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Euskal Herriko Unibertsitatea

Ph.D. Thesis

Topics in Harmonic Analysis; commutators and directional singular integrals.

Natalia Accomazzo Scotti

Supervised by Ioannis Parissis and Carlos Pérez Moreno

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Every day, paciencia y fe.[1]

[1] In the Heights

Abstract

This dissertation focuses on two main topics: commutators and maximal directional operators. Our first topic will also distinguish between two cases: commutators of singular integral operators and BMO functions and commutators of fractional integral operators and a BMO class that comes from changing the underlying measure. Commutators are not only interesting for its own sake, but they have been broadly studied because of their connection to PDEs.

Our first result gives us a new way of characterizing the class BMO. Assuming that the commutator of the Hilbert transform in dimension 1 (or a Riesz transform in dimensions 2 and higher) and the symbol b satisfy an $L \log L$-type of modular inequality on the endpoint with constant B, we can bound the BMO norm of the symbol by a fixed multiple of B; thus providing an endpoint version of the classical result of Coifman, Rochberg and Weiss for commutators of Calderón-Zygmund operators and BMO.

We also studied commutators of fractional integrals and BMO. In this case, we were interested in finding quantitave two-weights estimates for the iterated version of these operators. We extended the known sharp inequalities for the commutator of first order to the iterated case and also provided a new proof of the previous results.

Lastly, we studied maximal directional operators. Specifically, we considered a singular integral operator that commutes with translations and studied the maximal directional operator that arises from it. We proved that for any subset of cardinality N of a lacunary set of directions we can bound the $L^p(\mathbb{R}^n)$-norm of the operator by the sharp bound $\sqrt{\log N}$, thus completing some previous results on the Hilbert transform on low dimensions.

Resumen

Esta tesis está centrada en dos temas principales: conmutadores y operadores maximales direccionales. El primer tema a tratar a su vez distinguirá entre dos casos: conmutadores de operadores de integrales singulares y funciones BMO y conmutadores de operadores de integrales fraccionarias y una clase de funciones de BMO que formaremos al cambiar la medida considerada. La teoría de los conmutadores no sólo es interesante por sí misma, sino que también por su relación intrínseca con la teoría de las EDPs.

Nuestro primer resultado nos proveerá de una nueva forma de caracterizar el espacio BMO. Asumiendo que el conmutador de la transformada de Hilbert en dimensión 1 (o de una transformada de Riesz en dimensiones superiores a 2) y el símbolo b cumple una desigualdad de tipo $L \log L$ en el extremo con constante B, podemos acotar la norma BMO del símbolo por un múltiplo de la constante B; de esta manera proveyendo una versión en el extremo de un teorema clásico de Coifman, Rochberg y Weiss para conmutadores de operadores de Calderón-Zygmund y funciones BMO.

También estudiamos conmutadores de operadores de integrales fraccionarias y BMO. En este caso, estuvimos interesados en encontrar cotas cuantitativas con dos pesos para la versión iterada de estos operadores. Extendimos las cotas óptimas conocidas para el caso del conmutador de primer orden al caso iterado, también dando una nueva prueba de los resultados ya conocidos.

Finalmente, estudiamos operadores maximales direccionales. Específicamente, consideramos operadores de integrales singulares que conmutan con las traslaciones y estudiamos el operador maximal direccional que se forma a partir de éstos. Probamos que para cualquier subconjunto de cardinal N de un conjunto de direcciones lagunar podemos acotar la norma $L^p(\mathbb{R}^n)$ de este operador por la constante óptima en términos de la cardinalidad del conjunto de direcciones $\sqrt{\log N}$, de esta manera completando los resultados previos que se habían obtenido para la transformada de Hilbert en dimensiones bajas.

Introduction

This dissertation is roughly divided into two parts. The first chapters deal with the theory of commutators of some linear operators and their associated symbols, while the last one is about maximal directional operators.

Commutators

Take T to be a linear operator acting on some $L^p(\mathbb{R}^n)$ space, and b a function (which we will refer as the *symbol*), we define the commutator operator as $[b,T]f(x) = b(x)Tf(x) - T(bf)(x)$. By iterating this definition, recursively plugging as our operator T the commutator $[b,T]$, we can form what we are going to call an *iterated commutator* or *commutator of order* k for k an integer and denote T_b^k.

Commutators of Calderón-Zygmund operators and BMO

Chapter 2 is devoted to the study of the commutator that arises when we take T to be a Calderón-Zygmund operator and b a BMO function. This class of operators was considered by Coifman, Rochberg and Weiss as part of a study on factorization of the real Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. They proved that the commutator of a Calderón-Zygmund operator and a BMO function is a bounded map from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. They also proved that if we take the operator T to be the Riesz transforms, then the condition that the symbol belongs to BMO is necessary to get boundedness in any $L^p(\mathbb{R}^n)$, $1 < p < \infty$, thus providing a new way of characterizing the class BMO. In this dissertation, we focus on the endpoint version of this result: on the one hand, it was Pérez who realized that the commutator might fail to be of weak type $(1,1)$ and obtained the following endpoint inequality

$$|\{x \in \mathbb{R}^n : |[T,b]f(x)| > t\}| \leq \int_{\mathbb{R}^n} \phi\left(\frac{\|b\|_{\mathrm{BMO}}|f(x)|}{t}\right) \mathrm{d}x, \tag{0.1}$$

where $\phi(t) = t(1+\log^+(t))$ and $\log^+(t) = \max(t,0)$. This result opened up two natural questions: whether we could get the necessity of BMO in this endpoint inequality, and what do we need to ask of the function b to actually get the commutator to be of weak type $(1,1)$. The answer to the first question is the first original contribution of this dissertation and is presented as Theorem 2.3 of Chapter 2.

Theorem A. *Let b be a locally integrable function on $\mathbb{R}^n$ and $T = R_j$ a Riesz transform, $1 \leq j \leq n$. If there exists a constant B such that the commutator $[b, R_j]$ satisfy the inequality (0.1) for all $t > 0$ and all $f = \chi_E$ the characteristic function of a measurable set E, then b belongs to BMO.*

As for the second question that we posed, we obtained that what we need in order to get that the weak $(1, 1)$ condition on the commutator is that the function b belongs to $L^\infty(\mathbb{R}^n)$, and that is the second original contribution of this dissertation and is presented as Theorem 2.2 of Chapter 2.

Theorem B. *Let b be a locally integrable function on $\mathbb{R}^n$ and $T = R_j$ a Riesz transform, $1 \leq j \leq n$. If the commutator $[b, R_j]$ is of weak type $(1, 1)$ then $b \in L^\infty(\mathbb{R}^n)$.*

Commutators of fractional integrals and BMO

In Chapter 3 we are going to study two-weight estimates for the commutator of a fractional integral operator and a symbol b that we will take suitable BMO space. The study of weighted estimates for commutators of singular integral operators can be traced back to the work of Bloom, who proved that taking H to be the Hilbert transform, $\lambda, \mu \in A_p$ and $\nu = \left(\frac{\mu}{\lambda}\right)^{1/p}$, then

$$\|[H, b]f\|_{L^p(\lambda)} \leq C\|f\|_{L^p(\mu)}$$

if and only if b belongs to BMO_ν, namely b is a locally integrable function such that

$$\|b\|_{\mathrm{BMO}_\nu} = \sup_Q \frac{1}{\nu(Q)} \int_Q |b - b_Q| < \infty.$$

For fractional integral operators the class of weights that is natural to consider is the $A_{p,q}$ class, since we know that fractional integrals map $L^q(w^q)$ to $L^p(w^p)$ if and only if the weight w belongs to the $A_{p,q}$ class. Our contribution to this topic is the following result, which can be thought as the quantitative analogue of Bloom's theorem can be found as Theorem 3.1 of Chapter 3.

Theorem C. *Let $0 < \alpha < n$ and $1 < p < \frac{n}{\alpha}$, q defined by $\frac{1}{q} + \frac{\alpha}{n} = \frac{1}{p}$, k a positive integer. Assume μ, $\lambda \in A_{p,q}$ and that $\nu = \frac{\mu}{\lambda}$. If $b \in \mathrm{BMO}_{\nu^{1/k}}$ then*

$$\|(I_\alpha)_b^k f\|_{L^q(\lambda^q)} \leq C(m, n, \alpha, p, [\lambda]_{A_{p,q}}, [\mu]_{A_{p,q}})\|b\|^k_{\mathrm{BMO}_{\nu^{1/k}}}\|f\|_{L^p(\mu^p)}.$$

Conversely if for every set E of finite measure we have that

$$\|(I_\alpha)_b^k \chi_E\|_{L^q(\lambda^q)} \leq C\mu^p(E)^{1/p},$$

then $b \in \mathrm{BMO}_{\nu^{1/k}}$.

The constant that we have denoted as $C(m, n, \alpha, p, [\lambda]_{A_{p,q}}, [\mu]_{A_{p,q}})$ will be given explicit in terms of the $A_{p,q}$ constants of the weights λ and μ. Observe also that if we take the weights to be $\lambda = \mu$ we arrive at yet another characterization of the space BMO.

Maximal directional operators

In Chapter 4 we study a different class of operators that we will form by considering a one dimensional operator acting along a line in some n-dimensional Euclidean space, and then taking a supremum as the line changes through a set of directions. In this dissertation we are going to care about two types of questions of the theory of maximal directional operators: the first one is about getting some structure on the set of directions so as to have the operator L^p-bounded, and the second one is trying to get sharp bounds of the L^p-norm of the operator in terms of the number of directions (in this case we are implicitly assuming that the set of directions we are considering has finite cardinality). In this dissertation, we study the maximal directional operator that arises from translation invariant singular integrals; specifically if V is a set of directions in the sphere $\mathbb{S}^{n-1}$ we define

$$T_V f(x) \coloneqq \sup_{v\in V} \left| \int_{\mathbb{R}^n} m(v\cdot\xi)\hat{f}(\xi)e^{ix\cdot\xi}\mathrm{d}\xi \right|, \qquad x \in \mathbb{R}^n,$$

where m is a Mikhlin-Hörmander multiplier. If we try to address the two questions that we mentioned for this operator, we immediately bump into a result by Łaba, Marinelli and Pramanik that tells us that for the Hilbert transform and V any set of finite cardinality we have the lower bound $\|H_V\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)} \gtrsim (\log \#V)^{\frac{1}{2}}$. This result tells us at the same time that we cannot expect boundedness of the operator whenever the set of directions is infinite and it gives us a candidate for a sharp bound in terms of the cardinality of the set of directions. Our contribution is the following theorem, that proves that if we have some structure on the set of directions then we can bound the maximal directional singular integral operator by the sharp constant $(\log \#V)^{\frac{1}{2}}$ and that can be found as Theorem 4.1 of Chapter 4.

Theorem D. *For Ω a finite union of lacunary sets, and $V \subset \Omega$ any set of cardinality N it holds*

$$\|T_V f\|_{L^p(\mathbb{R}^n)} \lesssim \sqrt{\log N}\|f\|_{L^p(\mathbb{R}^n)},$$

with the implicit constant dependent on the lacunarity order and constant, and the dimension n.

Contents

Chapter 1

Preliminaries

We begin by defining the main operators that will be used throughout the thesis and stating some well known properties.

1.1 Some basic notations

Throughout this dissertation we will take $\mathbb{R}^n$ to be the ambient space, and we reserve the notation n for the dimension.
By Q we will denote a cube with sides parallel to the axes, by which we mean that there exists real numbers $a_1 < b_1, \ldots, a_n < b_n$ with $|b_i - a_i| \coloneqq \operatorname{side}(Q)$ for all $i = 1, \ldots, n$ so that $Q = \prod_{i=1}^{n}[a_i, b_i)$.
By $B(x, r)$ we will denote a ball with radious r and centered at the point $x \in \mathbb{R}^n$.
We will call w a *weight* if w is a nonnegative, locally integrable function, that is finite almost everywhere. We denote by $\|f\|_{L^p(w)}$ the usual weighted L^p norms, that is

$$\|f\|_{L^p(w)} \coloneqq \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) \, \mathrm{d}x \right)^{\frac{1}{p}}$$

and the weak weighted L^p norms by

$$\|f\|_{L^{p,\infty}(w)} \coloneqq \inf \left\{ C > 0 : w(\{x \in \mathbb{R}^n : |f(x)| > t\}) \leq \frac{C^p}{t^p}, \quad \text{for all } t > 0 \right\}.$$

We will use the notations $\lesssim$ and $\simeq$ to indicate that the implicit constants are numerical, we will indicate by subindexes whenever we want to make explicit the dependence of the constant on p or n, for example.
We will also use the notation

$$f_Q = \fint_Q f(x) \, \mathrm{d}x \coloneqq \frac{1}{|Q|} \int_Q f(x) \, \mathrm{d}x.$$

1.2 Classical theory

This section introduces some very well-known operators and some of their basic properties. These results and further analysis can be found easily in the literature, we refer to the reader to for example Chapters 2, 5 and 7 of [18].

1.2.1 Maximal functions

Definition 1.1. For a function f locally integrable in $\mathbb{R}^n$ we define the Hardy-Littlewood maximal function as

$$\mathrm{M}f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| \, \mathrm{d}y, \quad x \in \mathbb{R}^n$$

where the supremum is taken over all the cubes containing x.

It is very well known that we could define the maximal function by taking the supremum to be over balls instead of cubes, both centered and uncentered, and all these operators are comparable and hence they enjoy the same boundedness properties.

Theorem. *The maximal operator* M *satisfies*

$$|\{x \in \mathbb{R}^n : \mathrm{M}f(x) > t\}| \lesssim \frac{1}{t} \int_{\mathbb{R}^n} |f(x)| \, \mathrm{d}x, \quad f \in L^1(\mathbb{R}^n)$$

and for $1 < p \leq \infty$

$$\|\mathrm{M}f\|_{L^p(\mathbb{R}^n)} \lesssim_p \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

If we change the cubes for general rectangles with sides parallel to the axes, the resulting operator is not comparable the Hardy-Littlewood maximal operator.

Definition 1.2. For f a locally integrable function in $\mathbb{R}^n$, we define the strong maximal operator as

$$\mathrm{M}_s f(x) := \sup_{R \ni x} \frac{1}{|R|} \int_R |f(y)| \, \mathrm{d}y, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all the rectangles with sides parallel to the axes containing x.

Theorem. *The strong maximal function* M_s *maps boundedly* $L^p(\mathbb{R}^n)$ *to itself for all* $1 < p < \infty$. *Unlike the HL maximal function, the strong maximal function is not weak (1,1).*

Definition 1.3. We say that a weight w belongs to the A_p class for $1 < p < \infty$ if

$$[w]_{A_p} := \sup_Q \frac{1}{|Q|} \int_Q w(x) \, \mathrm{d}x \left(\frac{1}{|Q|} \int_Q w(x)^{-\frac{1}{p-1}} \, \mathrm{d}x \right)^{p-1} < \infty,$$

where the supremum is taken over all cubes Q with sides parallel to the axes.

We say that a weight w belongs to the A_1 class if

$$[w]_{A_1} := \sup_Q \frac{1}{|Q|} \int_Q w(x)\,\mathrm{d}x\, \|w^{-1}\|_{L^\infty(Q)} < \infty,$$

where the supremum is taken over all cubes Q with sides parallel to the axes.

A classical result tells us that the A_p classes characterizes the strong (p,p) and weak $(1,1)$ boundedness of the maximal operator. Specifically, we have the following result.

Theorem. *Let* M *be the Hardy-Littlewood operator and* w *a weight. Then* M *is bounded from* $L^p(w)$ *to* $L^p(w)$ *for some* $1 < p < \infty$ *if and only if* w *belongs to the* A_p *class. In the case* $p = 1$*, we have that* M *is* w*-weak* $(1,1)$ *if and only if* w *belongs to the* A_1 *class.*

1.2.2 Singular integral operators

Definition 1.4. A linear operator T will be a Calderón-Zygmund operator (CZO) if it maps $L^2(\mathbb{R}^n)$ into itself boundedly and admits a representation whenever f is a $C_0^\infty(\mathbb{R}^n)$ function and $x \notin \mathrm{supp}(f)$ of the form

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y) f(y)\,\mathrm{d}y,$$

where the kernel K is a function $K : \mathbb{R}^n \times \mathbb{R}^n \backslash \{(x,x) : x \in \mathbb{R}^n\} \to \mathbb{R}$ that satisfies:

(i) $|K(x,y)| \le \frac{C}{|x-y|^n}$, if $x \neq y$,

(ii) $|K(y,x) - K(y,z)| + |K(x,y) - K(z,y)| \le C\frac{|x-z|^\gamma}{|x-y|^{n+\gamma}}$, if $|x-z| < \frac{1}{2}|x-y|$,

for some constant C and exponent $0 < \gamma \le 1$.

Definition 1.5. For f a function in the Schwartz class $\mathcal{S}(\mathbb{R})$ we define the Hilbert transform of f as

$$Hf(x) := \mathrm{p.\,v.} \int_{\mathbb{R}} \frac{f(x-y)}{y}\,\mathrm{d}y = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{f(x-y)}{y}\,\mathrm{d}y, \quad x \in \mathbb{R}.$$

Definition 1.6. For f a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ we define the Riesz transforms of f as

$$R_j f(x) := \mathrm{p.\,v.} \int_{\mathbb{R}^n} f(y) \frac{x_j - y_j}{|x-y|^{n+1}}\,\mathrm{d}y, \quad x \in \mathbb{R}^n, j = 1, \ldots, n.$$

A classical result tells us that if T is a CZO, then it extends to a linear operator that is of weak type $(1,1)$ and of strong type (p,p) for all $1 < p < \infty$. This operator is also bounded in the weighted L^p, whenever the weight is in the right class.

Theorem. *Let* T *be a CZO and* w *a weight in the class* A_p *for some* $1 < p < \infty$*. Then,* T *maps boundedly from* $L^p(w)$ *into itself. If* w *is a weight in* A_1*, then* $T : L^1(w) \to L^{1,\infty}(w)$*.*

1.3 The space BMO

We say that a function $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ belongs to the class $\text{BMO}(\mathbb{R}^n)$ if

$$\|b\|_{\text{BMO}(\mathbb{R}^n)} := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| \,\mathrm{d}x < \infty$$

where the supremum is taken over cubes. In this space, we have an equivalent norm, defined by

$$\|b\|'_{\text{BMO}(\mathbb{R}^n)} := \sup_Q \inf_c \frac{1}{|Q|} \int_Q |b(x) - c| \,\mathrm{d}x.$$

For a cube Q, the infimum above is attained and the constants where this happens can be found among the *median values.*

Definition 1.7. A *median value* of b over a cube Q will be any real number $m_Q(b)$ that satisfies simultaneously

$$|\{x \in Q : b(x) > m_b(Q)\}| \leq \frac{1}{2}|Q|$$

and

$$|\{x \in Q : \, b(x) < m_b(Q)\}| \leq \frac{1}{2}|Q|.$$

The fact that the constant c in the definition of $\|b\|'_{\text{BMO}(\mathbb{R}^n)}$ can be chosen to be a median value of b can be found for instance in [43]. [Ch. 8, p. 199]
An equivalent description of $\text{BMO}(\mathbb{R}^n)$ was obtained by John [23] and by Strömberg [42]. These authors considered the following quantities for $0 < s < 1$ and b measurable

$$\|b\|_{\text{BMO}_s} := \sup_Q \inf_c \inf\{t \geq 0 : \, |\{x \in Q : \, |b(x) - c| > t\}| \leq s|Q|\}$$

and proved that $\|b\|_{\text{BMO}_s}$ is equivalent to the usual $\text{BMO}(\mathbb{R}^n)$-norm for $0 < s \leq 1/2$. Here we will understand that $\text{BMO}_\text{s} \equiv \text{BMO}_\text{s}(\mathbb{R}^n)$, we omit the dimension to simplify notation. They obtained the following more precise estimates.

Theorem (Strömberg, [42])**.** *For $0 < s \leq 1/2$ there exists a constant C depending only on n such that*

$$s\|b\|_{\text{BMO}_s} \leq \|b\|_{\text{BMO}} \leq C\|b\|_{\text{BMO}_s}.$$

For these "norms" it will be also useful to replace the general constant c by the median $m_Q(b)$.

Remark 1.8. Observe that to prove that a function b belongs to BMO it will be enough to find constants A and s $(0 < s \leq 1/2)$ such that, for every cube Q we have

$$|\{x \in Q : \, |b(x) - m_Q(b)| > A\}| \leq s|Q|.$$

Then we also have that $\|b\|_{\mathrm{BMO}} \lesssim_{n,s} A$.

1.4 Commutators of singular integrals and BMO

In this section we introduce an important class of operators, that will be the main focus of the next chapters. Given a singular integral T and a locally integrable function b, we can define the commutator operator as

$$[b,T]f(x) := b(x)T(f)(x) - T(fb)(x)$$

This class of operators was first studied by Coifman, Rochberg and Weiss [10] while they were looking for a new factorization of the real Hardy space $\mathcal{H}^1$. In order to achieve this the authors established that if T is a Calderón-Zygmund operator then the condition that b belongs to BMO is sufficient and necessary for L^p-boundedness, when $1 < p < \infty$.

Theorem (Coifman, Rochberg, Weiss (1976) [10])**.** *Take T to be a Calderón-Zygmund operator and b a function in $\mathrm{BMO}(\mathbb{R}^n)$. Then, the commutator $[b,T]$ is a bounded map from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ for every $1 < p < \infty$. Conversely, if R_j $(1 \leq j \leq n)$ are the Riesz transforms, b is a locally integrable function and there exists some $1 < p < \infty$ such that all the commutators $[b,R_j]$ are bounded from $L^p(\mathbb{R}^n)$ into $L^p(\mathbb{R}^n)$ then the function b belongs to $\mathrm{BMO}(\mathbb{R}^n)$. Moreover, we have*

$$\|b\|_{\mathrm{BMO}} \lesssim_n \sum_{j=1}^{n} \|[b,R_j]\|_{L^p(\mathbb{R}^n)\to L^p(\mathbb{R}^n)}.$$

Unlike the theory of singular integrals, this proof does not rely on a weak $(1,1)$ inequality, but instead on duality. We can actually check that in general the commutator can fail to be weak $(1,1)$: we just need to consider the case $T = H$ the Hilbert transform and $b = \log|1+x|$. This example was observed by Pérez in [39], where he also provided an endpoint theory for these operators. Before stating this endpoint estimate, we first introduce a generalization of the commutator.

Definition 1.9. The *commutator of order k* for $k = 2, 3, \dots$ is defined by the recursive formula $T_b^k \coloneqq [T_b^{k-1}, b]$. For $k = 1$ we define T_b^1 as the usual commutator $T_b^1 \coloneqq [b,T]$. We will also refer to this operator as an *iterated commutator*.

Theorem (Pérez, (1995) [39])**.** *For T a Calderón-Zygmund operator, $b \in \mathrm{BMO}(\mathbb{R}^n)$ and $k \geq 1$ an integer, we have the following estimate*

$$|\{x \in \mathbb{R}^n : |T_b^k f(x)| > t\}| \leq \int_{\mathbb{R}^n} \phi_k\left(\frac{\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}|f(x)|}{t}\right) \mathrm{d}x$$

for every smooth function with compact support f and $t > 0$; here the function ϕ_k is defined by $\phi_k(t) \coloneqq t(1+\log^+(t))^k$, and $\log^+(t) = \max(0, \log(t))$.

1.5 Fractional operators

Chapter 3 is devoted to proving a two-weight estimate for the commutator of a fractional operator and a symbol belonging to the 'right' BMO in this context. The study of weighted estimates for fractional operators is not interesting just for its own sake but also for its applications to partial differential equations, Sobolev embeddings or quantum mechanics (see for instance [17, Section 9] or [40]).

