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Injective modules over the Jacobson algebra $K\langle X, Y \mid XY = 1\rangle$

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Abstract. For a field K, let $\mathcal{R}$ denote the Jacobson algebra $K\langle X, Y \mid XY = 1\rangle$. We give an explicit construction of the injective envelope of each of the (infinitely many) simple left $\mathcal{R}$-modules. Consequently, we obtain an explicit description of a minimal injective cogenerator for $\mathcal{R}$. Our approach involves realizing $\mathcal{R}$ up to isomorphism as the Leavitt path K-algebra of an appropriate graph $\mathcal{T}$, which thereby allows us to utilize important machinery developed for that class of algebras.

1 Introduction

A unital ring A is called *directly finite* in case, for any $x, y \in A$, if $xy = 1$ then $yx = 1$. It is not hard to show that rings which satisfy various natural conditions (commutativity, some mild chain condition, and so on) are directly finite. On the other hand, examples abound of rings containing elements x, y for which $xy = 1$ but $yx \neq 1$. Perhaps the most natural 'concrete' example is found in the endomorphism ring of a countably-infinite-dimensional vector space V over a field. Here, if $\{e_i \mid i \in \mathbb{N}\}$ is a basis for V, then the right shift transformation y which takes e_i to e_{i+1} and the left shift transformation x which takes e_1 to 0 and e_i to e_{i-1} for $i \geq 2$ satisfy $xy = 1$ but $yx \neq 1$. A moment's reflection yields that there is an even more natural example of a ring which fails to be directly finite, to wit:

$$\mathcal{R} = K\langle X, Y \mid XY = 1\rangle,$$

the free associative K-algebra on two (noncommuting) generators, modulo the single relation $XY = 1$. A search of the literature suggests that this algebra was first explicitly studied by Jacobson in the late 1940s in [13]. Throughout the article we will refer to this algebra as the *Jacobson algebra* over K.[1] While the displayed description of $\mathcal{R}$ is quite straightforward, the structure of $\mathcal{R}$ is anything but.

Various ring-theoretic and module-theoretic properties of $\mathcal{R}$ have been analyzed during the seven decades since Jacobson's work, including in: Cohn [10] (1966);

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[1] Because of its close relationship to the well-studied Toeplitz C^*-algebra, the Jacobson K-algebra $\mathcal{R}$ has also been called the "algebraic Toeplitz K-algebra" elsewhere in the literature, see *e.g.* [1]. We prefer to call $\mathcal{R}$ the "Jacobson algebra" to further emphasize our focus on $\mathcal{R}$ as an algebra in and of itself, rather than on its connection to graph C^*-algebras.

Bergman [8] (1974); Gerritzen [11] (2000); Bavula [7] (2010); Ara and Rangaswamy [6] (2014); Iovanov and Sistko [12] (2017); and Lu, Wang and Wang [15] (2019).

For the directed graph $\mathcal{T} = \circlearrowleft \bullet \longrightarrow \bullet$, the Jacobson algebra $\mathcal{R}$ is isomorphic to the Leavitt path algebra $L_K(\mathcal{T})$ (see Proposition 2.1 below). This interpretation guides our investigation. We refer those readers who are unfamiliar with Leavitt path algebras to [1, Chapter 1].

Following [6], there are three classes of *Chen* simple modules for Leavitt path algebras $L_K(E)$ of a general (finite) graph E:

- simple modules associated to sinks;
- simple modules associated to infinite irrational paths, and
- simple modules associated to infinite rational paths and irreducible polynomials in $K[x]$ with constant term equal to -1.

By [6, Corollary 4.6] a complete list of nonisomorphic simple left $L_K(\mathcal{T})$-modules is given by

- the Chen simple module $L_K(\mathcal{T})w$ associated to the sink w, and
- the Chen simple modules V^f associated to the infinite rational path c^∞ (where c is the loop in $\mathcal{T}$), and to irreducible polynomials $f(x)$ in $K[x]$ with $f(0) = -1$.

Among other things, results regarding the Ext^1 groups of pairs of Chen simple modules, the Bézout property, the construction of "Prüfer-like" modules for Chen simple modules, and the construction of injective envelopes for some of these Chen simples have been achieved in previous collaborative work of the three coauthors ([2], [3], and [4]). In the current work we bundle some of the consequences of these results together with a new type of construction in the specific case where $E = \mathcal{T}$.

Our two main goals of the article are as follows. First, we explicitly construct the injective envelope of each simple left $L_K(\mathcal{T})$-module. For modules of the form V^f as described above, this is achieved in Corollary 6.3. For the module $L_K(\mathcal{T})w$ this is done in Corollary 6.12. Second, we use the information achieved in the first goal to describe a minimal injective cogenerator for the category $L_K(\mathcal{T})$-Mod (Theorem 6.14). This is the the first time in the literature that an injective cogenerator for a non-Noetherian Leavitt path algebra is completely described. In particular, the structure of all injective $L_K(\mathcal{T})$-modules, and hence of all representations of $L_K(\mathcal{T})$, is revealed.

2 Prerequisites

We set some notation. We denote by $\mathbb{N}$ the set of positive integers $\{1, 2, 3, \dots\}$, and by $\mathbb{Z}^+$ the set $\mathbb{N} \cup \{0\} = \{0, 1, 2, \dots\}$.

The word "module" will always mean "left module". For $f(x) \in K[x]$ and $n \in \mathbb{N}$ we denote $(f(x))^n$ by $f^n(x)$.

For any polynomial $g(x) = \sum_{i=0}^m k_i x^i \in K[x]$, and the cycle c in $\mathcal{T}$, we denote by $g(c)$ the element

$$g(c) := k_0 1_{L_K(\mathcal{T})} + k_1 c + \cdots + k_m c^m \in L_K(\mathcal{T}).$$

Rewritten, $g(c) = k_0 v + k_0 w + k_1 c + \cdots + k_m c^m \in L_K(\mathcal{T})$. This notation is well suited for our purposes, *but we note that this definition of $g(c)$ is different from that used for* *expressions of the form $g(c)$ elsewhere in the literature.* For $g(x) = \sum_{i=0}^{m} k_i x^i \in K[x]$ we denote by $g\big|_v(c)$ the element

$$g\big|_v(c) := k_0 v + k_1 c + \cdots + k_m c^m \in L_K(\mathcal{T}).$$

So $g(c) = k_0 w + g\big|_v(c)$ and $g\big|_v(c) = v g(c)$.

We denote by $\mathcal{P}$ the set of polynomials

$$\mathcal{P} := \{p(x) \in K[x] \mid p(0) \neq 0\},$$

and by $\mathcal{F} \subseteq \mathcal{P}$ the set of polynomials

$$\mathcal{F} := \{f(x) \in K[x] \mid f \text{ is irreducible in } K[x], \text{and } f(0) = -1_K\}.$$

We note that the family $\mathcal{F}$ is a set of pairwise nonassociate representatives of the irreducible elements in the ring of Laurent polynomials $K[x, x^{-1}]$.

