2}$ -MRF  $W_1 \leq W$  for  
 8  $\bar{f}, \bar{l}$ , which by [16, Theorem 1.1] yields the existence of a locally bounded feed-  
 9 back  $K : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  that sample stabilizes to  $\mathcal{C}$  the rescaled problem (16)–(17)  
 10 with  $(p_0/2, W)$ -regulated cost. Therefore,  $K$  sample stabilizes to  $\mathcal{C}$  with  $(p_0/2, W)$ -  
 11 regulated cost also the original problem (14)–(15), in view of Theorem 2.5.

12 If instead  $W$  is a Lipschitz continuous  $p_0$ -MRF for  $f, l$ , let us fix  $R_1 > 0$ . By  
 13 Theorem 4.3 there exists a Lipschitz continuous  $p_0$ -MRF  $\bar{W}_{R_1} \geq W$  for  $\bar{f}, \bar{l}$ , and  
 14 it verifies  $\bar{W}_{R_1} \equiv W$  on  $B_{R_1}(\mathcal{C})$ . Then the existence of a locally bounded feedback  
 15  $K : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  that sample stabilizes to  $\mathcal{C}$  the original problem (14)–(15) with  
 16  $(p_0/2, \bar{W}_{R_1})$ -regulated cost, can be obtained as in the previous case. The fact that the  
 17 cost is actually  $(p_0/2, W)$ -regulated, can be proven arguing as in the last part of  
 18 the proof of Theorem 4.4.

19 In both cases, the Euler and weak Euler stabilizability with the same regulated cost  
 20 then follows by Theorem 3.4.  $\square$

21 In the case of control-affine data, the previous results extend the sufficient condi-  
 22 tions for sample stabilizability with regulated cost introduced in [17], which require  
 23 the existence of a MRF for the *rescaled* problem.

## 24 4.2. Proofs of Proposition 5 and of Theorem 4.3.

*Proof of Proposition 5.* Let  $\{\mu_k\}_{k \in \mathbb{Z}} \subset (0, +\infty)$  be a bi-infinite, strictly increasing  
 sequence such that  $\mu_k \rightarrow 0$  as  $k \rightarrow -\infty$  and  $\mu_k \rightarrow +\infty$  as  $k \rightarrow +\infty$ . Since  $\gamma$  is  
 strictly increasing, by (60) one has for all  $k \in \mathbb{Z}$

$$H(x, p_0, \partial_L W(x)) < -\gamma(W(x)) \leq -\gamma(\mu_k) \quad \forall x \in W^{-1}([\mu_k, +\infty)).$$

25 In particular, for all  $\bar{x} \in W^{-1}([\mu_k, +\infty))$  and  $\bar{p} \in \partial_L W(\bar{x})$ , there exists  $\bar{u} \in U$  such  
 26 that

$$\langle \mathbf{f}(\bar{x}, \bar{u}), \bar{p} \rangle + p_0 \mathbf{I}(\bar{x}, \bar{u}) < -\gamma(\mu_k). \quad (68)$$

Fix  $k \in \mathbb{Z}$  and define

$$\Gamma_k := \{(x, p) \mid x \in W^{-1}([\mu_k, \mu_{k+1}]), p \in \partial_L W(x)\}.$$

Notice that the properties of  $W$  –in particular, the properness of  $W$  and the upper  
 semicontinuity of the set-valued map  $x \rightsquigarrow \partial_L W(x)$ – imply that  $\Gamma_k$  is a compact set.  
 Then the map  $h_k : [0, +\infty) \rightarrow \mathbb{R}$  given by

$$h_k(N) := \max_{(x, p) \in \Gamma_k} \min_{U \cap B(0, N)} \{\langle \mathbf{f}(x, u), p \rangle + p_0 \mathbf{I}(x, u)\}$$

27 is well defined.