We start this section by giving the definitions of our main operators. From this moment on we are going to take α a real number with $0 < \alpha < n$.

Definition 1.10. For f a locally integrable function we define the *fractional maximal function* as

$$\mathrm{M}_\alpha f(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| \, \mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

Definition 1.11. For $f \in L^p(\mathbb{R}^n)$, $1 \leq p < \infty$ we define the *fractional integral operator* as

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, \mathrm{d}y, \qquad x \in \mathbb{R}^n.$$

The first result that we are going to state is a classical theorem that can be found for example in [19, Chapter 6].

Theorem. *Let $1 \leq p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. We have that I_α maps $L^q(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ whenever $p > 1$ and that it is of weak type $(1, \frac{n-\alpha}{n})$.*

The class of weights that governs the behaviour of fractional operators is the $A_{p,q}$ class, which was introduced by Muckenhoupt and Wheeden [35].

Definition 1.12. Given $1 < p < q < \infty$, we will say a weight w belongs to the class $A_{p,q}$ if

$$[w]_{A_{p,q}} := \sup_Q \frac{1}{|Q|} \int_Q w^q \left(\frac{1}{|Q|} \int_Q w^{-p'} \right)^{\frac{q}{p'}} < \infty.$$

Since $1 < p < q < \infty$, using Hölder's inequality, it is a simple observation that

$$[w^p]_{A_p} \leq [w]_{A_{p,q}}^{\frac{p}{q}} \quad \text{and} \quad [w^q]_{A_q} \leq [w]_{A_{p,q}}. \tag{1.1}$$

In [35] the authors relate the boundedness of the fractional operator in the weighted setting with the class $A_{p,q}$, proving the following theorem.

Theorem (Muckenhoupt, Wheeden [35])**.** *Let $1 < p < \frac{n}{\alpha}$ and q such that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Let T be either the fractional operator I_α or the fractional maximal function M_α. Then, T maps $L^q(w^q)$ to $L^p(w^p)$ if and only if w belongs to the $A_{p,q}$ class.*

During the last decade, many authors have devoted plenty of works to the study of quantitative weighted estimates, in other words, estimates in which the quantitative dependence on the A_p

constant $[w]_{A_p}$ or, in its case, on the $A_{p,q}$ constant $[w]_{A_{p,q}}$, is the central point. The A_2 Theorem, namely the linear dependence on the A_2 constant for Calderón-Zygmund operators [21] can be considered the most representative result in this line. In the case of fractional integrals, the sharp dependence on the $A_{p,q}$ constant was obtained by Lacey, Moen, Pérez and Torres [28]. The precise statement is the following

Theorem (Lacey, Moen, Pérez, Torres [28])**.** *Let $1 < p < \frac{n}{\alpha}$ and q defined by $\frac{1}{q} + \frac{\alpha}{n} = \frac{1}{p}$. Then, if $w \in A_{p,q}$ we have that*

$$\|I_\alpha f\|_{L^q(w^q)} \leq c_{n,\alpha}[w]_{A_{p,q}}^{\left(1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{q}\right\}} \|f\|_{L^p(w^p)}$$

and the estimate is sharp in the sense that the inequality does not hold if we replace the exponent of the $A_{p,q}$ constant by a smaller one.

1.5.1 Commutators of fractional operators and BMO

The counterpart of the theorem by Lacey, Moen, Pérez and Torres for commutators was obtained by Cruz-Uribe and Moen [12]. The precise statement of their result is the following.

Theorem (Cruz-Uribe, Moen [12])**.** *Let $\alpha \in (0, n)$ and $1 < p < \frac{n}{\alpha}$ and q defined by $\frac{1}{q} + \frac{\alpha}{n} = \frac{1}{p}$. Then, if $w \in A_{p,q}$ and $b \in \mathrm{BMO}$ we have that*

$$\|[b, I_\alpha]f\|_{L^q(w^q)} \leq c_{n,\alpha}\|b\|_{\mathrm{BMO}}[w]_{A_{p,q}}^{\left(2-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{q}\right\}} \|f\|_{L^p(w^p)}$$

and the estimate is sharp in the sense that the inequality does not hold if we replace the exponent of the $A_{p,q}$ constant by a smaller one.

1.6 Maximal directional operators

Like we already mention at the Introduction, the last part of this thesis focuses on the study of maximal directional singular integrals, that is considering a singular integral operator along a line embedded in some higher dimensional space, and then taking a supremum over the lines. To introduce this topic, we are going to review some of the previous work on the directional maximal function.

Definition 1.13. For a set of directions $\Omega \subset \mathbb{S}n-1$ and f a locally integrable function in $\mathbb{R}^n$ we define the directional maximal function as

$$\mathrm{M}_\Omega f(x) := \sup_{v\in\Omega}\sup_{\varepsilon>0} \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} |f(x+tv)|\,\mathrm{d}t \qquad x \in \mathbb{R}^n.$$

The first clear observation is that if Ω consists of a single direction, then the operator will share the same boundedness properties of the maximal operator on the real line. We also note that the construction of Besicovitch sets on the plane tells us that the L^p-boundedness cannot hold independently of the set Ω. Indeed, for every $\varepsilon > 0$ we can find a collection of rectangles $\mathcal{R}_\varepsilon$ such that the measure $|\cup_{R\in\mathcal{R}_\varepsilon} R| < \varepsilon$ and $|\cup_{R\in\mathcal{R}_\varepsilon} R| \gtrsim 1$. If we take $f_\varepsilon = \mathbf{1}_{\cup_{R\in\mathcal{R}_\varepsilon} R}$, we then have that $\mathrm{M}_\Omega f_\varepsilon(x) \gtrsim 1$ for $x \in \cup_{R\in\mathcal{R}_\varepsilon} 3R$. This means that while the L^p norm of $\mathrm{M}_\Omega f_\varepsilon$ is bounded by below by some constant independent of ε, we have that $\|f_\varepsilon\|^p_{L^p(\mathbb{R}^2)} < \varepsilon$. This essentially opens up two types of questions:

1. If we take Ω to be finite we trivially get $\|M_\Omega\|_{L^p(\mathbb{R}^n)} \lesssim \#\Omega$, just by applying the triangle inequality. The problem in this scenario is try to get optimal bounds depending on the number of directions.

2. When Ω is infinite, the goal now shifts to finding some structure in Ω in order to guarantee the L^p boundedness of the operator. Like we already mentioned, not containing Besicovitch sets is a necessary condition for this to happen. It turns out that, in the plane, this condition is equivalent to Ω being a generalized lacunary set, which in turn implies that M_Ω is bounded on $L^p(\mathbb{R}^2)$, for $1 < p < \infty$ [4]. In higher dimensions, it is a priori not clear what a lacunary set should look like, since we lose the notion of order.

1.6.1 Lacunary sets of directions

We are going to follow the notion of *lacunary* set of directions as seen on [37]. From now on the ambient space will be $\mathbb{R}^n$ and we consider sets of directions $\Omega \subset \mathbb{S}^{n-1}$. If $\operatorname{span}(\Omega) = \mathbb{R}^d$ for some $d \le n$ then we define the sets of ordered pairs of indices

$$\Sigma = \Sigma(d) \coloneqq \{\sigma = (j,k) : 1 \le j < k \le d\};$$

we will typically drop the dependence on d from the notation.

For $\sigma \in \Sigma$ we now consider lacunary sequences $\{\theta_{\sigma,i}\}_{i\in\mathbb{Z}}$ that satisfy $0 < \theta_{\sigma,i+1} \le \lambda_\sigma \theta_{\sigma,i}$, with $0 < \lambda_\sigma < 1$. Take $\lambda \coloneqq \max_\sigma \lambda_\sigma$. From here on we will assume that the lacunarity constant $\lambda \in (0,1)$ has a fixed numerical value and all sequences considered below will be lacunary with respect to that fixed value λ.

Given an orthonormal basis (ONB) of $\operatorname{span}(\Omega) = \mathbb{R}^d$

$$\mathcal{B} \coloneqq (e_1, \ldots, e_d),$$

and a choice of lacunary sequences $\{\theta_{\sigma,\ell}\}$ as above we get for each $\sigma \in \Sigma$ a partition of the sphere into sectors

$$S_{\sigma,\ell} \coloneqq \left\{ v \in \mathbb{S}^{n-1} : \theta_{\sigma,\ell+1} < \frac{|v \cdot e_{\sigma(2)}|}{|v \cdot e_{\sigma(1)}|} \le \theta_{\sigma,\ell} \right\}, \qquad \mathbb{S}^{d-1} = \bigcup_{\ell\in\mathbb{Z}} S_{\sigma,\ell}.$$

Strictly speaking we need to complete the partition by adding the limit set $S_{\sigma,\infty} \coloneqq \mathbb{S}^{d-1} \cap (e_{\sigma(1)^\perp} \cup$

$e_{\sigma(2)^\perp}$). A convenient way to do so is to define $\mathbb{Z}^* := \mathbb{Z} \cup \{\infty\}$. We write any $\Omega \subseteq \mathbb{S}^{d-1}$ as a disjoint union as follows:

$$\Omega = \bigcup_{\ell \in \mathbb{Z}^*} \Omega \cap S_{\sigma,\ell} := \bigcup_{\ell \in \mathbb{Z}^*} \Omega_{\sigma,\ell}, \qquad \forall \sigma \in \Sigma.$$

The collection of $|\Sigma(d)| = d(d-1)/2$ partitions of Ω will be called a *lacunary dissection* of Ω with parameters $\mathcal{B}$ and $\{\theta_{\sigma,\ell}\}$. In particular we have that $\{S_{\sigma,\ell}\}$ as defined above is a lacunary dissection of the sphere $\mathbb{S}^{d-1}$. We will refer to the sets $\{\Omega_{\sigma,\ell}\}, \{S_{\sigma,\ell}\}$ as *sectors* of a dissection.
We will also need a finer partition of subsets of the sphere into *cells* which is generated as follows. Consider a lacunary dissection of $\Omega \subseteq \mathbb{S}^{d-1}$, namely an ONB $\mathcal{B}$ and sequences $\{\theta_{\sigma,\ell}\}$. Given $\boldsymbol{\ell} = \{\ell_\sigma : \sigma \in \Sigma(d)\} \in \mathbb{Z}^\Sigma$ we define

$$S_{\boldsymbol{\ell}} := \bigcap_{\sigma \in \Sigma} S_{\sigma,\ell_\sigma}, \qquad \Omega_{\boldsymbol{\ell}} := \bigcap_{\sigma \in \Sigma} \Omega_{\sigma,\ell_\sigma},$$

so that we get the partitions

$$\mathbb{S}^{d-1} = \bigcup_{\boldsymbol{\ell} \in \mathbb{Z}^\Sigma} S_{\boldsymbol{\ell}}, \qquad \Omega_{\boldsymbol{\ell}} = \bigcup_{\boldsymbol{\ell} \in \mathbb{Z}^\Sigma} \Omega_{\boldsymbol{\ell}}.$$

We can now define what we mean by lacunary set of directions.

Definition 1.14. Let $\Omega \subset \mathbb{S}^{n-1}$ be a set of directions and assume that $\operatorname{span}(\Omega) = \mathbb{R}^d$. Then Ω is called lacunary of order 0 if it consists of a single direction. If L is a positive integer then Ω is called lacunary of order L if there exists a dissection $\{\Omega_{\sigma,\ell}\}$ of Ω such that for each $\sigma \in \Sigma(d)$ and $\ell \in \mathbb{Z}^*$, the sector $\Omega_{\sigma,\ell} = S_{\sigma,\ell} \cap \Omega$ is a lacunary set of order $L-1$. A set Ω will be called *lacunary* if it is a finite union of lacunary sets of finite order.

Observe that a set Ω is lacunary of order 1 if there exists a dissection $\{\Omega_{\sigma,\ell}\}$ such that each sector $\Omega_{\sigma,\ell}$ contains at most one direction.

Theorem (Parcet, Rogers [37])**.** *Let $n \geq 2$, $p > 1$ and Ω a set of directions in $\mathbb{S}^{n-1}$ that is lacunary. We have*

$$\|\mathrm{M}_\Omega\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)} \leq C \sup_{\sigma \in \Sigma} \sup_{\ell \in \mathbb{Z}} \|\mathrm{M}_{\Omega_{\sigma,\ell}}\|_{L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)},$$

where the constant C depends on n, p and the lacunary constants.

Observe that by an inductive argument we can conclude that lacunarity is a sufficient condition to get the L^p boundedness of the directional maximal function.
In Chapter 4 we are going to find bounds for the L^p norms of another directional operator: the one that arises by taking a singular integral instead of the maximal function. For that, we are going to need a series of tools, which we introduce in the next two subsections.

1.6.2 Directional weighted norm inequalities

Given a closed set of directions $\Omega \subset \mathbb{S}^{n-1}$ and a non-negative, continuous function w on $\mathbb{R}^n$, we say that w belongs to A_p^Ω for $1 \leq p < \infty$ if w belongs to the one-dimensional class $A_p(\ell_v)$ for all lines ℓ_v, $v \in \Omega$, with uniform bounds. More precisely, if we define the segments

$$I(x,t,v) \coloneqq \{x + sv : |s| < t\} \subset \mathbb{R}^n, \qquad x \in \mathbb{R}^n, \quad t > 0, \quad v \in \Omega,$$

then

$$[w]_{A_p^v} \coloneqq \sup_{x\in\mathbb{R}^n, t>0} \left(\fint_{I(x,t,v)} w \right) \left(\fint_{I(x,t,v)} w^{-\frac{1}{p-1}} \right) \qquad [w]_{A_p^\Omega} \coloneqq \sup_{v\in\Omega} [w]_{A_p^v}, \quad 1 < p < \infty$$

and

$$[w]_{A_1^v} \coloneqq \sup_{x\in\mathbb{R}^n, t>0} \frac{w(I(x,t,v))}{|I(x,t,v)|} \|w^{-1}\|_{L^\infty(I(x,t,v))} \qquad [w]_{A_1^\Omega} \coloneqq \sup_{v\in\Omega} [w]_{A_1^v},$$

and $A_p^\Omega \coloneqq \{w \in C(\mathbb{R}^n) : [w]_{A_p^\Omega} < \infty\}$. Note that we need to consider continuous weights in order to make sense of their restrictions to line segments in $\mathbb{R}^n$. This turns out to be more of a technical nuisance rather than substantial limitation and it is inconsequential for our applications. Finally we write

$$A_\infty^\Omega \coloneqq \bigcup_{p>1} A_p^\Omega.$$

In the special case that $\Omega = \{e_1, \ldots, e_n\}$ is the standard coordinate basis we just write A_p^* for the corresponding A_p-class.

The following weighted version of the Marcinkiewicz multiplier theorem, due to Kurtz, can be used in several occasions where we need to prove weighted norm inequalities along lacunary sets of directions.

Proposition 1.15 (Kurtz [25])**.** *Let m be a C^∞ function in $\mathbb{R}^n$ away from the coordinate hyperplanes and assume that $\|m\|_\infty \leq B$. Suppose that for all $0 < k \leq n$ we have*

$$\sup_{x_{k+1},\ldots,x_n} \int_\rho \left| \frac{\partial^k m(x)}{\partial_{\xi_1} \cdots \partial_{\xi_k}} \right| \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_k \leq B$$

for all dyadic rectangles $\rho \subset \mathbb{R}^k$, and any permutation of the coordinates $(\xi_1, \ldots, \xi_n)$. Then for all $p \in (1, \infty)$ and all $w \in A_p^*$ the multiplier operator $T_m(f) \coloneqq (m\widehat{f})^\vee$ satisfies the weighted bounds

$$\|T_m\|_{L^p(w)} \lesssim [w]_{A_p^*}^\gamma$$

where $\gamma = \gamma(p,n,B)$ and the implicit constant is independent of w.

Proposition 1.16. *Let $\Omega \subset \mathbb{S}^{n-1}$ be a set of directions which is lacunary of order L, where L is a positive integer, and let $w \in A_p^\Omega$ be a directional weight with respect to Ω. For all $p \in (1,\infty)$ there* *exists a constant $\gamma = \gamma(p,n) > 0$ such that*

$$\|\mathrm{M}_\Omega\|_{L^p(w)} \lesssim [w]_{A_p^\Omega}^{\gamma L},$$

with implicit constant depending only on p, n and the lacunarity constant of Ω.

The boundedness of the directional maximal function M_Ω now allows us to extrapolate weighted norm inequalities from $L^2(w)$ as in [14, §4.2]. Namely the following holds.

Proposition 1.17. *Let $\Omega \subseteq \mathbb{S}^{n-1}$ be a (closed) lacunary set of directions of finite order. Suppose that there exists a $p_0 \in (1,\infty)$ and $\gamma > 0$ such that for some family of pairs of non-negative function (f,g) we have*

$$\|f\|_{L^{p_0}(w)} \lesssim [w]_{A_{p_0}^\Omega}^{\gamma} \|g\|_{L^{p_0}(w)}$$

with implicit constant independent of (f,g) and w. Then for all $p \in (1,\infty)$ and all $w \in A_p^\Omega$ we have

$$\|f\|_{L^p(w)} \lesssim [w]_{A_p^\Omega}^{\gamma_p} \|g\|_{L^p(w)}$$

where γ_p depends on γ, n, p and the order of lacunarity of Ω; the implicit constant depends only on p, n and the lacunarity constant of Ω.

1.6.3 The Chang-Wilson-Wolff inequality

A familiar tool that has been successfully used in several occasions in the theory of directional singular integrals is a consequence of the Chang-Wilson-Wolff inequality, [8]. This allows us to commute the supremum over N multipliers with a suitable Littlewood-Paley projection at a $\sqrt{\log N}$-loss. For our application it will be useful to have the weighted version of the Chang-Wilson-Wolff inequality, which we state below. For the details of the proof see for example [14, Proposition 5.4] and the references therein. In order to state this result we introduce a coordinate-wise Littlewood-Paley decomposition in the usual fashion.

Letting p be a smooth function on $\mathbb{R}$ such that

$$\sum_{t\in\mathbb{Z}} p(2^{-t}\xi) = 1, \qquad \xi \neq 0,$$

and such that p vanishes off the set $\{\xi \in \mathbb{R} : \frac{1}{2} < |\xi| < 2\}$, we define

$$(P_j^t f)^\wedge(\xi) := p(2^{-t}\xi_j), \qquad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \qquad t \in \mathbb{Z}.$$

Proposition 1.18. *Let $\{R_1, \ldots, R_N\}$ be Fourier multiplier operators on $\mathbb{R}^n$ satisfying uniform $L^2(w)$-bounds*

$$\sup_{1\le\tau\le N} \|R_\tau\|_{L^2(w)} \le [w]_{A_2^*}^{\gamma}$$

for some $\gamma > 0$. *Let* $\{P_t^j\}_{t\in\mathbb{Z}}$ *be a smooth Littlewood-Paley decomposition acting on the* j*-th frequency variable, where* $1 \leq j \leq n$. *For* $w \in A_p^*$ *and* $1 < p < \infty$ *we then have*

$$\Big\| \sup_{1\leq\tau\leq N} |R_\tau f| \Big\|_{L^p(w)} \lesssim [w]_{A_p^*}^{\gamma_p} \bigg(\|f\|_{L^p(w)} + \sqrt{\log(N+1)} \, \Big\| \big(\sum_{t\in\mathbb{Z}} \sup_{1\leq\tau\leq N} |R_\tau P_t^j f|^2 \big)^{\frac{1}{2}} \Big\|_{L^p(w)} \bigg)$$

for some exponent $\gamma_p = \gamma_p(\gamma, p, n)$ *and implicit constant independent of* w, f, N.

Chapter 2

A characterization of BMO in terms of endpoint bounds for the commutator

The main objective of this chapter is to prove the analogous necessity result from Coifman-Rochberg-Weiss theorem stated on Section 1.4 for the endpoint. As we already mentioned, the proof of the CRW theorem is based on estimating the BMO norm of the symbol via a duality argument, which is not a tool that we have at our disposition at the endpoint. The results presented here can be found in [1].

We begin with a result that relates the BMO norm of the symbol with the endpoint bound for the higher order commutator of the Hilbert transform and the symbol b. We recall that if b is a locally integrable function, $f \in C_0^\infty(\mathbb{R})$ and k is a positive integer we may write the commutator of order k of the Hilbert transform and b as

$$H_b^k f(x) = \text{p. v.} \int_{\mathbb{R}} \frac{(b(x) - b(y))^k}{x - y} f(y) \, \mathrm{d}y.$$

Theorem 2.1. *Let $b \in L^1_{\text{loc}}(\mathbb{R})$. If there exists a constant B and a positive integer k such that we have the following estimate*

$$|\{x \in \mathbb{R} : |H_b^k f(x)| > t\}| \leq \int_{\mathbb{R}} \phi_k \left(\frac{B|f(x)|}{t} \right) \mathrm{d}x,$$

where $\phi_k(t) = t(1 + \log^+(t))^k$ and $\log^+(t) = \max(\log(t), 0)$, then $b \in \text{BMO}(\mathbb{R})$ and $\|b\|_{\text{BMO}(\mathbb{R})} \lesssim B^{1/k}$.

Proof. As we noted at Remark 1.8, it is enough to find a constant A such that for every interval I,

$$|\{x \in I : |b(x) - m_I(b)|^k > A\}| \leq \frac{1}{2}|I|,$$

where $m_I(b)$ is a median of b on I.
Fix I an interval. We can find disjoint subsets of I, E_+ and E_- such that $|E_+| = |E_-| = |I|/2$,

$$E_+ \subset \{y \in I : b(y) \geq m_I(b)\}$$
$$E_- \subset \{y \in I : b(y) \leq m_I(b)\}.$$

Then,

$$|b(x) - m_I(b)|^k \chi_I(x) = (b(x) - m_I(b))^k \chi_{E_+}(x) + (m_I(b) - b(x))^k \chi_{E_-}(x).$$

For $y \in E_-$ and $z \in E_+$ we have

$$|b(x) - m_I(b)|^k \chi_I(x) \leq (b(x) - b(y))^k \chi_{E_+}(x) + (b(z) - b(x))^k \chi_{E_-}(x).$$

Now integrating for $y \in E_-$ and $z \in E_+$ and calling c_I the center of I, we get

$$|b(x) - m_I(b)|^k \chi_I(x) \leq \frac{1}{|E_-|}\int_{E_-} (b(x) - b(y))^k \chi_{E_+}(x)\,\mathrm{d}y + \frac{1}{|E_+|}\int_{E_+} (b(x) - b(z))^k \chi_{E_-}(x)\,\mathrm{d}z.$$

The first summand in the right hand side of the estimate above can be bounded above by

$$\begin{aligned}\frac{1}{|E_-|}\int_{E_-} (b(x) - b(y))^k \chi_{E_+}(x)\,\mathrm{d}y &\leq \frac{1}{|E_-|}\int_{\mathbb{R}} \frac{(b(x) - b(y))^k}{x - y}(x - c_I)\chi_{E_+}(x)\chi_{E_-}(y)\,\mathrm{d}y \\ &\quad + \frac{1}{|E_-|}\int_{\mathbb{R}} \frac{(b(x) - b(y))^k}{x - y}(c_I - y)\chi_{E_+}(x)\chi_{E_-}(y)\,\mathrm{d}y \\ &\leq 2\frac{|x - c_I|}{|I|}|H^k(\chi_{E_+})(x)| + 2\left|H^k\left(\frac{(\cdot - c_I)}{|I|}\chi_{E_-}\right)(x)\right|.\end{aligned}$$

Using a similar estimate for the second summand we get

$$\begin{aligned}&|\{x \in I : |b(x) - m_I(b)|^k > A\}| \leq |\{x \in \mathbb{R} : |H^k(\chi_{E_+})(x)| > A/8\}| \\ &+ |\{x \in \mathbb{R} : |H^k((\cdot - c_I)/|I|\chi_{E_-}(x) > A/8\}| \\ &+ |\{x \in \mathbb{R} : |H^k(\chi_{E_-})(x)| > A/8\}| + |\{x \in \mathbb{R} : |H^k((\cdot - c_I)/|I|\chi_{E_+})(x)| > A/8\}| \\ &=: (i) + (ii) + (iii) + (iv).\end{aligned}$$

We show the estimate for (i). The estimates for the other terms are similar.

$$(i) \leq \int_{\mathbb{R}} \chi_{E_+}(x)\frac{8B}{A}\left(1 + \log^+\left(\chi_{E_+}(x)\frac{8B}{A}\right)\right)^k \mathrm{d}x \leq |E_+|\frac{1}{4} = \frac{|I|}{8},$$

if we choose $A = 32B$. Summing,

$$|\{x \in I : |b(x) - m_I(b)|^k > A\}| \leq \frac{1}{2}|I|$$

as we wanted. $\square$

2.1 Higher dimensions

For this section we will be considering singular integral operators T of the form

$$Tf(x) \coloneqq \text{p. v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,\mathrm{d}y, \tag{2.1}$$

where $\Omega \in \operatorname{Lip}(\mathbb{S}^{n-1})$ is homogeneous of degree zero, satisfies $\int_{\mathbb{S}^{n-1}} \Omega = 0$, and the set $\{\Omega(x) = 0\}$ has zero measure. An important class of operators that satisfies these conditions are the Riesz transforms,

$$R_j f(x) \coloneqq \text{p. v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\,\mathrm{d}y.$$

First we prove that if the commutator of one of these operators with a symbol b is of weak type $(1,1)$ then b must satisfy a stronger condition than $\mathrm{BMO}(\mathbb{R}^n)$, namely $b \in L^\infty(\mathbb{R}^n)$.

