11

The Nash Problem from Geometric and Topological Perspective

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We survey the proof of the Nash conjecture for surfaces and show how geometric and topological ideas developed in previous articles by the authors influenced it. Later, we summarize the main ideas in the higher dimensional statement and proof by de Fernex and Docampo. We end the paper by explaining later developments on generalized Nash problem and on Kollár and Nemethi holomorphic arcs.

1. Introduction

The Nash problem [**19**] was formulated in the 1960s (but published later) in an attempt to understand the relation between the structure of resolution of singularities of an algebraic variety X over a field of characteristic 0 and the space of arcs (germs of parameterized curves) in the variety. He proved that the space of arcs centered at the singular locus (endowed with an infinite-dimensional algebraic variety structure) has finitely many irreducible components and proposed to study the relation of these components with the essential irreducible components of the exceptional set of a resolution of singularities.

An irreducible component E_i of the exceptional divisor of a resolution of singularities is called *essential*, if given any other resolution, the birational transform of E_i to the second resolution is an irreducible component of the exceptional divisor. Nash defined a mapping from the set of irreducible components of the space of arcs centered at the singular locus to the set of essential components of a resolution as follows: he assigns to each component W of the space of arcs centered at the singular locus the unique component of the exceptional set which meets the lifting of a generic arc of W to the resolution. Nash established the injectivity of this mapping. For the case of surfaces, it seemed plausible for him that the mapping is also surjective and posed the problem as an open question. He also proposed to study the mapping in the higher dimensional case. Nash resolved the question positively for the surface A_k singularities and in analyzing the higher dimensional A_k singularities, he could not prove the bijectivity for A_4.

As a general reference for the Nash problem, the reader may look at [**19**, **9**].

Bijectivity of the Nash mapping was shown for many classes of surfaces (see [**6**, **9–11**, **14**, **15**, **18–20**, **22–24**, **26**, **28**, **29**]). The techniques leading to the proof of each of these cases are different in nature, and the proofs are often complicated. It is worthwhile to note that even for the case of the rational double points not solved by Nash a complete proof had to be awaited until 2011: see [**20**], where the problem is solved for any quotient surface singularity and also [**23**, **26**] for the cases of D_n and E_6. In [**3**], it is shown that the Nash problem for surfaces only depends on the topological type of the singularity. In 2012, the authors established in the affirmative the Nash question for the general surface case [**4**]. The proof we found was of a topological nature, and it is essential to work with convergent arcs and their convergent deformations. This motivated Kollár and Nemethi to pursue the study of convergent arcs and deformations in [**13**]. The topological ideas of [**3**, **20**] also had an impact on the generalized Nash problem; in [**5**], Popescu-Pampu and the authors show that the generalized Nash problem is of topological nature and explore the relation and applications of this problem to Arnol'd classical adjacency problem.

It is well known that birational geometry of surfaces is much simpler than in higher dimension. This fact reflects on the Nash problem: Ishii and Kollár showed in [**9**] a four-dimensional example with a non-bijective Nash mapping. In the same paper, they showed the bijectivity of the Nash mapping for toric singularities of arbitrary dimension. Other advances in the higher dimensional case include [**25**, **6**, **16**]. In 2013, de Fernex [**1**] found the first counterexamples to the Nash question; further counterexamples and a deeper understanding of how they appear were provided by Johnson and Kollár in [**7**]. There it was proved that the threefold A_4

$$x^2 + y^2 + z^2 + w^5 = 0,$$

the example that Nash left unfinished, was indeed a counterexample! In 2016, de Fernex and Docampo [**2**] proved that terminal divisors are at the image of the Nash map. Since at the surface case, terminal and essential divisors are precisely the same, this seems to be the correct higher dimensional generalization. It would, however, remain to be characterized which essential non-terminal divisors are at the image of the Nash map.

For other modern review articles concerning the Nash problem, the reader may consult [**27**, **12**, **8**].

In this chapter, we explain how geometric and topological techniques contributed to the development of the proof of the Nash conjecture and how they relate with other viewpoints and further developments.

Sections 2–5 explain our proof of the two-dimensional case in a non-technical way, pointing to the main new ideas appearing in it. We present a proof for the case in which the minimal resolution has a strict normal crossings exceptional divisor. In this case, all new essential ideas already appear, but the amount of technicalities can be reduced drastically. We include enough pictures so that the reader can grasp what is going on in an easy and intuitive way.

In Section 6, we emphasize the notion of returns, which was discovered in [**20**] and was crucial for the development of the general proof. We also take the opportunity to comment on deformation techniques that were useful to establish the hard cases of E_6, E_7 and E_8.

In Section 7, we explain the relation of our proof with the higher dimensional one of [**2**]. We do it by giving a short exposition of their proof that we believe condense all main ideas.

In Section 8, we summarize our contribution with Popescu-Pampu on the generalized Nash problem [**5**] and its impact on Arnol'd classical adjacency problem. Here, we use the techniques of [**3**] to show that the generalized Nash problem is of topological nature.

Finally, in Section 9, we explain the relation of our ideas with further developments of more geometric–topological nature by Kollár and Nemethi [**13**].

2. The Idea of the Proof for Surfaces

Let (X, O) be a surface singularity defined over an algebraically closed field of 0 characteristic. Let

$$\pi : (\tilde{X}, E) \to (X, O)$$

be the minimal resolution of singularities, which is an isomorphism outside the exceptional divisor $E := \pi^{-1}(O)$. Consider the decomposition $E = \bigcup_{i=0}^{r} E_i$ of E into irreducible components. These irreducible components are the essential components of (X, O).

Given any irreducible component E_i, we denote by N_{E_i} the Zariski closure in the arc space of X of the set of non-constant arcs whose lifting to the resolution is centered at E_i. These Zariski closed subsets are irreducible and each irreducible component of the space of arcs is equal to some N_{E_i} for a certain component E_i. The Nash mapping is the map assigning to each irreducible component N_{E_i} the exceptional divisor E_i. Injectivity is immediate. The Nash problem is about determining whether the Nash mapping is bijective.

The Nash mapping is not bijective if and only if there exist two different irreducible components E_i and E_j of the exceptional divisor of the minimal resolution, such that we have the inclusion $N_{E_i} \subset N_{E_j}$ (see [**19**]). Such inclusions were called *adjacencies* in [**3**].

An application of the Lefschetz principle allows one to reduce to the case in which the base field is $\mathbb{C}$. Details are provided in [**4**]. We make this assumption for the rest of the paper. Moreover, the case of a non-normal surface follows from the normal surface case easily (see [**4**, Section 6]). Then, we assume (X, O) to be a complex normal surface singularity.

