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Topologies for the continuous representability of every nontotal weakly continuous preorder

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Abstract Necessary and sufficient conditions on a topology t on an arbitrary set X are presented, under which every not necessarily total preorder, which in addition satisfies a general continuity condition, namely *weak continuity*, admits a continuous order-preserving real-valued function. Some interesting properties associated to this notions are studied.

Keywords Complete separable system · Useful topology · Strongly useful topology · Normal Hausdorff-space

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1 Introduction

Let (X, t) be a topological space. Then, in order for a preorder $\precsim$ on (X, t) to being representable by a continuous order-preserving real-valued function, it is necessary that for every pair $(x, y) \in \prec$ there exists a continuous increasing real-valued function $u_{xy} : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$ such that $u_{xy}(x) < u_{xy}(y)$. Therefore, in this paper a preorder $\precsim$ on (X, t) is defined to be *weakly continuous* if it satisfies the just defined monotony behaviour.

The concept of weak continuity of a preorder $\precsim$ on a topological space (X, t), which was introduced by Herden and Pallack [19] (see also Bosi and Herden [3,4]), is equivalent to requiring, for every pair $(x, y) \in \prec$, to exist a *decreasing complete separable system* $\mathcal{E}$ on (X, t) such that there exist sets $E \subset \overline{E} \subset E'$ in $\mathcal{E}$ such that $x \in E$ and $y \notin E'$ (see Bosi and Herden [5, Definition 2.1 and Definition 2.2]).

In this paper we are interested in studying conditions on a topology t on a set X, according to which every weakly continuous preorder on the topological space (X, t) admits a continuous *order-preserving* real-valued function u (i.e., a continuous increasing function $u : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$ such that $u(x) < u(y)$ for all pairs $(x, y) \in \prec$). Therefore, a topology t on X is said to be *strongly useful* if every weakly continuous preorder $\precsim$ on (X, t) admits a continuous order-preserving real-valued function. It should be noted that the concept of a strongly useful topology includes the consideration of preorders which are not *total*. The importance of considering nontotal preorders as more realistic representations of individual preferences has been recognized long ago in the seminal papers of Aumann [1] and Peleg [20].

It is immediate to check that a strongly useful topology t on X is also *useful*, i.e., every *continuous* total preorder $\precsim$ on (X, t) is representable by a continuous utility function u. We recall that the concept of a useful topology (also referred to as a *continuously representable topology* by other authors like Campión et al. [8–11] and Candeal et al. [12]) was first introduced by Herden [18], and recently studied by Bosi and Herden [5], who proved a simple result according to which a topology t on a set X is useful if and only if the topology $t_{\mathcal{E}}$ generated by every complete separable system $\mathcal{E}$ on (X, t) has a countable basis of open sets (i.e., $t_{\mathcal{E}}$ is *second countable*). Based on this characterization (see Bosi and Herden [5, Theorem 3.1]), the classical theorems of Eilenberg [16] and Debreu [14,15], according to which every continuous total preorder on a connected and separable topological space, and respectively on a second countable topological space, are representable by a continuous utility function, become immediate corollaries. The Eilenberg theorem has been recently discussed by Rébillé [21].

While in a metric space the concepts of a useful, strongly useful and separable (or equivalently second countable) topology are all equivalent, there are strongly useful topologies which are not second countable (see Example 3.3 below). Nevertheless, every topology on a countable set is strongly useful.

By defining a topology t on X to be *weakly preorderable* if it coincides with the coarsest topology $t^w_{\precsim}$ on X with respect to which some preorder $\precsim$ is weakly continuous, we prove that a topology t on X is strongly useful if and only if every weakly preorderable subtopology of t is second countable. Considering the weak topology $\sigma(X, C(X, t, \mathbb{R}))$ on X, i.e. the coarsest topology on X for which the continuous real-valued functions on (X, t) remain being continuous, we show that if t is a strongly useful topology on a set X, then $(X_{|\sim_C}, \sigma(X, C(X, t, \mathbb{R}))_{|\sim_C})$ is a normal Hausdorff-space, where $\sim_C$ stands for the equivalence on X represented by the coincidence of continuous functions.