Theorem 2.2. *Let b be a locally integrable function and suppose $[b,T] : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$ is bounded. Then $b \in L^\infty(\mathbb{R}^n)$ and we have the bound $\|b\|_\infty \leq C(\Omega, n)\|[b,T]\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}}(\mathbb{R}^n)$.*

Proof. We begin by fixing a locally integrable function b. Note that this assumption implies that b is finite almost everywhere, and that almost every point $y \in \mathbb{R}^n$ is a Lebesgue point of b. Now recall that

$$[b,T]f(x) = \text{p. v.} \int_{\mathbb{R}^n} \frac{b(x) - b(y)}{|x-y|^n} \Omega(x-y) f(y)\,\mathrm{d}y.$$

By renormalizing b, we can assume $\|[b,T]\|_{L^1(\mathbb{R}^n)\to L^{1,\infty}(\mathbb{R}^n)} = 1$. Take f to be a $C^\infty(\mathbb{R}^n)$ function with compact support, even, $\operatorname{supp} f \subset B(0,1)$, $\int f = 1$, and $0 \leq f \leq 1$. For every $\varepsilon > 0$, set $f_\varepsilon(x) \coloneqq \frac{1}{\varepsilon^n} f\left(\frac{x}{\varepsilon}\right)$ and $f_\varepsilon^y(x) \coloneqq f_\varepsilon(y - x)$. Then, whenever y is a Lebesgue point of b, we have

$$\lim_{\varepsilon \to 0} |[b,T]f_\varepsilon^y(x)| = \frac{|b(x) - b(y)|}{|x-y|^n} |\Omega(x-y)|.$$

So we get that, for every $\lambda > 0$ and y a Lebesgue point for b,

$$|\{x \in \mathbb{R}^n : \frac{|b(x) - b(y)|}{|x-y|^n} |\Omega(x-y)| > \lambda\}| \leq \frac{\|[b,T]\|_{L^1 \to L^{1,\infty}}}{\lambda}. \tag{2.2}$$

Fix $\varepsilon > 0$ and take K to be a compact subset of $\mathbb{S}^{n-1}$ such that $\{x \in \mathbb{S}^{n-1} : \Omega(x) = 0\} \cap K = \emptyset$ and $|\mathbb{S}^{n-1} \setminus K| < \varepsilon$. Call $C_\Omega \coloneqq \inf\{|\Omega(x)| : x \in K\}$ and note that $C_\Omega > 0$, by the Lipschitz assumption on Ω. We now define the following sets

$$\Lambda_\lambda(y) := \left\{x \in \mathbb{R}^n : \frac{x-y}{|x-y|} \in K, \frac{|b(x)-b(y)|}{|x-y|^n} > \lambda\right\},$$
$$S_K(y) := \left\{x \in \mathbb{R}^n : \frac{x-y}{|x-y|} \in K\right\}.$$

Note that, with the definitions above, and the choice of K, we have for every $r > 0$ that

$$|B(0,r) \setminus S_K(0)| < \epsilon r^n/n. \tag{2.3}$$

By (2.2) and the definition of C_Ω we have

$$|\Lambda_\lambda(y)| \leq \frac{1}{C_\Omega \lambda}.$$

Since our hypothesis is invariant under replacing b by $b-c$, for any constant c, and since b is finite almost everywhere, we can assume that $b(0) = 0$ and we also have

$$|\Lambda_\lambda(0)| = \left|\left\{x \in \mathbb{R}^n : \frac{x}{|x|} \in K, \frac{|b(x)|}{|x|^n} > \lambda\right\}\right| \leq \frac{1}{C_\Omega \lambda}.$$

Let $y \neq 0$ and $x \notin \Lambda_{1/|y|^n}(y)$, $x \in B(y, \frac{1}{2}|y||b(y)|^{1/n}) \cap S_K(y)$

$$\begin{aligned}|b(x)| &\geq |b(y)| - \frac{|b(x)-b(y)|}{|x-y|^n}|x-y|^n \geq |b(y)| - \frac{1}{|y|^n}\left(\frac{1}{2}|y||b(y)|^{1/n}\right)^n \\ &\geq \big(1 - \frac{1}{2^n}\big)|b(y)| = c_n|b(y)|\end{aligned}$$

for almost every y. Suppose that $|b(y)| > 2^n$ (if we had that $|b(y)| \leq 2^n$ for all $y \neq 0$ that is also a Lebesgue point then we would be done). We conclude that

$$\begin{aligned}A(y,K) &:= |\{x \in [B(y, \frac{1}{2}|y||b(y)|^{1/n}) \cap S_K(y) \cap S_K(0)] \setminus \Lambda_{1/|y|^n}(y) : \frac{|b(y)|}{|x|^n} > \frac{1}{|y|^n}\}| \\ &\leq |\{x \in S_K(0) : \frac{c_n|b(x)|}{|x|^n} > \frac{1}{|y|^n}\}| = |\Lambda_{1/(c_n|y|^n)}(0)| \leq C_\Omega^{-1} c_n |y|^n.\end{aligned} \tag{2.4}$$

Since $|b(y)| > 2^n$,

$$|y||b(y)|^{1/n} = \frac{1}{2}|y||b(y)|^{1/n} + \frac{1}{2}|y||b(y)|^{1/n} \geq |y| + \frac{1}{2}|y||b(y)|^{1/n}.$$

This implies that $B(y, \frac{1}{2}|y||b(y)|^{1/n}) \subset B(0, |y||b(y)|^{1/n})$ and so

$$|A(y,K)| \geq \left|B(y, \frac{1}{2}|y||b(y)|^{1/n}) \cap S_K(y)\right| - \left|B(0, |y||b(y)|^{1/n}) \cap S_K(0)^c\right| - \left|\Lambda_{1/|y|^n}(y)\right|. \tag{2.5}$$

Let us observe that

$$\begin{aligned}|B(y,\frac{1}{2}|y||b(y)|^{1/n})\cap S_K(y)| &= |B(0,\frac{1}{2}|y||b(y)|^{1/n})\cap S_K(0)|\\ &= \frac{1}{2^n}|y|^n|b(y)|(|B(0,1)|-|B(0,1)\cap S_K(0)^c|) \geq \frac{1}{2^n}|y|^n|b(y)|\left(\omega_n - \frac{\varepsilon}{n}\right)\end{aligned}$$

by (2.3); here ω_n denotes the measure of the unit ball. We also have that

$$\begin{aligned}|B(0,|y||b(y)|^{1/n})\cap S_K(0)^c| + |\Lambda_{1/|y|^n}(y)| &\leq |y|^n|b(y)||B(0,1)\cap S_K(0)^c| + C_\Omega^{-1}|y|^n\\ &\leq |y|^n|b(y)|\frac{\varepsilon}{n} + C_\Omega^{-1}|y|^n,\end{aligned}$$

here we are using (2.3) again. Estimate (2.5) then yields

$$|A(y,K)| \geq |y|^n|b(y)|\left(\frac{1}{2^n}\omega_n - \frac{\varepsilon}{n}\frac{2^n+1}{2^n}\right) - C_\Omega^{-1}|y|^n.$$

Now take $\varepsilon = \frac{n}{2}\omega_n\frac{1}{2^n+1}$. Combining with the previous estimate we get that

$$c_nC|y|^n \geq \frac{\omega_n}{2^{n+1}}|y|^n|b(y)| - C_\Omega^{-1}|y|^n$$

and so

$$|b(y)| \leq \max\left\{2^n, \frac{2^{n+1}(c_n+1)}{C_\Omega\omega_n}\right\} = \max\left\{2^n, \frac{2}{\omega_n}(2^{n+1}-1)\frac{1}{C_\Omega}\right\} =: C(\Omega,n)$$

for almost all $y\in\mathbb{R}^n$ and thus b is bounded and $\|b\|_\infty \leq C(\Omega,n)$ as desired. □

Now we prove a higher dimensional analogue of Theorem 2.2 for the class of singular integral operators given in (2.1). We use a similar argument as the one given by Uchiyama in [44]. We impose a symmetric condition on the adjoint operator; indeed, since we are assuming an endpoint estimate, we can no longer rely on duality in order to conclude the boundedness of the adjoint commutator. Note however that for the Riesz transforms, as well as for more general odd kernels as in (2.1), it will be enough to assume the endpoint boundedness of $[b,T]$ at the endpoint in order to conclude that $b\in \mathrm{BMO}(\mathbb{R}^n)$ (we would get the condition on the adjoint for free, since $[b,T]^* = [b,T^*] = [b,-T]$ for odd convolution kernels).
For the statement of the theorem below we remember that $\phi_1(t) = t(1+\log^+ t)$.

Theorem 2.3. *Let b be a locally integrable function on $\mathbb{R}^n$. If there exists a constant B such that for every measurable set E and $t>0$ we have that*

$$|\{x\in\mathbb{R}^n : |[b,T]\chi_E(x)|>t\}| \leq \int_{\mathbb{R}^n}\phi_1\left(\frac{B\chi_E(x)}{t}\right)\mathrm{d}x,$$

and

$$|\{x\in\mathbb{R}^n : |[b,T^*]\chi_E(x)|>t\}| \leq \int_{\mathbb{R}^n}\phi_1\left(\frac{B\chi_E(x)}{t}\right)\mathrm{d}x,$$

then $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and* $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq C(\Omega, n)B$.

Proof. As we did in the proof of Theorem 2.2, we can assume $B = 1$. Define $M(b, Q) := \inf_{c\in\mathbb{R}} \frac{1}{|Q|} \int_Q |b(y) - c| dy$. We want to prove

$$\sup_Q M(b, Q) \leq C(\Omega, n). \tag{2.6}$$

By translation and dilation invariance it suffices to prove (2.6) for the cube $Q_1 = \{x \in \mathbb{R}^n : |x| < (2\sqrt{n})^{-1}\}$.

Let $M := M(b, Q_1) = |Q_1|^{-1} \int_{Q_1} |b(y) - m_{Q_1}(b)| dy$, where $m_{Q_1}(b)$ is a median of b over Q_1. Since $[b - m_{Q_1}(b), T] = [b, T]$ we may assume that $m_{Q_1}(b) = 0$. This means that we can find disjoint subsets of Q_1, $E_1 \supset \{x \in Q_1 : b(x) < 0\}$ and $E_2 \supset \{x \in Q_1 : b(x) > 0\}$ of equal measure. Define $\psi := \chi_{E_2} - \chi_{E_1}$. Then ψ satisfies: $\|\psi\|_\infty = 1$, $\operatorname{supp} \psi \subset Q_1$,

$$\int \psi(x)\, \mathrm{d}x = 0, \quad \psi(x)b(x) \geq 0, \quad \text{and} \quad |Q_1|^{-1} \int \psi(x)b(x)\, \mathrm{d}x = M.$$

Take $\Sigma \subset \mathbb{S}^{n-1}$ a compact set such that $\Omega(x) > 0$ for every $x \in \Sigma$. From now on, we will denote by A_i constants depending only on the dimension n and the kernel Ω. Take A_1 such that for every $x \in \Sigma$ and $z \in \mathbb{S}^{n-1}$ satisfying $|x - z| < A_1$, we have $|\Omega(x) - \Omega(z)| < \frac{1}{2}\Omega(x)$. Denote $x' = x/|x|$. Then, for $x \in G := \{x \in \mathbb{R}^n : |x| > A_2 = 2A_1^{-1} + 1 \text{ and } x' \in \Sigma\}$,

$$|[b, T]\psi(x)| = |T(b\psi)(x) - b(x)T\psi(x)| \geq |T(b\psi)(x)| - |b(x)||T\psi(x)|.$$

We bound these two terms separately. First, we bound

$$|T(b\psi)(x)| = \left| \text{p. v.} \int_{Q_1} \frac{\Omega(x - y)}{|x - y|^n} b(y)\psi(y)\, \mathrm{d}y \right|.$$

Observe that $|(x-y)' - x'| < A_1$ and so $\Omega(x-y) > \frac{1}{2}\Omega(x)$, which in particular means that $\Omega(x-y)$ is positive. Since we already have that $b(y)\psi(y)$ is nonnegative and we are taking $x \in G$, we get

$$\left| \text{p. v.} \int_{Q_1} \frac{\Omega(x - y)}{|x - y|^n} b(y)\psi(y)\, \mathrm{d}y \right| = \int_{Q_1} \frac{\Omega(x - y)}{|x - y|^n} |b(y)|\, \mathrm{d}y \geq A_3 M|x|^{-n}.$$

Now we have to deal with $|T\psi(x)|$. Since we have that $\int \psi = 0$ we can estimate

$$\begin{aligned}\left|\mathrm{p.v.}\int_{Q_1}\frac{\Omega(x-y)}{|x-y|^n}\psi(y)\,\mathrm{d}y\right| &= \left|\int_{Q_1}\left(\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n}\right)\psi(y)\,\mathrm{d}y\right|\\ &\le \int_{Q_1}\left|\frac{\Omega(x-y)}{|x-y|^n}-\frac{\Omega(x)}{|x|^n}\right|\mathrm{d}y\\ &\le A_4|x|^{-n-1}.\end{aligned}$$

Then, we have

$$|[b,T]\psi(x)| \ge A_3M|x|^{-n} - A_4|b(x)||x|^{-n-1}.$$

Letting $F := \{x \in G : |b(x)| > (MA_3/2A_4)|x| \text{ and } |x| < M^{1/n}\}$ we have

$$\begin{aligned}&|\{x\in\mathbb{R}^n : |[b,T]\psi(x)| > A_3/2\}| \ge |\{x \in (G\backslash F)\cap\{|x|<M^{1/n}\} : |[b,T]\psi(x)| > A_3/2\}|\\ &\ge |\{x\in(G\backslash F)\cap\{|x|<M^{1/n}\} : 2^{-1}A_3M|x|^{-n} > A_3/2\}|\\ &= |(G\backslash F)\cap\{|x|<M^{1/n}\}| = A_5(M-A_2^n) - |F|\end{aligned}$$

By our assumption, we have that

$$|\{x\in\mathbb{R}^n : |[b,T]\psi(x)| > A_3/2\}| \le \int_{Q_1}\phi(2A_3^{-1}|\psi(x)|)dx \le |Q_1|\phi(2A_3^{-1}).$$

Then

$$|F| \ge A_5(M-A_2^n) - \phi(2A_3^{-1})|Q_1| \ge A_5M/2$$

by assuming, as we may, that M is large enough.

Let $g(x) := \mathrm{sgn}(b(x))\chi_F(x)$ and T^* be the adjoint operator of T. Then, for $x \in Q_1$,

$$|[b,T^*]g(x)| \ge |T^*(bg)(x)| - |b(x)||T^*(g)(x)|.$$

By the definition of F, we have

$$\begin{aligned}|T^*(bg)(x)| &= \left|\mathrm{p.v.}\int_{\mathbb{R}^n}\Omega(y-x)|x-y|^{-n}b(y)g(y)\,\mathrm{d}y\right|\\ &= \int_F \Omega(y-x)|x-y|^{-n}|b(y)|\,\mathrm{d}y.\end{aligned}$$

Note that $y \in F$ means that $|y| \le M^{1/n}$ and thus

$$\begin{aligned}|T^*(bg)(x)| &\ge A_6\int_F\frac{MA_3}{2A_4}|y|^{-n+1}\mathrm{d}y\\ &\ge A_6A_3(2A_4)^{-1}A_2M^{1/n}|F| \ge A_7M^{1+1/n}.\end{aligned}$$

For the second summand in the estimate for $[b, T^*]$ we have for $x \in Q_1$ that

$$\begin{aligned} |T^* g(x)| &\leq \left| \text{p. v.} \int_F \Omega(y-x)|x-y|^{-n} g(y)\, \mathrm{d}y \right| \\ &\leq \int_F |\Omega(y-x)||x-y|^{-n}\, \mathrm{d}y \\ &\leq \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \int_F |y-x|^{-n}\, \mathrm{d}y \\ &\leq \|\Omega\|_{L^\infty(\mathbb{S}^{n-1})} \int_{A_2 \leq |y| \leq M^{1/n}} \frac{1}{|y|^n - 2^{-n}} \leq A_8 \log M. \end{aligned}$$

Then, for $x \in Q_1$,

$$|[b, T^*]g(x)| \geq A_7 M^{1+1/n} - A_8 |b(x)| \log M.$$

By our assumption on T^* we can now conclude that

$$\begin{aligned} &|\{x \in \mathbb{R}^n : |[b, T^*]g(x)| \geq (A_7/2) M^{1+1/n}\}| \leq \int_{\mathbb{R}^n} \phi\left(\frac{|g(x)|}{(A_7/2)M^{1+1/n}} \right) \mathrm{d}x \\ &= \int_F \phi([(A_7/2)M^{1+1/n}]^{-1}) dx = |F| \phi(A_9 M^{-1/n-1}) \\ &\leq M \phi(A_9 M^{-1/n-1}) = A_9 M^{-1/n}, \end{aligned}$$

where the last inequality follows by taking M large enough, since $\log^+ t$ vanishes for $|t| < 1$. On the other hand,

$$\begin{aligned} &|\{x \in \mathbb{R}^n : [b, T^*]g(x)| \geq A_7 M^{1+1/n}\}| \geq |\{x \in Q_1 : |[b, T^*]g(x)| \geq A_7/2 M^{1+1/n}\}| \\ &\geq |\{x \in Q_1 : A_7 M^{1+1/n} - A_8 \log M |b(x)| \geq A_7/2 M^{1+1/n}\}| \\ &= |\{x \in Q_1 : |b(x)| \leq A_{10} M^{1+1/n} (\log M)^{-1}\}| \\ &= |Q_1| - |\{x \in Q_1 : |b(x)| > A_{10} M^{1+1/n} (\log M)^{-1}\}| \\ &\geq |Q_1| - A_{10} |Q_1| \log M M^{-1-1/n} |Q_1|^{-1} \int_{Q_1} |b(x)|\, \mathrm{d}x \\ &= |Q_1| - A_{10} |Q_1| \log M M^{-1/n} \geq A_{11}, \end{aligned}$$

as $M^{-1/n} \log M$ is bounded for every $M > e^{1/n}$. Then, we have that

$$M \leq (A_9/A_{11})^n.$$

Summarizing the estimates above, we have proved that

$$M \leq \max\left\{ \frac{2}{A_5}(1 + A_5 A_2^n), A_9^{-1/n-1}, e^{1/n}, (A_9/A_{11})^n \right\} =: C(\Omega, n),$$

as all the constants A_i depend only on Ω and n. □

2.2 A Hardy space for the commutator

We already mentioned that the commutator can fail to be of weak type $(1,1)$, so it makes sense to ask whether, as in the case of classical singular integrals, we can find a subspace of the Hardy space $\mathcal{H}^1$ from which we can obtain boundedness into $L^1(\mathbb{R}^n)$. A natural candidate of such a subspace was mentioned in [39], where the author introduced the spaces $\mathcal{H}_b^{(1,p)}(\mathbb{R}^n)$, which were defined atomically in the following fashion. For $1 < p \leq \infty$ we will call a a $(1,p)$ b-atom if it satisfies:

(a) There exists a cube Q such that $\operatorname{supp}(a) \subset Q$,

(b) $\|a\|_{L^p(Q)} \leq |Q|^{-\frac{1}{p'}}$,

(c) $\int_{\mathbb{R}^n} a(x)dx = 0$,

(d) $\int_{\mathbb{R}^n} a(x)b(x)dx = 0$.

The space $\mathcal{H}_b^{(1,p)}(\mathbb{R}^n)$ is then defined as

$$\mathcal{H}_b^{(1,p)}(\mathbb{R}^n) := \left\{ f \in \mathcal{H}^1(\mathbb{R}^n) : f = \sum_{j=1}^{\infty} \lambda_j a_j, \text{ with } (a_j)_{j\in\mathbb{N}} \text{ } b\text{-atoms}, \ (\lambda_j)_{j\in\mathbb{N}} \in \mathbb{R} \text{ and } \sum_{j=1}^{\infty} |\lambda_j| < \infty \right\}.$$

These spaces endowed with their natural norm

$$\|f\|_{\mathcal{H}_b^{(1,p)}} := \inf \left\{ \sum_{j=1}^{\infty} |\lambda_j| : \ f = \sum_{j=1}^{\infty} \lambda_j a_j, \ (a_j) \text{ } b\text{-atoms} \right\}$$

are Banach spaces. Note that the $(1,p)$ b-atoms are classical $\mathcal{H}^1$ atoms with the extra cancellation condition (d), that involves the symbol b. It was suggested in [39] that this space is the "right" space for the endpoint boundedness of the commutator, that is: given a BMO function b and a Calderón-Zygmund operator T we have that $[b,T] : \mathcal{H}_b^{(1,p)}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$. Specifically, what was proven was the following.

Theorem (Pérez [39])**.** *Let T be a CZO and b a function in BMO. For $1 < p \leq \infty$, call $\mathcal{A}_b^{(1,p)}(\mathbb{R}^n)$ to the set of $(1,p)$ b-atoms that we defined before. Then, we have*

$$\sup_{a \in \mathcal{A}_b^{(1,p)}(\mathbb{R}^n)} \|[b,T](a)\|_{L^1(\mathbb{R}^n)} \leq C$$

for some constant C.

