On a class of automorphisms in $\mathbf{H}^2$ which resemble the property of preserving volume

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Abstract
We give a possible extension of definition of shears and overshears in the case of two non commutative (quaternionic) variables in relation with the associated vector fields and flows. We define the divergence operator and determine the vector fields with divergence. Given the non-existence of quaternionic volume form on $\mathbf{H}^2$, we define automorphisms with volume to be time-one maps of vector fields with divergence and volume preserving automorphisms to be time-one maps of vector fields with divergence 0. To these two classes the Andersen–Lempert theory applies. Finally, we exhibit an example of a quaternionic automorphism, which is not in the closure of the set of finite compositions of volume preserving quaternionic shears even though its restriction to the complex subspace $\mathbf{C} \times \mathbf{C}$ is in the closure of the set of finite compositions of complex shears.

KEYWORDS
Andersen–Lempert theory, bidegree full functions, quaternions, slice regular function

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1 | INTRODUCTION

Complex holomorphic shears and overshears represent the major tools for the description of the groups of automorphisms of $\mathbf{C}^n$ with $n > 1$. In this paper, we give a possible extension for shears and overshears in the case of two non-commutative variables. In particular, we investigate what are the minimal conditions to define good generalizations of the complex holomorphic shears and overshears in relation with the associated vector fields and flows in the non commutative (mainly quaternionic) setting. To this end, we restrict our research to mappings represented by convergent quaternionic power series.

Complex analytic shears are simple automorphisms with volume 1. Since there does not exist a quaternionic volume form on $\mathbf{H}^n$, and since the automorphisms with convergent power series as components are not necessarily regular in the sense of [7], the class of quaternionic automorphisms with volume 1 is not defined.

We present an alternative definition of partial derivative, divergence and rotor for the quaternionic setting, and determine the subclasses of vector fields with divergence or rotor. Then, we define automorphisms with volume to be deformations of identity by vector fields with divergence, and we show that they present a proper class of automorphisms for which the Andersen–Lempert theory applies. In particular, shears and overshears in this class are the quaternionic analogue of complex holomorphic shears and overshears.

Finally, we exhibit an example of a quaternionic automorphism, which is not in the closure of the set of finite compositions of volume preserving quaternionic shears while its restriction to the complex variables is approximable by a finite composition of (complex) shears.

The paper is structured as follows: Section 2 contains the description of our setting with basic definitions and notions, such as partial derivatives, divergence, and rotor. Bidegree full functions are introduced. Section 3 is devoted to vector fields and their properties, in particular it contains the main theorem (Theorem 3.4) on vector fields with divergence. Section 4 studies the connections between Jacobians of shears and overshears and properties of the corresponding vector fields. Section 5 presents the application of Andersen–Lempert theory in quaternionic setting with the above-mentioned example.

2 | PRELIMINARIES ON CONVERGENT QUATERNIONIC POWER SERIES

In this section we introduce the basic concepts and notions to deal with generalizations of complex holomorphic shears and overshears, flows, and vector fields in the corresponding quaternionic setting. We denote by $\mathbf{H}$ the algebra of quaternions. Let $\mathbf{S}$ be the sphere of imaginary quaternions, i.e. the set of quaternions I such that $I^2 = -1$. Given any quaternion $z \notin \mathbf{R}$, there exist (and are uniquely determined) an imaginary unit I, and two real numbers x, y (with $y \geq 0$) such that $z = x + Iy$. With this notation, the conjugate of z will be $\bar{z} := x - Iy$. We consider the graded algebra of polynomials in the non commutative variables $z_1, \dots, z_n$. This algebra of polynomials will be denoted by $\mathbf{H}[z_1, \dots, z_n]$. In other words

$$\mathbf{H}[z_1, \dots, z_n] = \bigoplus_d \mathbf{H}_d[z_1, \dots, z_n]$$

where $\mathbf{H}_d[z_1, \dots, z_n]$ consists of finite linear combinations of monomials in the variables $z_1, \dots, z_n$ of degree d over the quaternions, namely monomials of the form

$$a_0 * a_1 * \dots * a_d, \ a_m \in \mathbf{H}, \quad \text{for all } \ m, \tag{2.1}$$

where each $*$ is replaced by one of the variables $z_1, \dots, z_n$. The space $\mathbf{H}_d[z_1, \dots, z_n]$ consists of all homogeneous polynomials in the variables $z_1, \dots, z_n$ of degree d over the quaternions. Equivalently, it is the class of polynomials of degree d in $4n$ real variables with quaternionic coefficients (see the Introduction in [5]). Our basic assumption on regularity, for the definition of the class of quaternionic functions we are interested in, is that any such function f has a series expansion of the form

$$f(z_1, \dots, z_n) = \sum_d f_d(z_1, \dots, z_n) \tag{2.2}$$

with $f_d(z_1, \dots, z_n) \in \mathbf{H}_d[z_1, \dots, z_n]$ for any d, which converges absolutely. The set of all such functions, which turns out to be a right or left $\mathbf{H}$-module, will be denoted by $\mathcal{H}[z_1, \dots, z_n]$. Actually, we can restrict our considerations to the case in which any $f_d(z_1, \dots, z_n)$ is a sum of monomials of degree d in the variables $z_1, \dots, z_n$ whose coefficients $a_0, \dots, a_{d-1}$ (using the same notation as in (2.1)) are all in $\mathbf{R}P^3 = S^3/\{-1, 1\}$, which can be identified with

$$\{x = x_0 + x_1 i + x_2 j + x_3 k, \|x\| = 1, x_0 > 0 \text{ or } x_0 = 0, x_1 > 0 \text{ or } x_0, x_1 = 0, x_2 > 0 \text{ or } x = k\}.$$

This fact guarantees formal uniqueness of the expansion in the right $\mathbf{H}$ module $\mathcal{H}[z_1, \dots, z_n]$. We assume the formal uniqueness of power series expansion of the functions considered, namely, two such functions are the same if and only if the corresponding power series coincide. Furthermore $\mathcal{H}[z_1, \dots, z_n]$ can be considered as a ring with respect to standard (pointwise) sum and (non commutative) multiplication.

We remark that $\mathcal{H}[z_1, \dots, z_n]$ contains, as a particular case, the right submodule of slice-regular functions $\mathcal{SR}$ as introduced in [7]. Another interesting subclass of functions in $\mathcal{H}[z_1, \dots, z_n]$ (which also contains slice-regular functions) is the one whose elements are functions as in (2.2) such that each of the unitary coefficients $a_0, \dots, a_{d-1}$ of f_d is exactly 1, or, to put it differently, series with coefficients on the right. This class will be denoted by $\mathcal{H}_{rhs}[z_1, \dots, z_n]$. In the case of one variable $z_1 = z$ the class $\mathcal{H}_{rhs}[z] = \mathcal{SR}$; the notation $\mathcal{SR}(D)$ refers to slice-regular functions defined on the open set $D \subset \mathbf{H}$.

In general, there is no standard way of introducing a notion of (partial) derivative for quaternionic functions (see for instance [6, 7]). The recent development of the theory of slice-regular quaternionic functions of several variables doesn't fit well with our purposes, for example even a composition $f \circ A$ with f slice-regular function of 2 variables and A a real nondiagonal 2×2 matrix is not slice-regular (see [7]). Therefore the set of slice-regular shear vector fields is not very large (Corollary 3.5).