The idea of the proof is as follows. We reason by contradiction. Let (X, O) be a normal surface singularity and

$$\pi : \tilde{X} \to (X, O)$$

be the minimal resolution of singularities. Assume that the Nash mapping is not bijective. Then, by a theorem of [**3**], there exists a convergent wedge

$$\alpha : (\mathbb{C}^2, O) \to (X, O)$$

with certain precise properties (see Definition 3.1). As in [**20**], taking a suitable representative, we may view α as a uniparameteric family of mappings

$$\alpha_s : \mathcal{U}_s \to (X, O)$$

from a family of domains $\mathcal{U}_s$ to X with the property that each $\mathcal{U}_s$ is diffeomorphic to a disk. For any s, we consider the lifting

$$\tilde{\alpha}_s : \mathcal{U}_s \to \tilde{X}$$

to the resolution. Note that $\tilde{\alpha}_s$ is the normalization mapping of the image curve.

On the other hand, if we denote by Y_s the image of $\tilde{\alpha}_s$ for $s \neq 0$, then we may consider the limit divisor Y_0 in $\tilde{X}$ when s approaches 0. This limit divisor consists of the union of the image of $\tilde{\alpha}_0$ and certain components of the exceptional divisor of the resolution whose multiplicities are easy to compute. We prove an upper bound for the Euler characteristic of the normalization of any reduced deformation of Y_0 in terms of the following data: the topology of Y_0, the multiplicities of its components and the set of intersection points of Y_0 with the generic member Y_s of the deformation. Using this bound, we show that the Euler characteristic of the normalization of Y_s is strictly smaller than one. This contradicts the fact that the normalization is a disk.

In the following three sections, we fill the details of the above sketch.

3. Turning the Problem into a Problem of Convergent Wedges

The germ (X, O) is embedded in an ambient space $\mathbb{C}^N$. Denote by B_ϵ the closed ball of radius ϵ centered at the origin and by $\mathbb{S}_\epsilon$ its boundary sphere. Take a *Milnor radius* ϵ_0 for (X, O) in $\mathbb{C}^N$, i.e., we choose $\epsilon_0 > 0$, such that for a certain representative X and any radius $0 < \epsilon \leq \epsilon_0$, we have that all the spheres $\mathbb{S}_\epsilon$ are transverse to X and $X \cap \mathbb{S}_\epsilon$ is a closed subset of $\mathbb{S}_\epsilon$ (see [**17**] for a proof of its existence). In particular, $X \cap B_{\epsilon_0}$ has conical structure. From now on, we will denote by X_{ϵ_0} the *Milnor representative* $X \cap B_{\epsilon_0}$ and by $\tilde{X}_{\epsilon_0}$ the resolution of singularities $\pi^{-1}(X_{\epsilon_0})$ (see Figure 1).

We recall some terminology and results from [**3**]. Consider coordinates (t, s) in the germ $(\mathbb{C}^2, O)$. A *convergent wedge* is a complex analytic germ

$$\alpha : (\mathbb{C}^2, O) \to (X, O),$$

which sends the line $V(t)$ to the origin O. Given a wedge α and a parameter value s, the arc

$$\alpha_s : (\mathbb{C}, 0) \to (X, O)$$

is defined by $\alpha_s(t) = \alpha(t, s)$. The arc α_0 is called *the special arc* of the wedge. For small enough $s \neq 0$, the arcs α_s are called *generic arcs*.

Any non-constant arc

$$\gamma : (\mathbb{C}, 0) \to (X, O)$$

admits a unique lifting to $(\tilde{X}, O)$ that we denote by $\tilde{\gamma}$.

Definition 3.1 ([**3**]). A convergent wedge α *realizes an adjacency* $N_{E_i} \subset N_{E_j}$ (with $j \neq i$) if and only if the lifting $\tilde{\alpha}_0$ of the special arc

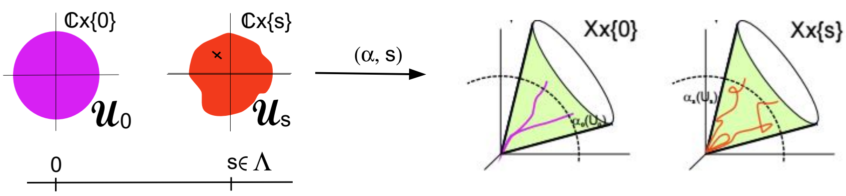

Figure 1. A wedge representative $\alpha : \mathcal{U} \to X \times \Lambda$ and the representatives $\alpha_0|_{\mathcal{U}_0}$ and $\alpha_s|_{\mathcal{U}_s}$.

meets E_i transversely at a non-singular point of E and the lifting $\tilde{\alpha}_s$ of a generic arc satisfies $\tilde{\alpha}_s(0) \in E_j$.

Our proof is based on the following theorem, which is the implication "(1)⇒(a)" of Corollary B of [**3**].

Theorem 3.2 ([**3**]). *An essential divisor E_i is in the image of the Nash mapping if there is no other essential divisor $E_j \neq E_i$, such that there exists a convergent wedge realizing the adjacency $N_{E_i} \subset N_{E_j}$.*

The proof in [**3**] of this theorem has two parts. The first consists of proving that if there is an adjacency, then there exists a *formal wedge*:

$$\alpha : \mathrm{Spec}(\mathbb{C}[[t, s]]) \to (X, O),$$

realizing the adjacency. For this, first, we use a theorem of Reguera [**30**], which produces wedges defined over large fields. Then, a specialization argument is performed to produce a wedge defined over the base field $\mathbb{C}$. This was done independently in [**16**]. The second part is an argument based on Popescu's Approximation Theorem, which produces the convergent wedge from the formal one.

Then, to prove the Nash Conjecture, we reason by contradiction and by Theorem 3.2, we assume that there exists a convergent wedge $\alpha : (\mathbb{C}^2, O) \to (X, O)$ that realizes some adjacency $N_{E_0} \subset N_{E_j}$.

4. Reduction to an Euler Characteristic Estimate

Following [**20**], we shall work with representatives rather than germs in order to get richer information about the geometry of the possible wedges.

Shrinking ϵ is necessary, we can choose a Milnor representative of α_0, say $\alpha_0|_U$, with U diffeomorphic to a disk, such that $\alpha_0|_U^{-1}(\partial X_\epsilon) = \partial U$ and the mapping $\alpha_0|_U$ is transverse to any sphere $\mathbb{S}_{\epsilon'}$ for any $0 < \epsilon' \leq \epsilon$.

Moreover, we can consider U, such that for a positive and small enough δ, the mapping α is defined in $U \times D_\delta$. Note also that we can assume $\alpha_0|_U$ injective and consequently $\alpha_s|_U$ generically one to one for s small enough (see [**4**] for details).

We consider the mapping

$$\beta : (\mathbb{C}^2, (0, 0)) \to (\mathbb{C}^N \times \mathbb{C}, (O, 0))$$

given by $\beta(t, s) := (\alpha(t, s), s)$ and its restriction

$$\beta|_{U \times D_\delta} : U \times D_\delta \to X \times D_\delta.$$

We denote by *pr* the projection of $U \times D_\delta$ onto the second factor.