Because the Leavitt path algebra $L_K(\mathcal{T})$ plays a central role in our investigations, we give a detailed description of it here. For the directed graph

$$\mathcal{T} = \begin{tikzcd} \bullet^v \arrow[loop left, "c"] \arrow[r, "d"] & \bullet^w \end{tikzcd},$$

we consider the *extended graph* $\widehat{\mathcal{T}}$ of $\mathcal{T}$, pictured as:

$$\widehat{\mathcal{T}} = \begin{tikzcd} \bullet^v \arrow[loop below, "c"'] \arrow[loop above, "c^*"] \arrow[r, bend right, "d"'] & \bullet^w \arrow[l, bend right, "d^*"'] \end{tikzcd}.$$

Then $L_K(\mathcal{T})$ is defined to be the standard path algebra $K\widehat{\mathcal{T}}$ of $\widehat{\mathcal{T}}$ with coefficients in K, modulo these relations:

$$c^*c = v;\ \ d^*d = w;\ \ c^*d = d^*c = 0;\ \ \text{and}\ \ cc^* + dd^* = v.$$

In particular, $v + w = 1_{L_K(\mathcal{T})}$.

Proposition 2.1 *[1, Proposition 1.3.7] Let K be any field, and let $\mathcal{T}$ be the graph $\begin{tikzcd} \bullet^v \arrow[loop left, "c"] \arrow[r, "d"] & \bullet^w \end{tikzcd}$. Then $\mathcal{R} \cong L_K(\mathcal{T})$ as K-algebras.*

Proof In $L_K(\mathcal{T})$ we have

$$(c^* + d^*)(c + d) = v + 0 + 0 + w = 1_{L_K(\mathcal{T})}, \ \ \text{and}$$
$$(c + d)(c^* + d^*) = cc^* + 0 + 0 + dd^* = v \neq 1_{L_K(\mathcal{T})}.$$

With this as context, one can show that the map

$$\varphi : \mathcal{R} \to L_K(\mathcal{T}) \ \ \text{given by the extension of } \ \varphi(X) = c^* + d^*, \ \varphi(Y) = c + d$$

is an isomorphism of K-algebras. ∎

In particular, note that the element c of $L_K(\mathcal{T})$ corresponds to the element Y^2X of $\mathcal{R}$ under this isomorphism.

Corollary 2.2 *The Jacobson algebra is (left and right) hereditary. Specifically, quotients of injective left $\mathcal{R}$-modules are injective.*

Proof By [1, Theorem 3.2.5], the Leavitt path algebra $L_K(E)$ for any finite graph E is hereditary. But hereditary rings have the specified property by [14, Theorem 3.22]. ■

An application of [1, Corollary 1.5.12] yields the following useful description of a K-basis of $L_K(\mathcal{T})$.

Lemma 2.3 *The following set forms a K-basis of $L_K(\mathcal{T})$:*

$$v,\ w,\ d,\ d^*,\ c^i,\ c^i d,\ c^i(c^*)^j,\ (c^*)^j,\ d^*(c^*)^j$$

where $i, j \geq 1$.

We conclude the Prerequisites section by giving some properties of the simple left $L_K(\mathcal{T})$-modules of the form V^f, where $f(x) \in \mathcal{F}$. Some of these properties follow from results which were established in [6]. We will develop here some additional information about these simple modules which will be needed in the sequel. Although we will not actually utilize the following piece of information until the final section of the article, we reiterate here that because there is a unique cycle in $\mathcal{T}$, [6, Corollary 4.6] applies. This yields that, up to isomorphism, all but one of the simple modules over $L_K(\mathcal{T})$ are of the form V^f, where $f(x) \in \mathcal{F}$. The only other simple $L_K(\mathcal{T})$-module is $L_K(\mathcal{T})w$.

We now make a detailed presentation of the construction of the modules V^f. Assume $f(x) \in \mathcal{F}$ has degree n. Denote by K' the field $K[x]/\langle f(x)\rangle$ and by $\overline{x}$ the element $x + \langle f(x)\rangle$. Clearly $\{1, \overline{x}, \dots, \overline{x}^{n-1}\}$ is a K-basis of K'. The class of infinite paths tail equivalent to c^∞ consists only of c^∞ itself. Let $V^{\overline{x}}$ be the one dimensional K'-vector space generated by c^∞. Setting

$$d \star c^\infty = d^* \star c^\infty = w \star c^\infty = 0;$$
$$v \star c^\infty = c^\infty;\ c \star c^\infty = \overline{x}c^\infty;\ \text{and } c^* \star c^\infty = \overline{x}^{-1}c^\infty,$$

$V^{\overline{x}}$ becomes a left $L_{K'}(\mathcal{T})$-module. Consider the linear maps

$$\sigma^{\overline{x}} : V^{\overline{x}} \to K', \quad h \cdot c^\infty \mapsto h, \text{ and}$$
$$\rho^{\overline{x}} : K' \to V^{\overline{x}}, \quad h \mapsto h \cdot c^\infty.$$

Clearly these maps are inverse isomorphisms of one-dimensional K'-vector spaces.Restricting the scalars to K, the abelian group $V^{\overline{x}}$ also has a left $L_K(\mathcal{T})$-module structure: we denote this left $L_K(\mathcal{T})$-module by V^f.

The set $\{c^\infty, \overline{x}c^\infty, \dots, \overline{x}^{n-1}c^\infty\}$ is a K-basis of V^f. Denote by G^f the K-subspace of $L_K(\mathcal{T})$ generated by $\{1, c, \dots, c^{n-1}\}$. We note that any element in G^f clearly commutes with $f(c)$. The linear maps

$$\sigma^f : V^f \to G^f, \quad \overline{x}^i c^\infty \mapsto c^i, \text{ and}$$
$$\rho^f : G^f \to V^f, \quad c^i \mapsto \overline{x}^i c^\infty$$

(for $0 \le i \le n-1$) define inverse isomorphisms of n-dimensional K-vector spaces. The map ρ^f is the restriction to G^f of the right multiplication map by c^∞:

$$\rho : L_K(\mathcal{T}) \to V^f,\ r \mapsto r \star c^\infty.$$

Clearly one has

$$\sigma^f(\overline{x}^i c^\infty) \star c^\infty = c^i \star c^\infty = \overline{x}^i c^\infty = \rho^f(c^i) = \rho(c^i).$$

Lemma 2.4 [6, Lemma 3.3] *Let $f(x) \in \mathcal{F}$. Then the left $L_K(\mathcal{T})$-module V^f is simple.*

Proof Let U be a nonzero $L_K(\mathcal{T})$-submodule of V^f. Since $\{1, \overline{x}, \ldots, \overline{x}^{n-1}\}$ is a K basis for K' and

$$\overline{x} \cdot u = c \star u \quad \forall u \in U,$$

U is also a K'-space. Since $V^{\overline{x}}$ is a one-dimensional K'-space, we have $U = V^{\overline{x}}$ as K'-spaces and hence $U = V^f$ as left $L_K(\mathcal{T})$-modules. ∎

Throughout the remainder of the article, we will often denote $L_K(\mathcal{T})$ simply by R.