We introduce new differential operators $\widehat{\partial}_{z_j}$ on $\mathcal{H}\left[z_1, \dots, z_n\right]$, which can be interpreted as new partial derivatives for a convergent power series as in (2.2) with respect to each of the variables $z_1, \dots, z_n$.

Definition 2.1. If f is a convergent power series of variables $z_1, \dots, z_n$, for a given $j \in \mathbf{N}, 1 \le j \le n$, and (sufficiently small) $h \in \mathbf{H}$, we say that $\widehat{\partial}_{z_j} f\left(z_1, \dots, z_n\right)[h]$ is to be defined by the position

$$f\left(z_1, \dots, z_j + h, \dots, z_n\right) - f\left(z_1, \dots, z_j, \dots, z_n\right) = \widehat{\partial}_{z_j} f\left(z_1, \dots, z_n\right)[h] + o(\|h\|), \tag{2.3}$$

or equivalently for $t \in \mathbf{R}$

$$\widehat{\partial}_{z_j} f\left(z_1, \dots, z_n\right)[h] = \lim_{t \to 0} \frac{1}{t}\left(f\left(z_1, \dots, z_j + th, \dots, z_n\right) - f\left(z_1, \dots, z_j, \dots, z_n\right)\right).$$

All the operators $\widehat{\partial}_{z_j}$ are additive and right-$\mathbf{H}$-linear as operators on functions, but $\widehat{\partial}_{z_j} f(z)[h] \neq h\, \widehat{\partial}_{z_j} f(z)[1]$. The partial derivative $\widehat{\partial}_{z_j} f(z)[h]$ is thus a first order nonlinear approximation (in h) of the difference in Equation (2.3). The Leibniz rule also holds.

In practice, each of the operators $\widehat{\partial}_{z_j}$ acts by replacing a prescribed variable in each monomial of f_d with $h \in \mathbf{H}$ as in the following example

$$\widehat{\partial}_{z_1}\left(z_1 z_2 z_1^2 z_2 a\right)[h] := \left(h z_2 z_1^2 z_2 + z_1 z_2 h z_1 z_2 + z_1 z_2 z_1 h z_2\right) a.$$

The following result, whose proof is somehow redundant, motivates the introduction of the differential operators $\widehat{\partial}_{z_j}$ on $\mathcal{H}\left[z_1, \dots, z_n\right]$.

Lemma 2.2. *If $\widehat{\partial}_{z_j} f\left(z_1, \dots, z_n\right) \equiv 0$, then $f\left(z_1, \dots, z_n\right)$ is (formally) independent of z_j.*

Remark 2.3. One can also define the (differential) operator

$$\widetilde{\partial}_{z_j} f\left(z_1, \dots, z_n\right) := \widehat{\partial}_{z_j} f\left(z_1, \dots, z_n\right)[1], \tag{2.4}$$

which coincides with the corresponding (Cullen) derivative, when f is a slice-regular functon. In short, the operator $\widetilde{\partial}_{z_j}$ replaces each z_j with 1.

However, a result like the one in Lemma 2.2 doesn't hold when considering $\widetilde{\partial}$ instead of $\widehat{\partial}$. Indeed,

$$\widetilde{\partial}_{z_1}\left(z_1 z_2 - z_2 z_1\right) = 0$$

but the function $f\left(z_1, z_2\right) = z_1 z_2 - z_2 z_1$ does not depend on z_2 only.

2.1 | Derivatives of mappings

Even though many of the following considerations can be given in a general formulation for $f \in \mathcal{H}\left[z_1, \dots, z_n\right]$, for the sake of clearness and to avoid complicated notations, we'll focus our attention to the two variable case and denote $z_1 = z, z_2 = w$.

Consider a mapping $F = \left(f_1, f_2\right)$, $f_1, f_2 \in \mathcal{H}[z, w]$ and define

$$DF(z, w)[h_1, h_2] := \begin{bmatrix} \widehat{\partial}_z f_1(z, w)[h_1] & \widehat{\partial}_w f_1(z, w)[h_2] \\ \widehat{\partial}_z f_2(z, w)[h_1] & \widehat{\partial}_w f_2(z, w)[h_2] \end{bmatrix}.$$

Let $G = \big(g_1, g_2\big)$, $g_1, g_2 \in \mathcal{H}[z, w]$, and write $(u, v) = G(z, w)$. If

$$DG(z,w)\big[h_1,h_2\big] = \begin{bmatrix} \widehat{\partial}_z g_1(z,w)\big[h_1\big] & \widehat{\partial}_w g_1(z,w)\big[h_2\big] \\ \widehat{\partial}_z g_2(z,w)\big[h_1\big] & \widehat{\partial}_w g_2(z,w)\big[h_2\big] \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$$

then we define the derivative of the composition as

$$D(F\circ G)(z,w)\big[h_1,h_2\big] = \begin{bmatrix} \widehat{\partial}_z f_1(u,v)\big[a_1\big] + \widehat{\partial}_w f_1(u,v)\big[a_2\big] & \widehat{\partial}_z f_1(u,v)\big[b_1\big] + \widehat{\partial}_w f_1(u,v)\big[b_2\big] \\ \widehat{\partial}_z f_2(u,v)\big[a_1\big] + \widehat{\partial}_w f_2(u,v)\big[a_2\big] & \widehat{\partial}_z f_2(u,v)\big[b_1\big] + \widehat{\partial}_w f_2(u,v)\big[b_2\big] \end{bmatrix}.$$

We introduce a new notation and write

$$D(F\circ G)(z,w)\big[h_1,h_2\big] = \begin{bmatrix} \widehat{\partial}_z f_1(u,v) & \widehat{\partial}_w f_1(u,v) \\ \widehat{\partial}_z f_2(u,v) & \widehat{\partial}_w f_2(u,v) \end{bmatrix} \diamond \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix} \tag{2.5}$$

so that

$$D(F\circ G)(z,w)\big[h_1,h_2\big] = DF(G(z,w)) \diamond DG(z,w)\big[h_1,h_2\big].$$

2.2 | Bidegree full functions (in two variables)

In each $\mathbf{H}_d[z,w]$, we consider the submodule $\mathbf{H}_{rhs,d}[z,w]$ whose elements are finite linear combinations of monomials of bidegree (p,q), $p,q \geq 0$, $p+q=d$ with respect to the variables z and w; they are all the monomials of total degree d formed considering p copies of z's and q copies of w's. There are $\binom{p+q}{p}$ such monomials and each of them can be represented by a string (called a *word*) $\alpha^{p,q} = \big(\alpha_1^{p,q}, \dots, \alpha_d^{p,q}\big) \in \big\{0,1\big\}^d$ such that

$$|\alpha^{p,q}| := \sum_{l=1}^{d} \alpha_l^{p,q} = p.$$

With this notation we can write

$$(z,w)^{\alpha^{p,q}} := \Big(z^{\alpha_1^{p,q}} w^{1-\alpha_1^{p,q}}\Big) \cdot \dots \cdot \Big(z^{\alpha_d^{p,q}} w^{1-\alpha_d^{p,q}}\Big).$$

Notice that, if $f(z,w) = \sum_d f_d(z,w) \in \mathcal{H}_{rhs}[z,w]$, then

$$f_d(z,w) = \sum_{\alpha^{p,q}} (z,w)^{\alpha^{p,q}} a_{\alpha^{p,q}}$$

with $p+q=d$.