28 *Step 1.* Given any  $k \in \mathbb{Z}$ , we show that there exists a sufficiently large  $N_k$   
 29 satisfying

$$h_k(N) < -\gamma(\mu_k) \quad \forall N \geq N_k. \quad (69)$$



Indeed, let  $\{N^j\}$  be a positive, strictly increasing, diverging sequence of real numbers. Consider a sequence  $\{(x^j, p^j)\} \subset \Gamma_k$  such that

$$(x^j, p^j) \in \operatorname{argmax}_{(x,p) \in \Gamma_k} \left\{ \min_{U \cap B(0, N^j)} \{ \langle \mathbf{f}(x, u), p \rangle + p_0 \mathbf{l}(x, u) \} \right\} \quad \forall j \in \mathbb{N},$$

so that

$$h_k(N^j) = \min_{U \cap B(0, N^j)} \{ \langle \mathbf{f}(x^j, u), p^j \rangle + p_0 \mathbf{l}(x^j, u) \}.$$

Since  $\Gamma_k$  is compact, then, by passing to a subsequence if necessary,  $(x^j, p^j)$  converges to some  $(\bar{x}, \bar{p}) \in \Gamma_k$  as  $j \rightarrow \infty$ . Choose  $\bar{u}$  like in (68). By the continuity of  $\mathbf{f}$  and  $\mathbf{l}$  there exists a sufficiently large  $J$  such that  $N^J > |\bar{u}|$  and

$$\langle \mathbf{f}(x^J, \bar{u}), p^J \rangle + p_0 \mathbf{l}(x^J, \bar{u}) < -\gamma(\mu_k).$$

Since by construction  $h_k$  is decreasing, then for all  $N \geq N^J$

$$\begin{aligned} h_k(N) &\leq h_k(N^J) = \min_{U \cap B(0, N^J)} \{ \langle \mathbf{f}(x^J, u), p^J \rangle + p_0 \mathbf{l}(x^J, u) \} \\ &\leq \langle \mathbf{f}(x^J, \bar{u}), p^J \rangle + p_0 \mathbf{l}(x^J, \bar{u}) < -\gamma(\mu_k). \end{aligned}$$

- 1 Therefore, setting  $N_k := N^J$  we have (69).

*Step 2.* Let  $\tilde{\gamma} : (0, +\infty) \rightarrow (0, +\infty)$  be a strictly increasing, continuous map such that, for every  $k \in \mathbb{Z}$ ,

$$\tilde{\gamma}(\mu) \leq \gamma(\mu_k) \quad \forall \mu \in [\mu_k, \mu_{k+1}]$$

- 2 (for instance,  $\tilde{\gamma}$  can be obtained by the linear interpolation of the point set  
 3  $\{(\mu_{k+1}, \gamma(\mu_k))\}_{k \in \mathbb{Z}}$ . Let  $N : (0, +\infty) \rightarrow (0, +\infty)$  be a continuous approximation  
 4 from above of the piecewise constant function  $\bar{N}(\mu) := N_k$  for all  $\mu \in [\mu_k, \mu_{k+1})$ ,  
 5  $k \in \mathbb{Z}$ . With this choice of  $N$  and  $\tilde{\gamma}$ , by (69) it follows that relation (61) is verified.

*Step 3.* Fixed a selection  $p(x) \in \partial_L W(x)$  for every  $x \in \mathbb{R}^n \setminus \mathcal{C}$ , consider a function  $\mathbf{K} : \mathbb{R}^n \setminus \mathcal{C} \rightarrow U$  such that

$$\mathbf{K}(x) \in \operatorname{argmin}_{U \cap B(0, N(W(x)))} \{ \langle \mathbf{f}(x, u), p(x) \rangle + p_0 \mathbf{l}(x, u) \} \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}.$$

Then  $|\mathbf{K}(x)| \leq N(W(x))$  and by (61) one has

$$\langle \mathbf{f}(x, \mathbf{K}(x)), p(x) \rangle + p_0 \mathbf{l}(x, \mathbf{K}(x)) < -\tilde{\gamma}(W(x)) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}.$$

- 6 Furthermore, for any compact set  $\mathcal{K} \subset \mathbb{R}^n \setminus \mathcal{C}$ , let  $\mu_{min} := \min_{x \in \mathcal{K}} W(x)$  and  $\mu_{max} :=$   
 7  $\max_{x \in \mathcal{K}} W(x)$ . Set  $N_{\mathcal{K}} := \max_{\mu \in [\mu_{min}, \mu_{max}]} N(\mu)$ . Therefore  $\mathbf{K}(x) \in U \cap B(0, N_{\mathcal{K}})$  for all  
 8  $x \in \mathcal{K}$ , so proving that  $\mathbf{K}$  is a locally bounded feedback.  $\square$