2 Notation and preliminary results

A *preorder* $\precsim$ on a nonempty set X is a *reflexive* and *transitive* binary relation on X. A preorder is said to be *total* if, for all $x, y \in X$, either $x \precsim y$ or $y \precsim x$. The *strict part* (or *asymmetric part*) of a preorder $\precsim$ on X is defined as follows for all $x, y \in X$: $x \prec y$ if and only if $(x \precsim y)$ *and not*$(y \precsim x)$. Further, the *symmetric part* $\sim$ of a preorder $\precsim$ on X is defined as follows for all $x, y \in X$: $x \sim y$ if and only if $(x \precsim y)$ *and* $(y \precsim x)$. We have that $\sim$ is an *equivalence* on

X, and we denote by $X_{|\sim}$ the *quotient set*, made up by the equivalence classes $[x] = \{z \in X | z \sim x\}$ $(x \in X)$.

A subset D of a *preorderd set* $(X \precsim)$ is said to be *decreasing* if $(x \in D)$ and $(z \precsim x)$ imply $z \in D$, for all $z \in X$.

If t is a *topology* on X, then a family $\mathcal{B}' \subset t$ is said to be a *subbasis* of t if the family $\mathcal{B}$ consisting of all possible intersections of finitely many elements of $\mathcal{B}'$ is a *basis* of t (i.e., every set $O \in t$ is the union of some sets of $\mathcal{B}$).

A topology t on X is said to be *second countable* if there is a countable basis $\mathcal{B} = \{B_n | n \in \mathbb{N}^+\}$ for t. Further, a topology t on X is said to be *separable* if there exists a countable subset D of X such that $D \cap O \neq \emptyset$ for every $O \in t$.

We recall that, for two topologies t', t on set X, t' is said to be *coarser* (respectively, *finer*) than t if it happens that $t' \subset t$ $(t \subset t')$. If t' is coarser than t, we also say that t' is a *subtopology* of t.

We recall that a topological space (X, t) is said to be

(i) *Hausdorff* if, given any two points $x, y \in X$ with $x \neq y$, there exist two open disjoint sets U, V such that $x \in U$ and $y \in V$;

(ii) *completely regular* if, for every $x \in X$ and every closed set $F \subset X$ not containing x, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for $y \in F$;

(iii) *normal* if, for every pair (A, B) of disjoint closed subsets of X, there exist two open disjoint sets U, V such that $A \subset U$, $B \subset V$.

Definition 2.1 A preorder $\precsim$ on a topological space (X, t) is said to be *continuous* if the sets $d_{\precsim}(x) = \{z \in X | z \precsim x\}$ and $i_{\precsim}(x) = \{z \in X | x \precsim z\}$ are closed subsets of X for every $x \in X$.

Clearly, a total preorder $\precsim$ on a topological space (X, t) is continuous if and only if the following subsets of X are both open for every point $x \in X$:

$$l_{\precsim}(x) = \{y \in X | y \prec x\} = X \setminus i_{\precsim}(x),$$

$$r_{\precsim}(x) = \{y \in X | x \prec y\} = X \setminus d_{\precsim}(x).$$

Equivalently, this is the case when t is *finer* than the *order topology* $t^{\precsim}$ on X associated to $\precsim$, which is precisely the topology *generated* by the family $\{l_{\precsim}(x)|x \in X\} \cup \{r_{\precsim}(x)|x \in X\}$ (i.e., $\{l_{\precsim}(x)|x \in X\} \cup \{r_{\precsim}(x)|x \in X\}$ is a subbasis of t). In other words, $t^{\precsim}$ is the coarsest topology on X such that the sets $l_{\precsim}(x)$ and $r_{\precsim}(x)$ are open for every $x \in X$.

A topology t on X is said to be *useful* (see Herden [18]) if every continuous total preorder $\precsim$ on the topological space (X, t) has a *continuous utility representation* (*order preserving function*) u, i.e., there exists a continuous real-valued function u on the *totally preordered topological space* $(X, \precsim, t)$ such that $x \precsim y$ is equivalent to $u(x) \leq u(y)$ for all $x, y \in X$.

A real-valued function u on a preordered set $(X, \precsim)$ is said to be

1. *increasing*, if, for all $x, y \in X$,

$$x \precsim y \Rightarrow u(x) \leq u(y);$$

2. *order-preserving*, if u is increasing and, for all $x, y \in X$,

$$x \prec y \Rightarrow u(x) < u(y).$$

We shall denote by t_{nat} the *natural* (*interval*) topology on the real line $\mathbb{R}$.

The following definition is found in Herden and Pallack [19].