Denote by

$$S_{p,q}(z,w) := \sum_{\substack{\alpha^{p,q}, \\ |\alpha^{p,q}|=p \\ p+q=d}} (z,w)^{\alpha^{p,q}}.$$

It is clear that $S_{p,q}(z,w) = S_{q,p}(w,z)$. We also have this important identity

$$\widehat{\partial}_z S_{p+1,q}(z,w)[h] = \widehat{\partial}_w S_{p,q+1}(z,w)[h]. \tag{2.6}$$

Let us remark that if z and w commute, then $S_{p,q}(z,w) = \binom{p+q}{p} z^p w^q$.

Proving that monomials of bidegree (p,q) are not just formally (right) linearly independent, but (right) linearly independent as functions, is a nontrivial problem. However, we can prove this fact for some cases.

Proposition 2.4. *Consider a polynomial of bidegree (p,q) with $p+q=d$ and either $q \leq 1$ or $p \leq 1$,*

$$P_{p,q}(z,w) = \sum_{\substack{\alpha^{p,q},\\ |\alpha^{p,q}|=p}} (z,w)^{\alpha^{p,q}} a_{\alpha^{p,q}}.$$

If $P_{p,q}(z,w) \equiv 0$ then necessarily $a_{\alpha^{p,q}} = 0$ for any $\alpha^{p,q}$.

Proof. The cases $p=0$ or $q=0$ are trivial. If $q=1$ then we can use a simpler notation and write

$$P_{p,1}(z,w) = \sum_{n=0}^{d-1} z^n w z^{d-n} a_n.$$

If $P_{p,1}(z,w) \equiv 0$, then in particular $P_{p,1}(z,w) = 0$ for $z = x + Iy$ and $w = J \in \mathbf{S}$ an imaginary unit orthogonal to I such that $\{I, J, IJ\}$ is an orthonormal basis of $\mathbf{R}^3$. In particular, this choice of J implies that $zw = w\bar{z}$. Hence

$$0 = P_{p,1}(z,w) = w \sum_{n=0}^{d} \bar{z}^n z^{d-n} a_n;$$

since $w = J \neq 0$, it follows that

$$\sum_{n=0}^{d} \bar{z}^n z^{d-n} a_n \equiv 0$$

for any choice of $x, y \in \mathbf{R}$ or $z \in \mathbf{C}_I := \{z = x + Iy \mid x, y \in \mathbf{R}\} \simeq \mathbf{C}$. Since for any n it turns out that $a_n = u_n + v_n J$ with $u_n, v_n \in \mathbf{C}_I$, then $\sum_{n=0}^{d} \bar{z}^n z^{d-n} a_n = 0$ splits into two independent conditions (on $\mathbf{C}_I$), namely $\sum_{n=0}^{d} \bar{z}^n z^{d-n} u_n = 0$ and $\sum_{n=0}^{d} \bar{z}^n z^{d-n} v_n = 0$; from the Identity Principle for complex polynomials, we conclude, that $u_n = 0$ and $v_n = 0$ for any n and so $a_n = 0$ for $n = 0, \dots, d$. For a general statement about linear dependence of monomials we refer the interested reader to [8] □

Definition 2.5. We define

$$\mathbf{H}_d^{BF}[z,w] := \left\{ \sum_{p+q=d} S_{p,q}(z,w) a_{p,q},\ a_{p,q} \in \mathbf{H} \right\}$$

and

$$\mathbf{H}^{BF}[z,w] := \bigoplus_d \mathbf{H}_d^{BF}[z,w].$$

We say that $\mathbf{H}^{BF}[z,w]$ is the right module of *bidegree full* (in short BF) polynomials in the variables z, w. Similarly, we define the right module of *bidegree full functions* to consist of converging power series of the form

$$f(z,w) = \sum_{d=0}^{\infty} f_d(z,w),$$

with $f_d(z,w) \in \mathbf{H}_d^{BF}[z,w]$ and denote it by $\mathcal{H}^{BF}[z,w]$.

The following result shows that bidegree full polynomials form an interesting class of polynomials.

Lemma 2.6. *For any real number μ and any $d \in \mathbf{N}$, the polynomial $(z-\mu w)^d := \overbrace{(z-\mu w)\cdots(z-\mu w)}^{d \text{ times}}$ is bidegree full. If*

$$P(z,w) = \sum_{d=0}^{l} \sum_{\substack{p,q\geq 0,\\ p+q=d}} S_{p,q}(z,w)a_{p,q}$$

is a bidegree full polynomial of degree l, then it also has a decomposition

$$P(z,w) = \sum_{d=0}^{l} \sum_{p+q=d} \left(\sum_{n=0}^{d} (z-nw)^d r_{p,d}(n) \right) a_{p,q}, \ with \ r_{p,d}(n) \in \mathbf{R}. \tag{2.7}$$

Proof. Indeed, from direct calculations, it follows that

$$(z-\mu w)^d = (z-\mu w)\cdot \ldots \cdot (z-\mu w) = \sum_{\substack{p,q\geq 0,\\ p+q=d}} S_{p,q}(z,w)(-\mu)^q.$$

The second statement follows from the fact (proved in [1] by induction on d with an argument which applies to our setting) that the polynomials $\{x^d, (x-1)^d, \ldots, (x-d)^d\}$ form a basis of real polynomials of order less or equal to d and consequently polynomials $z^d, (z-w)^d, \ldots, (z-dw)^d$ form a basis of $\mathbf{H}_d^{BF}[z,w]$. $\square$

Analogously as above, we define $(z-\mu w)^d := \overbrace{(z-\mu w)\cdots(z-\mu w)}^{d \text{ times}}$ also for arbitrary $\mu \in \mathbf{H}$. But the "chain rule"

$$\widehat{\partial}_w (z-\mu w)^d = -\mu \widehat{\partial}_z (z-\mu w)^d \tag{2.8}$$

holds if and only if $\mu \in \mathbf{R}$.

Remark 2.7. As a consequence of Lemma 2.6, from any convergent quaternionic power series in the variable u of the form

$$u \mapsto \sum_d u^d a_d$$

(which actually is a slice-regular function of u) one gets a bidegree full function by replacing u with $z-\mu w$, namely

$$f(z,w) = \sum_d (z-\mu w)^d a_d \in \mathcal{H}^{BF}[z,w];$$

this function is not a slice-regular function in the variables z and w.

2.3 | Generalizations of bidegree full functions

The generators $z^d, (z-w)^d, \ldots, (z-dw)^d$ of $\mathbf{H}_d^{BF}[z,w]$ were obtained by precomposing the monomial u^d by functions $u = z - nw$ for $n = 0, \ldots, d$.

Similarly, given $a = (a_1, \ldots, a_d)$, one can consider the monomial of degree d in variable u of the form

$$a_0 u a_1 u \ldots a_d u.$$

Precomposing it by functions $u = z - nw$ for $n = 0, \dots, d$, one obtains generators of the right module of generalized BF polynomials of degree d denoted by $\mathbf{H}_d^{BF,a}[z,w]$.

Another possible generalization is to consider the precompositions of the slice-regular functions $f(u) = \sum_d u^d a_d$ by $u = z - \mu w$ as in Lemma 2.6 with $\mu \in \mathbf{H}$,

$$f(z - \mu w) = \sum_d (z - \mu w)^n a_n, \quad a_n \in \mathbf{H}.$$

These functions have the geometric property of leaving invariant quaternionic parallel affine subsets along the direction $(\mu, 1)$ as explained in the next

Definition 2.8. Given $\mu \in \mathbf{H}$, we say that a quaternionic function f of the variables z, w is $(\mu, 1)$-*right-invariant* if

$$f(z,w) = f(z + \mu s, w + s) = f((z,w) + (\mu, 1)s) = f(z - \mu w, 0)$$

for any z, w and any $s \in \mathbf{H}$.