3 The Division Algorithm

The goal of this short section is to establish *The Division Algorithm*, Proposition 3.4. This result will subsequently be used to construct the injective envelope of each simple R-module of the form V^f. We start by showing that each V^f is finitely presented. We also determine the annihilator of each V^f.

Lemma 3.1 *Let $f(x) \in \mathcal{F}$. Denoting by $\hat{\rho}_{f(c)} : R \to R$ the right multiplication map by $f(c)$, we have the following short exact sequence of left R-modules:*

$$0 \longrightarrow R \xrightarrow{\hat{\rho}_{f(c)}} R \xrightarrow{\rho} V^f \longrightarrow 0\ .$$

In particular:

(1) *The kernel of $\rho : R \to V^f$ is $Rf(c)$.*
(2) *$Rf(c)$ coincides with the two-sided ideal $\operatorname{Ann}_R(V^f)$.*

Proof We have already observed that ρ^f is surjective, and thus ρ is surjective as well.

For the injectivity of $\hat{\rho}_{f(c)}$, we note that any element $f(x) \in \mathcal{F}$ can be written as $f(x) = xg(x) - 1$, and so $f(c) = cg(c) - 1 \in R$, for a suitable polynomial $g(x) \in K[x]$. Let $r \in R$ such that $\hat{\rho}_{f(c)}(r) = 0$. So $r(cg(c) - 1) = 0$, and thus $rcg(c) = r$, which recursively implies $r(cg(c))^j = r$ for any $j \ge 1$. Now write $r = \sum_{i=1}^n k_i \alpha_i {\beta_i}^*$, where the α_i and β_i are in $\operatorname{Path}(\mathcal{T})$. We note that, for any $\beta \in \operatorname{Path}(\mathcal{T})$, there exists a suitable m_β such that $\beta^*(cg(c))^{m_\beta}$ is either 0 or an element of $K\mathcal{T}$. Now let N be the maximum in the set $\{m_{\beta_1}, m_{\beta_2}, \ldots, m_{\beta_n}\}$. Then the above discussion shows that $r(cg(c))^N$ is an element of R of the form $\sum_{i=1}^n k_i \gamma_i$, where $\gamma_i \in K\mathcal{T}$ for $1 \le i \le n$. That is, $r(cg(c))^N \in K\mathcal{T}$. But $r(cg(c))^N = r$, so that $r \in K\mathcal{T}$. However, the equation $r(cg(c)) = r$ (i.e., $rf(c) = 0$) has only the zero solution in $K\mathcal{T}$ by a degree argument. So $r = 0$.

(1) We now show $\operatorname{Ker}\rho = Rf(c)$. Using [6, Lemma 3.2], we get that the annihilator of V^f is the two-sided ideal $I = \langle w, f\big|_v(c)\rangle$. Notice that $w = -w(cg(c)-1) = -wf(c)$. Therefore, in the notation used herein, we have $I = \langle f(c)\rangle$. Clearly I is contained in the kernel of ρ. Let $r \in \operatorname{Ker}\rho$. To prove that $r \in I$ we have to check that $r \star \overline{x}^i c^\infty = 0$ for $i = 0,\dots,n-1$; in other words, that left multiplication by r annihilates all the elements of a K-basis of V^f. We consider the left $L_{K'}(\mathcal{T})$-module $V^{\overline{x}}$. Since $\overline{x}^i$ is a scalar in $L_{K'}(\mathcal{T})$, and $r \star c^\infty = 0$ we have the following equality in $V^{\overline{x}}$:

$$r \star \overline{x}^i c^\infty = \overline{x}^i r \star c^\infty = 0.$$

Since $V^{\overline{x}} = V^f$ as abelian groups, the desired result follows.

(2) We prove now that $Rf(c) = \langle f(c)\rangle$. It is sufficient to check that the product of $f(c)$ on the right by each element of the K-basis of R highlighted in Lemma 2.3 belongs to $Rf(c)$. First of all observe that

$$w = -wf(c) \in Rf(c), \ d = dw \in Rf(c), \ d^* = -d^* f(c) \in Rf(c).$$

Then clearly each of

$$f(c)v = vf(c), \ f(c)w, \ f(c)d, \ f(c)d^*, \ f(c)c^i = c^i f(c), \text{ and } f(c)c^i d$$

is in $Rf(c)$. Assume $f(c) = -1 + k_1 c + \cdots + k_n c^n$. Then

$$\begin{aligned} f(c)c^* &= -c^* + k_1 cc^* + \cdots + k_n c^n c^* \\ &= -c^* + k_1(1-dd^*) + \cdots + k_n c^{n-1}(1-dd^*) \\ &= (-c^* + k_1 + \cdots + k_n c^{n-1}) - (k_1 d + \cdots + k_n c^{n-1} d)d^* \\ &= c^* f(c) + rd^* \in Rf(c). \end{aligned}$$

Then, by induction, $f(c)(c^*)^j \in Rf(c)$ for each $j \geq 0$. Finally, $f(c)c^i(c^*)^j = c^i f(c)(c^*)^j \in Rf(c)$ and $f(c)d^*(c^*)^j = d^* f(c)(c^*)^j \in Rf(c)$ for each $j \geq 0$. ∎

Remark 3.2 As mentioned in the Preliminaries section, for $f(x) = -1 + \sum_{i=1}^n k_i x^i$ in $\mathcal{F}$, we define $f(c) = -1_R + \sum_{i=1}^n k_i c^i \in R$. We established in Lemma 3.1(1) that right multiplication by $f(c)$ is injective. If one were to instead use the notation for $f(c)$ which appears elsewhere in the literature (namely, $f(c) := -v + \sum_{i=1}^n k_i c^i$), then the right multiplication map by $f(c)$ would not be injective.

Lemma 3.3 *For any $f(x) \in \mathcal{F}$, the intersection of $Rf(c)$ with G^f is 0.*