The following lemma is proved using transversality arguments, together with Ehresmann Fibration Theorem, and the method is nowadays classical in Singularity Theory. Details are provided in [**4**].

LEMMA 4.1. *After possibly shrinking δ, we have that there exists $\epsilon > 0$, such that, defining*

$$\mathcal{U} := \beta|_{U\times D_\delta}^{-1}(X_\epsilon \times D_\delta),$$

we have the following:

(a) *the restriction $\beta|_{\mathcal{U}} : \mathcal{U} \to X_\epsilon \times D_\delta$ is a proper and finite morphism of analytic spaces*;
(b) *the set $\beta(\mathcal{U})$ is a two-dimensional closed analytic subset of $X_\epsilon \times D_\delta$*;
(c) *for any $s \in D_\delta$, the restriction $\beta|_{U\times\{s\}}$ is transverse to $\mathbb{S}_\epsilon \times \dot{D}_\delta$*;
(d) *the set $\mathcal{U}$ is a smooth manifold with boundary $\beta|_{\mathcal{U}}^{-1}(\partial X_\epsilon \times D_\delta)$*;
(e) *for any $s \in D_\delta$, the intersection $\mathcal{U} \cap (\mathbb{C} \times \{s\})$ is diffeomorphic to a disk.*

We will denote by $\mathcal{U}_s$ the fiber $pr|_{\mathcal{U}}^{-1}(s)$. The fact that every $\mathcal{U}_s$ is a disk is a key in the proof as it was in the final step of the proof of the main result of [**20**].

Now, we consider the image $H := \beta(\mathcal{U})$. For every $s \in D_\delta$, the fiber H_s, by the natural projection onto D_δ, is the image of the representative

$$\alpha_s|_{\mathcal{U}_s} : \mathcal{U}_s \to X_\epsilon.$$

Given the minimal resolution of singularities

$$\pi : \tilde{X}_\epsilon \to X_\epsilon,$$

we consider the mapping

$$\sigma : \tilde{X}_\epsilon \times D_\delta \to X_\epsilon \times D_\delta$$

defined by $\sigma(x, s) = (\pi(x), s)$. Note that the mapping σ is an isomorphism outside $E \times D_\delta$. We denote by Y the strict transform of H by σ in $\tilde{X}_\epsilon \times D_\delta$ that is the analytic Zariski closure in $\tilde{X}_\epsilon \times D_\delta$ of

$$\sigma^{-1}(H \setminus (\{O\} \times D_\delta)). \tag{4.2}$$

The space (4.2) is an irreducible surface, thus so is its closure Y. Since $\tilde{X}_\epsilon \times D_\delta$ is a smooth threefold, the surface Y considered with its reduced structure is a Cartier divisor (that is, a codimension 1 analytic subset whose sheaf of ideals is locally principal). We denote by Y_s the intersection $Y \cap (\tilde{X} \times \{s\})$.

The indeterminacy locus of the mapping $\sigma^{-1} \circ \beta|_{\mathcal{U}}$ has codimension 2, hence reducing ϵ and δ if necessary, we can assume that the origin $(0, 0) \in \mathcal{U}$ is the only indeterminacy point. Denote by

$$\tilde{\beta} : \mathcal{U} \setminus \{(0, 0)\} \to \tilde{X}_\epsilon \times D_\delta$$

the restriction of $\sigma^{-1} \circ \beta|_{\mathcal{U}}$ to its domain of definition $\mathcal{U} \setminus \{(0, 0)\}$. Observe that we have the equality

$$\tilde{\beta}(\mathcal{U} \setminus \beta^{-1}(\{O\} \times D_\delta)) = \sigma^{-1}(H \setminus (\{O\} \times D_\delta)).$$

Consequently, Y is the analytic Zariski closure of $\tilde{\beta}(\mathcal{U} \setminus \{(0,0)\})$ and moreover, we have the equality

$$Y \cap (\tilde{X}_\epsilon \times (D_\delta \setminus \{0\})) = \tilde{\beta}(\mathcal{U} \setminus \mathcal{U}_0). \tag{4.3}$$

For any $s \in D_\delta$, there exists a unique lifting

$$\tilde{\alpha}_s : \mathcal{U}_s \to \tilde{X}_\epsilon,$$

such that $\alpha_s = \pi \circ \tilde{\alpha}_s$. Obviously, for $s \neq 0$, we have the equality $\tilde{\beta}(t) = (\tilde{\alpha}_s(t), s)$ for any $t \in \mathcal{U}_s$. This, together with equality (4.3), implies the equality

$$Y_s = \tilde{\alpha}_s(\mathcal{U}_s). \tag{4.4}$$

Since Y is reduced, perhaps shrinking δ, we can assume that Y_s is reduced. Since α_s is proper and generically one to one, and $\mathcal{U}_s$ is smooth, we have that the mapping

$$\tilde{\alpha}_s : \mathcal{U}_s \to Y_s$$

is the normalization of Y_s.

Now, we describe the divisor Y_0. It is clear that all the components except $\tilde{\alpha}_0(\mathcal{U}_0)$ live above the origin that is the only indeterminacy point of $\tilde{\beta}$, i.e., the divisor Y_0 decomposes as a sum

$$Y_0 = Z_0 + \sum_{i=0}^{r} a_i E_i, \tag{4.5}$$

where we have denoted $Z_0 := \tilde{\alpha}_0(\mathcal{U}_0)$. This divisor Y_0 has the following properties (Figure 2):

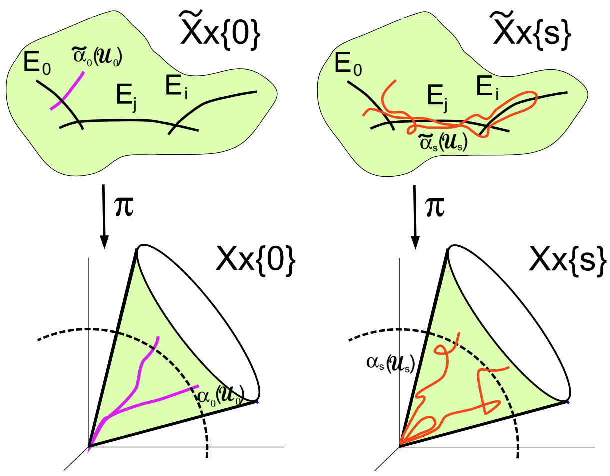

Figure 2. The lifting to the resolution of the special and generic arc of a wedge.

(i) it is reduced at $Z_0 \setminus E$ since σ is an isomorphism outside $E \times D_\delta$ and H_0 is reduced out of the origin;
(ii) Z_0 intersects transversely E_0 in a smooth point;
(iii) all the a_i's are non-negative since the divisor Y_0 is effective;
(iv) some a_i are strictly non-zero, in particular a_0, since α realizes an adjacency and then $\tilde{\beta}|_{\mathcal{U}}$ has indeterminacy.