3 | QUATERNIONIC VECTOR FIELDS IN TWO VARIABLES

In this section, using the definition of $\hat{\partial}$, we develop some analytic tools such as divergence, rotor, and flow for quaternionic vector fields in two variables. We show that there is a large class of vector fields with good analyticity properties.

Definition 3.1. Given $f, g \in \mathcal{H}[z,w]$, the mapping $X(z,w) = (f(z,w), g(z,w))$ is called a *vector field* in $\mathbf{H}^2$, in short we write $X \in \mathcal{VH}$. The subset of vector fields $X = (f,g)$ with $f, g \in \mathcal{H}_{rhs}[z,w]$ is denoted by $\mathcal{VH}_{rhs}$. In particular, we say, that a vector field $X = (f,g)$ is *bidegree full* (in short BF) if the functions f, g are bidegree full functions and use the notation $X \in \mathcal{VH}^{BF}$. We assume from now on that the vector fields and functions are all defined on $\mathbf{H}^2$.

Next we introduce the following definition.

Definition 3.2. Given the vector field $X(z,w) = (f(z,w), g(z,w))$, we define the differential operator

$$\mathrm{Div}X(z,w)[h] := \hat{\partial}_z f(z,w)[h] + \hat{\partial}_w g(z,w)[h]$$

and we say that the vector field X has *divergence* if $\mathrm{Div}X(z,w)[h]$ is left h-linear, i.e. if there exists a function, which will be denoted by $\mathrm{div}X(z,w)$, such that

$$\mathrm{Div}X(z,w)[h] = h\,\mathrm{div}X(z,w).$$

Example 3.3. The vector field $\left(zw + wz, -w^2\right)$ has divergence zero,

$$\mathrm{Div}\left(zw + wz, -w^2\right)[h] = hw + wh - (hw + wh) = 0,$$

while the vector field $\left(z^2 w, -zw^2\right)$ does not have divergence, since the operator

$$\mathrm{Div}\left(z^2 w, -zw^2\right)[h] = (hz + zh)w - z(hw + wh) = hzw - zwh$$

is not left linear in h.

One of the main reasons for the introduction of the operators $\hat{\partial}_z, \hat{\partial}_w$ and Div is the following.

Theorem 3.4. *Let $X(z,w) = (f(z,w), g(z,w)) \in \mathcal{VH}_{rhs}$ be a vector field with divergence. Then* $\mathrm{div}X(z,w)$ *is BF. If* $\mathrm{div}X(z,w) = 0$ *then X is BF.*

Proof. To simplify the notation write $\mathrm{div}X(z,w) = \Delta(z,w)$. Let $f(z,w) = \sum f_{p,q}(z,w), g(z,w) = \sum g_{p,q}(z,w)$ and $\Delta(z,w) = \sum \Delta_{p,q}(z,w)$ be the decompositions of f, g and Δ with respect to the bidegrees. Then $\mathrm{Div}X(z,w)[h] = h\Delta(z,w)$, if and only if

$$\widehat{\partial}_z f_{p+1,q}(z,w)[h] + \widehat{\partial}_w g_{p,q+1}(z,w)[h] = h\Delta_{p,q}(z,w) \tag{3.1}$$

for $p, q \geq 0$. We have two more equations, which always hold, namely,

$$\widehat{\partial}_z f_{0,q}(z,w)[h] = 0 \text{ and } \widehat{\partial}_w g_{p,0}(z,w)[h] = 0.$$

Write

$$f_{p+1,q}(z,w) = z \sum_{\substack{\alpha_1 \in \{0,1\}^{p+q}, \\ |\alpha_1| = p}} (z,w)^{\alpha_1} A_{\alpha_1} + w \sum_{\substack{\alpha_2 \in \{0,1\}^{p+q}, \\ |\alpha_2| = p+1}} (z,w)^{\alpha_2} A_{\alpha_2},$$

$$g_{p,q+1}(z,w) = z \sum_{\substack{\beta_1 \in \{0,1\}^{p+q}, \\ |\beta_1| = p-1}} (z,w)^{\beta_1} B_{\beta_1} + w \sum_{\substack{\beta_2 \in \{0,1\}^{p+q}, \\ |\beta_2| = p}} (z,w)^{\beta_2} B_{\beta_2}.$$

Since divergence is left linear in h, all the terms in the derivative coming from the second sum for $f_{p+1,q}$ (similarly for the first sum for $g_{p,q+1}$) should cancel out. Since the terms in the expression $\widehat{\partial}_z f_{p+1,q}(z,w)[h]$ are formally linearly independent, the only possibility is, that such a term is cancelled out by a term in $\widehat{\partial}_z g_{p,q+1}(z,w)[h]$. Consider a monomial from the second sum whose associated word is of the form $0\alpha_2 = 0\alpha_2^1 1\alpha_2^2 0\alpha_2^3$. Then $\widehat{\partial}_z(z,w)^{0\alpha_2}[h]$ has a monomial of the form $w \cdot (z,w)^{\alpha_2^1} \cdot h \cdot (z,w)^{\alpha_2^2} \cdot w \cdot (z,w)^{\alpha_2^3}$, so it can be canceled out only by a term in $\widehat{\partial}_w(z,w)^{\beta}[h]$ for $\beta = 0\alpha_2^1 0\alpha_2^2 0\alpha_2^3$. Since there is another zero, the above derivative contains also a term $w \cdot (z,w)^{\alpha_2^1} \cdot w \cdot (z,w)^{\alpha_2^2} \cdot h \cdot (z,w)^{\alpha_2^3}$, and this one can be canceled only by a term from $\widehat{\partial}_z(z,w)^{\alpha}[h]$ for $\alpha = 0\alpha_2^1 0\alpha_2^2 1\alpha_2^3 = 0\tilde{\alpha}_2$. The sequences α_2 and $\tilde{\alpha}_2$ differ only by a transposition. So, if both α_2 and $\tilde{\alpha}_2$ with $|\alpha| = |\alpha_2| = p+1$ contain at least one 1 (which is the case) and one 0, they differ by a sequence of transpositions and therefore $A_{\alpha_2} = A_{\tilde{\alpha}_2}$. So, there exist A such that

$$A = A_{\alpha_2} = A_{\tilde{\alpha}_2} = -B_{\beta_2}, \qquad \text{for all } \alpha_2, \beta_2,$$

provided $q \geq 2$ (and $p+1 \geq 1$). Analogously, there exist B such that

$$B = -B_{\beta_1} = A_{1\alpha_1}, \text{ for all } \alpha_1, \beta_1,$$

if $q \geq 1$ and $p \geq 1$. Then

$$\begin{aligned}
\big(f_{p+1,q}(z,w), g_{p,q+1}(z,w)\big) &= \big(zS_{p,q}(z,w)B + wS_{p+1,q-1}(z,w)A, -zS_{p-1,q+1}(z,w)B - wS_{p,q}(z,w)A\big) \\
&= z\big(S_{p,q}(z,w), -S_{p-1,q+1}(z,w)\big)B + w\big(S_{p+1,q-1}(z,w), -S_{p,q}(z,w)\big)A,
\end{aligned}$$

and

$$\begin{aligned}
h\Delta_{p,q}(z,w) = {} & hS_{p,q}(z,w)B + z\widehat{\partial}_z S_{p,q}(z,w)[h]B + w\widehat{\partial}_z S_{p+1,q-1}(z,w)[h]A \\
& -z\widehat{\partial}_w S_{p-1,q+1}(z,w)[h]B - hS_{p,q}(z,w)[h]A - w\widehat{\partial}_z S_{p,q}(z,w)[h]A \\
= {} & hS_{p,q}(z,w)(B - A),
\end{aligned}$$

since by (2.6) we have

$$\widehat{\partial}_z S_{p,q}(z,w)[h] = \widehat{\partial}_w S_{p-1,q+1}(z,w)[h], \;\; \widehat{\partial}_z S_{p+1,q-1}(z,w)[h] = \widehat{\partial}_w S_{p,q}(z,w)[h],$$

thus $\Delta_{p,q}$ is BF and $\operatorname{div}\big(S_{p,q}(z,w), -S_{p-1,q+1}(z,w)\big) = 0$ for all $p \geq 1, q \geq 0$. If divergence is 0, then also $A = B$ and

$$\big(f_{p+1,q}(z,w), g_{p,q+1}(z,w)\big) = \big(S_{p+1,q}(z,w), -S_{p,q+1}(z,w)\big)A.$$