Proof If ℓ belongs to $Rf(c) \cap G^f$, then $\rho(\ell) = 0$ by Lemma 3.1(1), so that

$$0 = \sigma^f(0) = \sigma^f(\rho(\ell)) = \sigma^f(\rho^f(\ell)) = \ell$$

(using $\rho^f(\ell) = \rho(\ell)$ since $\ell \in G^f$). ∎

Proposition 3.4 (The Division Algorithm) *Let $f(x) \in \mathcal{F}$. For any $\beta \in R$ there exists unique $q_\beta \in R$ and $r_\beta \in G^f$ such that*

$$\beta = q_\beta f(c) + r_\beta.$$

Proof Consider the element $r_\beta := \sigma^f(\rho(\beta))$. Clearly r_β belongs to $G^f \subseteq R$. Let us prove that the difference $\beta - r_\beta$ belongs to $\operatorname{Ker}\rho$. By Lemma 2.3, it is sufficient to prove that $\beta - r_\beta$ belongs to $\operatorname{Ker}\rho$ for $\beta \in \{v, w, d, c^i, c^i d, c^i(c^*)^j, (c^*)^j, d^*(c^*)^j\}$. Whenever $\rho(\beta) = 0$, then also $r_\beta = 0$ and hence $\beta - r_\beta$ belongs to $\operatorname{Ker}\rho$ in these cases. So the result immediately holds for $\beta = w, d, c^i d$, and $d^*(c^*)^j$. For the others:

$$r_v = \sigma^f(c^\infty) = 1_K,\ r_{c^i} = \sigma^f(\overline{x}^i c^\infty) = c^i,$$

$$r_{c^i(c^*)^j} = \sigma^f(\overline{x}^{i-j} c^\infty) = \begin{cases} c^{i-j} & \text{if } i > j \geq 0, \\ 1_K & \text{if } i = j \geq 0, \\ (c^*)^{j-i} & \text{if } 0 \leq i < j, \end{cases}$$

and clearly $v - 1_K$, $c^i - r_{c^i}$ (which is 0), and $c^i(c^*)^j - c^{i-j}$ for $i > j$, $c^i(c^*)^i - 1_K$, $c^i(c^*)^j - (c^*)^{j-i}$ for $i < j$, belong to $\operatorname{Ker}\rho$. By Lemma 3.1, $\operatorname{Ker}\rho = Rf(c)$. Therefore $\beta - r_\beta = q_\beta f(c)$ for a suitable $q_\beta \in R$.

We now prove that $q_\beta \in R$ and $r_\beta \in G^f$ are uniquely determined. Assume

$$\beta = q_1 f(c) + r_1 = q_2 f(c) + r_2.$$

Then we have $r_1 - r_2 = (q_2 - q_1)f(c) \in Rf(c) \cap G^f$, which is 0 by Lemma 3.3. Therefore $r_1 = r_2$ and $(q_1 - q_2)f(c) = r_1 - r_2 = 0$. Since by Lemma 3.1 right multiplication by $f(c)$ is injective, we have $q_1 = q_2$. ■

4 The Prüfer Modules U^f

For any simple R-module V^f there exists a uniserial R-module U^f of infinite length, all of whose composition factors are isomorphic to V^f. We call U^f *the Prüfer module* associated to V^f. The construction of U^f is a particular case of a method of building injective modules over general Leavitt path algebras described in [4].

Lemma 4.1 *For any $f(x) \in \mathcal{F}$, the element $f(c) \in R$ is neither a right zero divisor nor left-invertible.*

Proof The element $f(c) \in R$ is not a right zero divisor, since the right multiplication $\hat{\rho}_{f(c)} : R \to R$ is injective by Lemma 3.1. By that same Lemma we also have $f(c) \star c^\infty = 0$ in V^f, and so $f(c)$ is not left invertible in R. ■

The upshot of Lemma 4.1 is that we can apply the construction of the Prüfer module described in [4, Section 2] with $a = f(c)$. For each natural number $n \geq 1$, set

- $M_n^f := R/Rf^n(c)$, the nonzero cyclic left R-module generated by $1 + Rf^n(c)$.
- $\eta_n^f : R \to M_n^f$ the canonical projection.
- $\theta_n^f : Rf(c) \to M_n^f$, $f(c) \mapsto 1 + Rf^n(c)$.
- $\psi_{i,\ell} : M_i^f \to M_\ell^f, 1 + Rf^i(c) \mapsto f^{\ell-i}(c) + Rf^\ell(c)$ for each $i \leq \ell$; the cokernel of $\psi_{i,\ell}$ is isomorphic to $M_{\ell-i}^f$.

With this notation, the diagram

$$\begin{tikzcd} R \arrow[r, "\hat{\rho}_{f(c)}"] \arrow[d, "\eta^f_{n-1}"'] & R \arrow[d, "\eta^f_n"] \\ M^f_{n-1} \arrow[r, "\psi_{n-1,n}"] & M^f_n \end{tikzcd} \quad \text{or, equivalently, the diagram} \quad \begin{tikzcd} Rf(c) \arrow[r, hook, "i"] \arrow[d, "\theta^f_{n-1}"'] & R \arrow[d, "\eta^f_n"] \\ M^f_{n-1} \arrow[r, "\psi_{n-1,n}"] & M^f_n \end{tikzcd}$$

is a pushout diagram. By Lemma 2.4, $M^f_1 \cong V^f$ is a simple R-module.

We now establish the key property of the modules $\{M^f_i \mid i \in \mathbb{N}\}$ which will allow us to further apply additional machinery built in [4].

Lemma 4.2 *Let $f(x) \in \mathcal{F}$. Then the equation $f(c)\mathbb{X} = 1 + Rf^n(c)$ has no solutions in the left R-module M^f_n.*

Proof Let $m + Rf^n(c) \in M^f_n$, with $m \in R$. By a repeated application of Proposition 3.4, we have

$$m = q_1 f(c) + g_1,\ q_1 = q_2 f(c) + g_2,\ \ldots\ q_{n-1} = q_n f(c) + g_n,$$

where the elements g_i $(1 \le i \le n)$ belong to G^f. Therefore

$$m - \big(g_1 + g_2 f(c) + \cdots + g_n f^{n-1}(c)\big) \in Rf^n(c).$$

In particular we can assume that the representative m of the coset $m + Rf^n(c)$ is equal to $g_1 + g_2 f(c) + \cdots + g_n f^{n-1}(c)$. Assume $f(c)m + Rf^n(c) = 1 + Rf^n(c)$. Then $f(c)m - 1$ belongs to $Rf^n(c)$. Therefore

$$f(c)\big(g_1 + g_2 f(c) + \cdots + g_n f^{n-1}(c)\big) - 1$$

belongs to $Rf^n(c)$. Since as noted above $f(c)g_i = g_i f(c)$ for each $1 \le i \le n$, we get

$$-1 + g_1 f(c) + g_2 f^2(c) + \cdots + g_n f^n(c) \in Rf^n(c).$$

Then $-1 = rf(c)$ for a suitable $r \in R$ and hence $f(c)$ would be left invertible in R, which contradicts Lemma 4.1. ■

With Lemma 4.2 established, we may apply [4, Proposition 2.2] to conclude that each left R-module M^f_n $(n \in \mathbb{N})$ is uniserial of length n. We define

$$U^f := \varinjlim \left\{M^f_i, \psi_{i,j}\right\}_{i \le j},$$

and, for each $i \in \mathbb{N}$, the induced monomorphism

$$\psi_i : M^f_i \to U^f.$$

By [4, Proposition 2.4], U^f is uniserial and artinian.