Assuming the existence of a wedge realizing an adjacency, we have found a deformation Y_s of some Y_0 (as in (4.5) and satisfying (i)–(iv)) that has the following properties:

(a) Y_s is reduced for $s \neq 0$ small enough;
(b) its normalization, i.e., $\mathcal{U}_s$, is diffeomorphic to a disk;
(c) its boundary, i.e., $\tilde{\alpha}_s(\partial \mathcal{U}_s)$, is an $\mathbb{S}^1$ that degenerates to the boundary of Y_0, i.e., $\tilde{\alpha}_0(U) \cap \partial \tilde{X}_\epsilon$;
(d) Y_s meets $E_j \neq 0$.

The remaining part of the proof consists in proving that the Euler characteristic of the normalization of such a deformation Y_s of Y_0 is less than or equal to 0, which contradicts (b).

5. The Euler Characteristic Estimates

To simplify the computation of the Euler characteristic estimates, we assume the minimal resolution of (X, O) has as exceptional divisor a simple normal crossings divisor. This is the first case that we discovered. The general case is technically more elaborate, but follows essentially the same ideas. It may be checked in [**4**].

Let Y_0 be a Cartier divisor as in (4.5) that satisfies (i)–(iv). Consider a deformation Y_s of Y_0 satisfying (a)–(d). Let $n : \mathcal{U}_s \to Y_s$ be its normalization.

We consider a tubular neighborhood of Y_0 inside $\tilde{X}$ as a union of the following sets (Figures 3 and 4):

[h!]

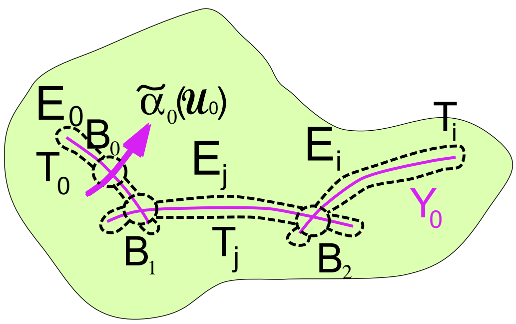

Figure 3. Adapted tubular neighborhood of Y_0.

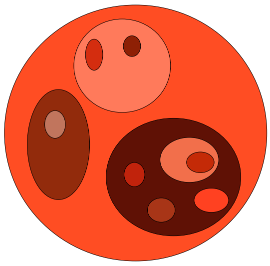

Figure 4. The normalization $\mathcal{U}_s$ of Y_s inside the tubular neighborhood of Y_0 is a disk. In the picture, we see the result of cutting Y_s along the boundary of the Milnor balls B_i around the normal crossings of Y_0. Each piece is either $n^{-1}(B_i)$ or $n^{-1}(T_j)$. The exterior piece that we call A satisfies that $n(A)$ is contained in B_0.

- a Milnor ball $B_0 := B(Z_0 \cap E, \epsilon_0)$ for Y_0 around the meeting point of Z_0 and E;
- Milnor balls $B_1, \ldots, B_k$ centered at each of the singular points of E^{red};
- tubular neighborhoods $T_1, \ldots, T_r$ contained in $\tilde{X} \setminus \bigcup_{j=0}^{k} B_j$ around each $E_i \setminus \bigcup_{j=0,\ldots,k} B_j$ such that there exist strong deformation retracts
$$\zeta_i : T_i \to E_i \Big\backslash \bigcup_{j=0,\ldots,k} B_j$$
(see [**4**] for technical details).

For s small enough, we have
$$Y_s \subset B_0 \cup \bigcup_{j=1}^{k} B_j \cup \bigcup_{i=0}^{r} T_i$$
By the choice of the Milnor balls, we have that for s small enough, we have transversality of Y_s and the boundaries of the B_j's and T_j's (see [**4**] for technical details).

We are going to give an estimate for $\chi(\mathcal{U}_s)$ splitting $\mathcal{U}_s$ as the union of $n^{-1}(B_j)$ and $n^{-1}(T_i)$. Note that $n^{-1}(B_j)$ and $n^{-1}(T_j)$ are respectively the normalization of $Y_s \cap B_j$ and $Y_s \cap T_j$. In particular, they are disjoint unions of Riemann surfaces with boundary. Since we know that $\mathcal{U}_s$ is a disk and B_i has transversal boundary with Y_s, we have a decomposition of $\mathcal{U}_s$ as in Figure 4. Furthermore,

$$\chi(\mathcal{U}_s) = \chi(n^{-1}(B_0)) + \sum_{j=1}^{k} \chi(n^{-1}(B_j)) + \sum_{i=0}^{r} \chi(n^{-1}(T_j)). \tag{5.1a}$$

We will separately give estimates for each of the summands on the right-hand side of the equality (Figure 6).

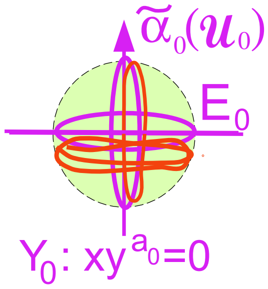

Figure 5. Counting the maximal number of disk images in $Y_s \cap B_0$ as a reduced deformation of $Y_0 \cap B_0$ of equation $xy^{a_0} = 0$.

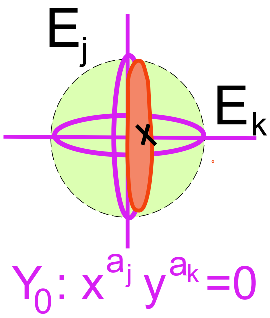

Figure 6. Counting the maximal number of disk images in $Y_s \cap B_i$ as a reduced deformation of $Y_0 \cap B_i$ of equation $x^{a_j} y^{a_k} = 0$.

5.1. Bound in B_0

The set $Y_0 \cap B_0$ is defined by $f_0(x,y) = xy^{a_0} = 0$, where x and y are the coordinates of B_0. The divisor $Y_s \cap B_0$ is defined by some deformation $f_s(x,y) = 0$, where f_s is a 1-parameter holomorphic deformation of f_0, such that f_s is reduced for $s \neq 0$.

We observe that $n^{-1}(B_0)$ is a disjoint union of Riemann surfaces with boundary. The only connected orientable surface with boundary which has positive Euler characteristic is the disk. Hence, $\chi(\mathcal{U}_s)$ is bounded above by the number of connected components of $n^{-1}(B_0)$ that are disks.