We have three remaining cases to check separately, $p = 0, q = 0$ and $q = 1$. In the first case, we have a degree $q+1$ vector field $X(z,w) = \big(f_{1,q}(z,w), g_{0,q+1}(z,w)\big)$,

$$f_{1,q}(z,w) = zw^q A_q + w \sum_{\alpha \in \{0,1\}^{q-1}, |\alpha|=1} (z,w)^\alpha A_\alpha, \quad g_{0,q+1} = w^{q+1}B.$$

Since there is only one element in the second component, it follows that $B = -A_\alpha$ for all α and so the vector field is of the form

$$\big(zw^q A_q - wS_{1,q-1}(z,w)B, w^{q+1}B\big) = (zw^q, 0)A_q + \big(-wS_{1,q-1}(z,w), S_{0,q+1}\big)B$$

with divergence equal to $w^q\big(A_q + B\big)$. Again, if divergence is 0, then $A_q = -B$ and the vector field is of the form

$$\big(zw^q A_q - wS_{1,q-1}(z,w)B, w^{q+1}B\big) = \big(-S_{1,q}(z,w), S_{0,q+1}\big)B.$$

The second is the case of degree $p+1$ vector fields of the form $X(z,w) = \big(f_{p+1,0}(z,w), g_{p,1}(z,w)\big)$ and is treated similarly as the first case. In the third case we have vector fields of the form $X(z,w) = \big(f_{p+1,1}(z,w), g_{p,2}(z,w)\big)$ and because the case $p = 0$ is already proved we assume $p > 0$. Then there is only one $A_{\alpha_2} = A$ and so $B_{\beta_2} + A = 0$, therefore the vector fields are of the form

$$\big(f_{p+1,1}, g_{p,2}\big)(z,w) = z\left(\sum_{\alpha_1 \in \{0,1\}^{p+1}, |\alpha_1|=p} (z,w)^{\alpha_1} A_{\alpha_1}, \sum_{\beta_1 \in \{0,1\}^{p+1}, |\beta_1|=p-1} (z,w)^{\beta_1} B_{\beta_1}\right) + \big(wz^p, -wS_{p,1}(z,w)\big)A.$$

Since there are two zeroes in β_1 and one zero in α_1, we can apply the same transposition argument as above, but to the word of the form $1\alpha_1 = 1\alpha_1^1 1\alpha_1^2 0\alpha_1^3$ and conclude, that for any two words α_1, α_2 we have

$$A_{\alpha_1} = A_{\alpha_2} = -B_{\beta_1} = B,$$

so

$$\big(f_{p+1,1}, g_{p,2}\big)(z,w) = z\big(S_{p,1}(z,w), -S_{p-1,2}(z,w)\big)B + w\big(S_{p,0}(z,w), -S_{p,1}(z,w)\big)A$$

with divergence equal to

$$\operatorname{div}\big(f_{p+1,1}, g_{p,2}\big)(z,w) = S_{p,1}(z,w)(B-A).$$

If divergence is 0, then the vector field is of the form

$$\begin{aligned}\big(f_{p+1,1}, g_{p,2}\big)(z,w) &= \big(zS_{p,1}(z,w) + wS_{p,0}(z,w), -zS_{p-1,2}(z,w) - wS_{p,1}(z,w)\big)A \\ &= \big(S_{p+1,1}(z,w), -S_{p,2}(z,w)\big)A,\end{aligned}$$

so it is BF. □

An immediate consequence of the proof is the following corollary.

Corollary 3.5. *Let $X(z,w) \in \mathcal{VH}_r$ be a vector field with divergence. Then it has a form*

$$X(z,w) = \Big(z\sum_{p\geq 1}\big(S_{p,q}(z,w), -S_{p-1,q+1}(z,w)\big)a_{p,q} + w\sum_{q\geq 1}\big(S_{p+1,q-1}(z,w), -S_{p,q}(z,w)\big)b_{p,q}\Big) + \big(g_0(w), f_0(z)\big)$$

and its divergence is $\mathrm{div}X(z,w) = \sum_{p,q\geq 0} S_{p,q}(z,w)\big(a_{p,q} - b_{p,q}\big)$. *The vector fields with divergence 0 are of the form*

$$X(z,w) = \sum X_{p,q}(z,w)a_{p,q} + \big(g_0(w), f_0(z)\big),$$

where $X_{p,q}(z,w) = \big(S_{p+1,q}(z,w), -S_{p,q+1}(z,w)\big)$. *The slice-regular divergence 0 vector fields are only* $\big(g_0(w), f_0(z)\big)$.

Definition 3.6. Given the vector field $X(z,w) = \big(f(z,w), g(z,w)\big)$, we define the differential operator

$$\mathrm{Rot}\, X(z,w)[h] := -\widehat{\partial}_z g(z,w)[h] + \widehat{\partial}_w f(z,w)[h]$$

and we say that the vector field X has *rotor* if $\mathrm{Rot}\, X(z,w)[h]$ is left h linear, in other words if there exists a function, which will be denoted by $\mathrm{rot}\, X(z,w)$, such that

$$\mathrm{Rot}\, X(z,w)[h] = h\, \mathrm{rot}\, X(z,w).$$

Since $\mathrm{Rot}\,(f,g) = \mathrm{Div}(-g,f)$, we have the following:

Theorem 3.7. *Let $X(z,w) = (f(z,w), g(z,w)) \in \mathcal{VH}_r$ be a vector field with rotor. Then $\mathrm{rot}\, X(z,w)$ is BF. If $\mathrm{rot}\, X(z,w) = 0$, then X is BF and has the form*

$$X(z,w) = \sum_{p,q\geq 1}\big(S_{p-1,q}(z,w), S_{p,q-1}(z,w)\big)\, a_{p,q} + \left(\sum_{p\geq 0} z^p a_p, \sum_{q\geq 0} w^q b_q\right).$$

Define

$$\chi(z,w) := \sum_{p,q\geq 1} S_{p,q}(z,w)\frac{a_{p,q}}{p+q} + \sum_{p\geq 0}\left(z^{p+1}\frac{a_p}{p+1} + w^{p+1}\frac{b_p}{p+1}\right) + C,$$

where $C \in \mathbf{H}$ is an arbitrary constant. Then

$$X(z,w) = \big(\widetilde{\partial}_z \chi(z,w), \widetilde{\partial}_w \chi(z,w)\big).$$

Proof. By Definition (2.4) of derivatives $\widetilde{\partial}_z$ and $\widetilde{\partial}_w$ we have

$$\widetilde{\partial}_z S_{p,q}(z,w) = (p+q)S_{p-1,q}(z,w),$$
$$\widetilde{\partial}_w S_{p,q}(z,w) = (p+q)S_{p,q-1}(z,w).$$

□

Definition 3.8. Let $D \subset \mathbf{H}^2 \times \mathbf{R}$ be an open set containing $\mathbf{H}^2 \times \{0\}$. A function $\Phi^X : D \to \mathbf{H}^2$ is a *flow* of the vector field X if

$$\frac{d}{dt}\Phi^X(z,w,t) = X\big(\Phi^X(z,w,t)\big), \quad \text{for all } (z,w,t) \in D,$$
$$\Phi^X(z,w,0) = (z,w), \qquad \text{for all } (z,w) \in \mathbf{H}^2.$$

If $D = \mathbf{H}^2 \times \mathbf{R}$, we say that a vector field X is *complete*.