For each $n \in \mathbb{N}$, the element

$$\alpha_{n,f} := \psi_n(1 + Rf^n(c))$$

is a generator of the submodule $\psi_n(M_n^f)$ of U^f. In the sequel, to simplify the notation, we will denote by M_n^f the submodule $\psi_n(M_n^f)$ of U^f, in fact identifying M_n^f with its image in U^f through the monomorphism ψ_n. Let $r\alpha_{n,f} = r + Rf^n(c)$ be a generic element of U^f. Applying the Division Algorithm (Proposition 3.4) $n-1$ times, we get

$$\begin{aligned} r\alpha_{n,f} = r + Rf^n(c) &= g_0^f + g_1^f f(c) + \cdots + g_{n-1}^f f^{n-1}(c) + Rf^n(c) \\ &= \left(g_0^f + g_1^f f(c) + \cdots + g_{n-1}^f f^{n-1}(c)\right)\alpha_{n,f} \end{aligned}$$

for suitable $g_0^f, \ldots, g_{n-1}^f \in G^f$.

Remark 4.3 As an immediate consequence of Lemma 4.2, we see that any R-module of the form M_i^f is *not* injective, because the map $\psi : R \to M_i^f$ defined by setting $\psi(1) = 1 + Rf^i(c)$ does not factor through the monomorphism $\hat{\rho}_{f(c)} : R \to R$. In particular, the simple module $M_1^f \cong V^f$ is not injective. However, in the next section, we will show that each $U^f = \varinjlim\{M_i^f\}$ *is* an injective left R-module.

5 The Left Ideals in $R = L_K(\mathcal{T})$

In order to test whether a module is injective by applying Baer's criterion, we must have available a complete description of the left ideals in R. We will show that any ideal of R is either a direct summand of a left ideal of the form $Rp(c)$ (where $p(x) \in K[x]$ has $p(0) = 1$), or a direct summand of $\mathrm{Soc}(R)$. We recall that $\mathcal{P}$ denotes the set of polynomials $p(x) \in K[x]$ with $p(0) \neq 0$.

Remark 5.1 We collect up in this remark some properties of $J := \mathrm{Soc}(R)$, the socle of R. It is well known (or see [1, Theorem 2.6.14]) that $J = \langle w \rangle$ as a two-sided ideal. Further, as left R-ideals,

$$J = Rw \oplus (\oplus_{i\in\mathbb{Z}^+} Rc^i dd^*(c^*)^i) = Rw \oplus (\oplus_{i\in\mathbb{Z}^+} Rd^*(c^*)^i).$$

Moreover, each summand of the form $Rc^i dd^*(c^*)^i$ is isomorphic to the simple module Rw.

It has been noted elsewhere in the literature (see e.g. [16, Example 4.5]) that $R/J \cong K[x, x^{-1}]$ as K-algebras. This isomorphism is also as left R-modules (and left R/J-modules), which is not hard to see directly. Indeed, the standard monomials in R end (on the right) with a term having one of the forms $v, w, d, d^*, c^i, c^i d, c^i(c^*)^j$, $(c^*)^j$, or $d^*(c^*)^j$. Moreover, we have

$$w \equiv d \equiv d^* \equiv c^i d \equiv d^*(c^*)^j \equiv 0 \bmod J, \quad \text{while} \quad v \equiv cc^* \equiv c^* c \equiv 1 \bmod J.$$

So the only terms which survive mod J are powers of c (positive or negative).

The standard bijective correspondence between left ideals of R which contain J and submodules of R/J, together with the well-known principal ideal structure of $K[x, x^{-1}]$, yields that every left ideal of R which properly contains J is of the form $Rp\big|_v(c)$ for some $p(x) \in \mathcal{P}$. But $w \in J$ and $J \subseteq Rp\big|_v(c)$ together yield that $Rp\big|_v(c) = Rp(c)$. The upshot is that every left ideal of R which properly contains J is of the form $Rp(c)$ for some $p(x) \in \mathcal{P}$. ■

Proposition 5.2 *Let $f(x) \in \mathcal{F}$. Then $Hom_R(J, U^f) = \{0\}$.*

Proof For any $f(x) \in \mathcal{F}$ we have $\mathrm{Hom}_R(Rw, V^f) \cong wV^f = \{0\}$, because V^f is generated as a K-space by elements of the form $\overline{x}^i c^\infty$ ($0 \le i \le \deg(f) - 1$), and $w\overline{x}^i c^\infty = \overline{x}^i wc^\infty = 0$.

By [4, Proposition 2.2], the composition factors of the finitely generated submodules of U^f are isomorphic to V^f. This together with the previous paragraph implies $\mathrm{Hom}_R(Rw, U^f) = \{0\}$.

As noted in Remark 5.1, $J = Rw \oplus (\oplus_{i \in \mathbb{Z}^+} Rc^i dd^*(c^*)^i) \cong \oplus_{i\in\mathbb{Z}^+} Rw$. Then $\mathrm{Hom}_R (J, U^f) \cong \mathrm{Hom}_R(\oplus_{i\in\mathbb{Z}^+} Rw, U^f) \cong \prod_{i\in\mathbb{Z}^+} \mathrm{Hom}_R(Rw, U^f) = \prod_{i\in\mathbb{Z}^+}\{0\} = \{0\}$. ∎

Proposition 5.3 *Let I be a left ideal of R. Then either:*

1) *There exists $p(x) \in \mathcal{P}$ for which I is a direct summand of $Rp(c)$, or*
2) *I is a direct summand of $J = \mathrm{Soc}(R)$.*

Proof Case 1. J is properly contained in I. By Remark 5.1, we have $I = Rp(c)$ for some $p(x) \in \mathcal{P}$, and so we are done in this case.

Case 2. Suppose I is not contained in J, and I does not contain J. Consider the left ideal $A = I + J$. Then A properly contains J, so we may apply the Case 1 analysis to A, so that $A = Rq(c)$ for some $q(x) \in \mathcal{P}$. Since the socle J is a direct sum of simple left R-modules, we have $J = (I \cap J) \oplus B$ for some left ideal B of R contained in J. It is straightforward to show that this implies $A = I \oplus B$. But then I has been shown to be a direct summand of $A = Rq(c)$, as desired.

Case 3. Suppose I is contained in J. Then the semisimplicity of J immediately implies that I is a direct summand of J. ∎

Remark 5.4 We note that Gerritzen in [11, Proposition 3.4] established that all one-sided ideals of the Jacobson algebra $\mathcal{R}$ are either principal, or contained in the socle of $\mathcal{R}$. Similarly, Iovanov and Sistko in [12, Theorem 2 and Corollary 1] establish the same type of result in $\mathcal{R}$, in terms of polynomials in the element x of $\mathcal{R}$. By a previous observation, the element c of R corresponds to the element Y^2X of $\mathcal{R}$. The point to be made here is that while these two results from [11] and [12] are clearly related to the conclusion of Proposition 5.3, Proposition 5.3 yields a more explicit description of these left ideals, in a form which will be quite useful for us in the sequel.