It is relevant to note that both the commutator and the atoms involve the same symbol b. When dealing with a general linear operator, asking to get boundedness of the operator on the whole space from boundedness on the atoms is the same as asking for equivalence of the norm with the *finite* norm, which we define now. Consider the subspace $\mathcal{H}_{b,\text{fin}}^{(1,p)}(\mathbb{R}^n) := \mathcal{H}_b^{(1,p)}(\mathbb{R}^n) \cap L_c^p(\mathbb{R}^n)$, which consists of all the functions on $\mathcal{H}_b^{(1,p)}(\mathbb{R}^n)$ that admit a finite atomic decomposition. In this subspace we can define the norm

$$\|f\|_{\mathcal{H}_{b,\text{fin}}^{(1,p)}} := \inf \left\{ \sum_{j=1}^{N} |\lambda_j| : f = \sum_{j=1}^{N} \lambda_j a_j, (a_j)\, b\text{-atoms} \right\}.$$

Observe that, for a function $f \in \mathcal{H}_{b,\text{fin}}^{(1,p)}(\mathbb{R}^n)$, we have the trivial inequalities

$$\|f\|_{\mathcal{H}^1} \leq \|f\|_{\mathcal{H}_b^{(1,p)}} \leq \|f\|_{\mathcal{H}_{b,\text{fin}}^{(1,p)}}.$$

Here we want to draw a parallel with the case of the classical Hardy space $\mathcal{H}^1(\mathbb{R}^n)$. We know that we can define this space using any $(1,p)$ atoms with $p \in [1,\infty]$; note that here the case $p = \infty$ is included. We can also define the finite norm, and we know that for $p < \infty$, the usual and the finite norm are equivalent [34], that is

$$\|f\|_{\mathcal{H}^{(1,p)}} \approx \|f\|_{\mathcal{H}_{\text{fin}}^{(1,p)}}, \qquad f \in \mathcal{H}_{\text{fin}}^{(1,p)},$$

and so to prove that a linear operator is bounded on $\mathcal{H}^1$ it is enough to test our operator on the atoms. We also know that this equivalence might fail: this is the case when $p = \infty$. Indeed, in [6] Bownik gives an example of a linear operator that is bounded uniformly on $(1,\infty)$-atoms but doesn't admit a bounded extension to all $\mathcal{H}^1$. It is important to note that for some special cases, like when our operator is a CZO, boundedness on atoms is enough to extrapolate to the whole space. For the commutator we don't have an analog result at hand. This was observed by Ky in [26], where the author also obtained the following theorem.

Theorem (Ky [26])**.** *Let T be a CZO, b a BMO function and $1 < p \leq \infty$. Then, the commutator $[b,T] : \mathcal{H}_b^{(1,p)}(\mathbb{R}^n) \to L^1(\mathbb{R}^n)$.*

An easy observation about these spaces is that $\mathcal{H}_b^{(1,p_2)}(\mathbb{R}^n) \subseteq \mathcal{H}_b^{(1,p_1)}(\mathbb{R}^n)$ when $p_2 > p_1$, but the equality is unclear. An interesting open question is trying to characterize the *biggest* space contained in $\mathcal{H}^1(\mathbb{R}^n)$ that maps into $L^1(\mathbb{R}^n)$. Getting a clear picture of this space would also allows us, via duality, to define the commutator on the other endpoint, namely from $L^\infty(\mathbb{R}^n)$. As a first step in this direction, we computed the dual space of the $\mathcal{H}_{b,\text{fin}}^{(1,p)}$, from which we obtained the following spaces.

Definition 2.4. The space $\text{BMO}_b^q(\mathbb{R}^n)$ consists of the functions $f \in L_{\text{loc}}^q(\mathbb{R}^n)$ such that the quantity

$$\|f\|_{\text{BMO}_b^q} := \sup_B \left(\inf_{c_0,c_1} \frac{1}{|B|} \int_B |f(x) - c_0 - c_1 b(x)|^q \, \mathrm{d}x \right)^{1/q} < \infty,$$

where the supremum is taken over all the balls B.

Observe that $\|f\|_{\text{BMO}_b^q} = 0$ if and only if $f = \alpha + \beta b$ and so, in order to have a norm, we have to take the quotient by the subspace

$$S \coloneqq \{\alpha + \beta b,\ \alpha, \beta \in \mathbb{R}\} =< 1, b > .$$

It is clear from the definition that $\mathrm{BMO}(\mathbb{R}^n) \subset \mathrm{BMO}_b^q(\mathbb{R}^n)$ for any $1 < q < \infty$. We will show that this inclusion is proper with an easy example. Take $b \in \mathrm{BMO}(\mathbb{R}^n)$ and consider the function b^2, which is typically not a $\mathrm{BMO}(\mathbb{R}^n)$ function. We can compute for any ball B

$$\frac{1}{|B|}\int_B |b^2(x) - 2b_B b(x) + (b_B)^2|^q \,\mathrm{d}x = \frac{1}{|B|}\int_B |b(x) - b_B|^{2q} \,\mathrm{d}x \leq \|b\|_{\mathrm{BMO}}^{2q} < \infty,$$

and so b^2 belongs to $\mathrm{BMO}_b^q(\mathbb{R}^n)$. For example, taking $b(x) = \log(x)$, we have that $b^2(x) = \log^2(x)$ is not a BMO function.

An interesting question is whether we have an analog of the John-Nirenberg for the spaces $\mathrm{BMO}_b^q(\mathbb{R}^n)$. The same example shows that if yes, then the decay must depend on the symbol b and will in general be worse than the exponential one that we have in the traditional theorem.

Following the proof of the classical duality theorem of $\mathcal{H}^1(\mathbb{R}^n)$ and $\mathrm{BMO}(\mathbb{R}^n)$ [41, Chapter IV] we get the following theorem.

Theorem 2.5. *Let b be non-constant* BMO *function, $1 < p < \infty$ and q such that $1/p + 1/q = 1$. Then the dual space of $(\mathcal{H}_{b,\mathrm{fin}}^{(1,p)}(\mathbb{R}^n), \|\cdot\|_{\mathcal{H}_{b,\mathrm{fin}}^{(1,p)}})$ is* $\mathrm{BMO}_b^q(\mathbb{R}^n)$.

Proof. Let $g \in \mathrm{BMO}_b^q(\mathbb{R}^n)$ and define the functional l_g on $\mathcal{H}_{b,\mathrm{fin}}^{(1,p)}(\mathbb{R}^n)$ by

$$l_g(f) \coloneqq \int_{\mathbb{R}^n} g(x) f(x) \,\mathrm{d}x.$$

Observe that, since f has compact support, the integral above is absolutely convergent. We also have

$$|l_g(f)| = |\int_{\mathbb{R}^n} \sum_{j=1}^N \lambda_j a_j(x) g(x)\, dx| = |\sum_{j=1}^N \lambda_j \int_{B_j} a_j(x)(g(x) - c_0^j - c_1^j b(x))\,\mathrm{d}x|$$
$$\leq \sum_{j=1}^N |\lambda_j| (\int_{B_j} |a_j(x)|^p \,\mathrm{d}x)^{1/p} (\int_{B_j} |g(x) - c_0^j - c_1^j b(x)|^q \,\mathrm{d}x)^{1/q} \leq \|g\|_{\mathrm{BMO}_b^q} \sum_{j=1}^N |\lambda_j|.$$

Taking infimum on both sides with respect to all finite representations of f, we get

$$|l_g(f)| \leq \|g\|_{\mathrm{BMO}_b^q} \|f\|_{\mathcal{H}_{b,\mathrm{fin}}^{(1,p)}},$$

and this shows that every function in $\mathrm{BMO}_b^q(\mathbb{R}^n)$ can be identified with a functional in $(\mathcal{H}_{b,\mathrm{fin}}^{(1,p)}(\mathbb{R}^n))^*$, proving the first inclusion.

To prove the converse inclusion take $l \in (\mathcal{H}_{b,\mathrm{fin}}^{(1,p)}(\mathbb{R}^n))^*$ of norm 1. We are going to construct a function $g \in \mathrm{BMO}_b^p(\mathbb{R}^n)$ of norm less or equal than 1 such that $l = l_g$. To that end fix a ball B

and consider the closed subspace

$$L_0^p(B) = \{f \in L^p(B) : \int_B f(x)\,\mathrm{d}x = 0, \int_B f(x)b(x)\,\mathrm{d}x = 0\}.$$

Observe that any function in $L_0^p(B)$ belongs to the space $\mathcal{H}_{b,\mathrm{fin}}^{(1,p)}$ and, moreover, satifies the norm estimate $\|f\|_{\mathcal{H}_{b,\mathrm{fin}}^{(1,p)}} \leq |B|^{1/q}\|f\|_{L^p(B)}$. Then l extends to a linear functional on $L_0^p(B)$ with norm at most $|B|^{1/q}$. Define $S(B) :=< \chi_B, b\chi_B > \subset L^p(B)$. It is not difficult to see that we can identify

$$\begin{aligned}(L_0^p(B))^\perp &= \{\phi \in (L^p(B))^* : \phi(L_0^p(B)) = 0\}\\ &= \{g \in L^q(B) : \int_B g(x)f(x)\,\mathrm{d}x = 0 \text{ for all } f \in L_0^p(B)\} = S(B)\end{aligned}$$

which means that $(L_0^p(B))^* = L^q(B)/S(B)$. So there exists an element $G^B \in L^q(B)/S(B)$ such that

$$\begin{aligned}&l(f) = \int_B f(x)G^B(x)\,\mathrm{d}x \text{ for all } f \in L_0^p(B)\\ &\text{and} \quad \|G^B\|_{L^q(B)/S(B)} = \inf_{c_0,c_1}\Big(\int_B |G^B(x) - c_0 - c_1 b(x)|^q\,\mathrm{d}x\Big)^{1/q} \leq |B|^{1/q}.\end{aligned}$$

Observe that if $B_1 \subset B_2$ are balls, then $G^{B_1} - G^{B_2} \in S(B_1)$ since they both define the same functional on B_1. Without loss of generality we can suppose that b is non-constant on the ball of radius 1 and center at the origin. Now consider the sequence of balls $(B_i) = (B(0,i))_{i\in\mathbb{N}}$ and for each ball define $g^{B_i} = G^{B_i} + \alpha_{B_i} + \beta_{B_i} b\chi_{B_i}$, with the constants chosen such that

$$\int_{B(0,1)} g^{B_i} = \int_{B(0,1)} g^{B_i} b = 0.$$

Now if we take two balls of the sequence, $B_j \subset B_k$, we have that $g^{B_j} - g^{B_k} \in S(B_j)$, say $g^{B_j} - g^{B_k} = \alpha + \beta b$ in B_j. But, since $B(0,1)$ is contained in B_j, we also have that $\int_{B(0,1)} \alpha + \beta b(x)dx = \int_{B(0,1)}(\alpha + \beta b(x))b(x)dx = 0$ from where we can deduce that $\alpha = \beta = 0$. We can now define $g(x) = g^{B_i}(x)$ for all $x \in B_i$. We are left to prove that g belongs to $\mathrm{BMO}_b^q(\mathbb{R}^n)$. Given any ball B we can find an i such that $B \subset B_i$. Since $g = g^{B_i}$ on B, we have that $g|_B = G^{B_i}|_B + \alpha_{B_i} + \beta_{B_i} b\chi_B$ which is an element of the equivalence class of G^B as we observed before. Hence

$$\inf_{c_0,c_1}\left(\frac{1}{|B|}\int_B |g(x) - c_0 - c_1 b(x)|^q\,\mathrm{d}x\right)^{1/q} = \frac{1}{|B|^{1/q}}\|G^B\|_{L^q(B)/S(B)} \leq 1.$$

Since B was any ball, we can now take supremum over B to conclude the proof. □

Chapter 3

Commutators of fractional integrals

The purpose of this chapter is to prove a quantitative two-weight estimate for the commutator of fractional integrals and BMO functions. All the results of this chapter were part of joint work with J. Martínez Perales and I. Rivera Ríos [3].

We recall that for $\alpha \in (0,n)$, a non-negative integer m and $b \in L^m_{\text{loc}}(\mathbb{R}^n)$, the commutator of order m is defined by

$$(I_\alpha)^m_b f(x) := \int_{\mathbb{R}^n} \frac{(b(x)-b(y))^m}{|x-y|^{n-\alpha}} f(y)\,\mathrm{d}y, \quad x \in \mathbb{R}^n.$$

Combining a sparse domination result that will be presented in Section 3.1 with techniques in [32] we obtain the following result.

Theorem 3.1. *Let $\alpha \in (0,n)$ and $1 < p < \frac{n}{\alpha}$, q defined by $\frac{1}{q} + \frac{\alpha}{n} = \frac{1}{p}$ and m a positive integer. Assume that $\mu, \lambda \in A_{p,q}$ and that $\nu = \frac{\mu}{\lambda}$. If $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$, then*

$$\|(I_\alpha)^m_b f\|_{L^q(\lambda^q)} \leq c_{m,n,\alpha,p} \|b\|^m_{\mathrm{BMO}_{\nu^{\frac{1}{m}}}} \kappa_m \|f\|_{L^p(\mu^p)}, \tag{3.1}$$

where

$$\kappa_m := \sum_{h=0}^{m} \binom{m}{h} \left([\lambda]^{\frac{h}{m}}_{A_{p,q}} [\mu]^{\frac{m-h}{m}}_{A_{p,q}}\right)^{\left(1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{q}\right\}} P(m,h) Q(m,h)$$

and

$$P(m,h) \leq \left([\lambda^q]^{\frac{m+(h+1)}{2}}_{A_q} [\mu^q]^{\frac{m-(h+1)}{2}}_{A_q}\right)^{\frac{m-h}{m}\max\{1,\frac{1}{q-1}\}}$$

$$Q(m,h) \leq \left([\lambda^p]^{\frac{h-1}{2}}_{A_p} [\mu^p]^{m-\frac{h-1}{2}}_{A_p}\right)^{\frac{h}{m}\max\{1,\frac{1}{p-1}\}}.$$

Conversely if for every set E of finite measure we have that

$$\|(I_\alpha)^m_b \chi_E\|_{L^q(\lambda^q)} \leq c\mu^p(E)^{\frac{1}{p}}, \tag{3.2}$$

then $b \in \mathrm{BMO}_{\nu^{\frac{1}{m}}}$.

In the case $m = 1$ a qualitative version of this result was established by Holmes, Rahm and Spencer [20]. Besides providing a new proof of the result in [20], our theorem improves that result in several directions. We provide quantitative bounds instead of qualitative ones and we extend the result to iterated commutators.
If we restrict ourselves to the case $\mu = \lambda$ we have the following result.

Corollary 3.2. *Let* $\alpha \in (0, n)$ *and* $1 < p < \frac{n}{\alpha}$, q *defined by* $\frac{1}{q} + \frac{\alpha}{n} = \frac{1}{p}$ *and* m *a non-negative integer. Assume that* $w \in A_{p,q}$ *and that* $b \in \mathrm{BMO}$. *Then*

$$\|(I_\alpha)_b^m f\|_{L^q(w^q)} \le c_{n,p,q}\|b\|_{\mathrm{BMO}}^m [w]_{A_{p,q}}^{\left(m+1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{q}\right\}} \|f\|_{L^p(w^p)}, \tag{3.3}$$

and the estimate is sharp in the sense that the inequality does not hold if we replace the exponent of the $A_{p,q}$ *constant by a smaller one.*
Conversely if $m > 0$ *and for every set* E *of finite measure we have that*

$$\|(I_\alpha)_b^m \chi_E\|_{L^q(w^q)} \le c w^p(E)^{\frac{1}{p}},$$

then $b \in \mathrm{BMO}$.

In the case $m = 0$ the preceding result is due to Lacey, Moen, Pérez and Torres [28]. The case $m = 1$ was settled in [12] but using a different proof based on a suitable combination of the so called conjugation method, that was introduced in [10] (see [9] for the first application of the method to obtain sharp constants), and an extrapolation argument. The case $m > 1$ was recently established in [5] also relying upon the conjugation method. We observe that Corollary 3.2 provides a new proof of the results in [5, 12]. Additionally we settle the sharpness of the iterated case and provide a new characterization of BMO in terms of iterated commutators.

3.1 A sparse domination result for iterated commutators of fractional integrals

We begin this section recalling the definitions of the dyadic structures we will rely upon. These definitions and a profound treatise on dyadic calculus can be found in [30].
Given a cube $Q \subset \mathbb{R}^n$, we denote by $\mathcal{D}(Q)$ the family of all dyadic cubes with respect to Q, that is, the cubes obtained subdividing repeatedly Q and each of its descendants into 2^n subcubes of the same sidelength.
Given a family of cubes $\mathcal{D}$, we say that it is a dyadic lattice if it satisfies the following properties:

1. If $Q \in \mathcal{D}$ then $\mathcal{D}(Q) \subset \mathcal{D}$,

2. For every pair of cubes $Q', Q'' \in \mathcal{D}$ there exists a common ancestor, namely we can find $Q \in \mathcal{D}$ such that $Q', Q'' \in \mathcal{D}(Q)$,

3. For every compact set $K \subset \mathbb{R}^n$ there exists a cube $Q \in \mathcal{D}$ such that $K \subset Q$.

Given a dyadic lattice $\mathcal{D}$ we say that a family $\mathcal{S} \subset \mathcal{D}$ is an η-sparse family with $\eta \in (0,1)$ if there exists a family $\{E_Q\}_{Q\in\mathcal{S}}$ of pairwise disjoint measurable sets such that, for any $Q \in \mathcal{S}$, the set E_Q is contained in Q and satisfies $\eta|Q| \leq |E_Q|$.
Since the first simplification of the proof of the A_2 theorem provided by Lerner [29], sparse domination theory has experienced a fruitful and fast development. However in the case of fractional integrals, the sparse domination philosophy, via dyadic discretizations of the operator, had been already implicitly exploited in [40], [38], and a dyadic type expression for commutators had also shown up in [12]. We remit the reader to [11] for a more detailed insight on the topic.
Relying upon ideas in [22] and [31], it is possible to obtain a pointwise sparse domination that covers the case of iterated commutators of fractional integrals. The precise statement is the following.

Theorem 3.3. *Let $0 < \alpha < n$. Let m be a non-negative integer. For every $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ and $b \in L^m_{loc}(\mathbb{R}^n)$, there exist a family $\{\mathcal{D}_j\}_{j=1}^{3^n}$ of dyadic lattices and a family $\{\mathcal{S}_j\}_{j=1}^{3^n}$ of sparse families such that $\mathcal{S}_j \subset \mathcal{D}_j$, for each j, and*

$$|(I_\alpha)_b^m f(x)| \leq c_{n,m,\alpha} \sum_{j=1}^{3^n} \sum_{h=0}^{m} \binom{m}{h} \mathcal{A}^{m,h}_{\alpha,\mathcal{S}_j}(b,f)(x), \qquad a.e.\ x \in \mathbb{R}^n,$$

where, for a sparse family $\mathcal{S}$, $\mathcal{A}^{m,h}_{\alpha,\mathcal{S}}(b,\cdot)$ is the sparse operator given by

$$\mathcal{A}^{m,h}_{\alpha,\mathcal{S}}(b,f)(x) = \sum_{Q\in\mathcal{S}} |b(x) - b_Q|^{m-h} |Q|^{\frac{\alpha}{n}} |f(b-b_Q)^h|_Q \chi_Q(x).$$

To establish the preceding theorem we need to prove that the grand maximal truncated operator $\mathcal{M}_{I_\alpha}$ defined by

$$\mathcal{M}_{I_\alpha} f(x) = \sup_{Q\ni x} \operatorname*{ess\,sup}_{\xi\in Q} \left| I_\alpha(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)\right|,$$

where the supremmum is taken over all the cubes $Q \subset \mathbb{R}^n$ containing x, maps $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$. We will also use a local version of this operator which is defined, for a cube $Q_0 \subset \mathbb{R}^n$, as

$$\mathcal{M}_{I_\alpha,Q_0} f(x) = \sup_{x\in Q\subset Q_0} \operatorname*{ess\,sup}_{\xi\in Q} \left| I_\alpha(f\chi_{3Q_0\setminus 3Q})(\xi)\right|.$$

3.1.1 Lemmata

The purpose of this subsection is to provide two lemmas that will be needed to establish Theorem 3.3. We start by presenting the first of them.

Lemma 3.4. *Let $0 < \alpha < n$. Let $Q_0 \subset \mathbb{R}^n$ be a cube. The following pointwise estimates hold:*

1. *For a.e.* $x \in Q_0$,

$$|I_\alpha(f\chi_{3Q_0})(x)| \leq \mathcal{M}_{I_\alpha,Q_0}f(x).$$

2. *For all* $x \in \mathbb{R}^n$

$$\mathcal{M}_{I_\alpha}f(x) \leq c_{n,\alpha}\left(M_\alpha f(x) + I_\alpha|f|(x)\right).$$

From this last estimate it follows that $\mathcal{M}_{I_\alpha}$ *is bounded from* $L^1(\mathbb{R}^n)$ *to* $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$.

Proof of Lemma 3.4. To prove (1), let $Q(x,s)$ be a cube centered at x and such that $Q(x,s) \subset Q_0$. Then,

$$\begin{aligned} |I_\alpha(f\chi_{3Q_0})(x)| &\leq |I_\alpha(f\chi_{3Q(x,s)})(x)| + |I_\alpha(f\chi_{3Q_0\setminus 3Q(x,s)})(x)| \\ &\leq |I_\alpha(f\chi_{3Q(x,s)})(x)| + \mathcal{M}_{I_\alpha,Q_0}f(x) \\ &\leq C_{n,\alpha}s^\alpha Mf(x) + \mathcal{M}_{I_\alpha,Q_0}f(x), \end{aligned} \tag{3.4}$$

where the last estimate for the first term follows by standard computations involving a dyadic annuli-type decomposition of the cube $Q(x,s)$. The estimate in (1) is then settled letting $s \to 0$ in (3.4).

For the proof of the pointwise inequality in (2), let x be a point in $\mathbb{R}^n$ and Q a cube containing x. Denote by B_x the closed ball centered at x of radius $2\operatorname{diam} Q$. Then $3Q \subset B_x$, and, for every $\xi \in Q$ we obtain

$$\begin{aligned} |I_\alpha(f\chi_{\mathbb{R}^n\setminus 3Q})(\xi)| &= |I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(\xi) + I_\alpha(f\chi_{B_x\setminus 3Q})(\xi)| \\ &\leq |I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(\xi) - I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \\ &\quad + |I_\alpha(f\chi_{B_x\setminus 3Q})(\xi)| + |I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)|. \end{aligned}$$

For the first term, by using the mean value theorem and adapting [18, Theorem 2.1.10] to our setting, we get

$$\begin{aligned} |I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(\xi) - I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)| &\leq \int_{\mathbb{R}^n\setminus B_x} \left|\frac{1}{|y-\xi|^{n-\alpha}} - \frac{1}{|y-x|^{n-\alpha}}\right| |f(y)|\,\mathrm{d}y \\ &\leq c_{n,\alpha}\int_{\mathbb{R}^n\setminus B_x} \frac{|x-\xi|}{(|x-y|+|y-\xi|)^{n-\alpha+1}}|f(y)|\,\mathrm{d}y \\ &\leq c_{n,\alpha}M_\alpha f(x). \end{aligned}$$

For the second term, taking into account the definition of B_x, we can write

$$\begin{aligned}|I_\alpha(f\chi_{B_x\setminus 3Q})(\xi)| &= \left|\int_{B_x\setminus 3Q}\frac{1}{|y-\xi|^{n-\alpha}}f(y)\,\mathrm{d}y\right|\\ &\leq \int_{B_x\setminus 3Q}\frac{1}{|y-\xi|^{n-\alpha}}\,|f(y)|\,\mathrm{d}y\\ &\leq c_{n,\alpha}\frac{1}{\ell(Q)^{n-\alpha}}\int_{B_x}|f(y)|\,\mathrm{d}y\\ &\leq c_{n,\alpha}M_\alpha f(x).\end{aligned}$$

To end the proof of this pointwise estimate we observe that

$$|I_\alpha(f\chi_{\mathbb{R}^n\setminus B_x})(x)| \leq I_\alpha|f|(x),$$

which finishes the proof of (2). Now, taking into account the pointwise estimate we have just obtained, and the boundedness properties of the operators I_α and M_α, it is clear that $\mathcal{M}_{I_\alpha}$ is bounded from $L^1(\mathbb{R}^n)$ to $L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)$, and we are done. □

The second lemma that we will need for the proof of Theorem 3.3 is the so-called 3^n dyadic lattices trick. A proof can be found for example in [30] and essentially says that given a dyadic lattice $\mathcal{D}$, if we consider the family of cubes $\{3Q : Q \in \mathcal{D}\}$ it is possible to arrange them in 3^n dyadic lattices.