Whenever it is clear from the context which vector field we are referring to, we omit the superscript X.

Example 3.9. Consider the vector fields

$$X(z,w) = (f(w), 0) \ \text{ and } \ Y(z,w) = (zg(w), 0)$$

with f and g slice-regular functions defined on $\mathbf{H}$. We have

$$\operatorname{div} X(z,w) = 0 \ \text{ and } \ \operatorname{div} Y(z,w) = g(w).$$

The corresponding flows are

$$\Phi^X(z,w,t) = (z,w) + t(f(w),0) \ \text{ and } \ \Phi^Y(z,w,t) = (z,w) + \left(z\left(e^{tg(w)} - 1\right), 0\right) \tag{3.2}$$

and the vector fields are complete. The exponential function is defined by series expansion: for any function $\phi(z,w,t)$, let

$$e^{\phi(z,w,t)} = \sum_0^\infty \frac{\phi(z,w,t)^n}{n!}. \tag{3.3}$$

Notice that the function $e^{tg(w)} = \sum_0^\infty t^n g(w)^n/n!$ is not a slice-regular function in general.

Example 3.10. The vector field $X(z,w) = \left(z^2 w, -zw^2\right)$ is complete with a flow

$$\Phi^X(z,w,t) = \left(ze^{tzw}, e^{-tzw}w\right) = (u,v)$$

$$\begin{aligned}
\frac{d}{dt}\left(ze^{tzw}, e^{-tzw}w\right) &= \left(ze^{tzw}zw, -zwe^{-tzw}w\right) \\
&= \left(\left(ze^{tzw}\right)\left(ze^{tzw}\right)\left(e^{-tzw}w\right), -\left(ze^{tzw}\right)\left(e^{-tzw}w\right)\left(e^{-tzw}w\right)\right) \\
&= \left(u^2 v, -uv^2\right),
\end{aligned}$$

where by (3.3) the exponential function is defined as power series expansion for $\phi(z,w,t) = zwt$. Since t is real, we have $e^{tzw} = \sum_0^\infty t^n (zw)^n/n!$. Because $\operatorname{Div} X(z,w)[h] = hzw - zwh$, the vector field X does not have divergence.

4 | QUATERNIONIC DETERMINANTS AND APPLICATIONS TO VECTOR FIELDS OF SHEAR AND OVERSHEAR AUTOMORPHISMS

This chapter is mainly devoted to the study of special classes of vector fields which are generalizations of the two vector fields from Example (3.9). We focus, in particular, on the geometric properties of the divergence of the flows of these vector fields.

If A is an invertible real matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in GL(2, \mathbf{R})$$

and $f \in \mathcal{SR}\mathbf{H}$, we consider the vector field

$$X(z,w) = \frac{1}{ad - bc}(d, -c) f(cz + dw).$$

If $\pi_2 : \mathbf{H}^2 \to \mathbf{H}$ is the projection onto the second coordinate, one can write

$$X(z,w) = A^{-1}\big(f \circ \pi_2, 0\big)^T \big(A \cdot (z,w)^T\big).$$

Notice that if $d = 0$, the vector field is of the form $(0, g(z))$ and if $c = 0$ is of the form $(g(w), 0)$ for a suitable entire slice-regular function g. In both cases, the vector field X has divergence 0.

Assume now that $c \neq 0$. Then

$$\begin{aligned}
\mathrm{Div} X(z,w)[h] &= \frac{1}{ad-bc}\big(\widehat{\partial}_z f(cz+dw)[h]d + \widehat{\partial}_w f(cz+dw)[h](-c)\big) \\
&= \frac{1}{ad-bc}\big(\widehat{\partial}_z f(cz+dw)[h]d + \widehat{\partial}_z f(cz+dw)[h]c^{-1}d(-c)\big) \\
&= 0.
\end{aligned}$$

If $c \neq 0$, we may assume that $c = -1$. If we write $d = \mu$, the vector field X can be written in a form

$$X(z,w) = (\mu, 1)\tilde{f}(z - \mu w)$$

for some other slice-regular function $\tilde{f}$. Notice that the vector field X is in the kernel of the functional $\Lambda(z,w) = z - \mu w$, i.e. $\Lambda(X) = 0$.

If $\pi_1 : \mathbf{H}^2 \to \mathbf{H}$ is the projection onto the first coordinate, consider the vector field

$$Y(z,w) = A^{-1}\big(\pi_1 \cdot f \circ \pi_2, 0\big)^T\big(A \cdot (z,w)^T\big) = \frac{1}{ad-bc}(d,-c)(az+bw)f(cz+dw). \tag{4.1}$$

It has divergence

$$\begin{aligned}
\mathrm{Div} Y(z,w)[h] &= \frac{1}{ad-bc}\Big[(az+bw)\big(\widehat{\partial}_z f(cz+dw)[h]d + \widehat{\partial}_w f(cz+dw)[h](-c)\big) + (ad-bc)hf(cz+dw)\Big] \\
&= hf(cz+dw).
\end{aligned}$$

Similarly as before, $\Lambda(Y) = 0$ for $\Lambda(z,w) = z - \mu w$ for $\mu = -d/c$.

Definition 4.1. Let π_1, π_2 denote the projections of $\mathbf{H}^2$ on the first and second coordinate respectively. We define the following two classes of vector fields:

$$SV_{\mathbf{R}} = \Big\{X,\, X(z,w) = A^{-1}\big(f \circ \pi_2, 0\big)^T\big(A \cdot (z,w)^T\big),\ A \in SL(2,\mathbf{R}), f \in \mathcal{SR}(\mathbf{H})\Big\},$$

$$OV_{\mathbf{R}} = \Big\{Y,\, Y(z,w) = A^{-1}\big(\pi_1 \cdot f \circ \pi_2, 0\big)^T\big(A \cdot (z,w)^T\big),\ A \in GL(2,\mathbf{R}), f \in \mathcal{SR}\mathbf{H}\Big\}.$$

The classes $SV_{\mathbf{R}}$ and $OV_{\mathbf{R}}$ are called *shear* and *overshear vector fields* respectively.