**Remark 5.** Given any  $\sigma > 0$ , one can assume without loss of generality the function  $N$  in Proposition 5 decreasing in  $(0, \sigma]$ . It suffices, for instance, to replace  $N$  with a continuous approximation from above of the map

$$\tilde{N}_\sigma(\mu) := \begin{cases} \max_{r \in [\mu, 2\sigma]} N(r) & \mu \in (0, 2\sigma], \\ N(\mu) & \mu > 2\sigma, \end{cases}$$

- 9 which is clearly decreasing on  $(0, \sigma]$ .

*Proof of Theorem 4.3.* Let us prove (i). Given  $W$  as in the statement of Theorem 4.3, fix  $R > 0$  and let  $\sigma = \sigma(R) := \inf \{ \sigma > 0 : \{z : W(z) \leq \sigma\} \supseteq B_R(\mathcal{C}) \}$ , so that when  $\mathbf{d}(x) \leq R$  one has  $W(x) \leq \sigma$ . Fix an arbitrary  $\bar{\sigma} > 2\sigma$ . Let  $\tilde{\gamma}$ ,  $N$  and  $K$  be as in Proposition 5 for  $f$  and  $l$ , and let us assume  $N$  decreasing on  $(0, \bar{\sigma}]$ , as it is possible thanks to Remark 5. Let  $\nu_0 : (0, +\infty) \rightarrow [0, +\infty)$  be given by

$$\nu_0(\mu) := \begin{cases} \max_{(x,u) \in W^{-1}([\mu, \bar{\sigma}] \times (U \cap B(0, N(\mu))))} \{ \nu(x, u) \} & \mu \in (0, \bar{\sigma}], \\ \max_{(x,u) \in W^{-1}(\{\mu\} \times (U \cap B(0, N(\mu))))} \{ \nu(x, u) \} & \mu > \bar{\sigma}. \end{cases}$$

- 1 The function  $\nu_0$  is well defined and locally bounded, because  $W$  is proper and  
 2  $\nu$ ,  $N$  are continuous. Moreover,  $\nu_0$  is decreasing on  $(0, \bar{\sigma}]$ . Let  $\nu_1$  be a smooth  
 3 approximation from above of  $\nu_0$ , decreasing on  $(0, 2\sigma]$  – like  $N$ . Then

$$\nu_1(W(x)) \geq \nu(x, u) \quad \forall x \in \mathbb{R}^n \setminus \mathcal{C}, \quad u \in U \cap B(0, N(W(x))). \quad (70)$$

- 4 Consider a nonnegative, smooth map  $\bar{\nu} = \bar{\nu}_R : [0, +\infty) \rightarrow [0, +\infty)$  such that  $\bar{\nu} \equiv 0$   
 5 in  $[0, \sigma]$  and  $\bar{\nu} := \nu_1$  in  $[2\sigma, +\infty)$ .  
 6 We set

$$\begin{aligned} \xi_R(\mu) &:= \mu + \int_0^\mu \bar{\nu}(r) dr \quad \forall \mu \in [0, +\infty), \quad \bar{W}_R := \xi_R \circ W, \\ \tilde{\gamma}_R(\mu) &:= \begin{cases} \frac{\tilde{\gamma}}{1+\nu_1} \circ \xi_R^{-1}(\mu) & \mu \in (0, 2\sigma], \\ \tilde{\gamma} \circ \xi_R^{-1}(\mu) & \mu > 2\sigma. \end{cases} \end{aligned} \quad (71)$$