Lemma 3.5 (Lerner, Nazarov [30])**.** *Given a dyadic lattice $\mathcal{D}$ there exist 3^n dyadic lattices $\mathcal{D}_j$ such that*

$$\{3Q : Q \in \mathcal{D}\} = \bigcup_{j=1}^{3^n}\mathcal{D}_j$$

and for every cube $Q \in \mathcal{D}$ we can find a cube R_Q in each $\mathcal{D}_j$ such that $Q \subset R_Q$ and $3\ell(Q) = \ell(R_Q)$

Remark 3.6. Fix a dyadic lattice $\mathcal{D}$. For an arbitrary cube $Q \subset \mathbb{R}^n$ there is a cube $Q' \in \mathcal{D}$ such that $\frac{\ell(Q)}{2} < \ell(Q') \leq \ell(Q)$ and $Q \subset 3Q'$. We can take a cube with that property since every generation of cubes in $\mathcal{D}$ tiles $\mathbb{R}^n$. From this and the preceding lemma it follows that $3Q' = P \in \mathcal{D}_j$ for some $j \in \{1,\dots,3^n\}$. Therefore, for every cube $Q \subset \mathbb{R}^n$ there exists some $j \in \{1,\dots,3^n\}$ and some $P \in \mathcal{D}_j$ such that $Q \subset P$ and $\ell(P) \leq 3\ell(Q)$ and consequently $|Q| \leq |P| \leq 3^n|Q|$.

3.1.2 Proof of Theorem 3.3

From Remark 3.6 it follows that there exist 3^n dyadic lattices such that for every cube Q of $\mathbb{R}^n$ there is a cube $R_Q \in \mathcal{D}_j$ for some j for which $3Q \subset R_Q$ and $|R_Q| \leq 9^n|Q|$.
We claim that there is a positive constant $c_{n,m,\alpha}$ verifying that, for any cube $Q_0 \subset \mathbb{R}^n$, there exists

a $\frac{1}{2}$-sparse family $\mathcal{F} \subset \mathcal{D}(Q_0)$ such that for a.e. $x \in Q_0$

$$|(I_\alpha)_b^m(f\chi_{3Q_0})(x)| \leq c_{n,m,\alpha} \sum_{h=0}^{m} \binom{m}{h} \mathcal{B}_{\mathcal{F}}^{m,h}(b,f)(x), \tag{3.5}$$

where

$$\mathcal{B}_{\mathcal{F}}^{m,h}(b,f)(x) := \sum_{Q\in\mathcal{F}} |b(x) - b_{R_Q}|^{m-h} |3Q|^{\frac{\alpha}{n}} |f(b - b_{R_Q})^h|_{3Q} \chi_Q(x).$$

Suppose that we have already proved the claim. Let us take a partition of $\mathbb{R}^n$ by a family $\{Q_k\}_{k\in\mathbb{N}}$ of cubes Q_k such that $\mathrm{supp}(f) \subset 3Q_k$ for each $j \in \mathbb{N}$. We can do it as follows. We start with a cube Q_0 such that $\mathrm{supp}(f) \subset Q_0$. And cover $3Q_0 \setminus Q_0$ by $3^n - 1$ congruent cubes Q_k. Each of them satisfies $Q_0 \subset 3Q_k$. We do the same for $9Q_0 \setminus 3Q_0$ and so on. The union of all those cubes, including Q_0, will satisfy the desired properties.
Fix $k \in \mathbb{N}$ and apply the claim to the cube Q_k. Then we have that since $\mathrm{supp}\, f \subset 3Q_k$ the following estimate holds for almost every $x \in \mathbb{R}^n$:

$$|(I_\alpha)_b^m f(x)|\, \chi_{Q_k}(x) = |(I_\alpha)_b^m(f\chi_{3Q_k})(x)|\, \chi_{Q_k}(x) \leq c_{n,m,\alpha} \sum_{h=0}^{m} \binom{m}{h} \mathcal{B}_{\mathcal{F}_k}^{m,h}(b,f)(x),$$

where $\mathcal{F}_k \subset \mathcal{D}(Q_k)$ is a $\frac{1}{2}$-sparse family. Taking $\mathcal{F} = \bigcup_{k\in\mathbb{N}} \mathcal{F}_k$ we have that $\mathcal{F}$ is a $\frac{1}{2}$-sparse family and

$$|(I_\alpha)_b^m f(x)| \leq c_{n,m,\alpha} \sum_{h=0}^{m} \binom{m}{h} \mathcal{B}_{\mathcal{F}}^{m,h}(b,f)(x), \qquad \text{a.e. } x \in \mathbb{R}^n.$$

Fix $Q \subset \mathcal{F}$. Since $3Q \subset R_Q$ and $|R_Q| \leq 3^n|3Q|$, we have that $|3Q|^{\frac{\alpha}{n}} |f(b - b_{R_Q})^h|_{3Q} \leq 3^n|R_Q|^{\frac{\alpha}{n}} |f(b - b_{R_Q})^h|_{R_Q}$. Setting

$$\mathcal{S}_j := \{R_Q \in \mathcal{D}_j \,:\, Q \in \mathcal{F}\}$$

and using that $\mathcal{F}$ is $\frac{1}{2}$-sparse, we obtain that each family $\mathcal{S}_j$ is $\frac{1}{2\cdot 9^n}$-sparse. Then we have that

$$|(I_\alpha)_b^m f(x)| \leq c_{n,m,\alpha} \sum_{j=1}^{3^n} \sum_{h=0}^{m} \binom{m}{h} \mathcal{A}_{\alpha,\mathcal{S}_j}^{m,h}(b,f)(x), \qquad \text{a.e. } x \in \mathbb{R}^n,$$

and we are done.

Proof of the Claim (3.5)

To prove the claim it suffices to prove the following recursive estimate: there is a positive constant $c_{n,m,\alpha}$ verifying that there exists a countable family $\{P_j\}_j$ of pairwise disjoint cubes in $\mathcal{D}(Q_0)$ such that $\sum_j |P_j| \leq \frac{1}{2}|Q_0|$ and

$$\begin{aligned}&|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\chi_{Q_0}(x)\\&\quad\leq c_{n,m,\alpha}\sum_{h=0}^{m}\binom{m}{h}|b(x)-b_{R_{Q_0}}|^{m-h}|3Q_0|^{\frac{\alpha}{n}}|f(b-b_{R_{Q_0}})^h|_{3Q_0}\chi_{Q_0}(x)\\&\quad+\sum_j|(I_\alpha)_b^m(f\chi_{3P_j})(x)|\chi_{P_j}(x),\qquad \text{a.e. } x\in Q_0.\end{aligned}$$

Iterating this estimate, we obtain (3.5) with $\mathcal{F}=\{P_j^k\}_{j,k}$ where $\{P_j^0\}_j:=\{Q_0\}$, $\{P_j^1\}_j:=\{P_j\}_j$ and $\{P_j^k\}_j$ is the union of the cubes obtained at the k-th stage of the iterative process from each of the cubes P_j^{k-1} of the $(k-1)$-th stage. Clearly $\mathcal{F}$ is a $\frac{1}{2}$-sparse family, since the conditions in the definition hold for the sets

$$E_{P_j^k}=P_j^k\setminus\bigcup_j P_j^{k+1}.$$

Let us prove then the recursive estimate.

For any countable family $\{P_j\}_j$ of disjoint cubes $P_j\in\mathcal{D}(Q_0)$ we have that

$$\begin{aligned}&|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\,\chi_{Q_0}(x)\\&=|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\,\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\,\chi_{P_j}(x)\\&\leq|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\,\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|(I_\alpha)_b^m(f\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)\\&\quad+\sum_j\left|(I_\alpha)_b^m(f\chi_{3P_j})(x)\right|\chi_{P_j}(x)\end{aligned}$$

for almost every $x\in\mathbb{R}^n$. So it suffices to show that we can find a positive constant $c_{n,m,\alpha}$ in such a way we can choose a countable family $\{P_j\}_j$ of pairwise disjoint cubes in $\mathcal{D}(Q_0)$ with $\sum_j|P_j|\leq\frac{1}{2}|Q_0|$ and such that, for a.e. $x\in Q_0$,

$$\begin{aligned}&|(I_\alpha)_b^m(f\chi_{3Q_0})(x)|\,\chi_{Q_0\setminus\bigcup_j P_j}(x)+\sum_j\left|(I_\alpha)_b^m(f\chi_{3Q_0\setminus 3P_j})(x)\right|\chi_{P_j}(x)\\&\leq c_{n,m,\alpha}\sum_{h=0}^{m}\binom{m}{h}|b(x)-b_{R_{Q_0}}|^{m-h}|3Q_0|^{\frac{\alpha}{n}}|f(b-b_{R_{Q_0}})^h|_{3Q_0}\chi_{Q_0}(x).\end{aligned}\tag{3.6}$$

Using that $(I_\alpha)_b^m f=(I_\alpha)_{b-c}^m f$ for any $c\in\mathbb{R}$, and also that

$$(I_\alpha)_{b-c}^m f=\sum_{h=0}^{m}(-1)^h\binom{m}{h}I_\alpha((b-c)^h f)(b-c)^{m-h},$$

it follows that

$$
\begin{aligned}
&|(I_\alpha)_b^m(f\chi_{3Q_0})|\chi_{Q_0\setminus\cup_j P_j}(x) + \sum_j |(I_\alpha)_b^m(f\chi_{3Q_0\setminus 3P_j})|\chi_{P_j}(x) \\
&\le \sum_{h=0}^m \binom{m}{h} |b(x)-b_{R_{Q_0}}|^{m-h} |I_\alpha((b-b_{R_{Q_0}})^h f\chi_{3Q_0})(x)|\chi_{Q_0\setminus\cup_j P_j}(x) \qquad (3.7)\\
&+ \sum_{h=0}^m \binom{m}{h} |b(x)-b_{R_{Q_0}}|^{m-h} \sum_j |I_\alpha((b-b_{R_{Q_0}})^h f\chi_{3Q_0\setminus 3P_j})(x)|\chi_{P_j}(x). \qquad (3.8)
\end{aligned}
$$

Now we define the set $E := \cup_{h=0}^m E_h$, where

$$
E_h := \left\{x \in Q_0 \,:\, \mathcal{M}_{I_\alpha,Q_0}\left((b-b_{R_{Q_0}})^h f\right)(x) > c_{n,m,\alpha}|3Q_0|^{\frac{\alpha}{n}}|(b-b_{R_{Q_0}})^h f|_{3Q_0}\right\},
$$

with $c_{n,m,\alpha}$ being a positive number to be chosen.
As we proved in Lemma 3.4, we have that

$$
c_{n,\alpha} := \|\mathcal{M}_{I_\alpha}\|_{L^1(\mathbb{R}^n)\to L^{\frac{n}{n-\alpha},\infty}(\mathbb{R}^n)} < \infty,
$$

so, since $\mathcal{M}_{I_\alpha,Q_0} g \le \mathcal{M}_{I_\alpha}(g\chi_{3Q_0})$, we can write, for each $h \in \{0,1,\dots,m\}$,

$$
\begin{aligned}
|E_h| &\le \left(\frac{c_{n,\alpha}\int_{3Q_0}|f(b-b_{R_{Q_0}})^h|}{c_{n,m,\alpha}|3Q_0|^{\frac{\alpha}{n}}|f(b-b_{R_{Q_0}})^h|_{3Q_0}}\right)^{\frac{n}{n-\alpha}} \\
&= \left(\frac{c_{n,\alpha}|3Q_0|^{\frac{\alpha}{n}-1}\int_{3Q_0}|f(b-b_{R_{Q_0}})^h|}{c_{n,m,\alpha}|3Q_0|^{\frac{\alpha}{n}}|f(b-b_{R_{Q_0}})^h|_{3Q_0}}\right)^{\frac{n}{n-\alpha}} |3Q_0|^{\left(1-\frac{\alpha}{n}\right)\frac{n}{n-\alpha}} \\
&= \left(\frac{c_{n,\alpha}}{c_{n,m,\alpha}}\right)^{\frac{n}{n-\alpha}} |3Q_0|^{\left(1-\frac{\alpha}{n}\right)\frac{n}{n-\alpha}} = 3^n\left(\frac{c_{n,\alpha}}{c_{n,m,\alpha}}\right)^{\frac{n}{n-\alpha}} |Q_0|,
\end{aligned}
$$

and we can choose $c_{n,m,\alpha}$ such that

$$
|E| \le \sum_{h=0}^m |E_h| \le \frac{1}{2^{n+2}}|Q_0|, \qquad (3.9)
$$

this choice being independent from Q_0 and f.
Now we apply Calderón-Zygmund decomposition to the function χ_E on Q_0 at height $\lambda = \frac{1}{2^{n+1}}$. We obtain a countable family $\{P_j\}_j$ of pairwise disjoint cubes in $\mathcal{D}(Q_0)$ such that

$$
\chi_E(x) \le \frac{1}{2^{n+1}}, \qquad \text{a.e. } x \notin \bigcup_j P_j.
$$

From this it follows that $\left|E \setminus \bigcup_j P_j\right| = 0$. The family $\{P_j\}_j$ also satisfies that

$$\sum_j |P_j| \leq 2^{n+1}|E| \leq \frac{1}{2}|Q_0|$$

and

$$\frac{|P_j \cap E|}{|P_j|} = \frac{1}{|P_j|}\int_{P_j} \chi_E(x) \leq \frac{1}{2} \qquad \text{for all } j,$$

from which it readily follows that $|P_j \cap E^c| > 0$ for every j. Indeed, given j,

$$|P_j| = |P_j \cap E| + |P_j \cap E^c| \leq \frac{1}{2}|P_j| + |P_j \cap E^c|,$$

and from this it follows that $0 < \frac{1}{2}|P_j| < |P_j \cap E^c|$.
Fix some j. Since we have $P_j \cap E^c \neq \emptyset$, we observe that

$$\mathcal{M}_{I_\alpha,Q_0}\left((b - b_{R_{Q_0}})^h f\right)(x) \leq c_{n,m,\alpha}|3Q_0|^{\frac{\alpha}{n}}|(b - b_{R_{Q_0}})^h f|_{3Q_0}$$

for some $x \in P_j$ and this implies that, for any $Q \subset Q_0$ containing x, we have

$$\operatorname*{ess\,sup}_{\xi \in Q}\left|I_\alpha((b - b_{R_{Q_0}})^h f\chi_{3Q_0\setminus 3Q})(\xi)\right| \leq c_{n,m,\alpha}|3Q_0|^{\frac{\alpha}{n}}|(b - b_{R_{Q_0}})^h f|_{3Q_0},$$

which allows us to control the summation in (3.8) by considering the cube P_j.
Now, by (1) in Lemma 3.4, we know that

$$\left|I_\alpha((b - b_{R_{Q_0}})^h f\chi_{3Q_0})(x)\right| \leq \mathcal{M}_{I_\alpha,Q_0}\left((b - b_{R_{Q_0}})^h f\right)(x), \qquad \text{a.e. } x \in Q_0.$$

Since $\left|E \setminus \bigcup_j P_j\right| = 0$ we have, by the definition of E, that

$$\mathcal{M}_{I_\alpha,Q_0}\left((b - b_{R_{Q_0}})^h f\right)(x) \leq c_{n,m,\alpha}|3Q_0|^{\frac{\alpha}{n}}|(b - b_{R_{Q_0}})^h f|_{3Q_0}, \quad \text{a.e. } x \in Q_0 \setminus \bigcup_j P_j.$$

Consequently,

$$\left|I_\alpha((b - b_{R_{Q_0}})^h f\chi_{3Q_0})(x)\right| \leq c_{n,m,\alpha}|3Q_0|^{\frac{\alpha}{n}}|(b - b_{R_{Q_0}})^h f|_{3Q_0}, \quad \text{a.e. } x \in Q_0 \setminus \bigcup_j P_j.$$

These estimates allow us to control the remaining terms in (3.7), so we are done.

3.2 Proofs of Theorem 3.1 and Corollary 3.2

The proof of Theorem 3.1 is presented in the two first subsections. First we deal with the upper bound and then with the necessity.

We will end up this section with a subsection devoted to establish Corollary 3.2.

3.2.1 Proof of the upper bound

To settle the upper bound in Theorem 3.1 we argue as in [31, Theorem 1.4] or, to be more precise as in [32, Theorem 1.1]. To do that we need to borrow the following estimate that was obtained in the case $j = 1$ in [31] and for $j > 1$ in [32] and that can be stated as follows.

Lemma 3.7 (Lerner, Ombrosi, Rivera-Ríos [31, 32])**.** *Let $\mathcal{S}$ be a sparse family contained in a dyadic lattice $\mathcal{D}$, η a weight, $b \in \mathrm{BMO}_\eta$ and $f \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. There exists a possibly larger sparse family $\tilde{\mathcal{S}} \subset \mathcal{D}$ containing $\mathcal{S}$ such that, for every positive integer j and every $Q \in \tilde{\mathcal{S}}$*

$$\int_Q |b - b_Q|^j |f| \leq c_n \|b\|_{\mathrm{BMO}_\eta}^j \int_Q A^j_{\tilde{\mathcal{S}},\eta} f$$

where $A^j_{\tilde{\mathcal{S}},\eta} f$ stands for the j-th iteration of $A_{\tilde{\mathcal{S}},\eta}$, which is defined by $A_{\tilde{\mathcal{S}},\eta} f := A_{\tilde{\mathcal{S}}}(f)\eta$, with $A_{\tilde{\mathcal{S}}}$ being the sparse operator given by

$$A_{\tilde{\mathcal{S}}} f(x) = \sum_{Q \in \tilde{\mathcal{S}}} \frac{1}{|Q|} \int_Q |f| \chi_Q(x).$$

We will also make use of the following quantitative estimates. Let $1 < p < \infty$ and $\mathcal{S}$ a γ-sparse family. If $w \in A_p$ then

$$\|A_{\mathcal{S}}\|_{L^p(w)} \leq c_{n,p} [w]_{A_p}^{\max\left\{1, \frac{1}{p-1}\right\}}. \tag{3.10}$$

If p, q, α are as in the hypothesis of Theorem 3.1 and $w \in A_{p,q}$, then

$$\|I_{\mathcal{S}}^\alpha\|_{L^q(w^q) \to L^p(w^p)} \leq c_{n,p,q,\alpha} [w]_{A_{p,q}}^{\left(1 - \frac{\alpha}{n}\right) \max\left\{1, \frac{p}{q'}\right\}}, \tag{3.11}$$

where

$$I_{\mathcal{S}}^\alpha f(x) = \sum_{Q \in \mathcal{S}} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f| \chi_Q(x).$$

We observe that the proof of (3.11) is implicit in one of the proofs of [28, Theorem 2.6] that relies essentially on computing the norm of the operator $I_{\mathcal{S}}^\alpha$ by duality.
At this point we are in the position to prove the estimate (3.1).
Assume preliminary that $b \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$, we'll remove this assumption by the end. Taking into account Theorem 3.3, it suffices to prove the estimate for the sparse operators

$$A_{\alpha,\mathcal{S}}^{m,h}(b, f) := \sum_{Q \in \mathcal{S}} |b - b_Q|^{m-h} |Q|^{\alpha/n} |f(b - b_Q)^h|_Q \chi_Q, \qquad h \in \{0, 1, \dots, m\}.$$

Assume that $b \in \mathrm{BMO}_\eta$ with η to be chosen. We observe that, using Lemma 3.7,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} A^{m,h}_{\alpha,\mathcal{S}}(b,f)g\lambda^q\right| &\leq \sum_{Q\in\mathcal{S}}\left(\int_Q |g||b-b_Q|^{m-h}\lambda^q\right)\frac{1}{|Q|^{1-\alpha/n}}\int_Q |b-b_Q|^h|f|.\\ &\leq c_n\|b\|^m_{\mathrm{BMO}_\eta}\sum_{Q\in\mathcal{S}}\left(\int_Q A^{m-h}_{\tilde{\mathcal{S}},\eta}(|g|\lambda^q)\right)\frac{1}{|Q|^{1-\alpha/n}}\int_Q A^h_{\tilde{\mathcal{S}},\eta}(|f|)\\ &\leq c_n\|b\|^m_{\mathrm{BMO}_\eta}\int_{\mathbb{R}^n}\sum_{Q\in\mathcal{S}}\frac{1}{|Q|^{1-\alpha/n}}\left(\int_Q A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right)\chi_Q A^{m-h}_{\tilde{\mathcal{S}},\eta}(|g|\lambda^q)\\ &= c_n\|b\|^m_{\mathrm{BMO}_\eta}\int_{\mathbb{R}^n} I^\alpha_{\mathcal{S}}\left[A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right](x)A^{m-h}_{\tilde{\mathcal{S}},\eta}(|g|\lambda^q)(x)dx.\end{aligned}$$

Let us call $I^\alpha_{\mathcal{S},\eta}f := I^\alpha_{\mathcal{S}}(f)\eta$. Now, the self-adjointness of $A_{\tilde{\mathcal{S}}}$ yields

$$\int_{\mathbb{R}^n} I^\alpha_{\mathcal{S}}\left(A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right)A^{m-h}_{\tilde{\mathcal{S}},\eta}(|g|\lambda^q) = \int_{\mathbb{R}^n} A_{\tilde{\mathcal{S}}}\left\{A^{m-h-1}_{\tilde{\mathcal{S}},\eta}\left[I^\alpha_{\mathcal{S},\eta}\left(A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right)\right]\right\}|g|\lambda^q.$$

Combining the preceding estimates we have that

$$\left|\int_{\mathbb{R}^n} A^{m,h}_{\alpha,\mathcal{S}}(b,f)g\lambda^q\right| \leq c_n\|b\|^m_{\mathrm{BMO}_\eta}\left\|A_{\tilde{\mathcal{S}}}A^{m-h-1}_{\tilde{\mathcal{S}},\eta}I^\alpha_{\mathcal{S},\eta}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^q(\lambda^q)}\|g\|_{L^{q'}(\lambda^q)}$$

and consequently, taking supremum over $g \in L^{q'}(\lambda^q)$ with $\|g\|_{L^{q'}(\lambda^q)} = 1$,

$$\|A^{m,h}_{\alpha,\mathcal{S}}(b,f)\|_{L^q(\lambda^q)} \leq c_n\|b\|^m_{\mathrm{BMO}_\eta}\left\|A_{\tilde{\mathcal{S}}}A^{m-h-1}_{\tilde{\mathcal{S}},\eta}I^\alpha_{\mathcal{S},\eta}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^q(\lambda^q)}.$$

Taking into account (3.10) we can estimate

$$\begin{aligned}&\left\|A_{\tilde{\mathcal{S}}}A^{m-h-1}_{\tilde{\mathcal{S}},\eta}\ I^\alpha_{\mathcal{S},\eta}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^q(\lambda^q)}\\ &\qquad\leq c_{n,q}\left(\prod_{l=0}^{m-h-1}[\lambda^q\eta^{lq}]_{A_q}\right)^{\max\left\{1,\frac{1}{q-1}\right\}}\left\|I^\alpha_{\mathcal{S},\eta}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^q(\lambda^q\eta^{q(m-h-1)})}.\end{aligned}$$