The space of all shears $SV_{\mathbf{R}}$ can be also described as

$$SV_{\mathbf{R}} = \{(r,1)f(z - rw),\ r \in \mathbf{R}, f \in \mathcal{SR}(\mathbf{H})\} \cup \{(g(w), 0), g \in \mathcal{SR}(\mathbf{H})\}.$$

Lemma 4.2. *For each p, q there exists a vector field $Y_{p,q}$ with* $\mathrm{div} Y_{p,q}(z,w) = S_{p,q}(z,w)$ *and it is a sum of overshear vector fields.*

Proof. Since $S_{p,q}(z,w) = \sum_{n=0}^{p+q}(z-nw)^{p+q} r_n, r_n \in \mathbf{R}$ by formula (2.7), the vector field is

$$Y_{p,q}(z,w) = \sum_{n=0}^{p+q}(n,1)(nz+w)(z-nw)^{p+q}\frac{r_n}{n^2+1}.$$

$\square$

Proposition 4.3. *Any polynomial vector field $X \in \mathcal{VH}_{rhs}$ with divergence is a finite sum of shear and overshear vector fields. If* $\operatorname{div} X = 0$*, then X can be written as a sum of shear vector fields.*

Proof. Let $X = \sum_d X_d$ be the homogenous expansion of a vector field X. Since divergence of X is bidegree full, by Lemma 4.2 there exists a vector field Y, which is a sum of overshear vector fields, such that $\operatorname{div} X = \operatorname{div} Y$, so it is sufficient to prove that every divergence zero vector field is a sum of shear vector fields. Since the operator Div respects the degree in the expansion, it suffices to prove the assertion for each fixed degree. Now assume that $\operatorname{div} X_d = 0$. Because of Lemma 2.6, we can write X_d as

$$X_d(z,w) = \left(\sum_{n=0}^{d}(z-nw)^d a_{n,d}, \sum_{n=0}^{d}(z-nw)^d b_{n,d}\right).$$

Therefore

$$\begin{aligned}\operatorname{Div} X_d(z,w)[h] &= \sum_{n=0}^{d}\widehat{\partial}_z(z-nw)^d[h]a_{n,d} - \sum_{n=0}^{d}\widehat{\partial}_z(z-nw)^d[h]nb_{n,d}\\ &= \widehat{\partial}_z\left(\sum_{n=0}^{d}(z-nw)^d(a_{n,d}-nb_{n,d})\right)[h],\end{aligned}$$

so the condition $\operatorname{Div} X_d(z,w)[h] = 0$ and Lemma 2.2 imply

$$\sum_{n=0}^{d}(z-nw)^d\left(a_{n,d}-nb_{n,d}\right) = w^d q$$

for some $q \in \mathbf{H}$. Since the monomials $(z-nw)^d, n = 0, \dots, d$, are generators of all BF polynomials of degree d, there exist constants $\lambda_0, \dots \lambda_d$ such that

$$w^d = \sum_{n=0}^{d}(z-nw)^d \lambda_n.$$

So we have $\lambda_n q = a_{n,d} - nb_{n,d}$ and then $a_{n,d} = \lambda_n q + nb_{n,d}$. In other words,

$$\begin{aligned}X_d(z,w) &= \left(\sum_{n=0}^{d}(z-nw)^d(nb_{n,d}+\lambda_n q), \sum_{n=0}^{d}(z-nw)^d b_{n,d}\right)\\ &= \sum_{n=0}^{d}(n,1)(z-nw)^d b_{n,d} + (1,0)\sum_{n=0}^{d}(z-nw)^d\lambda_n q\\ &= \sum_{n=0}^{d}(n,1)(z-nw)^d b_{n,d} + (1,0)w^d q.\end{aligned}$$

As easily checked, all vector fields in the last sum have divergence 0. $\square$

Passing from a real to a quaternionic matrix, we have to point out that there is no canonical way to define the determinant of such a matrix. We consider only 2×2 matrices but we refer the reader to [3] and [9] for further references on general linear groups and determinants. There are several possibilities of introducing a generalization of the standard notion of determinant according to the properties one is looking at. For example, the real determinant $\det_{\mathbf{R}}$ and the complex determinant $\det_{\mathbf{C}}$ of a quaternionic matrix are defined when a quaternionic matrix is considered as the corresponding real or complex matrix obtained via the identification of $\mathbf{H}$ with $\mathbf{R}^4$ or with $\mathbf{C}^2$ respectively.

If

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

$(a, b, c, d \in \mathbf{H})$, we define the *Cayley determinant* of A to be

$$\det_C A = ad - cb.$$

If $b = a$ and $c = d$, the rank of the matrix is 1 and the determinant is $ac - ca$, which is 0 if and only if a and c commute. Another interesting definition is Dieudonné determinant $\det_D$. The *Dieudonné determinant* is defined as a mapping from $M(2, \mathbf{H})$ to a quotient Q of the multiplicative subgroup $\mathbf{H}^*$ of $\mathbf{H}$ to its quotient by a commutator subgroup, $Q = \mathbf{H}^*/[\mathbf{H}^*, \mathbf{H}^*]$. The group Q is isomorphic to $\mathbf{R}_+$, because the commutator subgroup consists precisely of all quaternionic units. For example, the representative of $\det_D A$ in Q is defined as

$$\det_D A = \begin{cases} -cb & \text{if } a = 0, \\ ad - aca^{-1}b & \text{if } a \neq 0. \end{cases}$$

The quaternionic determinants $\det_D$, $\det_{\mathbf{R}}$ and $\det_{\mathbf{C}}$ satisfy the three following axioms: the determinant is 0 if and only if the matrix is singular, the determinant of a product of matrices is a product of determinants and a particular Gaussian elimination is allowed.

Therefore the following two groups of transformations

$$SL(2, \mathbf{H}), \quad \text{and} \quad GL(2, \mathbf{H})$$

can be properly and correctly defined.

It is important to observe that the operator $\diamond$ as in (2.5) is not a product and therefore in general, no matter which definition of the determinant we adopt, the determinant of a composed mapping introduced by using $\diamond$ is not necessarily a product of determinants.

Definition 4.4. Let π_1, π_2 denote the projections of $\mathbf{H}^2$ on the first and second coordinate respectively. We define the following two classes of vector fields:

$$SV_{\mathbf{H}} = \left\{X,\, X(z,w) = A^{-1}\left(f \circ \pi_2, 0\right)^T\left(A \cdot (z,w)^T\right),\ A \in SL(2, \mathbf{H}), f \in \mathcal{SR}(\mathbf{H})\right\},$$

$$OV_{\mathbf{H}} = \left\{Y,\, Y(z,w) = A^{-1}\left(\pi_1 \cdot f \circ \pi_2, 0\right)^T\left(A \cdot (z,w)^T\right),\ A \in GL(2, \mathbf{H}), f \in \mathcal{SR}\mathbf{H}\right\}.$$

The classes $SV_{\mathbf{H}}$ and $OV_{\mathbf{H}}$ are called *generalized shear* and *generalized overshear vector fields* respectively.