There are at the most as many disks in $n^{-1}(B_0)$ as boundary components in $n^{-1}(B_0)$ which are at the most $a_0 + 1$ since they degenerate to the boundary components of $\{xy^{a_0} = 0\} \cap B_0$. But by (c) the component of $n^{-1}(B_0)$ whose boundary degenerates to the boundary of $x = 0$ in B_0, say A, is in fact the exterior component of $\mathcal{U}_s$ (see Figure 4) which cannot be a disk, unless it is the whole disk (and this is not possible because Y_s goes outside B_0 and meets E_j by (d)). Then, A has more than one boundary component. Then, $n^{-1}(B_0)$ has at the most $a_0 - 1$ disks and we have

$$\chi(n^{-1}(B_0)) \leq \#disks \leq a_0 - 1. \tag{5.1b}$$

5.2. Bound in the balls B_i, $i \neq 0$

We have $Y_0 \cap B_i$ defined by $f_0(x, y) = x^{a_j} y^{a_k} = 0$, where x and y are the coordinates in B_i.

Again, the Euler characteristic of $n^{-1}(B_j)$ which is a disjoint union of Riemann surfaces with boundary is bounded from above by the number of disks. Whenever there is a disk D in $n^{-1}(B_i)$, since its boundary degenerates either to the boundary of $x^{a_j} = 0$ or $y^{a_k} = 0$ in B_i, we have that $n(D)$ will meet at least once either $y = 0$ or $x = 0$ in B_0. Then,

$$\chi(n^{-1}(B_i)) \leq \#Y_s \cap Y_0 \cap B_i \leq \sum_{p \in B_i} I_p(Y_s, E).$$

Summing up the estimates of all the balls B_i with $i \neq 0$, we have

$$\sum_{i=1}^{k} \chi(n^{-1}(B_i)) \leq Y_s \cdot E = \sum_{k} Y_s \cdot E_k.$$

Note that $Y_s \cdot E$ counts the returns (see Section 6) with multiplicity.

Now, we can use that the intersection multiplicity is stable by deformation, i.e., $Y_s \cdot E = Y_0 \cdot E$ to get

$$\sum_{i=1}^{k} \chi(n^{-1}(B_i)) \leq Y_s \cdot E = Y_0 \cdot E = \left(Z_0 + \sum_{i=0}^{r} a_i E_i\right) \cdot E$$

$$= 1 + \sum_{i,k=0,\ldots,r} a_i E_i \cdot E_k. \tag{5.1c}$$

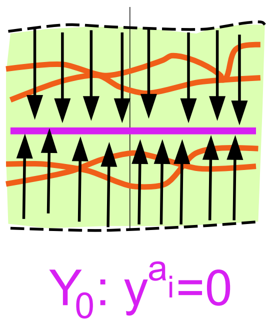

Figure 7. Bounding the Euler characteristic of the normalization of $Y_s \cap T_i$ as a reduced deformation of $Y_0 \cap T_i$ of equation $y^{a_i} = 0$.

5.3. Bound in every T_i

To estimate $\chi(n^{-1}(Y_s \cap T_i))$, we consider the composition

$$\zeta_i \circ n : n^{-1}(Y_s \cap T_i) \to E_i \setminus \bigcup_j B_j.$$

Although there are some technicalities to be taken into account, for Euler characteristic computations, one may think that it is a holomorphic branched cover of Riemann surfaces of degree a_i (the reader may find in [**4**] a completely detailed proof of this). Then, by the Riemann–Hurwitz formula, we get

$$\chi(n^{-1}(T_i)) \leq a_i \chi\left(E_i \setminus \bigcup_j B_j\right). \tag{5.1d}$$

We denote by g_i the genus of E_i. For $i \neq 0$, it is clear that $\sum_{k\neq j} E_i \cdot E_k$ counts the number of boundary components of $E_i \setminus \bigcup_j B_j$. The number of boundary components of $E_0 \setminus \bigcup B_i$ is $1 + \sum_{k\neq 0} E_0 \cdot E_k$ since E_0 meets also B_0.

Then, summing up the estimates (5.2) for all $i = 0, \dots, r$, we have

$$\begin{aligned} \sum_{i=0}^{r} \chi(n^{-1}(T_i)) \quad &\leq \quad a_0 \left(2 - 2g_0 - 1 - \sum_{k\neq 0} E_0 \cdot E_k\right) \\ &\quad + \sum_{j\neq 0} a_i \left(2 - 2g_i - \sum_{k\neq i} E_i \cdot E_k\right). \end{aligned} \tag{5.2}$$

5.4. Final estimate

Putting in (5.1a) the estimates (5.1b), (5.1c and (5.2), we get that

$$\chi(\mathcal{U}_s) \leq \sum_i a_i(2 - 2g_i + E_i \cdot E_i). \tag{5.1e}$$

By negative definiteness, for any $0 \leq i \leq r$, the self-intersection $E_i \cdot E_i$ is a negative integer. Observe that, since $\pi : \tilde{X} \to X$ is the minimal resolution, for any $0 \leq i \leq r$, if $E_i \cdot E_i$ is equal to -1, then either the divisor E_i is singular or it has positive genus (otherwise, it is a smooth rational divisor with self-intersection equal to -1 and the resolution is non-minimal by the Castelnuovo contractibility Criterion). Since we are assuming that every E_i is smooth, we get that $2 - 2g_i + E_i \cdot E_i \leq 0$ for all $i = 0, \dots, r$.

Remark 5.2. Note that the right-hand side is the adjunction formula in the surface case, which computes the degree of the relative canonical sheaf at each irreducible component of the exceptional divisor. This serves as an inspiration for the higher dimensional proof of de Fernex and Docampo [**2**].

Suppose that α is a wedge that does not lift to the minimal resolution. This is equivalent to the existence of indeterminacy of the mapping $\pi^{-1} \circ \alpha$. This implies the inequality $a_0 > 0$ (which is fact is equivalent) and bound (5.1b).

On the other hand, the reader may observe that if the wedge α lifts to the minimal resolution, then the arguments leading to the estimate (5.1b) break down.

The rest of the estimates appearing in our proof are valid in complete generality. So, we conclude the following.

Remark 5.3. Our proof shows that if α is a wedge such that the lifting to the minimal resolution of the special arc meets the exceptional divisor in a transverse way, then there is a lifting of α to the minimal resolution.

6. The Returns of a Wedge and Deformation Theoretic Ideas

Before the proof of the Nash conjecture in [**4**], the second author proved in her PhD the conjecture for the quotient surface singularities [**21**, **20**]. In that proof, it is shown that, despite the local nature of arcs, at least semi-local techniques were needed in order to study the arc space and the existence of certain families of arcs or wedges.

In particular, it was observed that for a representative $\alpha|_U$ of a wedge, returns might be non-avoidable. We recall that a *return* is a point in $\alpha_s^{-1}(0)$ different from the origin for $s \neq 0$. A return $p \in \alpha_s^{-1}(0)$ is identified with the associated arc that consists in viewing $\alpha_s|_{U_s}$ as a germ at p. If one thinks of an arc as the image of a parameterization "starting at the singular point", the returns are the points where the parameterization passes again through the singular point. The contribution of the returns is crucial in the Euler characteristic estimates needed in our proof of the Nash conjecture (see Section 5.2).