Using (3.11), we have that

$$\begin{aligned}&\left\|I^\alpha_{\mathcal{S},\eta}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^q(\lambda^q\eta^{q(m-h-1)})} = \left\|I^\alpha_{\mathcal{S}}A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^q(\lambda^q\eta^{q(m-h)})}\\ &\leq c_{n,p,\alpha}[\lambda\eta^{m-h}]_{A_{p,q}}^{\left(1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{q}\right\}}\left\|A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^p(\lambda^p\eta^{p(m-h)})}\end{aligned}$$

and applying again (3.10),

$$\left\|A^h_{\tilde{\mathcal{S}},\eta}(|f|)\right\|_{L^p(\lambda^p\eta^{p(m-h)})} \leq c_{n,p}\left(\prod_{l=m-h+1}^{m}[\lambda^p\eta^{lp}]_{A_p}\right)^{\max\left\{1,\frac{1}{p-1}\right\}}\|f\|_{L^p(\lambda^p\eta^{mp})}.$$

Gathering the preceding estimates we have that

$$\|A^{m,h}_{\alpha,\mathcal{S}}(b,f)\|_{L^q(\lambda^q)} \le c_{n,\alpha,p}\|b\|^m_{\mathrm{BMO}_\eta} PQ[\lambda\eta^{m-h}]_{A_{p,q}}^{\left(1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{q}\right\}}\|f\|_{L^p(\lambda^p\eta^{mp})},$$

where

$$P := \left(\prod_{l=0}^{m-h-1}[\lambda^q\eta^{lq}]_{A_q}\right)^{\max\left\{1,\frac{1}{q-1}\right\}} \qquad Q := \left(\prod_{l=m-h+1}^{m}[\lambda^p\eta^{lp}]_{A_p}\right)^{\max\left\{1,\frac{1}{p-1}\right\}}.$$

Now we observe that choosing $\eta=\nu^{1/m}$, it readily follows from Hölder's inequality

$$[\lambda\nu^{\frac{m-h}{m}}]_{A_{p,q}} \le [\lambda]^{\frac{h}{m}}_{A_{p,q}}[\mu]^{\frac{m-h}{m}}_{A_{p,q}} \quad \text{and} \quad [\lambda^r\nu^{r\frac{l}{m}}]_{A_r} \le [\lambda^r]^{\frac{m-l}{m}}_{A_r}[\mu^r]^{\frac{l}{m}}_{A_r}, \quad r=p,q.$$

Thus, we can write

$$P \le \left(\prod_{l=0}^{m-h-1}[\lambda^q]^{\frac{m-l}{m}}_{A_q}[\mu^q]^{\frac{l}{m}}_{A_q}\right)^{\max\{1,\frac{1}{q-1}\}} \qquad Q \le \left(\prod_{l=m-h+1}^{m}[\lambda^p]^{\frac{m-l}{m}}_{A_p}[\mu^p]^{\frac{l}{m}}_{A_p}\right)^{\max\{1,\frac{1}{p-1}\}}$$

and, computing the products, we obtain

$$P \le \left([\lambda^q]_{A_q}^{\frac{m+(h+1)}{2}}[\mu^q]_{A_q}^{\frac{m-(h+1)}{2}}\right)^{\frac{m-h}{m}\max\{1,\frac{1}{q-1}\}}$$

and

$$Q \le \left([\lambda^p]_{A_p}^{\frac{h-1}{2}}[\mu^p]_{A_p}^{m-\frac{h-1}{2}}\right)^{\frac{h}{m}\max\{1,\frac{1}{p-1}\}}.$$

Combining all the preceding estimates leads to the desired estimate.
To complete the proof we are going to show that $b\in L^m_{\mathrm{loc}}(\mathbb{R}^n)$. Indeed, for any compact set K we choose a cube Q such that $K\subset Q$. Then

$$\int_K |b|^m \le \int_Q |b|^m \le c_m\int_Q |b-b_Q|^m + c_m\left(\int_Q |b|\right)^m.$$

Since b is locally integrable, we only have to the deal with the first term. We observe that by Lemma 3.7,

$$\begin{aligned}\int_Q |b-b_Q|^m\chi_Q &\le c_n\|b\|^m_{\mathrm{BMO}_{\nu^{1/m}}}\int_Q A^m_{\tilde{\mathcal{S}},\nu^{1/m}}(\chi_Q)\\ &\le c_n\|b\|^m_{\mathrm{BMO}_{\nu^{1/m}}}\left(\int_{\mathbb{R}^n}A^m_{\tilde{\mathcal{S}},\nu^{1/m}}(\chi_Q)^p\lambda^p\right)^{\frac{1}{p}}\left(\int_Q\lambda^{\frac{p}{1-p}}\right)^{\frac{1}{p'}}\end{aligned}$$

and arguing analogously as above we are done.

3.2.2 Proof of the necessity

We are left to prove the converse statement of Theorem 3.1, that is that if for every set of finite measure E we have

$$\|(I_\alpha)_b^m \chi_E\|_{L^q} \leq c\mu^p(E)^{\frac{1}{p}},$$

then the symbol b must necessarily belong to the class $\mathrm{BMO}_{\nu^{\frac{1}{m}}}$, where $\nu = \frac{\mu}{\lambda}$.
We are going to follow ideas in [32]. First we recall [32, Lemma 2.1]

Lemma 3.8 (Lerner, Ombrosi, Rivera-Ríos [32])**.** *Let $\eta \in A_\infty$. Then*

$$\|b\|_{\mathrm{BMO}_\eta} \leq \sup_Q \omega_\lambda(b,Q)\frac{|Q|}{\eta(Q)} \qquad 0<\lambda<\frac{1}{2^{n+1}}.$$

where $\omega_\lambda(f,Q) = \inf_{c\in\mathbb{R}} \left((f-c)\chi_Q\right)^* (\lambda|Q|)$ and

$$((f-c)\chi_Q)^*(\lambda|Q|) = \sup_{\substack{E\subset Q\\ |E|=\lambda|Q|}} \inf_{x\in E} |(f-c)(x)|.$$

We are ready now to give the proof. Let $Q \subset \mathbb{R}^n$ be an arbitrary cube. There exists a subset $E \subset Q$ with $|E| = \frac{1}{2^{n+2}}|Q|$ such that for every $x \in E$

$$\omega_{\frac{1}{2^{n+2}}}(b,Q) \leq |b(x) - m_b(Q)|$$

where $m_b(Q)$ is a not necessarily unique number that satisfies

$$\max\{|\{x\in Q : b(x) > m_b(Q)\}|, |\{x \in Q : b(x) < m_b(Q)\}|\} \leq \frac{|Q|}{2}.$$

Now let $A \subset Q$ with $|A| = \frac{1}{2}|Q|$ and such that $b(x) \geq m_b(Q)$ for every $x \in A$. We call $B = Q \setminus A$. Then $|B| = \frac{1}{2}|Q|$ and $b(x) \leq m_b(Q)$ for every $x \in B$.
As Q is the disjoint union of A and B, at least half of the set E is contained either in A or in B. We may assume, without loss of generality, that half of E is in A, so we have

$$|E\cap A| = |E| - |E\cap(E\cap A)^c| \geq |E| - \frac{|E|}{2} = \frac{1}{2^{n+3}}|Q|.$$

So choosing $A' = A\cap E$ we have that if $y \in A'$ and $x \in B$ then $\omega_{\frac{1}{2^{n+2}}}(b,Q) \leq b(y) - m_b(Q) \leq b(y) - b(x)$. Consequently, taking into account that A' and B are disjoint subsets of Q, using Hölder's inequality and the hypothesis on $(I_\alpha)_b^m$,

$$\begin{aligned}
\omega_{\frac{1}{2^{n+2}}}(b,Q)^m|A'||B| &\le \int_{A'}\int_B (b(y)-b(x))^m \,\mathrm{d}x\mathrm{d}y\\
&\le \ell(Q)^{n-\alpha}\int_{A'}\int_B \frac{(b(y)-b(x))^m}{|x-y|^{n-\alpha}}\,\mathrm{d}x\mathrm{d}y\\
&= \ell(Q)^{n-\alpha}\int_{A'} (I_\alpha)_b^m(\chi_B)(x)\,\mathrm{d}x\\
&\le \ell(Q)^{n-\alpha}\left(\int_Q \lambda^{-q'}\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^n}(I_\alpha)_b^m(\chi_B)(x)^q\lambda(x)^q\,\mathrm{d}x\right)^{\frac{1}{q}}\\
&\le c\ell(Q)^{n-\alpha}\left(\int_Q \lambda^{-q'}\right)^{\frac{1}{q'}}\left(\int_Q \mu^p\right)^{\frac{1}{p}}\\
&= c|Q|^2\left(\frac{1}{|Q|}\int_Q \lambda^{-q'}\right)^{\frac{1}{q'}}\left(\frac{1}{|Q|}\int_Q \mu^p\right)^{\frac{1}{p}},
\end{aligned}$$

where we used that

$$\frac{1}{q}+\frac{\alpha}{n}=\frac{1}{p} \iff \frac{1}{q'}+\frac{1}{p}=1+\frac{\alpha}{n}.$$

Taking into account that $|A'|, |B| \simeq |Q|$, it readily follows from the estimate above that

$$\omega_{\frac{1}{2^{n+2}}}(b,Q)^m \le c\left(\frac{1}{|Q|}\int_Q \lambda^{-q'}\right)^{\frac{1}{q'}}\left(\frac{1}{|Q|}\int_Q \mu^p\right)^{\frac{1}{p}}.$$

Since $\mu \in A_{p,q}$ we have that $\mu^p \in A_p$. Hence (see for example [15])

$$\frac{1}{|Q|}\int_Q \mu^p \le c\left(\frac{1}{|Q|}\int_Q \mu^{\frac{p}{r}}\right)^r, \qquad \text{for any } r>1.$$

Using that inequality, for some $r>1$ to be chosen, combined with Hölder's inequality with $\beta = \frac{r}{p}\frac{1}{m}$, we have that

$$\begin{aligned}
\left(\frac{1}{|Q|}\int_Q \mu^p\right)^{\frac{1}{p}} &\le c\left(\frac{1}{|Q|}\int_Q \mu^{\frac{p}{r}}\lambda^{-\frac{p}{r}}\lambda^{\frac{p}{r}}\right)^{\frac{r}{p}} \le c\left(\frac{1}{|Q|}\int_Q \nu^{\frac{1}{m}}\right)^m\left(\frac{1}{|Q|}\int_Q \lambda^{\frac{p}{r}\beta'}\right)^{\frac{r}{p\beta'}}\\
&= c\left(\frac{1}{|Q|}\int_Q \nu^{\frac{1}{m}}\right)^m\left(\frac{1}{|Q|}\int_Q \lambda^{\frac{p}{r-pm}}\right)^{\frac{r-pm}{p}}
\end{aligned}$$

and choosing $r = pm+1$ we obtain

$$\left(\frac{1}{|Q|}\int_Q \mu^p\right)^{\frac{1}{p}} \le c\left(\frac{1}{|Q|}\int_Q \nu^{\frac{1}{m}}\right)^m\left(\frac{1}{|Q|}\int_Q \lambda^p\right)^{\frac{1}{p}}.$$

This yields

$$\omega_{\frac{1}{2^{n+2}}}(b,Q)^m \le c\left(\frac{1}{|Q|}\int_Q \nu^{\frac{1}{m}}\right)^m\left(\frac{1}{|Q|}\int_Q \lambda^{-q'}\right)^{\frac{1}{q'}}\left(\frac{1}{|Q|}\int_Q \lambda^p\right)^{\frac{1}{p}}.$$

Now we observe that since $q > p$ then by Hölder's inequality

$$\left(\frac{1}{|Q|}\int_Q \lambda^p\right)^{\frac{1}{p}} \leq \left(\frac{1}{|Q|}\int_Q \lambda^q\right)^{\frac{1}{q}} \quad \text{and} \quad \left(\frac{1}{|Q|}\int_Q \lambda^{-q'}\right)^{\frac{1}{q'}} \leq \left(\frac{1}{|Q|}\int_Q \lambda^{-p'}\right)^{\frac{1}{p'}}.$$

Thus

$$\left(\frac{1}{|Q|}\int_Q \lambda^{-q'}\right)^{\frac{1}{q'}} \left(\frac{1}{|Q|}\int_Q \lambda^p\right)^{\frac{1}{p}} \leq \left[\left(\frac{1}{|Q|}\int_Q \lambda^q\right)\left(\frac{1}{|Q|}\int_Q \lambda^{-p'}\right)^{\frac{q}{p'}}\right]^{\frac{1}{q}}.$$

Consequently, since $\lambda \in A_{p,q}$, we finally get

$$\omega_{\frac{1}{2^{n+2}}}(b,Q) \leq c\frac{1}{|Q|}\int_Q \nu^{\frac{1}{m}}$$

and we are done in view of Lemma 3.8.

3.2.3 Proof of Corollary 3.2

To prove the corollary it suffices to estimate each term in κ_m in Theorem 3.1 for $\mu = \lambda$. Indeed, we observe first that taking $\mu = \lambda$

$$\kappa_m = [\mu]_{A_{p,q}}^{\left(1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{q}\right\}} \sum_{h=0}^{m} \binom{m}{h} [\mu^q]_{A_q}^{(m-h)\max\{1,\frac{1}{q-1}\}} [\mu^p]_{A_p}^{h\max\{1,\frac{1}{p-1}\}}.$$

Now, taking into account (1.1), we get

$$[\mu^q]_{A_q}^{(m-h)\max\left\{1,\frac{1}{q-1}\right\}} \leq [\mu]_{A_{p,q}}^{(m-h)\max\left\{1,\frac{1}{q-1}\right\}}$$

and

$$[\mu^p]_{A_p}^{h\max\left\{1,\frac{1}{p-1}\right\}} \leq [\mu]_{A_{p,q}}^{h\frac{p}{q}\max\left\{1,\frac{1}{p-1}\right\}}.$$

Consequently

$$\kappa_m \leq c_m[\mu]_{A_{p,q}}^{\left(1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{q}\right\}+m\max\left\{1,\frac{1}{q-1},\frac{p}{q},\frac{p'}{q}\right\}}.$$

Note that since $p < q$ we have that $\frac{p}{q} < 1$ and also

$$p < q \iff p' > q' \iff \frac{1}{q-1} < \frac{p'}{q}.$$

This yields that $\max\left\{1,\frac{1}{q-1},\frac{p}{q},\frac{p'}{q}\right\} = \max\left\{1,\frac{p'}{q}\right\}$ and we have that

$$\kappa_m \leq c_m[\mu]_{A_{p,q}}^{\left(m+1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{q}\right\}},$$

as we wanted to prove.

To establish the sharpness of the exponent in (3.3) we will use the adaption of Buckley's example [7] to the fractional setting that was devised in [12]. First we observe that we can restrict ourselves to the case in which $p'/q \geq 1$, since the case $p'/q < 1$ follows at once by duality, taking into account the fact that $(I_\alpha)_b^m$ is essentially self-adjoint (in this case, $[(I_\alpha)_b^m]^* = (-1)^m (I_\alpha)_b^m$) and the fact that if $w \in A_{p,q}$, then $w^{-1} \in A_{q',p'}$ and $[w^{-1}]_{A_{q',p'}} = [w]_{A_{p,q}}^{p'/q}$.
Suppose then that $p'/q \geq 1$, and take $\delta \in (0,1)$. Define the weight $w_\delta(x) = |x|^{(n-\delta)/p'}$ and the power functions $f_\delta(x) = |x|^{\delta-n}\chi_{B(0,1)}(x)$. Easy computations yield

$$\|f_\delta\|_{L^p(w_\delta^p)} \asymp \delta^{-1/p}, \quad \text{and} \quad [w_\delta]_{A_{p,q}} \asymp \delta^{-q/p'}.$$

Let b be the BMO function $b(x) = \log|x|$. For $x \in \mathbb{R}^n$ with $|x| \geq 2$, we have that

$$\begin{aligned}
(I_\alpha)_b^m f_\delta(x) &= \int_{B(0,1)} \frac{\log^m(|x|/|y|)}{|x-y|^{n-\alpha}} |y|^{\delta-n}\,\mathrm{d}y \\
&\geq |x|^{\delta-n+\alpha} \int_{B(0,|x|^{-1})} \frac{\log^m(1/|z|)}{(1+|z|)^{n-\alpha}} |z|^{\delta-n}\,\mathrm{d}z \\
&\geq c_n \frac{|x|^\delta}{(1+|x|)^{n-\alpha}} \int_0^{|x|^{-1}} \log^m(1/r) r^{\delta-1}\,\mathrm{d}r.
\end{aligned}$$

Integration by parts yields

$$\begin{aligned}
\int_0^{|x|^{-1}} \log^m(1/r) r^{\delta-1} dr &= \delta^{-1}|x|^{-\delta} \log^m|x| \sum_{k=0}^m \frac{m!}{(m-k)!\delta^k \log^k|x|} \\
&\geq \delta^{-1}|x|^{-\delta} \log^m|x| \sum_{k=0}^m \binom{m}{k} (\delta^{-1}\log^{-1}|x|)^k \\
&= \delta^{-1}|x|^{-\delta} \log^m|x| (\delta^{-1}\log^{-1}|x| + 1)^m \\
&\geq \delta^{-m-1}|x|^{-\delta}.
\end{aligned}$$

Then,

$$(I_\alpha)_b^m f_\delta(x) \geq \frac{c_n}{\delta^{m+1}|x|^{n-\alpha}}, \qquad |x| \geq 2.$$

Hence, taking into account that $\frac{1}{q} + \frac{\alpha}{n} = \frac{1}{p}$, we have that

$$\begin{aligned}
\|(I_\alpha)_b^m f_\delta\|_{L^q(w_\delta^q)} &\geq c_n \delta^{-(m+1)} \left(\int_{|x|\geq 2} \frac{|x|^{(n-\delta)\frac{q}{p'}}}{|x|^{(n-\alpha)q}}\,\mathrm{d}x \right)^{1/q} \\
&= c_n \delta^{-(m+1)} \left(\int_{|x|\geq 2} |x|^{-\delta q/p' - n}\,\mathrm{d}x \right)^{1/q} \\
&= c\delta^{-(m+1)-\frac{1}{q}} \\
&= c[w_\delta]_{A_{p,q}}^{\left(m+1-\frac{\alpha}{n}\right)\frac{p'}{q}} \|f_\delta\|_{L^p(w_\delta^p)} \\
&= c[w_\delta]_{A_{p,q}}^{\left(m+1-\frac{\alpha}{n}\right)\max\left\{1,\frac{p'}{q}\right\}} \|f_\delta\|_{L^p(w_\delta^p)},
\end{aligned}$$

so the exponent in (3.3) is sharp.

Remark 3.9. We would like to point out that an alternative argument to settle the sharpness we have just obtained follows from the combination of arguments in Sections 3.1 and 3.4 in [33].

3.3 Some further remarks

Connection with the boundedness of commutators of singular integrals

Theorem 3.1 combined with the characterization recently obtained in [32] allows to connect the boundedness of commutators of singular integrals and of commutators of fractional integrals in the unweighted setting via BMO. To be more precise, as a straightforward consequence of the result above and [32, Remark 1.2] we obtain the following corollary.

Corollary 3.10. *Let m_1, m_2 be positive integers and $m = \max\{m_1, m_2\}$ and assume that $b \in L^m_{loc}(\mathbb{R}^n)$. The following statements are equivalent.*

1. $b \in \mathrm{BMO}$.
2. *Given $\alpha \in (0,n)$, $1 < p < \frac{n}{\alpha}$ and q defined by $\frac{1}{q} + \frac{\alpha}{n} = \frac{1}{p}$,*
$$(I_\alpha)^{m_1}_b : L^q(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n).$$
3. *Let $1 < p < \infty$. Given Ω a continuous function on $\mathbb{S}^{n-1}$, not identically zero with $\int_{\mathbb{S}^n} \Omega(\theta) d\theta = 0$ and such that*
$$\omega(t) = \sup_{|\theta-\theta'|\leq t} |\Omega(\theta) - \Omega(\theta')|$$
satisfies the Dini condition, namely $\int_0^1 \omega(t)\frac{\mathrm{d}t}{t} < \infty$, if we define
$$T_\Omega f(x) := \lim_{\varepsilon\to 0} \int_{|x-y|>\varepsilon} \frac{\Omega\left(\frac{x-y}{|x-y|}\right)}{|x-y|^n} f(y)\,\mathrm{d}y,$$
then,
$$(T_\Omega)^{m_2}_b : L^p(\mathbb{R}^n) \longrightarrow L^p(\mathbb{R}^n).$$

Notice that, for instance, Hilbert and Riesz transforms are particular cases of the statement above.

A_p-A_∞ constants

Also, we recall that $w \in A_\infty$ if and only if

$$[w]_{A_\infty} := \sup_Q \frac{1}{w(Q)} \int_Q M(w\chi_Q) < \infty$$

and that is, up until now (see [16]), the smallest constant characterizing the A_∞ class. We would like to point out that it would be possible to provide mixed estimates for $(I_\alpha)_b^m$ in terms of this A_∞-constant. For that purpose it suffices to follow the same argument used to establish Theorem 3.1, but taking into account that, if $w \in A_p$ and we call $\sigma := w^{\frac{1}{1-p}}$, then

$$\|A_{\mathcal{S}}\|_{L^p(w)} \leq c_{n,p}[w]_{A_p}^{\frac{1}{p}} \left([w]_{A_\infty}^{\frac{1}{p'}} + [\sigma]_{A_\infty}^{\frac{1}{p}} \right).$$

Also, if $\alpha \in (0, n)$, $1 < p < \frac{n}{\alpha}$, q is defined by $\frac{1}{q} + \frac{\alpha}{n} = \frac{1}{p}$ and $w \in A_{p,q}$ then taking σ as above,

$$\|I_{\mathcal{S}}^\alpha\|_{L^p(w^p) \to L^q(w^q)} \leq c_{n,p}[w]_{A_{p,q}} \left([w^q]_{A_\infty}^{\frac{1}{p'}} + [\sigma^p]_{A_\infty}^{\frac{1}{q}} \right).$$

The preceding estimate was established in [13] and is also contained in the paper [16].

Chapter 4

Maximal directional operators

This last chapter deals with maximal directional operators. Our main operator to study is going to be defined via the Fourier transform as follows. Take m to be a Mikhlin-Hörmander multiplier on $\mathbb{R}$, that is, m is a $C^\infty(\mathbb{R}\backslash\{0\})$ that satisfies

$$\sup_{\xi\in\mathbb{R}\backslash\{0\}} |\xi|^\alpha |\partial^\alpha m(\xi)| \lesssim_\alpha 1, \quad \text{for all non-negative integers } \alpha.$$

For $f \in C_0^\infty(\mathbb{R}^n)$ and V a set of directions in $\mathbb{S}^{n-1}$ we define

$$T_V f(x) \coloneqq \sup_{v\in V} \left| \int_{\mathbb{R}^n} m(v\cdot\xi)\hat{f}(\xi)e^{i\xi\cdot x}\mathrm{d}\xi \right|, \quad x \in \mathbb{R}^n.$$

As we saw on Section 1.6, when dealing with the first example that we gave -the maximal directional operator that arises from the maximal function- we usually focus on two types of questions. Recall that basically the first question was about the *structure* of the set in order to get L^p-boundedness and the second one was about trying to find *sharp bounds* of the L^p-norms of the operator in terms of the cardinality of the set of directions. Now observe that one example that fits our setting is the maximal directional Hilbert transform, this is the operator generated by considering $m(\xi) = -i\,\mathrm{sgn}(\xi)$. A result from Łaba, Marinelli and Pramanik [27], elaborating on a previous example by Karagulyan [24], tells us that if V is any set of directions in $\mathbb{S}^{n-1}$ of cadinality N it holds $\|H_V\|_{L^p\to L^p} \gtrsim (\log N)^{\frac{1}{2}}$, and thus we can immediately drop the first question and focus on the case of sets of directions of finite cardinality. In a joint work with F. Di Plinio and I. Parissis [2] we obtained the following.