Definition 2.2 A preorder $\precsim$ on a topological space (X, t) is said to be *weakly continuous* if, for every pair $(x, y) \in \prec$, there exists a continuous and increasing real-valued function u_{xy} on X such that $u_{xy}(x) < u_{xy}(y)$.

Herden and Pallack [19, Proposition 2.8] proved that a preorder $\precsim$ on a topological space (X, t) is weakly continuous provided that it is continuous and, for every $x \in X$, $l_{\precsim}(x) = \{y \in X|y \prec x\}$ and $r_{\precsim}(x) = \{y \in X|x \prec y\}$ are open subsets of X. By the way, a famous Theorem by Schmeidler [22] guarantees that a preorder $\precsim$ on a connected topological space (X, t), which satisfies the previous conditions, and in addition is *non trivial* (i.e., there exist points $x, y \in X$ such that $x \prec y$), is total.

However, needless to say, there are weakly continuous preorders which are not continuous, as the following example shows (see also Bosi and Zuanon [7, Example 2.30], and Herden and Pallack [19, Example 2.9]).

Example 2.3 Let X be an arbitrary uncountable set. Then we consider the topology t on X that contains the empty set and the sets $X \setminus F$ where F runs through the empty set and all finite subsets of X. Define $Y := X_1 \bigoplus X_2$, the topological sum of the spaces X_1, X_2, both homeomorphic to X. Choose arbitrary infinite sets S_1 in X_1, and S_2 in X_2, whose complements in X_1, respectively X_2, are also infinite. Then, define the following preorder $\precsim$ on Y:

$$\precsim := \{(x,x) \mid x \in X\} \cup \{(u,v) \mid (u \in S_1) \textit{ and } (v \in S_2)\}.$$

We have, directly from the definitions of Y and $\precsim$, that $\precsim$ is weakly continuous, while, for every point $v \in S_2$, neither $d_{\precsim}(v)$ is closed nor $l_{\precsim}(v)$ is open, and for every point $u \in S_1$, neither $i_{\precsim}(u)$ is closed, nor $r_{\precsim}(u)$ is open. Hence, $\precsim$ is weakly continuous and not continuous.

Let us now present the basic definition of the *order topology* $t^{\precsim}$ corresponding to a preorder $\precsim$ on a set X.

Definition 2.4 The *order topology* $t^{\precsim}$ on X associated with a preorder $\precsim$ on X is defined to be the coarsest topology on X for which the sets $l_{\precsim}(x)$ and $r_{\precsim}(x)$ are open for every $x \in X$.

It is clear that the above definition generalizes the classical definition of the order topology associated to a total preorder.

We now recall the definition of a *complete separable system* on a topological space (X, t).

The following definition is found in Bosi and Herden [5].

Definition 2.5 Let a topology t on X be given. A family $\mathcal{E}$ of open subsets of the topological space (X, t) such that $\bigcup_{E \in \mathcal{E}} E = X$ is said to be a *complete separable system* on (X, t) if it satisfies the following conditions:

S1 : There exist sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$ such that $\overline{E_1} \subset E_2$.

S2 : For all sets $E_1 \in \mathcal{E}$ and $E_2 \in \mathcal{E}$ such that $\overline{E_1} \subset E_2$ there exists some set $E_3 \in \mathcal{E}$ such that $\overline{E_1} \subset E_3 \subset \overline{E_3} \subset E_2$.

S3 : For all sets $E \in \mathcal{E}$ and $E' \in \mathcal{E}$ at least one of the following conditions $E = E'$ or $\overline{E} \subset E'$ or $\overline{E'} \subset E$ holds.

In the case when $\precsim$ is a preorder on X, a complete separable system $\mathcal{E}$ on (X, t) is said to be *complete decreasing separable system* on $(X, \precsim, t)$ as soon as every set $E \in \mathcal{E}$ is required to be decreasing.

The following theorem holds, presenting different conditions all equivalent to the continuity of a total preorder on a topological space (see Bosi and Zuanon [7, Theorem 2.23]).