- 7 The function  $\xi_R$  is the identity in  $[0, \sigma]$  and  $\xi_R(\mu) \geq \mu$  in  $(\sigma, +\infty)$ , so that  $\bar{W}_R \geq W$   
 8 in the whole set  $\mathbb{R}^n \setminus \mathcal{C}$  and  $\bar{W}_R \equiv W$  on  $W^{-1}([0, \sigma])$ . Hence, in particular,  $\bar{W}_R(x) =$   
 9  $W(x)$  when  $\mathbf{d}(x) \leq R$ . Moreover,  $\bar{W}_R$  is locally Lipschitz continuous, proper and  
 10 positive definite on  $\mathbb{R}^n \setminus \mathcal{C}$ ; it is also locally Lipschitz continuous on  $\overline{\mathbb{R}^n \setminus \mathcal{C}}$  or, by [4,  
 11 Proposition 2.1.12], locally semiconcave on  $\mathbb{R}^n \setminus \mathcal{C}$ , when  $W$  is. By the properties  
 12 of  $\xi_R$ , the decrease rate  $\tilde{\gamma}_R : (0, +\infty) \rightarrow (0, +\infty)$  is well defined, strictly increasing,  
 13 continuous except at point  $2\sigma$  and  $\tilde{\gamma}_R \leq \tilde{\gamma}$ . Since  $\tilde{\gamma} \leq \gamma$  by Proposition 5, then  
 14 there exists a positive, strictly increasing, smooth approximation from below  $\gamma_R$  of  
 15  $\tilde{\gamma}_R$  such that  $\gamma_R \leq \gamma$ : this concludes the proof of statement (i).

- 16 In order to prove (ii), namely that  $\bar{W}_R, \gamma_R$  verify the decrease condition (63), we  
 17 make use of the following result:

- 18 **Lemma 4.7.** *Let  $\Omega \subset \mathbb{R}^n$  be an open subset and let  $W : \Omega \rightarrow (0, +\infty)$  be a locally*  
 19 *Lipschitz continuous function. If  $\xi : (0, +\infty) \rightarrow (0, +\infty)$  is a strictly increasing,  $C^2$*   
 20 *function with  $\xi' > 0$ , then for every  $x \in \Omega$  one has*

$$\partial_L(\xi \circ W)(x) = \xi'(W(x)) \partial_L W(x). \quad (72)$$

- 21 *Proof.* Let us show that, given  $x \in \Omega$ ,

$$\partial_P(\xi \circ W)(x) = \xi'(W(x)) \partial_P W(x). \quad (73)$$

- 22 Then it immediately follows the thesis (72).

Let us begin by showing that  $\partial_P(\xi \circ W)(x) \subseteq \xi'(W(x)) \partial_P W(x)$ . Let  $p \in \partial_P W(x)$ . Then there exist a neighborhood of  $x$  and some  $\bar{\rho} > 0$  and  $\bar{\epsilon} > 0$ , such that

$$W(y) - W(x) + \bar{\rho}|y - x|^2 \geq \langle p, y - x \rangle \quad \forall y \in B(x, \bar{\epsilon}).$$

Since  $\xi \in C^2(0, +\infty)$ , by the Taylor expansion of  $\xi$  at  $\mu = W(x)$ , for any  $\tilde{\mu}$  in some neighbourhood of  $\mu$ , one has  $\xi(\tilde{\mu}) - \xi(\mu) = \xi'(\mu)(\tilde{\mu} - \mu) + \frac{\xi''(\mu)}{2} |\tilde{\mu} - \mu|^2 + o(|\tilde{\mu} - \mu|^2)$ .

For the local Lipschitz continuity of  $W$ , possibly reducing  $\bar{\varepsilon}$ , for every  $y \in B(x, \bar{\varepsilon})$ , one has

$$\begin{aligned} & \xi(W(y)) - \xi(W(x)) \\ &= \xi'(W(x))(W(y) - W(x)) + \frac{\xi''(W(x))}{2}|W(y) - W(x)|^2 + o(|W(y) - W(x)|^2). \end{aligned}$$

Since  $\xi' > 0$ , for every  $p \in \partial_P W(x)$  we derive that

$$\begin{aligned} \xi(W(y)) - \xi(W(x)) &\geq \xi'(W(x)) [\langle p, y - x \rangle - \bar{\rho}|y - x|^2] \\ &\quad + L_W^2 \frac{\xi''(W(x)) \wedge 0}{2} |y - x|^2 + o(|y - x|^2), \end{aligned}$$

where  $L_W > 0$  denotes the (local) Lipschitz constant of  $W$ . Hence

$$\xi(W(y)) - \xi(W(x)) + \rho|y - x|^2 \geq \langle \xi'(W(x)) p, y - x \rangle,$$

1 as soon as  $\rho > 0$  verifies  $\rho \geq \left[ \xi'(W(x)) \bar{\rho} - L_W^2 \frac{\xi''(W(x)) \wedge 0}{2} + \frac{o(|y-x|^2)}{|y-x|^2} \right]$ , so that  $\bar{p} :=$   
 2  $\xi'(W(x)) p \in \xi'(W(x)) \partial_P W(x)$  and the inclusion  $\xi'(W(x)) \partial_P W(x) \subseteq \partial_P(\xi \circ W)(x)$   
 3 is proved.