Theorem 4.1. *For Ω a finite union of lacunary sets, $1 < p < \infty$ and $V \subset \Omega$ any set of cardinality N it holds*

$$\|T_V f\|_{L^p(\mathbb{R}^n)} \lesssim (\log N)^{1/2}\|f\|_{L^p(\mathbb{R}^n)},$$

with the implicit constant dependent on the lacunarity order and constant, and on the dimension

n.

Recall that we have given the definition of lacunary sets of directions in Section 1.6.1. We immediately simplify the definition of lacunarity by assuming -without loss of generality- that all dissections are given with respect to lacunary sequences $\theta_{\sigma,\ell} = 2^{-\ell}$ for all $\sigma \in \Sigma$, corresponding to $\lambda = 1/2$. Furthermore by a standard approximation argument we can dispose of the final set of the partition $\Omega_{\sigma,\infty}$ and work with $\mathbb{Z}$ instead of $\mathbb{Z}^*$. Also, by finite splitting, we can and will assume that $\Omega \subset \{x \in \mathbb{R}^n : x_i > 0, \ i = 1, \ldots, n\}$.

4.1 The $L^2(\mathbb{R}^2)$-case

In this section we are going to present the proof for Theorem 4.1 in $L^2(\mathbb{R}^2)$, since the structure of the proof is similar to the one in the general case but the notation is much simpler. Take Ω a set of directions in $\mathbb{S}^1$ lacunary of order L. We recall the definition of the sectors S_ℓ that we have given in Section 1.6.1. Since we are assuming that $\Omega \subset \{x \in \mathbb{R}^2 : x_1, x_2 > 0\}$ and that the lacunarity constant is $1/2$, we have for $\ell \in \mathbb{Z}$

$$S_\ell = \Big\{\omega \in \mathbb{S}^1 : 2^{-(\ell+1)} \leq \frac{\omega_1}{\omega_2} < 2^{-\ell}\Big\}.$$

Whenever we are given a set of directions V, we will call $V_\ell \coloneqq V \cap S_\ell$.

Proof. Take V to be any subset of Ω of cardinality N, and v a direction of V. Observe that the singularity of the operator T_v lies on the line $\{\xi \in \mathbb{R}^2 : v \cdot \xi = 0\}$. For $\ell \in \mathbb{Z}$ we are going to cover all the singular lines of the operators T_v with v any direction in V_ℓ, with a cone. Explicitly, we define

$$\Psi_\ell \coloneqq \left\{\xi \in \mathbb{R}^2 \backslash e_2^\perp : \frac{2^{-(\ell+1)}}{2} \leq \frac{-\xi_1}{\xi_2} < 2 \cdot 2^{-\ell}\right\}.$$

For a direction $v \in V_\ell$ we are going to split T_v into two operators: one that has its frequency support inside the cone and another that has its frequency support outside the cone. To that end, take ψ to be a bump function, $\psi \equiv 1$ on $(1/4, 2)$ and $\psi \equiv 0$ on $(1/6, 3)^c$, and K_ℓ to be the Fourier multipliers with symbols $\widehat{K_\ell}(\xi) = \psi_\ell(\xi) \coloneqq \psi\big(\frac{-\xi_1}{2^{-\ell}\xi_2}\big)$. Observe that $\psi_\ell \equiv 1$ on Ψ_ℓ and $\psi_\ell \equiv 0$ outside a slightly larger cone. We now split

$$T_v f = T_v K_\ell f + T_v(\mathrm{Id} - K_\ell) f \coloneqq T_v^{\mathrm{in}} f + T_v^{\mathrm{out}} f;$$

We will deal with the two parts separately.

The inner part:

Define

$$T_V^{\mathrm{in}} f \coloneqq \sup_{\ell \in \mathbb{Z}} |T_{V_\ell}^{\mathrm{in}} f| = \sup_{\ell \in \mathbb{Z}} \sup_{v \in V_\ell} |T_v K_\ell f|.$$

We have

$$\|T_V^{\text{in}} f\|^2_{L^2(\mathbb{R}^2)} \leq \Big\|(\sum_{\ell\in\mathbb{Z}} |T_{V_\ell}^{\text{in}} K_\ell f|^2)^{1/2}\Big\|^2_{L^2} \leq \sup_{\ell\in\mathbb{Z}} \|T_{V_\ell}^{\text{in}}\|^2_{L^2(\mathbb{R}^2)} \Big\|(\sum_{\ell\in\mathbb{Z}} |K_\ell f|^2)^{\frac{1}{2}}\Big\|^2_{L^2(\mathbb{R}^2)}.$$

Using Plancharel's theorem and the fact that the cones Ψ_ℓ have bounded overlap,

$$\Big\|(\sum_{\ell\in\mathbb{Z}} |K_\ell f|^2)^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^2)} = \sum_{\ell\in\mathbb{Z}} \int_{\mathbb{R}^2} |K_\ell f(x)|^2 dx = \sum_{\ell\in\mathbb{Z}} \int_{\mathbb{R}^2} |\psi_\ell(\xi)\hat{f}(\xi)|^2 \mathrm{d}\xi \leq 5\|f\|^2_{L^2(\mathbb{R}^2)}.$$ [1]

Finally, we can bound

$$\|T_V^{\text{in}} f\|^2_{L^2(\mathbb{R}^2)} \lesssim \sup_{\ell\in\mathbb{Z}} \|T_{V_\ell}\|^2_{L^2(\mathbb{R}^2)} \|f\|^2_{L^2(\mathbb{R}^2)},$$

with V_ℓ lacunary of order $L-1$ by assumption.

The outer part:

We are going to further split the outer part into two pieces. Take another bump function η such that $\eta \equiv 1$ on $(-1/4, 1/4)$ and $\eta \equiv 0$ on $(-3/4, 3/4)^c$. Now,

$$T_v(\text{Id} - K_\ell)f = T_v N_\ell(\text{Id} - K_\ell)f + T_v(\text{Id} - N_\ell)(\text{Id} - K_\ell)f;$$

where $(N_\ell)^\wedge(\xi) = \eta_\ell(\xi) := \eta\big(\frac{-\xi_1}{2^{-\ell}\xi_2}\big)$. Observe that v is morally $(1, 2^{-\ell})$ so what this multiplier is doing is telling us which is the leading term in $|v\cdot\xi| \simeq |\xi_1 + 2^{-\ell}\xi_2|$.

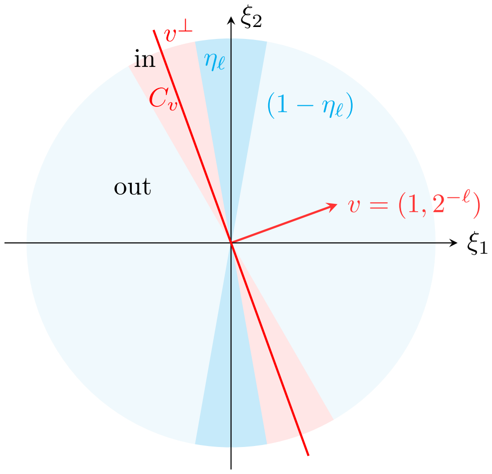

Figure 4.1: The splitting of the operator T_v.

[1] The constant 5 is irrelevant for us and included here because it is very easy to compute. We are going to omit explicit constant going forward.

We are going to use the Chang-Wilson-Wolff inequality that we presented in Section 1.6.3, with the weight $w = 1$. To that end, take p to be a smooth function on $\mathbb{R}$ such that

$$\sum_{t\in\mathbb{Z}} p(2^{-t}\xi) = 1, \qquad \xi \neq 0,$$

and such that p vanishes off the set $\{\xi \in \mathbb{R} : \frac{1}{2} < |\xi| < 2\}$ and define

$$(P_j^t f)^\wedge(\xi) := p(2^{-t}\xi_j), \qquad j = 1, 2, \qquad t \in \mathbb{Z}.$$

We will need to superimpose another Littlewood-Paley decomposition on top of this one, we do so by taking a smooth function such that

$$\operatorname{supp} q \subset \big\{\xi \in \mathbb{R} : \frac{1}{4} < |\xi| < 4\big\}, \qquad \sum_{t\in\mathbb{Z}} q(2^{-t}\xi) = 1 \quad \xi \neq 0$$

$$q = 1 \text{ on } \operatorname{supp} p.$$

We define the operator

$$(Q_t^j f)^\wedge(\xi) = \hat{f}(\xi) q(2^{-t}\xi_j) \quad j = 1, 2.$$

Lemma 4.2. *Take M_s to be the strong maximal operator defined in 1.2. For $v \in S_\ell$ we have*

$$|T_v N_\ell(\mathrm{Id} - K_\ell) P_t^2 f(x)| \lesssim \mathrm{M}_s f(x), \quad x \in \mathbb{R}^2$$

and

$$|T_v(\mathrm{Id} - N_\ell)(\mathrm{Id} - K_\ell) P_t^1 f(x)| \lesssim \mathrm{M}_s f(x), \quad x \in \mathbb{R}^2.$$

Proof. Define

$$\Phi_v(x_1, x_2) = \int_{\mathbb{R}^2} m(v \cdot \xi)\eta_\ell(\xi)(1 - \psi_\ell(\xi)) q(2^{-t}\xi_2) e^{ix\cdot\xi}\, \mathrm{d}\xi.$$

Observe that we have the pointwise identity $T_v N_\ell(\mathrm{Id} - K_\ell)P_t^2 f(x) = \Phi * (P_t^2 f)(x)$, so to get the desired estimate it is enough to control the derivatives of Φ.

Remember that we are taking $v \in S_\ell$, which means that $v_2 \simeq 2^{-\ell} v_1$. Note that the support of the symbol $\eta_\ell(\xi)(1 - \psi_\ell)(\xi) q(2^{-t}\xi_2)$ is contained in the set $\{-3/4 \le -\xi_1/(2^{-\ell}\xi_2) \le 1/4\} \cap \{|\xi_2| \sim 2^t\}$, so we trivially get that $|\Phi(x_1, x_2)| \lesssim 2^{2t-\ell}$. We can bound the derivatives of m by

$$|\partial_{\xi_1}^\alpha m(v \cdot \xi)| = |m^{(\alpha)}(v \cdot \xi)| v_1^\alpha \lesssim \frac{v_1^\alpha}{|v \cdot \xi|^\alpha} \lesssim \frac{1}{(2^{-\ell}|\xi_2|)^\alpha} \lesssim 2^{\alpha(\ell - t)}.$$

and

$$|\partial^\beta_{\xi_2} m(v\cdot\xi)| = |m^{(\beta)}(v\cdot\xi)|v_2^\beta \lesssim \frac{v_2^\beta}{|v\cdot\xi|^\beta} \lesssim \big(\frac{v_2}{2^{-\ell}v_1}\big)^\beta \frac{1}{|\xi_2|^\beta} \lesssim 2^{\beta(-t)}$$

Indeed, we have that $|\xi_1| \le \frac{1}{4}2^{-\ell}|\xi_2|$ and so

$$|v_1\xi_1 + v_2\xi_2| \ge v_1(\frac{v_2}{v_1}|\xi_2| - |\xi_1|) \gtrsim v_1(2^{-\ell}|\xi_2| - |\xi_1|) \gtrsim v_1 2^{-\ell}|\xi_2|.$$

We have to also consider the derivatives of η_ℓ and ψ_ℓ. We do one, the other one is similar:

$$|\partial^\beta_{\xi_2}\partial^\alpha_{\xi_1}\psi(\frac{-\xi_1}{2^{-\ell}\xi_2})| \le |\psi^{(\alpha+\beta)}(\frac{-\xi_1}{2^{-\ell}\xi_2})|\frac{1}{(2^{-\ell}|\xi_2|)^\alpha}\big(\frac{|\xi_1|}{2^{-\ell}|\xi_2|^2}\big)^\beta \lesssim 2^{-\alpha(t-\ell)}2^{-\beta t}.$$

Taking $\alpha, \beta = 0, 1, 2$ we can bound

$$|\Phi(x_1, x_2)| \lesssim 2^{2t-\ell}\frac{1}{(1+2^{t-\ell}|x_1|)^2}\frac{1}{(1+2^t|x_2|)^2}$$

and so $|\Phi * (P^2_t f)(x)| \lesssim \mathrm{M}_s(P^2_t f)(x)$ for all $x \in \mathbb{R}^2$.

The estimates for $T_v(\mathrm{Id} - N_\ell)(\mathrm{Id} - K_\ell)f$ are similar. Call $\varphi = (1-\psi)(1-\eta)$, and we have $\varphi \equiv 0$ on $(-1/4, 2)$ and $\varphi \equiv 1$ on $(-3/4, 3)^c$. We now take Φ to be

$$\Phi(x_1, x_2) = \int_{\mathbb{R}^2} m(v\cdot\xi)\varphi(\frac{-\xi_1}{2^{-\ell}\xi_2})q(2^{-t}|\xi_1|)e^{ix\cdot\xi}\,\mathrm{d}\xi$$

and we have that $\Phi * (P^1_t f) = T_v(\mathrm{Id} - N_\ell)(\mathrm{Id} - \psi_\ell)(P^1_t f)$. By support considerations we have that

$$|\Phi(x_1, x_2)| \lesssim 2^{2t+\ell}.$$

Now observe that φ_ℓ only has derivatives when $|\xi_1| \sim 2^{-\ell}|\xi_2|$ and so we get

$$|\partial^\beta_{\xi_2}\partial^\alpha_{\xi_1}\varphi(\frac{-\xi_1}{2^{-\ell}\xi_2})| \lesssim \frac{1}{(2^{-\ell}|\xi_2|)^\alpha}\frac{|\xi_1|^\beta}{(2^{-\ell}|\xi_2|^2)^\beta} \lesssim 2^{-\alpha t}2^{-\beta(t+\ell)}.$$

And also

$$|\partial^\alpha_{\xi_1} m(v\cdot\xi)| \lesssim \frac{v_1^\alpha}{|v\cdot\xi|^\alpha} \lesssim \frac{1}{|\xi_1|^\alpha} \sim 2^{-\alpha t}$$

$$|\partial^\beta_{\xi_2} m(v\cdot\xi)| \lesssim \frac{v_2^\beta}{|v\cdot\xi|^\beta} \lesssim \frac{v_2^\beta}{v_1^\beta|\xi_1|^\beta} \sim 2^{-\beta(t+\ell)}.$$

Like before, taking $\alpha, \beta = 0, 1, 2$

$$|\Phi(x_1, x_2)| \lesssim 2^{2t+\ell}\frac{1}{(1+2^t|x_1|)^2}\frac{1}{(1+2^{t+\ell}|x_2|)^2},$$

and so $|\Phi * (P^1_t f)(x)| \lesssim \mathrm{M}_s(P^1_t f)(x)$ for all $x \in \mathbb{R}^2$. $\square$

To complete the bound of the outer part, we apply both the Chang-Wilson-Wolff inequality from Proposition 1.18 and Lemma 4.2 to obtain

$$\begin{aligned}
&\Big\|\sup_{v\in V}|T_v^{\mathrm{out}}f|\Big\|_{L^2(\mathbb{R}^2)} \leq \big\|\sup_{v\in V}T_vN_\ell(\mathrm{Id}-K_\ell)f\big\|_{L^2(\mathbb{R}^2)} + \big\|\sup_{v\in V}T_v(\mathrm{Id}-N_\ell)(\mathrm{Id}-K_\ell)f\big\|_{L^2(\mathbb{R}^2)}\\
&\lesssim \|f\|_{L^2(\mathbb{R}^2)} + (\log(N+1))^{\frac{1}{2}}\Bigg(\Big\|\Big[\sum_{t\in\mathbb{Z}}\sup_{v\in V}|T_vN_\ell(\mathrm{Id}-K_\ell)(P_t^2f)|^2\Big]^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^2)}\\
&\qquad\qquad + \Big\|\Big[\sum_{t\in\mathbb{Z}}\sup_{v\in V}|T_v(\mathrm{Id}-N_\ell)(\mathrm{Id}-K_\ell)(P_t^1f)|^2\Big]^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^2)}\Bigg)\\
&\lesssim \|f\|_{L^2(\mathbb{R}^2)} + (\log N)^{\frac{1}{2}}\Bigg(\Big\|\Big[\sum_{t\in\mathbb{Z}}|\mathrm{M}_s(P_t^2f)|^2\Big]^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^2)} + \Big\|\Big[\sum_{t\in\mathbb{Z}}|\mathrm{M}_s(P_t^1f)|^2\Big]^{\frac{1}{2}}\Big\|_{L^2(\mathbb{R}^2)}\Bigg)\\
&\lesssim (\log N)^{\frac{1}{2}}\|f\|_{L^2(\mathbb{R}^2)}.
\end{aligned}$$

Completing the proof of theorem 4.1:

Combining both the estimates we obtained for the outer and the inner part we get

$$\begin{aligned}
&\big\|\sup_{v\in V}|T_vf|\big\|_{L^2(\mathbb{R}^2)} \leq \big\|\sup_{v\in V}|T_v^{\mathrm{in}}f|\big\|_{L^2(\mathbb{R}^2)} + \big\|\sup_{v\in V}|T_v^{\mathrm{out}}f|\big\|_{L^2(\mathbb{R}^2)}\\
&\lesssim \left(\sup_{\ell\in\mathbb{Z}}\|T_{V_\ell}\|_{L^2(\mathbb{R}^2)} + (\log N)^{\frac{1}{2}}\right)\|f\|_{L^2(\mathbb{R}^2)},
\end{aligned}$$

with V_ℓ lacunary of order $L-1$. We can now deduce the theorem by induction on L. Indeed, this estimate provides us with the inductive step and if $L=1$, then V_ℓ consists of only one direction and so $\sup_{\ell\in\mathbb{Z}}\|T_{V_\ell}\|_{L^2(\mathbb{R}^2)}$ is trivially bounded by the one-dimensional theory. □

4.2 The $L^p(\mathbb{R}^n)$-case

As we mentioned before, the core of the proof for the general case is similar to the $L^2(\mathbb{R}^2)$-case. Trying to replicate this proof comes with two main difficulties: the first one is figuring out the right splitting to take when dealing with the geometry of $\mathbb{R}^n$, and the second one is bounding the square function that we plugged in the proof of the inner part (4.1) when we lose L^2.

We review the set up of the problem. Let $\Omega\subset\mathbb{S}^{n-1}$ be a lacunary set of directions of order L. Recall we will assume $\Omega\subset\mathbb{R}^n_+$ and that all dissections are given with respect to the lacunary sequences $\theta_{\sigma,\ell}=2^{-\ell}$, $\ell\in\mathbb{Z}$, for all $\sigma\in\Sigma$. Let m be a Mikhlin-Hörmander multiplier on $\mathbb{R}$ and define for a subset $V\subset\Omega$,

$$T_Vf(x)=\sup_{v\in V}\left|\int_{\mathbb{R}^n}m(v\cdot\xi)\hat{f}(\xi)e^{i\xi\cdot x}\mathrm{d}\xi\right|,\quad x\in\mathbb{R}^n.$$

Observe that for a fixed direction $v\in V$, the singularity of the operator $T_vf=(m(v\cdot .)\hat{f})^\vee$ lies on the $(n-1)$ dimensional hyperplane $v^\perp$. In order to isolate the singularity, we introduce the following frequency cutoffs that were first considered by Nagel, Stein and Wainger in [36].

Definition 4.3. Let $\omega(\xi)$ denote a function that is homogeneous of degree zero and C^∞ away from the origin in $\mathbb{R}^n$, and which satisfies

$$\omega \equiv \begin{cases} 1, & \text{if} \quad |\xi_1 + \cdots + \xi_n| < \frac{1}{2n^2}\|\xi\|, \\ 0, & \text{if} \quad |\xi_1 + \cdots + \xi_n| \geq \frac{1}{n^2}\|\xi\|. \end{cases}$$

For a direction $v \in \mathbb{S}^{n-1}$ we define the smooth frequency projections

$$W_v f(x) := \int_{\mathbb{R}^n} \omega(v_1\xi_1, \ldots, v_n\xi_n)\widehat{f}(\xi)e^{ix\cdot\xi}\,\mathrm{d}\xi, \qquad x \in \mathbb{R}^n. \tag{4.1}$$

Note that the operator $\mathrm{Id} - W_v$ is a smooth frequency projection onto a cone with axis along v. In particular the frequency support of the symbol of $\mathrm{Id} - W_v$ only intersects the hyperplane $v^\perp$ at the origin. We will further split the "inner part" with the two-dimensional wedges that were introduced by Parcet and Rogers in [37].

Definition 4.4. We define for $\sigma \in \Sigma$ and $\ell \in \mathbb{Z}$

$$\Psi_{\sigma,\ell} := \left\{ \xi \in \mathbb{R}^n \backslash e_{\sigma(2)}^\perp : \frac{2^{-(\ell+1)}}{n} \leq \frac{-\xi_{\sigma(1)}}{\xi_{\sigma(2)}} < 2^{-\ell} n \right\},$$

and

$$\widetilde{\Psi}_{\sigma,\ell} := \left\{ \xi \in \mathbb{R}^n \backslash e_{\sigma(2)}^\perp : \frac{2^{-(\ell+1)}}{n+1} \leq \frac{-\xi_{\sigma(1)}}{\xi_{\sigma(2)}} < 2^{-\ell}(n+1) \right\}.$$

Take κ to be a bump function such that

$$\kappa \equiv \begin{cases} 1 & \text{on} \quad [1/2n, n], \\ 0 & \text{on} \quad [1/2(n+1), n+1]^c, \end{cases}$$

and define the Fourier multiplier operators $K_{\sigma,\ell}$ with symbols

$$\kappa_{\sigma,\ell}(\xi) := \kappa\left(-\frac{\xi_{\sigma(1)}}{2^{-\ell}\xi_{\sigma(2)}} \right), \qquad K_{\sigma,\ell} f := (\kappa_{\sigma,\ell}\hat{f})^\vee.$$

Note that $\kappa_{\sigma,\ell}$ is smooth, identically 1 on the wedge $\Psi_{\sigma,\ell}$, and identically 0 off $\widetilde{\Psi}_{\sigma,\ell}$. For a subset $\varnothing \neq U \subseteq \Sigma(d)$ we define

$$K_{U,\ell} := \prod_{\sigma\in U} K_{\sigma,\ell_\sigma}$$

with the product symbol being used to denote for compositions of operators in the display above.

The main geometric observation relating the Nagel-Stein-Wainger cones with the Parcet-Rogers wedges is contained in the following lemma, which is an elaboration of a similar statement from

[37, Proof of Theorem A].