Corollary 5.5 *In order to apply the Baer criterion to determine the injectivity of a left R-module, we need only check injectivity with respect to J, and with respect to left ideals of the form $Rp(c)$ for $p(x) \in \mathcal{P}$.*

6 A (Minimal) Injective Cogenerator for $R = L_K(\mathcal{T})$

In this final section we use the machinery developed above to achieve the main goal of this article; namely, to identify a minimal injective cogenerator for R. In the first portion of the section we show that the injective envelope of each of the simple modules V^f is the Prüfer module U^f. We then proceed to construct, using completely different methods, the injective envelope of the simple module Rw. We finish the section by appropriately combining these two types of injective modules.

In previous work by the three authors [4], modules of the form U^{x-1} over general Leavitt path algebras $L_K(E)$ were shown to be injective, in case the corresponding cycle c is maximal. Establishing injectivity of such U^{x-1} over the Leavitt path algebra $L_K(E)$ of an arbitrary finite graph E required an analysis of the structure of U^{x-1} viewed as a right module over its endomorphism ring. In the present setting, we need not invoke this right module structure, the reason being that in the particular case $R = L_K(\mathcal{T})$ we have a complete description of the left R-ideals, and therefore we are in position to productively use Baer's criterion to establish injectivity of left R-modules.

6.1 The injective envelope of V^f

We start by establishing that U^f is injective for any $f(x) \in \mathcal{F}$. By Proposition 5.2 we have $\mathrm{Hom}_R(J, U^f) = 0$. By Corollary 5.5, in order to check the injectivity of U^f it is enough to check the Baer criterion with respect to left ideals of the form $Rp(c)$ for $p(x) \in \mathcal{P}$.

Lemma 6.1 *Let $f(x) \in \mathcal{F}$, and let $g(x) \in K[x]$ which is not divisible by $f(x)$. Then there exists a polynomial $\beta(x) \in K[x]$ such that $\beta(c)g(c) \in 1 + Rf^n(c)$. In particular, $g(c) + Rf^n(c)$ is a generator of the uniserial module M_n^f.*

Proof Since $f(x)$ is irreducible, nondivisibility implies $\gcd(f^n(x), g(x)) = 1$. Then there exist polynomials $\alpha(x)$, $\beta(x) \in K[x]$ such that $1 = \alpha(x)f^n(x) + \beta(x)g(x)$. Therefore $\beta(c)g(c) = 1 - \alpha(c)f^n(c)$ and hence $\beta(c)g(c) \in 1 + Rf^n(c)$. ■

Proposition 6.2 *Let $f(x) \in \mathcal{F}$. Then the uniserial left R-module U^f is injective.*

Proof By Proposition 5.2 and Corollary 5.5, it suffices to show, for any $p(x) \in \mathcal{P}$ and $\varphi : Rp(c) \to U^f$, that φ extends to a map $\overline{\varphi} : R \to U^f$. Clearly the zero map extends to R. So suppose $\varphi \neq 0$. Let $n \in \mathbb{N}$ be minimal such that $\mathrm{Im}\,\varphi \subseteq M_n^f$, and write $\varphi(p(c)) = m + Rf^n(c)$ for some $m \in R$. As noted in the proof of Lemma 4.2, we can choose

$$m = g_1 + g_2 f(c) + \cdots + g_n f^{n-1}(c),$$

where $g_i \in G^f$ $(1 \le i \le n)$. In particular, m commutes with all polynomials in c.

By the construction of the direct limit U^f, for each $i \ge 0$ we have

$$\varphi(p(c)) = m + Rf^n(c) = mf^i(c) + Rf^{n+i}(c) = f^i(c)m + Rf^{n+i}(c).$$

Let $p(x) = f^\ell(x)p_0(x)$ with $\ell \ge 0$ and $f(x) \nmid p_0(x)$. By Lemma 6.1 there exists $\beta_0(x) \in K[x]$ such that $\beta_0(c)p_0(c) = p_0(c)\beta_0(c)$ belongs to $1 + Rf^{n+\ell}(c)$. Therefore

$$\begin{aligned} p(c)(\beta_0(c)m + Rf^{n+\ell}(c)) &= f^\ell(c)p_0(c)\beta_0(c)m + Rf^{n+\ell}(c) \\ &= f^\ell(c)m + Rf^{n+\ell}(c) = \varphi(p(c)). \end{aligned}$$

Thus the morphism $\overline{\varphi} : R \to U^f$ defined by setting $\overline{\varphi}(1) = \beta_0(c)m + Rf^{n+\ell}(c)$ extends φ. ■

Corollary 6.3 *Let $f(x) \in \mathcal{F}$. Then U^f is the injective envelope of V^f.*

Proof The simple module V^f is essential in U^f, since U^f is uniserial. The injective envelope of any module is an injective module in which the given module sits as an essential submodule. ■

In general the direct sum of infinitely many injective modules need not be injective. (Over an arbitrary ring S, any infinite direct sum of injectives is injective if and only if S is Noetherian; and clearly $R = L_K(\mathcal{T})$ is non-Noetherian, because, for example, J is a non-finitely-generated left ideal of R.) This observation notwithstanding, we close this subsection with the following.

Proposition 6.4 *Let $U = \oplus_{\lambda \in \Lambda} I_\lambda$ where, for each $\lambda \in \Lambda$, there exists $f(x) \in \mathcal{F}$ such that I_λ is an injective module isomorphic to U^f. Then U is injective.*

Proof We again invoke Corollary 5.5, and so we need only establish two steps.

Step 1: Consider the ideal J and let $\varphi : J \to U$. We show that $\varphi = 0$. Suppose otherwise. The image of φ is a semisimple module, isomorphic to a direct sum of copies of Rw. But each U^f has essential socle isomorphic to V^f and so the socle of U is isomorphic to the direct sum of copies of the V^fs. Since $Rw \ncong V^f$ for any f, we get a contradiction.

Step 2: Let $p(x)$ be a polynomial in $\mathcal{P}$. If $\varphi : Rp(c) \to U$ then the image of φ is finitely generated, and so is contained in $\hat{U} \cong \oplus_{i=1}^n U^{f_i}$ for some appropriate f_is. But $\hat{U}$ is injective because each U^{f_i} is (and the sum is finite), and so f extends. ■

6.2 The injective envelope of Rw

Having identified the injective envelope of each of the simple modules V^f ($f(x) \in \mathcal{F}$), we now turn our attention to identifying the injective envelope of the simple module Rw.

Lemma 6.5 *The set $\{w, d, cd, c^2d, \ldots, c^id, \ldots\}$ is a K-basis of the simple module Rw. That is, any element of Rw can be written uniquely as $kw + \sum_{i=0}^n k_i c^i d = kw + (\sum_{i=0}^n k_i c^i)d$, with $k, k_i \in K$.*

Proof It is easily shown that $Rw = Rd^*d = Rd$. By Lemma 2.3, the elements

$$v, w, d, d^*, c^i, c^i d, c^i(c^*)^j, (c^*)^j, d^*(c^*)^j \quad i, j \geq 1$$

form a K-basis of R. Since

$$0 = wd = dd = c^i dd = c^i(c^*)^j d = (c^*)^j d = d^*(c^*)^j d \quad \forall i, j \geq 1$$

we conclude that a basis of $Rw = Rd$ is formed by multiplying the remaining elements of the K-basis for R on the right by d, namely

$$vd = d, \;\; d^*d = w, \;\; \text{and } c^i d \;\; (i \geq 1),$$

which gives the result. ■

In the following sense, the simple module Rw behaves similarly to the simple modules V^f (see Remark 4.3).