Theorem 2.6 *Let $(X, \precsim, t)$ be a totally preordered topological space. Then the following conditions are equivalent:*

(i) *$\precsim$ is continuous;*

(ii) *The order topology $t^{\precsim}$ is coarser than t;*

(iii) *$l_{\precsim}(x) = \{y \in X | y \prec x\}$ and $r_{\precsim}(x) = \{y \in X | x \prec y\}$ are both open subsets of X for every point $x \in X$;*

(iv) *$\precsim$ is weakly continuous;*

(v) *For every pair $(x, y) \in \prec$ a complete decreasing separable system $\mathcal{E}_{xy}$ on X can be chosen in such a way that there exist sets $E \subset \overline{E} \subset E'$ in $\mathcal{E}_{xy}$ such that $x \in E$ and $y \notin E'$.*

We recall that a topology t on a set X is said to be *preorderable* if $t = t^{\precsim}$ for some continuous total preorder $\precsim$ on (X, t) (see e.g. Campión et al. [10]).

Let us introduce the definition of a *weakly preorderable topology.*

Definition 2.7 A topology t on a set X is said to be *weakly preorderable* if it coincides with the coarsest topology $t^{w}_{\precsim}$ on X with respect to which some preorder $\precsim$ is weakly continuous on (X, t).

The following proposition is an immediate consequence of Theorem 2.6.

Proposition 2.8 *If a topology t on X is preorderable, then it is weakly preorderable.*

Let us now present the fundamental definition of a *strongly useful topology*, which was first introduced by Bosi [2], and Bosi and Isler [6].

Definition 2.9 A topology t on a nonempty set X is said to be *strongly useful* if every weakly continuous preorder on the topological space (X, t) admits a continuous order-preserving function $u : (X, \precsim, t) \to (\mathbb{R}, \leq, t_{nat})$.

Remark 2.10 It is simple to realize that a strongly useful topology is necessarily a useful topology. Indeed, it is clear that an order-preserving function $u : (X, \precsim) \to (\mathbb{R}, \leq)$ is actually a utility function in case that the preorder $\precsim$ on X is total. Further, the fact that a weakly continuous total preorder is continuous is guaranteed by the equivalence"(i) $\Leftrightarrow$ (iv)" of Theorem 2.6.

Herden and Pallack [19, Theorem 2.15] proved that every weakly continuous preorder on a second countable topological space admits a continuous order-preserving real-valued function, so that the famous *Debreu Theorem* (see Debreu [14,15]), according to which every second countable topology is useful, can be generalized as follows.

GDT: *Every second countable topology t on X is strongly useful.*

The following theorem holds, presenting a characterization of the existence of a continuous order-preserving function.

Theorem 2.11 *Let $\precsim$ be a preorder on a topological space (X, t). Then the following conditions are equivalent:*

(i) *There exists a continuous order-preserving function* $u : (X, \precsim, t) \rightarrow (\mathbb{R}, \leq, t_{nat})$*;*

(ii) *There exists a second countable topology* t' *on* X*, which is coarser than* t *and such that* $\precsim$ *is weakly continuous on* (X, t')*;*

(iii) $\precsim$ *is weakly continuous on* (X, t) *and the coarsest topology* $t^w_{\precsim}$ *on* X *with respect to which* $\precsim$ *is weakly continuous is second countable.*

Proof (i) $\Rightarrow$ (ii). See Bosi and Isler [6, Theorem 4.1, "(i) $\Leftrightarrow$ (ii)"].
(ii) $\Rightarrow$ (iii). Obvious.
(iii) $\Rightarrow$ (i). From the Generalized Debreu Theorem **GDT**, we have that there exists a continuous order-preserving function $u : (X, \precsim, t^w_{\precsim}) \rightarrow (\mathbb{R}, \leq, t_{nat})$, and clearly such function u is a also a continuous order-preserving function, $u : (X, \precsim, t) \rightarrow (\mathbb{R}, \leq, t_{nat})$. This consideration completes the proof. □

3 The basic structural results

From the above Generalized Debreu Theorem, and from Estévez-Hervés' Theorem (**EHT**) (see Estévez and Hervés [17] and Candeal et al. [12, Theorem 1]), according to which a metrizable topology t on X is useful if and only if t is second countable, or equivalently separable, (see also Bosi and Herden [5]), we immediately get the following characterization of metrizable strongly useful topologies (see Bosi [2, Theorem 3.1]).

Theorem 3.1 *The following conditions are equivalent on a metrizable topology* t *on a set* X*.*

(i) t *is separable.*

(ii) t *is strongly useful.*

(iii) t *is useful.*

Proof (i) $\Rightarrow$ (ii). Since a separable metrizable space is second countable, this implication is a direct consequence of the Generalized Debreu Theorem **GDT**.