The study of returns had a direct impact in the application of the valuative criterion to rule out adjacencies $N_E \subset N_F$ in [**20**]. The valuative criterion was first studied in [**28**, **22**].Given an exceptional prime divisor D over $(X, 0)$, we denote by ord_D the associated divisorial valuation. The valuative criterion says that if there exists a germ $g \in \mathcal{O}_X$, such that $\mathrm{ord}_E(g) < \mathrm{ord}_F(g)$ where E and F are exceptional prime divisors, then the adjacency $N_E \subset N_F$ is not possible. Now, taking into account the returns, we can say that if we have an inequality $\mathrm{ord}_E(g) < \mathrm{ord}_F(g) + \mathrm{ord}_{F'}(g)$, then there is no wedge realizing the adjacency $N_E \subset N_F$ with a return with lifting by F' (nor a wedge realizing the adjacency $N_E \subset N_{F'}$ with a return by F).

This idea is applied in [**20**] more conveniently for the pullback of the wedges by the quotient map $q : (\mathbb{C}^2, 0) \to (X, 0)$ for a quotient surface singularity.

In [**4**], this criterion is completely understood in Section 3.2 as follows. Consider a wedge realizing an adjacency $N_{E_0} \subset N_{E_j}$ as in Section 4. Since the divisor Y_s defined in (4.3)–(4.4) is a deformation of the divisor Y_0, we have the equality

$$Y_0 \cdot E_i = Y_s \cdot E_i \tag{6.1}$$

for any i. Recall notation $Y_0 = Z_0 + \sum_i a_i E_i$. Denote by b_i the intersection product of $Y_s \cdot E_i$ and by M the matrix of the intersection form in $H_2(\tilde{X}_\epsilon, \mathbb{Z})$ with respect to the basis $\{[E_0], \ldots, [E_r]\}$. Then, (6.1) can be expressed as follows:

$$M(a_0, \ldots, a_r)^t = (1 - b_0, b_1, \ldots, b_r)^t. \tag{6.2}$$

The number b_i is the number of *returns* of the wedge through the divisor E_i counted with appropriate multiplicity.

An important observation is that all the entries of the inverse matrix M^{-1} are non-positive (see [**4**, Lemma 10]).

The equality (6.2) can be used to prove that wedges realizing certain adjacencies with certain prescribed returns b_i do not exist: the existence of such a wedge is impossible if the solution $a_0, \ldots, a_n$ of (6.2) has either a negative or a non-integral entry.

Moreover, to finish the proof in [**20**] for the E_8 singularities, further arguments using deformation theory were needed. There, wedges realizing an adjacency $N_{E_0} \subset N_{E_j}$ with a given special arc are seen as δ-constant deformations of the curve parameterized by the special arc. Then, the versal deformation of the curve parameterized by the special arc is computed. The codimensions of the δ-constant stratum and the codimension of the stratum of curves with the topological type of the generic curve of a family parameterized by a wedge representative with prescribed returns are computed. The inequality that these codimensions satisfy is not compatible with the existence of such a wedge (see [**20**, Proposition 4.5]).

7. The Proof by de Fernex and Docampo for the Higher Dimensional Case

De Fernex and Docampo figured out an algebro-geometric proof of the Nash conjecture based on bounds of coefficients of suitable relative canonical sheaves [**2**]. This enabled them to formulate and prove a correct statement of the Nash correspondence in higher dimension.

A *terminal valuation* is a divisorial valuation on X, such that there exists a terminal minimal model $\pi : Y \to X$ of X, such that the center of the valuation is a divisor in Y. The centers of terminal valuations are essential divisors. The main theorem of de Fernex and Docampo is as follows.

Theorem 7.1 (de Fernex, Docampo). *Terminal valuations are at the image of the Nash map.*

The beginning of the proof is similar to the surface case: assuming that the result is false, they derive the existence of a wedge, such that its special arc lifts to Y in a transverse way to the center of a terminal valuation, but that cannot be lifted to Y. Afterwards, assuming the existence of such a wedge, they bound a coefficient for a relative canonical sheaf in two different ways and produce a contradiction. So, in fact, they prove the following.

Theorem 7.2 (de Fernex, Docampo). *Let $\pi : Y \to X$ be a terminal model of X. Any wedge α, such that its special arc lifts to Y in a transverse way to the center of a terminal valuation, admits a lifting to Y.*

We refer to the original paper for a complete explanation of their proof. Here, instead, we explain the main ideas of the proof in the context of surfaces. In this case, $Y = \tilde{X}$ where $\tilde{X}$ is the unique terminal model which is the minimal resolution $\pi : \tilde{X} \to X$.

On the one hand, it is easier to digest, and all main ideas appear in this case. On the other hand, by doing it, we derive a precise set of numerical equalities (see (7.5)) that are satisfied for any non-constant wedge $\alpha : (\mathbb{C}^2, O) \to X$ not lifting to the minimal resolution and such that $\pi^{-1}\circ\alpha$ is a meromorphic map with an only indeterminacy point, regardless of whether this wedge has special arc lifting transversely or not (this means any wedge that is used in practice). This set of equalities have not been observed before. If one assumes that the special arc of the wedge lifts transversely, one may derive a chain of inequalities giving the contradiction in a straightforward way from this set of equalities.

Let $\alpha : (\mathbb{C}^2, O) \to X$ be any wedge so that $\pi^{-1}\circ\alpha$ is a rational map from $(\mathbb{C}^2, O)$ to $\tilde{X}$ and such that the special arc of the wedge. Let $\sigma : Z \to (\mathbb{C}^2, O)$ be the minimal sequence of blow-ups at points resolving the indetermination of $\pi^{-1}\circ\alpha$. Let $\beta : Z \to \tilde{X}$ be the map, such that $\pi\circ\beta = \alpha\circ\sigma$. De Fernex and Docampo shift the computation from $\tilde{X}$ to Z.

Let $F = \sum_{i=1}^{m} F_i$ and $E = \sum_{i=1}^{n} E_i$ be the decomposition in irreducible components of σ and π, respectively. In $(\mathbb{C}^2, O)$, we consider coordinates (t, s) so that t is the arc variable and s is the deformation parameter. The special arc of the wedge is then $\alpha(t, 0)$. We order the components so that F_1 is the unique component where the strict transform of $V(s)$ meets.

Denote by $K_Z = \sum_i a_i F_i$ the canonical divisor of Z. It is the only representative of the canonical class K_Z supported at the exceptional divisor. Each a_i is positive, and a simple observation on the behavior of the canonical divisor under blow-up shows the following important remark.

Remark 7.3. The number a_1 is the number of blowing-up centers touching the strict transform of $V(s)$.