4 The assumption  $\xi' > 0$  implies that the inverse function  $\xi^{-1}$  is strictly increasing  
 5 and  $C^2$ , as  $\xi$ . Hence the opposite inclusion  $\partial_P(\xi \circ W)(x) \subseteq \xi'(W(x)) \partial_P W(x)$ , can  
 6 be obtained by applying the previous arguments to  $\xi \circ W$  and  $\xi^{-1}$  in place of  $W$   
 7 and  $\xi$ , respectively. This yields the equality (73) and the proof of the lemma is  
 8 concluded.  $\square$

9 By Lemma 4.7, for every  $x \in \mathbb{R}^n \setminus \mathcal{C}$  we have  $\partial_L \bar{W}_R(x) = (1 + \bar{\nu}(W(x))) \partial_L W(x)$ ,  
 10 so that given an arbitrary  $\bar{p} \in \partial_L \bar{W}_R(x)$ , there exists some  $p \in \partial_L W(x)$  such that

$$\bar{p} = (1 + \bar{\nu}(W(x))) p. \quad (74)$$

Let  $\bar{u} \in U \cap B(0, N(W(x)))$  satisfy

$$\langle f(x, \bar{u}), p \rangle + p_0 l(x, \bar{u}) = \min_{U \cap B(0, N(W(x)))} \{ \langle f(x, u), p \rangle + p_0 l(x, u) \}.$$

11 By (61), (70), (71), and (74), when  $x \in W^{-1}((0, 2\sigma])$  one has  $\langle f(x, \bar{u}), p \rangle < 0$  and

$$\begin{aligned} H_{\bar{f}, \bar{l}}(x, p_0, \bar{p}) &\leq \langle \bar{f}(x, \bar{u}), \bar{p} \rangle + p_0 \bar{l}(x, \bar{u}) = (1 + \bar{\nu}(W(x))) \langle \bar{f}(x, \bar{u}), p \rangle + p_0 \bar{l}(x, \bar{u}) \\ &= \frac{1}{1 + \nu(x, \bar{u})} [(1 + \bar{\nu}(W(x))) \langle f(x, \bar{u}), p \rangle + p_0 l(x, \bar{u})] \\ &< -\frac{\tilde{\gamma}(W(x))}{1 + \nu(x, \bar{u})} \leq -\gamma_R(\bar{W}_R(x)). \end{aligned}$$

If otherwise  $x \in W^{-1}((2\sigma, +\infty))$ , then  $\bar{\nu}(W(x)) = \nu_1(W(x))$  and, recalling that  
 $\langle f(x, \bar{u}), p \rangle < 0$  and  $p_0 l(x, \bar{u}) \geq 0$ , we get by (70)

$$\begin{aligned} H_{\bar{f}, \bar{l}}(x, p_0, \bar{p}) &\leq \langle \bar{f}(x, \bar{u}), \bar{p} \rangle + p_0 \bar{l}(x, \bar{u}) \\ &= (1 + \nu_1(W(x))) \langle \bar{f}(x, \bar{u}), p \rangle + p_0 \bar{l}(x, \bar{u}) \\ &\leq \frac{1 + \nu_1(W(x))}{1 + \nu(x, \bar{u})} [\langle f(x, \bar{u}), p \rangle + p_0 l(x, \bar{u})] \\ &< -\tilde{\gamma}(W(x)) \leq -\gamma_R(\bar{W}_R(x)), \end{aligned}$$

12 and this implies the validity of (ii).

13 The proof of statement (iii) follows by the arguments above, by simply replacing  
 14  $\bar{u}$  with  $K(x)$ , where  $K$  is a feedback as in Proposition 5.  $\square$

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