Proposition 6.6 *The left R-module Rw is not injective. In particular, the map $\chi = \rho_d : R \to Rw$ (via $1 \mapsto d$) does not factor through the monomorphism $\hat{\rho}_{f(c)} : R \to R$ for any $f(x) \in \mathcal{F}$.*

Proof Write $f(c) = -1 + h_1 c + \cdots + h_m c^m$ with $h_m \neq 0$ ($m \geq 1$). The existence of a map $\xi : R \to Rw$ such that $\xi \circ \hat{\rho}_{f(c)} = \chi$ is equivalent to the solvability of the equation $f(c)x = d$ in Rw. We show that no such $x \in Rw$ exists. Assume to the contrary that there is such a solution, so necessarily $x \neq 0$. By Lemma 6.5 we may write $x = kw + \sum_{i=0}^n k_i c^i d$ for some (unique) $k, k_0, \ldots, k_n \in K$, where not all of these are 0. Then $f(c)x = d$ implies

$$f(c)\left(kw + \sum_{i=0}^{n} k_i c^i d\right) = d.$$

Multiplying both sides of this equation on the left by w we get $-kw = 0$, so $k = 0$. This yields that there are nonzero terms among the elements $k_0, k_1, \ldots, k_n$. We may assume $k_n \neq 0$. Now we have

$$f(c)\left(\sum_{i=0}^{n} k_i c^i d\right) = d.$$

But this is impossible, as the following shows. Expanding $f(c)\big(\sum_{i=0}^n k_i c^i d\big)$, we see the coefficient on the $c^{m+n}d$ term is $h_m k_n$. But the equation $f(c)\big(\sum_{i=0}^n k_i c^i d\big) = d$ implies that the coefficient on the $c^{m+n}d$ term is 0. So we get $0 = h_m k_n$ which, as $h_m \neq 0$, gives $k_n = 0$, a contradiction. ■

We seek to describe the injective envelope of Rw. With Proposition 6.6 in hand, this process will require us to build a module which is strictly larger than Rw.

Definition 6.1 Let $\hat{W}$ denote the K-space whose elements are "formal series" of the form

$$\hat{W} := \{k_{-1}w + k_0 d + k_1 cd + \cdots + k_i c^i d + \cdots \mid k_i \in K\}.$$

The K-space $\hat{W}$ has a natural structure as a left R-module, where for $y = k_{-1}w + k_0 d + k_1 cd + \cdots + k_i c^i d + \cdots$ one defines

$$c \cdot y = k_0 cd + k_1 c^2 d + \cdots + k_i c^{i+1} d + \cdots; \quad c^* \cdot y = k_1 d + \cdots + k_i c^{i-1} d + \cdots;$$
$$d \cdot y = k_{-1} d; \quad d^* \cdot y = k_0 w; \quad \text{and} \quad v \cdot y = k_1 cd + \cdots + k_i c^i d + \cdots, \ w \cdot y = k_{-1} w.$$

By Lemma 6.5, Rw is the R-submodule of $\hat{W}$ consisting of those elements for which $k_i = 0$ for all $i > N$ for some $N \in \mathbb{N}$, *i.e.*, Rw consists of the "standard polynomials" in $\hat{W}$.

Lemma 6.7 *Let $y = k_{-1}w + k_0 d + k_1 cd + \cdots + k_i c^i d + \cdots \in \hat{W}$.*

1) $wy = k_{-1}w$.
2) $d^*(c^*)^j y = k_j w$ *for all* $j \geq 0$.

Proof (1) is obvious, and (2) follows directly from the observation that $d^*(c^*)^j c^i d = w$ if $i = j$, and is 0 otherwise. ∎

Lemma 6.8 *The simple module Rw is essential in $\hat{W}$. In particular, $Rw = \mathrm{Soc}(\hat{W})$.*

Proof Consider an element $0 \neq y = k_{-1}w + k_0 d + k_1 cd + \cdots + k_i c^i d + \cdots \in \hat{W}$. There exists $\ell \in \mathbb{Z}^+ \cup \{-1\}$ such that $k_\ell \neq 0$. If $k_{-1} \neq 0$, then by Lemma 6.7(1) $wy = k_{-1}w \neq 0$ is in Rw. If $k_\ell \neq 0$ for $\ell \geq 0$, then by Lemma 6.7(2) $d^*(c^*)^\ell y = k_\ell w \neq 0$ is in Rw. ∎

Lemma 6.9 *Any R-homomorphism from $J = Rw \oplus Rd^* \oplus Rd^*c^* \oplus Rd^*(c^*)^2 \oplus \cdots$ to $\hat{W}$ extends to an R-homomorphism from R to $\hat{W}$.*

Proof Let $\varphi : J \to \hat{W}$ be a homomorphism of left R-modules. For each $i \geq 0$ let k_i denote the K-coefficient of w in the formal power series expression for $\varphi(d^*(c^*)^i)$, and let k_{-1} be the K-coefficient of w in $\varphi(w)$. Since $\varphi(w) = \varphi(w^2) = w\varphi(w)$ and $\varphi(d^*(c^*)^i) = \varphi(wd^*(c^*)^i) = w\varphi(d^*(c^*)^i)$, Lemma 6.7 implies that $\varphi(w) = k_{-1}w$ and $\varphi(d^*(c^*)^i) = k_i w$ for all $i \geq 0$.

Now consider the R-homomorphism $\Phi : R \to \hat{W}$ obtained by setting

$$\Phi(1) := k_{-1}w + k_0 d + k_1 cd + k_2 c^2 d + \cdots .$$

Since $\Phi(w) = w\Phi(1) = k_{-1}w$ and $\Phi(d^*(c^*)^i) = d^*(c^*)^i\Phi(1) = k_i w$ for each $i \geq 0$ (again by Lemma 6.7), Φ extends φ. ∎

It is well known that the invertible elements in the ring of formal power series $K[[x]]$ are precisely those formal power series $y(x) = \sum_{i=0}^{\infty} k_i x^i$ for which $k_0 \neq 0$, i.e., for which $y(0) \neq 0$.