Since β is a morphism between smooth spaces, the relative canonical class $K_{Z/\tilde{X}} := K_Z - \beta^* K_{\tilde{X}}$ may be represented by the divisor associated with the Jacobian of β. This is an effective divisor. When we write $K_{Z/\tilde{X}}$, we mean such a divisor. We decompose it as

$$K_{Z/\tilde{X}} = K^{\text{exc}}_{Z/\tilde{X}} + K^{\text{hor}}_{Z/\tilde{X}},$$

where $K^{\text{exc}}_{Z/\tilde{X}}$ is the part with support on the exceptional divisor of π and $K^{\text{hor}}_{Z/\tilde{X}}$ the complement. We have the equality

$$K_Z - K^{\text{exc}}_{Z/\tilde{X}} = K^{\text{hor}}_{Z/\tilde{X}} + \beta^* K_{\tilde{X}}. \tag{7.4}$$

The left-hand side is a divisor concentrated in the exceptional set of σ.

In order to express the right-hand side as a divisor concentrated in the exceptional set, we let M be the intersection matrix of the collection of divisors $\{F_i\}$ in Z. Since σ is a sequence of blow-ups, we have that M is unimodular and that its inverse M^{-1} is the matrix whose ith column $(m_{1,i}, \ldots, m_{n,i})^t$ is obtained as follows: let the C_i be the curve in $(\mathbb{C}^2, O)$ given by the image of σ of a cuvette transverse to F_i. Then, its total transform to Z is

$$\sigma^* C_i = \sum_j -m_{j,i} F_j.$$

As a consequence, we obtain that all the entries of M^{-1} are strictly negative (this is a general phenomenon which is well known, see, for example, [**4**, Lemma 10] for the proof for general normal surface singularities).

If we express $K^{\text{exc}}_{Z/\tilde{X}} = \sum_i b_i F_i$, and define the intersection numbers,

$$c_i := K^{\text{hor}}_{Z/\tilde{X}} \cdot F_i,$$

$$d_i := \beta^* K_{\tilde{X}} \cdot F_i,$$

Equality (7.4) becomes the following system of numerical equalities:

$$(a_1 - b_1, \ldots, a_n - b_n)^t = M^{-1}(c_1 + d_1, \ldots, c_n + d_n). \tag{7.5}$$

This numerical equality works for any resolution $\pi : \tilde{X} \to X$ (non-necessarily minimal) and any wedge α, so it may have applications in other problems. One instance could be the generalized Nash problem explained in Section 8.

Observe that, since $K^{\text{hor}}_{Z/\tilde{X}}$ has no component included in the exceptional divisor, each c_i is non-negative.

If we assume now that π is the minimal resolution, we have

$$d_i = \beta^* K_{\tilde{X}} \cdot F_i = K_{\tilde{X}} \cdot \beta_*(F_i),$$

which is non-negative by adjunction formula, using the fact that $\pi : \tilde{X} \to X$ is the minimal resolution.

This means that the right-hand side in equation (7.5) is *non-positive.*

In order to prove Theorem 7.2 for the surface case, we assume that the wedge has the special arc lifting transversely to the exceptional divisor and estimate the coefficient $a_1 - b_1$ on the left-hand side of equation (7.5).

By Remark 7.3, if the wedge α does not lift to $\tilde{X}$, then a_1 is strictly positive and integral. Since the right-hand side of equation (7.5) is non-positive, in order to finish the proof, it is enough to prove the strict inequality

$$b_1 < 1.$$

Let $\rho : Z \to Z'$ and $\beta' : Z' \to \tilde{X}$ be such that the factorization $\beta = \beta' \circ \rho$ consists in collapsing all non-dicritical components of the exceptional divisor in Z (a component F_i is non-dicritical if $\beta(F_i)$ is a point). The surface Z' has sandwiched singularities, which are rational and $\mathbb{Q}$-Gorenstein. Then, the canonical divisor $K_{Z'}$ is $\mathbb{Q}$-Cartier. Therefore, we may define the relative canonical class $K_{Z/Z'}$. This class has a unique representative as a $\mathbb{Q}$-divisor supported in the exceptional set of ρ, such that all its coefficients are non-positive.

We have the equality $K_{Z/\tilde{X}} = K_{Z/Z'} + \rho^* K_{Z'/\tilde{X}}$. By the non-positivity of the coefficients of $K_{Z/Z'}$ we get

$$b_1 = \mathrm{ord}_{F_1}(K_{Z/\tilde{X}}) = \mathrm{ord}_{F_1}(K_{Z/Z'} + \rho^* K_{Z'/\tilde{X}}) \leq \mathrm{ord}_{F_1}(\rho^* K_{Z'/\tilde{X}}).$$

We make an abuse of language and denote the components of the exceptional divisor of Z' by the same name that they have in Z. In order to estimate b_1, we enumerate $\{F_{i_1}, \ldots, F_{i_l}\}$ the components of the exceptional set of Z', which contain the image by ρ of F_1. This is a subset of the components of F not collapsed by ρ. Observe that if F_1 is dicritical, then this set of components has F_1 as a unique element, and the estimate that we will prove right away becomes much easier.

The special arc of the wedge α lifts transversely through an irreducible component of E. We enumerate the components so that this component is E_1. Then, for each of the components F_{i_j}, we have that $\beta(F_{i_j}) = E_1$.

Before proving our final estimate, we need the following observation:

$$\text{The following equality holds}: \ \mathrm{ord}_{F_{i_j}}(K_{Z'/\tilde{X}}) = \mathrm{ord}_{F_{i_j}}((\beta')^* E_1) - 1.$$

This holds because $K_{Z'/\tilde{X}}$ is given at smooth points by the divisor associated with the jacobian of β', and at a generic point of F_{i_j}, the mapping β can be expressed in local coordinates as $\beta(u, v) = (u^a, v)$, where $a = \mathrm{ord}_{F_{i_j}}((\beta')^* E_1)$.

The last estimate we need is

$$\begin{aligned} \mathrm{ord}_{F_1}(\rho^* K_{Z'/Y}) &= \mathrm{ord}_{F_1}\left(\sum_{j=1}^{l} \mathrm{ord}_{F_{i_j}}(K_{Z'/\tilde{X}}) \rho^* F_{i_j})\right) \\ &= \sum_{j=1}^{l} \mathrm{ord}_{F_1}(\mathrm{ord}_{F_{i_j}}((\beta')^* E_1) - 1) \rho^* F_{i_j}) \end{aligned}$$

$$< \sum_{j=1}^{l} \mathrm{ord}_{F_1}(\mathrm{ord}_{F_{i_j}}((\beta')^* E_1))\rho^* F_{i_j})$$
$$= \mathrm{ord}_{F_1}\rho^*(\beta')^* E_1 = \mathrm{ord}_{F_1}\beta^* E_1 = 1.$$

The last equality holds because the special arc lifts transversely by E_1. This concludes the proof.

8. The Generalized Nash Problem and the Classical Adjacency Problem

Let X be a normal surface singularity. *The Generalized Nash Problem* consists in characterizing the pair of divisors E, F appearing in resolutions of X, such that the adjacency $N_F \subset N_E$ holds.