(ii) $\Rightarrow$ (iii). See Remark 2.10.

(iii) $\Rightarrow$ (i). Candeal et al. [12, Theorem 1] proved that separability is a necessary and sufficient condition for usefulness of a metric topology. □

Proposition 3.2 *A topology t on a countable set X is strongly useful.*

Proof Let $\precsim$ be any weakly continuous preorder on (X,t) . Then the countability of X implies that there exist at most countably many pairs $(x_n, y_n) \in X \times X$ $(n \in \mathbb{N}^+)$ such that $x_n \prec y_n$. For every pair $(x_n, y_n) \in \prec$ there exists some continuous increasing function $u_n : (X, \precsim, t) \longrightarrow ([0,1], \leq, t_{nat})$ such that $u_n(x_n) < u_n(y_n)$. Hence, the function

$$u := \sum_{n \in \mathbb{N} \setminus \{0\}} \frac{1}{2^n} \cdot u_n,$$

$$u : (X, \precsim, t) \longrightarrow (\mathbb{R}, \leq, t_{nat})$$

is continuous and order-preserving. This means that t is strongly useful. □

Example 3.3 An example of a strongly useful topology t that is not second countable is in order now. Such an example is given by any normal Hausdorff-topology that is defined on a countable set X but fails to be second countable. The reader may think, for instance, of *Appert's topology* t_A that is defined on the positive integers $X := \mathbb{Z}^+$. For a detailed description of t_A the reader may consult Steen and Seebach [23]. The countability of X, of course, guarantees that t_A is strongly useful (see Proposition 3.2).

The following theorem holds, characterizing strongly useful topologies.

Theorem 3.4 *Let t be a topology on a set X. Then the following conditions are equivalent:*

(i) *t is strongly useful;*

(ii) *Every weakly preorderable subtopology of t is second countable.*

Proof (i) $\Rightarrow$ (ii). By contraposition, if there exists a weakly preorderable subtopology t' of t which is not second countable, then there exists some weakly continuous preorder $\precsim$ on (X,t) such that the coarsest topology $t^w_{\precsim}$ on X with respect to which $\precsim$ is weakly continuous is not second countable. Therefore Theorem 2.11 guarantees that there exists no continuous order-preserving function $u:(X,\precsim,t)\rightarrow(\mathbb{R},\leq,t_{nat})$, so that t is not strongly useful.

(ii) $\Rightarrow$ (i). Consider any weakly continuous preorder $\precsim$ on (X,t). Then the coarsest subtopology $t^w_{\precsim}$ of t on X with respect to which $\precsim$ is weakly continuous on (X,t) is a second countable, and therefore, by the Generalized Debreu Theorem, there exists a continuous order-preserving function $u:(X,\precsim,t^w_{\precsim})\rightarrow(\mathbb{R},\leq,t_{nat})$. Hence, $u:(X,\precsim,t)\rightarrow(\mathbb{R},\leq,t_{nat})$ is a continuous order-preserving function, and the proof is complete. □

Remark 3.5 Since every strongly useful topology is useful, and every total preorder is continuous if and only if it is weakly continuous, we have that the above Theorem 3.4 generalizes Theorem 5.1 in Campión et al. [10]. Indeed, this latter theorem states that a topology is useful if and only if every preorderable subtopology of it is second countable.

The reader may recall that, when we consider the space $C(X,t,\mathbb{R})$ of all continuous real-valued function on the topological space (X,t), the *weak topology* on X, $\sigma(X,C(X,t,\mathbb{R}))$, is the coarsest topology on X satisfying the property that every continuous real-valued function on (X,t) remains being continuous. Two points $x,y\in X$ are considered as being *equivalent* if $f(x)=f(y)$ for all functions $f\in C(X,t,\mathbb{R})$. For two equivalent points $x,y\in X$, we write $x\sim_{C(X,t,\mathbb{R})} y$ ($x\sim_C y$ for the sake of brevity).

It is well known that $(X_{|\sim_C},\sigma(X,C(X,t,\mathbb{R}))_{|\sim_C})$ is a completely regular Hausdorff-space (cf., for instance, Cigler, J., Reichel [13, Satz 10, page 101]). It is clear that $(X_{|\sim_C},\sigma(X,C(X,t,\mathbb{R}))_{|\sim_C})$ is the quotient space of $\sigma(X,C(X,t,\mathbb{R}))$ that is induced by the equivalence relation $\sim_C$.