Lemma 4.5. *Let* $\{\Omega_{\sigma,\ell}\}$ *be a lacunary dissection of* $\Omega \subset \mathbb{S}^{d-1}$ *and suppose that* $v \in \Omega_{\boldsymbol{\ell}}$ *for some* $\boldsymbol{\ell} \in \mathbb{Z}^{\Sigma}$ *with* $\boldsymbol{\ell} = \{\ell_\sigma : \sigma \in \Sigma\}$*. Then*

$$W_v f = \sum_{\varnothing \neq U \subseteq \Sigma(d)} (-1)^{|U|+1} W_v K_{U,\boldsymbol{\ell}} f.$$

Proof. Writing $(W_v f)^\wedge =: \omega_v \widehat{f}$ we note that the support of ω_v satisfies

$$\operatorname{supp} \omega_v \subseteq \left\{ \xi \in \mathbb{R}^n : |\xi \cdot v| < \frac{1}{n} \max_{1 \le k \le n} |\xi_k v_k| \right\} =: C_v.$$

We read from [14, Proof of Lemma 3.2], together with the assumption that $v \in \Omega_{\boldsymbol{\ell}}$, that

$$C_v \subseteq \bigcup_{\sigma \in \Sigma} \Psi_{\sigma,\ell_\sigma}.$$

The conclusion of the lemma follows from the display above, the inclusion-exclusion formula, and the fact that for each $\sigma \in \Sigma$ and $\ell \in \mathbb{Z}$ the operator $K_{\sigma,\ell}$ has symbol $\kappa_{\sigma,\ell}$ which is identically 1 on $\Psi_{\sigma,\ell}$. □

4.2.1 Bounding the symbols

Lemma 4.6. *Let* Σ *be associated with a given ONB on* $\mathbb{S}^{n-1}$ *and denote by* A_p^* *the class of weights corresponding to its coordinate directions. Then for all* $w \in A_p^*$ *we have*

$$\sup_{U \subseteq \Sigma} \left\| \left(\sum_{\boldsymbol{\ell} \in \mathbb{Z}^U} \left| K_{U,\boldsymbol{\ell}} f \right|^2 \right)^{\frac{1}{2}} \right\|_{L^p(w)} \lesssim [w]_{A_p^*}^{\gamma} \|f\|_{L^p(w)}$$

for some $\gamma = \gamma(p,n)$ *and implicit constant independent of* f *and* w*.*

Proof. By Khintchine's inequality, it is enough to bound in $L^p(w)$ the operator

$$f \mapsto \sum_{\boldsymbol{\ell} \in \mathbb{Z}^U} \varepsilon_{\boldsymbol{\ell}} K_{U,\boldsymbol{\ell}} f,$$

where $\{\varepsilon_{\boldsymbol{\ell}}\}$ is an arbitrary choice of signs. Choose $m = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^U} \varepsilon_{\boldsymbol{\ell}} \kappa_{U,\boldsymbol{\ell}}$ in the theorem by Kurtz stated in 1.15. The derivative estimates needed to apply the theorem are deduced by observing that the wedges $\{\widetilde{\Psi}_{\sigma,\ell}\}$ have bounded overlap. □

As another direct application of Kurtz's theorem one can easily provide weighted norm inequalities for the conical multipliers W_v associated with a fixed direction $v \in \mathbb{R}^n$.

Lemma 4.7. *For $v \in \mathbb{S}^{n-1}$ let W_v be defined as in (4.1). Then for all $p \in (1,\infty)$ and all $w \in A_p^*$ we have*

$$\sup_{v\in\mathbb{S}^{n-1}} \|W_v\|_{L^p(w)} \lesssim [w]_{A_p^*}^{\gamma}$$

for some $\gamma = \gamma(n,p)$ and implicit constant independent of w.

A maximal inequality for Nagel-Stein-Wainger cones

In the proof of our main theorem we will need a maximal version of Lemma 4.7. For this let us consider a set $\Omega \subset \mathbb{S}^{n-1}$ and define the maximal cone multiplier operator

$$W_\Omega f(x) := \sup_{v\in\Omega} |W_v f(x)|, \qquad x \in \mathbb{R}^n.$$

Lemma 4.8. *Let $\Omega \subset \mathbb{S}^{n-1}$ be a lacunary set and $w \in A_p^\Omega$. Then*

$$\|W_\Omega\|_{L^p(w)} \lesssim [w]_{A_p^\Omega}^{\gamma}$$

for some γ depending on p, n, and the lacunarity order of Ω.

Proof. By the extrapolation result of Proposition 1.17 it will be enough to proof the $L^2(w)$-version of the conclusion whenever $w \in A_2^\Omega$. We will do so by proving the recursive formula

$$\|W_\Omega f\|_{L^2(w)} \leq B[w]_{A_2^\Omega}^{\gamma} \sup_{\sigma\in\Sigma}\sup_{\ell\in\mathbb{Z}} \|W_{\Omega_{\sigma,\ell}}\|_{L^2(w)}$$

with γ as in the conclusion of the lemma and $B > 0$ a numerical constant depending only upon dimension. The proof then follows by an inductive application of the formula above, repeated as many times as the order of lacunarity L of Ω. The base step of the induction corresponds to lacunary sets of order 0 in which case the desired estimate is the content of Lemma 4.7.
To prove the recursive formula let $v \in \Omega$ so that $v \in \Omega_{\boldsymbol{\ell}}$ for some unique $\boldsymbol{\ell} \in \mathbb{Z}^\Sigma$. By Lemma 4.5 we have that

$$|W_v f(x)| \leq \sum_{\varnothing\neq U\subseteq\Sigma} \Big(\sum_{\boldsymbol{\ell}\in\mathbb{Z}^\Sigma} |W_{\Omega_{\boldsymbol{\ell}}} K_{U,\boldsymbol{\ell}} f|^2 \Big)^{\frac{1}{2}}.$$

Taking $L^2(w)$-norms and using the $L^2(w)$ vector-valued bound for $\{K_{U,\boldsymbol{\ell}}\}$ of Lemma 4.6 yields

$$\|W_v f\|_{L^2(w)}^2 \leq B \sup_{\boldsymbol{\ell}\in\mathbb{Z}^\Sigma} \|W_{\Omega_{\boldsymbol{\ell}}}\|_{L^2(w)}^2 \leq B[w]_{A_2^\Omega}^{2\gamma} \sup_{\sigma\in\Sigma}\sup_{\ell\in\mathbb{Z}} \|W_{\Omega_{\sigma,\ell}}\|_{L^2(w)}^2$$

which proves the desired recursive estimate and thus the lemma. □

4.2.2 The proof of Theorem 4.1

This section is dedicated to the proof of our main theorem. We remember that $m \in C^\infty(\mathbb{R}^n \backslash \{0\})$ and T_v is the directional multiplier operator

$$T_v f(x) = \int_{\mathbb{R}^n} \widehat{f}(\xi) m(\xi \cdot v) e^{ix\cdot\xi} \, \mathrm{d}\xi, \qquad x \in \mathbb{R}^n,$$

while for any $V \subset \mathbb{S}^{n-1}$ we have defined $T_V f = \sup_{v\in V} |T_v f|$. By the extrapolation result of Proposition 1.17 the proof of the statement

$$\sup_{\substack{V\subset\Omega \\ \#V=N}} \|T_V f\|_p \lesssim (\log N)^{1/2} \|f\|_p, \quad p \in (1, \infty),$$

is reduced to proving that for all $\Omega \subset \mathbb{S}^{n-1}$ which are lacunary of some order $L \geq 1$ and all directional weights $w \in A_2^\Omega$ we have

$$\sup_{\substack{V\subset\Omega \\ \#V=N}} \|T_V f\|_{L^2(w)} \lesssim [w]^\gamma_{A_2^\Omega} (\log N)^{1/2} \|f\|_{L^2(w)} \tag{4.2}$$

for some $\gamma > 0$ depending upon dimension and the order of lacunarity of Ω.
We note that, although we have allowed for the possibility that $\operatorname{span}(\Omega) = d \leq n$, we can safely reduce to the case $d = n$ by an application of Fubini's theorem. In what follows we thus work in $\mathbb{R}^n$ with $\Omega \subset \mathbb{S}^{n-1}$ and $\operatorname{span}(\Omega) = \mathbb{R}^n$. We will just write Σ for $\Sigma(n)$.

The main splitting

The whole proof is guided by the following splitting of the operator T_v into two pieces. The first contains the singularity of $\xi \mapsto m(\xi \cdot v)$, with the complementary piece given by a Nagel-Stein-Wainger cone as in Definition 4.3

$$|T_v f(x)| \leq |T_v W_v f(x)| + |T_v(\mathrm{Id} - W_v) f(x)| =: |T_v^{\mathrm{in}} f(x)| + |T_v^{\mathrm{out}} f(x)|, \qquad x \in \mathbb{R}^n. \tag{4.3}$$

Recall that W_v is defined in Definition 4.3. We deal first with the inner part.

The inner part

For fixed $v \in V \subset \Omega$ there exists a unique $\boldsymbol{\ell} \in \mathbb{Z}^\Sigma$ such that $v \in \Omega_{\boldsymbol{\ell}}$. Fixing such v and $\boldsymbol{\ell}$ and using Lemma 4.5 we readily see that

$$\begin{aligned} |T_v^{\mathrm{in}} f(x)| &\leq \Big| \sum_{\varnothing \neq U \subseteq \Sigma} (-1)^{|U|+1} T_v W_v K_{U,\boldsymbol{\ell}} f(x) \Big| \lesssim \sum_{\varnothing \neq U \subseteq \Sigma} \sup_{v \in V\cap\Omega_{\boldsymbol{\ell}}} \Big(\sum_{\boldsymbol{\ell}} |T_v W_v K_{U,\boldsymbol{\ell}} f(x)|^2 \Big)^{\frac{1}{2}} \\ &\leq \sup_{\varnothing \neq U \subseteq \Sigma} \Big(\sum_{\boldsymbol{\ell}} |T^{\mathrm{in}}_{V\cap\Omega_{\boldsymbol{\ell}}} K_{U,\boldsymbol{\ell}} f(x)|^2 \Big)^{\frac{1}{2}}. \end{aligned}$$

Recall

$$T_V^{\text{in}} f := \sup_{v\in V} |T_v^{\text{in}} f| = \sup_{v\in V} |T_v W_v f|. \tag{4.4}$$

Taking $L^2(w)$-norms and using the weighted vector-valued bound of Lemma 4.6

$$\begin{aligned}\left\|T_V^{\text{in}} f\right\|_{L^2(w)}^2 &\lesssim [w]_{A_2^\Omega}^{2\gamma_1} \sup_{\boldsymbol{\ell}\in\mathbb{Z}^\Sigma} \left\|T_{V\cap\Omega_{\boldsymbol{\ell}}}^{\text{in}}\right\|_{L^2(w)}^2 \|f\|_{L^2(w)}^2 \\ &\lesssim [w]_{A_2^\Omega}^{\gamma_2} \left\|W_\Omega\right\|_{L^2(w)}^2 \sup_{\sigma\in\Sigma}\sup_{\ell\in\mathbb{Z}} \|T_{V_{\sigma,\ell}}\|_{L^2(w)}^2 \|f\|_{L^2(w)}^2.\end{aligned}$$

Inserting the maximal inequality of Lemma 4.8 in the display above proves the recursive estimate

$$\|T_V^{\text{in}}\|_{L^2(w)} \lesssim [w]_{A_2^\Omega}^{\tilde\gamma} \sup_{\sigma\in\Sigma}\sup_{\ell\in\mathbb{Z}} \|T_{V_{\sigma,\ell}}\|_{L^2(w)} \tag{4.5}$$

for some exponent $\tilde\gamma$ depending only on the lacunarity order of Ω and the dimension.

The outer part

Let φ to be a bump function on $\mathbb{R}$ such that $\varphi \equiv 0$ on $[-1/4, 1/4]$ and $\varphi \equiv 1$ on $(-1/2, 1/2)^c$, and define

$$\varphi_v^j(\xi) := \varphi\left(\frac{n v_j \xi_j}{\|(v\xi)\|}\right), \qquad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n \backslash \{0\};$$

from here on, $(v\xi)$ denotes the vector $(v_1\xi_1, \dots, v_n\xi_n)$. Observe that on $\mathbb{R}^n\backslash\{0\}$ we have

$$1 = \varphi_v^1 + \left(\sum_{j=2}^{n-1} \varphi_v^j \prod_{1\le\ell<j} (1-\varphi_v^\ell)\right) + \prod_{1\le\ell<n} (1-\varphi_v^\ell) =: \eta_v^1 + \left(\sum_{j=2}^{n-1} \eta_v^j\right) + \eta_v^n. \tag{4.6}$$

Therefore, we can further split the operator $T_v^{\text{out}} = T_v(\text{Id} - W_v)$ into n pieces,

$$T_v^{\text{out}} f = \sum_{j=1}^n T_v^{\text{out}} N_v^j f,$$

where each N_v^j is the Fourier multiplier with symbol $\eta_{j,v}$.
The heart of the proof for the outer part is the content of the following lemma which provides a pointwise control of the operators $T_v^{\text{out}} N_v^j P_t^j$ by suitable averages which are independent of the direction. Here P_j^t is a coordinate-wise Littlewood-Paley projection which is defined as in the discussion preceding Lemma 1.18. That is,

$$\widehat{P_t^j f} = p(2^{-t}\xi_j)\widehat{f}, \qquad \xi = (\xi_1, \dots, \xi_n) \in \mathbb{R}^n\backslash\{0\}, \quad t \in \mathbb{Z},$$

with $\text{supp}(p) \subseteq \{\xi \in \mathbb{R} : \frac{1}{2} < |\xi| < 2\}$. We will need to superimpose another Littlewood-Paley decomposition on top of $\{P_t^j\}$. To this aim, consider a smooth function q on $\mathbb{R}$ such that

$$\operatorname{supp}(q) \subseteq \{\xi \in \mathbb{R} : \frac{1}{4} < |\xi| < 4\}, \qquad q \equiv 1 \quad \text{on} \quad \{\frac{1}{2} < |\xi| < 2\},$$

and

$$\sum_{t \in \mathbb{Z}} q(2^{-t}\xi) \approx 1, \qquad \xi \in \mathbb{R}^n \backslash \{0\}.$$

In the statement of the lemma below, M_s denotes the strong maximal function in $\mathbb{R}^n$ defined as in 1.2, with respect to our fixed choice of coordinates .

Lemma 4.9. *For $v \in \mathbb{S}^{n-1}$ and $j = 1, \ldots, n$, we have the pointwise estimate*

$$|T_v^{\text{out}} N_v^j P_t^j f(x)| \lesssim \mathrm{M}_s(P_t^j f)(x)$$

with implicit constant depending only upon dimension.

Proof. For $v \in \mathbb{S}^{n-1}$ call

$$\Phi_v(x) \coloneqq \int_{\mathbb{R}^n} m(v \cdot \xi)(1 - \omega_v(\xi))\eta_v^j(\xi) q(2^{-t}\xi_j) e^{ix\cdot\xi} \, \mathrm{d}\xi, \qquad x \in \mathbb{R}^n.$$

Remember that $v \in \Omega_\ell$ means that for every pair $\sigma = (k, j)$ with $1 \le k < j \le n$ we have that $v_j/v_k \sim 2^{-\ell_{(k,j)}}$. Now for a general pair (k, j), call $\ell_{kj} \coloneqq \ell_{(k,j)}$ if $k < j$ and $\ell_{kj} \coloneqq -\ell_{(j,k)}$ if $k > j$. Set also $\ell_{kk} = 0$.

From the construction of φ_v^j, and the definition (4.6) of η_v^j, it follows that

$$\xi \in \operatorname{supp} \eta_v^j \implies \|(v\xi)\| \lesssim |v_j \xi_j|.$$

Then, for $k = 1, \ldots, n$,

$$|\xi_k| \le \frac{\|(v\xi)\|}{v_k} \lesssim \frac{v_j}{v_k} |\xi_j| \lesssim 2^{t - \ell_{kj}},$$

which shows that $|\Phi_v(x)| \lesssim \prod_{k=1}^n 2^{t-\ell_{kj}}$.

We proceed to show suitable derivative estimates for the Fourier transform of Φ. Without further mention, estimates (4.7), (4.8), and (4.9) are meant to hold for $\xi \in \operatorname{supp} \widehat{\Phi}$, and $\alpha_1, \ldots, \alpha_n$ will denote non negative integers with $\alpha = \alpha_1 + \cdots + \alpha_n$. Firstly,

$$|\partial_{\xi_1}^{\alpha_1} \ldots \partial_{\xi_n}^{\alpha_n} \eta_j^v(\xi)| \lesssim \left(\frac{v_1}{v_j}\right)^{\alpha_1} \cdots \left(\frac{v_n}{v_j}\right)^{\alpha_n} \frac{1}{|\xi_j|^\alpha} \lesssim \prod_{k=1}^n 2^{\alpha_k(t-\ell_{kj})}. \tag{4.7}$$

It is not difficult to see that ω_v will satisfy the same derivative estimates, namely

$$|\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_n}^{\alpha_n}\omega_v(\xi)| \lesssim \left(\frac{v_1}{\|(v\xi)\|}\right)^{\alpha_1}\dots\left(\frac{v_n}{\|(v\xi)\|}\right)^{\alpha_n} \lesssim \prod_{k=1}^{n} 2^{\alpha_k(t-\ell_{kj})}. \tag{4.8}$$

Note that estimate (4.8) above was already implicitly used in the proof of Lemma 4.7. Finally, we have to consider the derivatives of $\xi \mapsto m(\xi\cdot v)$:

$$|\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_n}^{\alpha_n} m(\xi\cdot v)| \le |m^{(\alpha)}(v\cdot\xi)| v_1^{\alpha_1}\dots v_n^{\alpha_n} \lesssim \left(\frac{v_1}{|v\cdot\xi|}\right)^{\alpha_1}\dots\left(\frac{v_n}{|v\cdot\xi|}\right)^{\alpha_n}.$$

Observe that, since we are taking $\xi\in\operatorname{supp}(1-\omega_v)$, we have that

$$|v\cdot\xi| \ge \frac{1}{2n^2}\|(v\xi)\| \gtrsim |v_j\xi_j|$$

so that as before

$$|\partial_{\xi_1}^{\alpha_1}\dots\partial_{\xi_n}^{\alpha_n} m(v\cdot\xi)| \lesssim \prod_{k=1}^{n} 2^{\alpha_k(t-l_{kj})}. \tag{4.9}$$

Combining (4.7), (4.8), and (4.9) leads to the bound

$$|\Phi_v(x)| \lesssim \prod_{k=1}^{n} \frac{2^{t-\ell_{kj}}}{(1+2^{t-\ell_{kj}}|x_k|)^2},$$

whence

$$|T_v^{\text{out}} P_t^j f(x)| = |T_v^{\text{out}} Q_t^j P_t^j f(x)| = |\Phi_v * (P_t^j f)(x)| \lesssim \mathrm{M}_s(P_t^j f)(x)$$

as desired. □

Completing the proof of theorem 4.1

Recall the main splitting for T_v and the estimate for the inner part. We can then write, for each $V\subset\Omega$ with $\#V = N$, the estimate

$$\|T_V f\|_{L^2(w)} \le B[w]_{A_2^\Omega}^{\gamma} \sup_{\sigma\in\Sigma}\sup_{\ell\in\mathbb{Z}} \|T_{V_{\sigma,\ell}}\|_{L^2(w)} + \left\|\sup_{v\in V}|T_v^{\text{out}} f|\right\|_{L^2(w)},$$

where B denotes the implicit constant in the bound (4.5). Using weighted Littlewood-Paley theory, the Chang-Wilson-Wolff reduction of Lemma 4.7 and the pointwise estimate of Lemma 4.9, the second summand can be further estimated as follows:

$$
\begin{aligned}
\big\|\sup_{v\in V}|T_v^{\mathrm{out}}f|\big\|_{L^2(w)} &\lesssim [w]_{A_2^\Omega}^{\beta}\sup_{1\le j\le n}\Bigg\|\Big(\sum_{t\in\mathbb{Z}}P_t^j\Big(\sup_{v\in V}|T_v^{\mathrm{out}}N_v^j f|\Big)^2\Big)^{\frac12}\Bigg\|_{L^2(w)}\\
&\lesssim \sqrt{\log N}[w]_{A_2^\Omega}^{\beta'}\sup_{1\le j\le n}\Bigg\|\Big(\sum_{t\in\mathbb{Z}}\sup_{v\in V}|P_t^j(T_v^{\mathrm{out}}N_v^j f)|^2\Big)^{\frac12}\Bigg\|_{L^2(w)} \qquad (4.10)\\
&\lesssim \sqrt{\log N}[w]_{A_2^\Omega}^{\beta'}\sup_{1\le j\le n}\Bigg\|\Big(\sum_{t\in\mathbb{Z}}\mathrm{M}_s(P_t^j f)^2\Big)^{\frac12}\Bigg\|_{L^2(w)} \lesssim \sqrt{\log N}[w]_{A_2^\Omega}^{\beta'}\|f\|_{L^2(w)}.
\end{aligned}
$$

In the last approximate inequality we used the weighted vector-valued estimates for M_s and another application of weighted Littlewood-Paley theory.
Combining the estimates (4.5), (4.10), we realize that we have proved the following almost orthogonality principle for the maximal directional multiplier T_V.

Theorem 4.10. *Let $\Omega \subset \mathbb{S}^{n-1}$ be a set of directions which contains the coordinate directions. Then for all $w \in A_p^\Omega$ and every lacunary dissection $\{S_{\sigma,\ell}\}$ of $\mathbb{S}^{n-1}$ we have*

$$
\sup_{\substack{V\subseteq\Omega\\ \#V\le N}}\|T_V f\|_{L^2(w)} \le B[w]_{A_2^\Omega}^{\gamma}\Big(\sup_{\sigma\in\Sigma}\sup_{\ell\in\mathbb{Z}}\|T_{V_{\sigma,\ell}}\|_{L^2(w)}+\sqrt{\log N}\Big)\|f\|_{L^2(w)}
$$

for constants $B,\gamma>0$ depending upon dimension and the order of the lacunary dissection.

Our main result Theorem 4.1 may be easily derived from Theorem 4.10 by means of the following steps. First, Theorem 4.10 upgrades to the $L^2(w)$-estimate

$$
\sup_{\substack{V\subseteq\Omega\\ \#V\le N}}\|T_V f\|_{L^2(w)} \lesssim_L [w]_{A_2^\Omega}^{L\gamma}\sqrt{\log N}\|f\|_{L^2(w)}
$$

when $\Omega \subset \mathbb{S}^{n-1}$ is a lacunary set of order $L \ge 1$. This is obtained by induction on the order of lacunarity L. Indeed, the case $L = 0$ is immediate, as a 0-th order lacunary set contains exactly one direction. The inductive step follows by using the definition of lacunarity and the almost orthogonality principle of Theorem 4.10. Finally the $L^p(w)$-estimate of Theorem 4.1 for $p \in (1,\infty)$ is a consequence of the $L^2(w)$-estimate just proved and the extrapolation result of Proposition 1.17.

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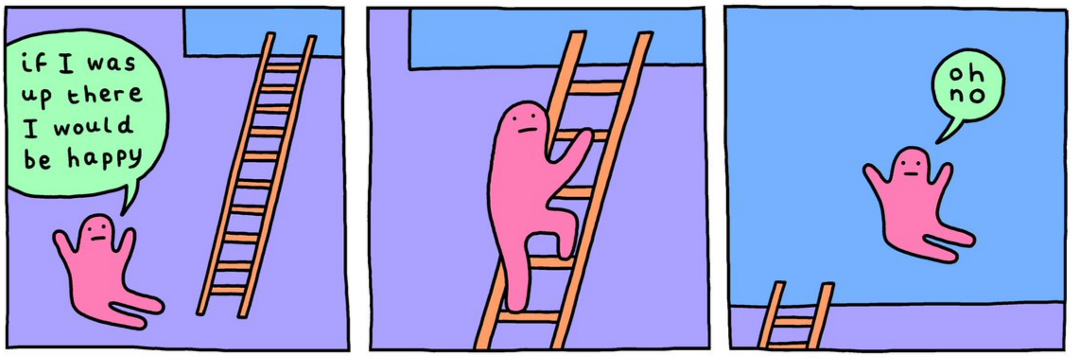
GOAL



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