For our proof of the Nash conjecture, it is essential to construct a holomorphic wedge α, as we have explained before. This was achieved in [**3**]. The technique developed to achieve this gave, as a byproduct, a proof of the fact that the validity of the Nash conjecture only depends on the topology of the link of the surface singularity, or equivalently, in the combinatorics of the minimal good resolution. The same technique could be adapted to prove that the generalized Nash problem is a topological problem in the following sense.

Since the generalized Nash problem is wide open even in the case in which X is smooth, we concentrate on this case. To any two exceptional divisors E and F of a sequence of blow-ups at the origin of X, we may associate a decorated graph as follows: consider the minimal sequence of blow-ups of $\pi : Y \to X$, where both E and F appear. Decorate the dual graph of the exceptional divisor of π attaching to each vertex the weight given by the self-intersection of the corresponding divisor. Finally, add labels E and F to the vertices corresponding to the divisors E and F, respectively. In [**5**], we proved the following.

Theorem 8.1. *Let (E_1, F_1) and (E_2, F_2) be two pairs of divisors having the same associated graph. Then, the adjacency $N_{F_1} \subset N_{E_1}$ is satisfied if and only if the adjacency, $N_{F_2} \subset N_{E_2}$ is satisfied.*

As a consequence, we could improve the discrepancy obstruction for adjacencies, see [**5**, Corollary 4.17 and 4.19]. Furthermore, we get a nice structure of nested Nash sets in the arc space of $\mathbb{C}^2$ and we made some conjectures about it, see [**5**, Conjectures 1 and 2].

For the sake of completeness, we summarize very briefly the other main result of [**5**].

Given a prime divisor E over the origin of X, we consider its associated valuation ν_E. It is easy to see that if we have the adjacency $N_F \subset N_E$, then the inequality $\nu_E \leq \nu_F$ holds. However, this criterion is not enough to characterize the Nash adjacencies (see Section 6). Our second main result is a characterization of the previous inequality in terms of deformations of plane curves.

We say that a plane curve germ C is *associated with* E in a model $\pi : Y \to \mathbb{C}^2$, where E appears if its strict transform by π meets E transversely at a point which does not meet the singular set of the exceptional divisor of π. A deformation g_s of function germs is a holomorphic function depending holomorphically on a parameter s. It is linear if it is of the form $g_0 + sh$ for g_0 and h holomorphic.

THEOREM 8.2. *Let E and F be prime divisors over the origin of $\mathbb{C}^2$ and let S be the minimal model containing both divisors. The following conditions are equivalent:*

(1) $\nu_E \leq \nu_F$.
(2) *There exists a deformation g_s with g_0 associated with F in S and g_s associated with E in S, for $s \neq 0$ small enough.*
(3) *There exists a linear deformation g_s with g_0 associated with F in S and g_s associated with E in S, for $s \neq 0$ small enough.*

In fact, in [**5**], we prove a more general version which allows non-prime divisors E and F.

This theorem provides a very easy way of producing adjacencies of plane curve singularities. Using this, we were able to recover most Arnol'd adjacencies, see [**5**, Section 3.4] for detailed explanations.

9. Holomorphic Arcs

In the proof of the Nash conjecture for surfaces, we study arcs and wedges from a convergent viewpoint and take representatives. In this sense, a wedge is for us a deformation of holomorphic maps from a disc to a representative of the singularity. At a generic parameter, the preimage of the singular point in general contains more points in the disc than just the origin. These points are unavoidable and we call them *returns* following [**20**] (see Section 6).

Kollár and Nemethi started in [**13**] the systematic study of convergent arc spaces as opposed to the classical formal arc spaces. We briefly summarize their main results and questions.

Let D denote the closed unit disk. A holomorphic map defined on D is the restriction to D of a holomorphic map in an open neighborhood of D. Let X be a singularity.

DEFINITION 9.1. A *complex analytic* arc is a holomorphic map $\gamma : D \to X$, such that the preimage of the singular set does not intersect the boundary ∂D. A *short complex analytic arc* is a complex analytic arc, such that the preimage of the singular set is just 1 point. A *deformation of a complex analytic arc* parameterized by an analytic space Λ is a holomorphic map

$$\alpha : D \times \Lambda \to X,$$

such that for any $s \in \Lambda$, the restriction $\alpha_s := \alpha|_{D\times\{s\}}$ is a complex analytic arc. If all the arcs appearing are short complex analytic arcs, we say that α is a *deformation of short complex analytic arcs*.

Remark 9.2. What we do in Section 4 is to derive from a convergent wedge a deformation of a complex analytic arc parameterized by a disk Λ. The special arc at this deformation is a short arc, but the generic arcs α_s are not short arcs, in general, due to the existence of returns. The topological analysis of that deformation of complex analytic arcs yields the proof of the Nash conjecture.

Denote by $\mathrm{Arc}(X)$ and $\mathrm{ShArc}(X)$ the sets of convergent analytic arcs in X. Kollár and Nemethi give natural metrics on these spaces, which endow them with a topology. Given an arc $\gamma : D \to X$, since the preimage of the singular set $\mathrm{Sing}(X)$ is disjoint from the circle ∂D, the restriction $\gamma|_{\partial D}$ defines an element of the fundamental group modulo conjugation $\pi_1(X \setminus \mathrm{Sing}(X))/(conjugation)$. Since this element does not change by continuous deformation of the arc γ, we have defined "winding number maps":

$$\pi_0(\mathrm{Arc}(X)) \to \pi_1(X \setminus \mathrm{Sing}(X))/(conjugation),$$

$$\pi_0(\mathrm{ShArc}(X)) \to \pi_1(X \setminus \mathrm{Sing}(X))/(conjugation).$$

The main result in [**13**] concerns short arcs:

Theorem 9.3 (Kollár, Nemethi). *The winding number map*

$$\pi_0(\mathrm{ShArc}(X)) \to \pi_1(X \setminus \mathrm{Sing}(X))/(conjugation)$$

is injective for any normal surface singularity X. It is bijective for quotient surface singularities.

For general normal surface singularities, the winding number map is far from being surjective, but its image is described in [**13**] in terms of the combinatorics of the resolution, or what is the same, the topology of the link.

On the other hand, the winding number map for long arcs, which is the one that is more related with the original Nash question, is not well understood.

Problem 9.4 (Kollár, Nemethi). Is the winding number map

$$\pi_0(\mathrm{Arc}(X)) \to \pi_1(X \setminus \mathrm{Sing}(X))/(conjugation)$$

injective?

In [**13**], many other open problems are proposed. Some interesting ones are concerned with the definition of a "finite type" holomorphic atlas in $\mathrm{Arc}(X)$ (see [**13**, Conjecture 72]) and with the existence of a curve selection lemma in $\mathrm{Arc}(X)$.

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