The following lemma holds true.

Lemma 3.6 *The coarsest topology on X satisfying the property that all weakly continuous preorders on (X,t) remain being continuous is $\sigma(X, C(X,t,\mathbb{R}))$.*

Proof Since weak continuity of a preorder $\precsim$ on (X,t) is described by continuous (increasing) real-valued functions, the validity of the lemma is immediate. □

Of course, Lemma 3.6 is equivalent to the statement that a preorder $\precsim$ on (X,t) is weakly continuous if and only if it is weakly continuous with respect to $\sigma(X, C(X,t,\mathbb{R}))$.

From Bosi and Herden [4] (see also Bosi and Herden [3]), a topology t on a set X is said to satisfy the *Szpilrajn property* if every weakly continuous preorder $\precsim$ on the topological space (X,t) has a *continuous total refinement* (i.e., there exists a continuous total preorder $\lesssim$ on (X,t) such that $\precsim \subset \lesssim$ and $\prec \subset <$).

The immediate proof of the following proposition is left to the reader.

Proposition 3.7 *If a topology t on a set X is strongly useful, then it satisfies the Szpilrajn property.*

We now present an important property that any strongly useful topology must satisfy.

Proposition 3.8 *In order that t is strongly useful, it is necessary that $(X_{|\sim_C}, \sigma(X, C(X,t,\mathbb{R}))_{|\sim_C})$ is a normal Hausdorff-space.*

Proof: Let A and B respectively be two disjoint closed subsets of the topological space $(X_{|\sim}, \sigma(X, C(X))_{\sim_C})$. Then we may assume, without loss of generality, that both sets A and B are not open. Indeed, otherwise, nothing has to be shown. Since neither A nor B is open we may choose some fixed point $z \in X \setminus (A \cup B)$. We have already recalled that $(X_{|\sim_C}, \sigma(X, C(X,t,\mathbb{R}))_{|\sim_C})$ is a completely regular Hausdorff-space. This means that for every $a \in A$ there exists some continuous function $f_a : (X_{|\sim_C}, \sigma(X, C(X,t,\mathbb{R}))_{|\sim_C}) \rightarrow ([0,1], t_{nat})$ such that $f_a(a) = 0$ and $f_{a|B \cup \{z\}} = 1$ and that, conversely, for every $b \in B$

there exists some continuous function $f_b : (X_{|\sim_C}, \sigma(X, C(X,t,\mathbb{R}))_{|\sim_C}) \rightarrow ([0,1], t_{nat})$ such that $f_{b|A\cup\{z\}} = 0$ and $f_b(b) = 1$. The continuity of the functions f_a and f_b respectively implies that the preorder $\precsim$ on $(X_{|\sim_C}, \sigma(X, C(X,t,\mathbb{R}))_{|\sim_C})$ that is defined by setting $x \precsim y \Leftrightarrow f_a(x) \leq f_a(y)$ and $f_b(x) \leq f_b(y)$ for all $x \in X$ and all $y \in X$ and every $a \in A$ and every $b \in B$ is continuous. Since t and, thus, $\sigma(X, C(X,t,\mathbb{R}))_{|\sim_C}$ is strongly useful (cf. Lemma 3.6) it follows that there exists a total continuous preorder $\trianglelefteq$ on $(X_{|\sim_C}, \sigma(X, C(X,t,\mathbb{R}))_{|\sim_C})$ such that $\precsim \subset \trianglelefteq$ and $\prec \subset \triangleleft$. Hence, the definition of $\precsim$ allows us to conclude that $a \triangleleft z \triangleleft b$ for every $a \in A$ and every $b \in B$. The sets $U := \{u \in X | u \triangleleft z\}$ and $V := \{v \in X | z \triangleleft v\}$, therefore, are disjoint open subsets of $(X_{|\sim_C}, \sigma(X, C(X,t,\mathbb{R}))_{|\sim_C})$ such that $A \subset U$ and $B \subset V$. This last conclusion finishes the proof of the lemma. □

4 Conclusions

In this paper we have been concerned with the continuous representability of all weakly continuous preorders on a topological space. A characterization has been given of a topology t on a set X, namely a *strongly useful topology*, satisfying this property. We have presented some properties associated to such concept, whose interest is related to the fundamental observation according to which nontotal preorders (in general, nontotal binary relations) are more realistic representations of the individual preferences.

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