Lemma 6.10 *Let $p(x) = p_0 + p_1 x + \cdots + p_n x^n \in \mathcal{P}$. Any R-homomorphism from $Rp(c)$ to $\hat{W}$ extends to an R-homomorphism from R to $\hat{W}$.*

Proof Let $\psi : Rp(c) \to \hat{W}$ be a homomorphism of left R-modules. Let $\psi(p(c)) = y$, where $y = k_{-1}w + k_0 d + k_1 cd + \cdots + k_i c^i d + \cdots$. We need to find an R-homomorphism $\Psi : R \to \hat{W}$ such that

$$\Psi(p(c)) = p(c)\Psi(1) = y.$$

Because $p_0 \neq 0$, viewing $p(x) \in K[[x]]$ there exists $\alpha(x) \in K[[x]]$ for which $p(x)\alpha(x) = 1$ in $K[[x]]$. Write $\alpha(x) = a_0 + a_1 x + a_2 x^2 + \cdots$. Set $p(x) = \sum_{i=0}^{\infty} p_i x^i$, with $p_i = 0 \, \forall i > n$; then

$$p_0 a_0 = 1, \quad \text{and} \quad \sum_{j=0}^{N} p_j a_{N-j} = 0 \quad \text{for all } N \geq 1 . \qquad (\star)$$

Now define the following elements of K:

$$z_{-1} := a_0 k_{-1}, \quad \text{and, for each } M \geq 0, \; z_M := \sum_{i=0}^{M} a_i k_{M-i}.$$

We construct $z \in \hat{W}$ by setting

$$z := z_{-1}w + z_0 d + z_1 cd + z_2 c^2 d + \cdots,$$

so that $p(c)z = (p_0 1_R + p_1 c + p_2 c^2 + \cdots + p_n c^n)(z_{-1}w + z_0 d + z_1 cd + z_2 c^2 d + \cdots)$. We already know that $p_0 a_0 = 1$, so that $p_0 z_{-1} = p_0 a_0 k_{-1} = k_{-1}$. Moreover, by standard computations and using the previous relations $(\star)$, one can show that for any $i \geq 0$, the coefficient of the term $c^i d$ in $p(c)z$ equals the coefficient of the term $c^i d$ in y. This implies that $p(c)z = y$ in $\hat{W}$. (Intuitively, the idea here is to "define informally" the expression $\alpha(c) = a_0 1_R + a_1 c + a_2 c^2 + \cdots$, and subsequently the element $z \in \hat{W}$ as $z = \alpha(c)y$, so that $p(c)\alpha(c)y = 1 \cdot y = y$.)

Finally, consider the R-homomorphism $\Psi : R \to \hat{W}$ obtained by setting

$$\Psi(1) = z.$$

Then $\Psi(p(c)) = p(c)\Psi(1) = p(c)z = y$, as desired. ■

Proposition 6.11 *The left R-module $\hat{W}$ is injective.*

Proof We use Corollary 5.5 again, which yields that we only need to test the injectivity of $\hat{W}$ with respect to the two indicated types of left R-ideals. But this is precisely what has been achieved in Lemmas 6.9 and 6.10. ■

Corollary 6.12 *$\hat{W}$ is the injective envelope of Rw.*

Proof As noted in Corollary 6.3, the injective envelope of any module is an injective module in which the given module sits as an essential submodule. So the result follows from Lemma 6.8 and Proposition 6.11. ■

We now describe the quotient $\hat{W}/Rw$ as an extension of a direct summand of a product of copies of the U^fs by the simple module Rw.

Proposition 6.13 *The module $\hat{W}/Rw$ is a direct summand of a product of copies of the U^fs.*

Proof Consider the short exact sequence $0 \to Rw \to \hat{W} \to \hat{W}/Rw \to 0$. First notice that $\mathrm{Hom}(Rw, \hat{W}/Rw) = 0$, as follows. To the contrary, suppose there exists $0 \neq f : Rw \to \hat{W}/Rw$. Then by the simplicity of Rw, the map f must be a monomorphism. Further, since Rw is projective, there then exists $\tilde{f} : Rw \to \hat{W}$ such that $\pi \circ \tilde{f} = f$. In particular $\mathrm{Im}\,\tilde{f} \cap Rw = 0$. But this is a contradiction since Rw is the essential socle of $\hat{W}$.

Now let $0 \neq x \in \hat{W}/Rw$, and consider the cyclic module $Rx \cong R/\mathrm{Ann}(x)$. Let M be a maximal left ideal of R containing $\mathrm{Ann}(x)$, so that $Rx \to R/M \to 0$. If $R/M \cong Rw$, since Rw is projective we would get that Rw is a summand of Rx, in particular is a submodule of Rx and thereby also of $\hat{W}/Rw$, contrary to the result of the previous paragraph. So R/M is a simple module of type V^f, and hence it embeds in U^f. In such a way, for any $0 \neq x \in \hat{W}/Rw$, there is a suitable $f(x) \in \mathcal{F}$ and a morphism $\varphi_x : Rx \to U^f$, such that $\varphi_x(x) \neq 0$. Since U^f is injective, φ_x extends to a morphism $\tilde{\varphi}_x : \hat{W}/Rw \to U^f$. So we get that $\hat{W}/Rw$ embeds in a product of copies of the U^f ($f(x) \in$

$\mathcal{F}$). But $\hat{W}$ is injective, and so $\hat{W}/Rw$ is also injective by Corollary 2.2. Thus $\hat{W}/Rw$ is indeed a direct summand of the product of copies of the U^fs. ∎

6.3 Consequences of Subsections 6.1 and 6.2

Every ring has (up to isomorphism) a unique minimal injective cogenerator (see *e.g.* [5, Section 18] for a full description of this concept). Since any representation of the ring embeds in a product of copies of a cogenerator, we can describe the entire category of modules over the ring once we know such a cogenerator. Using the previous results, we are able to determine a minimal injective cogenerator for the algebra $R = L_K(\mathcal{T})$.

Theorem 6.14 *The left R-module*

$$C = \hat{W} \oplus (\oplus_{f(x)\in\mathcal{F}} U^f)$$

is a minimal injective cogenerator for R.

Proof By combining Proposition 6.4 with Proposition 6.12, we directly obtain that C is injective. Because there is a unique cycle in $\mathcal{T}$, [6, Corollary 4.6] applies, and yields that (up to isomorphism) the set of all the simple modules over R consists of Rw together with the pairwise nonisomorphic modules of the form $\{V^f \mid f(x) \in \mathcal{F}\}$. Thus C contains a copy of every simple left R-module, and so it is a cogenerator for the module category [5, Proposition 18.15]. Since any injective cogenerator has to contain a copy of the injective envelope of any simple module, we get that C is a minimal injective cogenerator for R. ∎

When S is any Noetherian ring, then the minimal injective cogenerator is precisely the direct sum of the injective envelopes of the simple modules. We have reached the same conclusion for the non-Noetherian ring $R = L_K(\mathcal{T})$ in Theorem 6.14. Moreover, we have described each of these injective envelopes explicitly.

With Theorem 6.14 in hand, we achieve a description of all the injective left R-modules.

Corollary 6.15 *Let C denote $\hat{W} \oplus (\oplus_{f(x)\in\mathcal{F}} U^f)$. Then a left R-module M is injective if and only if M is isomorphic to a direct summand of a direct product of copies of C.*

Proof This follows immediately from the definition of a cogenerator, together with the facts that direct products and direct summands of injective modules are injective, and an injective submodule of a module is necessarily a direct summand. ∎

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