Example 4.5. Consider the matrix

$$A = \begin{bmatrix} \bar{\mu} & 1 \\ 1 & -\mu \end{bmatrix} \left(1 + |\mu|^2\right)^{-1}, \ \mu \in \mathbf{H}.$$

Since the entries commute, the formula for the inverse A^{-1} is the same as in the commutative case and so the conjugation by such A defines a $OV_{\mathbf{H}}$ vector field in the same manner as in (4.1). Unfortunately these vector fields do not have divergence unless μ is real. In fact, from the previous computation we have

$$\begin{aligned} Y(z,w) &= (\mu,1)\left(\bar\mu\left(1+|\mu|^2\right)^{-1}z+\left(1+|\mu|^2\right)^{-1}w\right)f\left(\left(1+|\mu|^2\right)^{-1}z-\mu\left(1+|\mu|^2\right)^{-1}w\right)\\ &= (\mu,1)(az+bw)f(bz-aw)\end{aligned}$$

where $a := \bar\mu\left(1+|\mu|^2\right)^{-1} = \left(1+|\mu|^2\right)^{-1}\bar\mu$ and $b := \left(1+|\mu|^2\right)^{-1}$. Notice that $\mu a + b = 1$. Then

$$\begin{aligned} \mathrm{Div}Y(z,w)[h] &= \left[\mu(az+bw)\left(\widehat{\partial}_z f(bz-aw)[h]\right) + (az+bw)\widehat{\partial}_w f(bz-aw)[h]) + (\mu ah+bh)f(bz-aw)\right]\\ &= \left[\mu(az+bw)\left(\widehat{\partial}_z f(bz-aw)[h]\right) + (az+bw)\widehat{\partial}_w f(bz-aw)[h])\right] + hf(bz-aw),\end{aligned}$$

since $\mu ah + bh = h$. The term in the brackets is not necessarily 0 since the chain rule does not apply and μ is not real. For example, a suitable choice of f gives $Y(z,w) = (\mu,1)(\bar\mu z + w)(z-\mu w)$ and then

$$\begin{aligned} \mathrm{Div}Y(z,w)[h] &= h\left(1+|\mu|^2\right)(z-\mu w) + \mu(\bar\mu z+w)(h) - (\bar\mu z+w)(\mu h)\\ &= h\left(1+|\mu|^2\right)(z-\mu w) + \mu wh - w\mu h + |\mu|^2 zh - \bar\mu z\mu h\end{aligned}$$

so Y does not have divergence. Similarly, the vector field of the form $X(z,w) = (\mu,1)f(z-\mu w)$ does not have divergence and actually $\mathrm{Div}X(z,w)[h] = \mu\widehat{\partial}_z f(z-\mu w) - \widehat{\partial}_w f(z-\mu w)$. This is 0 if and only if μ commutes with w and z, i.e. $\mu \in \mathbf{R}$.

The generalized shear and overshear vector fields, however, are complete. Indeed,

Lemma 4.6. *Let X be a vector field with a (real) flow Φ^X. Let $A \in GL(2,\mathbf{H})$ and consider the* conjugate *of X i.e. $Y = A^{-1}X\circ A$. Then the flow of Y is*

$$\Phi^Y = \Phi^{A^{-1}X\circ A} = A^{-1}\Phi^X\circ A.$$

Proof. Since in the flow the time t is real, the derivation with respect to t commutes with multiplication by a quaternionic matrix and so

$$\frac{d}{dt}A^{-1}\Phi^X\circ A = A^{-1}\left(\frac{d}{dt}\Phi^X\right)\circ A = A^{-1}X\circ\Phi^X\circ A = A^{-1}X\circ A\circ A^{-1}\Phi^X\circ A = \left(A^{-1}X\circ A\right)\circ\left(A^{-1}\Phi^X\circ A\right),$$

which proves that $A^{-1}\Phi^X\circ A$ is a flow of the vector field $A^{-1}X\circ A$. □

Example 4.7. The vector fields

$$\begin{aligned} X(z,w) &= (\mu,1)f(z-\mu w),\\ Y(z,w) &= (\mu,1)\left(|\mu|^2+1\right)^{-1}(\bar\mu z+w)f(z-\mu w)\end{aligned}$$

are obtained from vector fields in the example (3.9) by conjugation by suitable matrices, and therefore have the flows

$$\begin{aligned} \Phi^X(z,w,t) &= (z,w) + t(\mu,1)f(z-\mu w),\\ \Phi^Y(z,w,t) &= (z,w) + (\mu,1)\left(|\mu|^2+1\right)^{-1}(\bar\mu z+w)\left(e^{tf(z-\mu w)}-1\right).\end{aligned}$$

Definition 4.8. Let $\Lambda : \mathbf{H}^2 \to \mathbf{H}$ be a right $\mathbf{H}$-linear functional. Assume $v = (v_1,v_2) \in \ker\Lambda$, $\left\|(v_1,v_2)\right\| = 1$. For $f \in \mathcal{H}$, any mapping of the form

$$(z,w) \mapsto (z,w) + \left(v_1,v_2\right)f(\Lambda(z,w))$$

is called a *generalized shear*. A generalized shear is a *shear* if $v_1, v_2 \in \mathbf{R}$, Λ is represented by a real matrix, and f is slice-regular. We denote the class of generalized shears as $\mathcal{S}_{\mathbf{H}}$ and the class of shears as $\mathcal{S}_{\mathbf{R}}$.

Analogously, a mapping of the form

$$(z,w) \to (z,w) + \big(v_1, v_2\big)\big(\bar{v}_1 z + \bar{v}_2 w\big)\big(e^{f(\Lambda(z,w))} - 1\big),$$

$f \in \mathcal{H}$, is called a *generalized overshear*. A generalized overshear is an *overshear* if $v_1, v_2 \in \mathbf{R}$, Λ is represented by a real matrix, and f is slice-regular. We denote the class of generalized overshears as $\mathcal{O}_{\mathbf{H}}$ and the class of overshears as $\mathcal{O}_{\mathbf{R}}$.

For each fixed t the flows of (generalized) shear or overshear vector fields are (generalized) shears or overshears.

Lemma 4.9. *(Generalized) shears and overshears are time one maps of complete flows and therefore are automorphisms with (generalized) shears and overshears as inverses.*

Proof. The time one map of the flow $\Phi_t^X(z,w) = (z,w) + \big(v_1, v_2\big) t f(\Lambda(z,w))$ of the vector field $\big(v_1, v_2\big) f(\Lambda(z,w))$ is the generalized shear $F(z,w) = (z,w) + \big(v_1, v_2\big) f(\Lambda(z,w))$. Similarly, the generalized overshear

$$G(z,w) = (z,w) + \big(v_1, v_2\big)\big(\bar{v}_1 z + \bar{v}_2 w\big)\big(e^{f(\Lambda(z,w))} - 1\big)$$

is a time-one map of the vector field $Y(z,w) = \big(v_1, v_2\big)\big(\bar{v}_1 z + \bar{v}_2 w\big) f(\Lambda(z,w))$ with the flow

$$\Phi_t^Y(z,w) = (z,w) + \big(v_1, v_2\big)\big(\bar{v}_1 z + \bar{v}_2 w\big)\big(e^{t f(\Lambda(z,w))} - 1\big).$$

□

4.1 | Derivatives of shears and overshears

Consider a shear $F^{\mu}(z,w) = (z,w) + (\mu, 1) f(z - \mu w)$, with $f \in \mathcal{H}_{rhs}[u]$ and μ real. Then, using the notation as in (2.5), we have

$$DF^{\mu}(z,w)[h_1, h_2] := \begin{bmatrix} h_1 + \mu \widehat{\partial}_z f(z - \mu w)[h_1] & \mu \widehat{\partial}_w f(z - \mu w)[h_2] \\ \widehat{\partial}_z f(z - \mu w)[h_1] & h_2 + \widehat{\partial}_w f(z - \mu w)[h_2] \end{bmatrix}.$$

We would like to calculate the Jacobian, i.e. the Dieudonné determinant of the above matrix and see if it is, as in the complex or real case, proportional to $h_1 h_2$ with constant factor 1. We may assume that $|h_1| = |h_2| = 1$ because of real linearity. Since Gaussian elimination of rows by using left multiplication is allowed and μ is real, we have (by a slight abuse of notation we write $\det_D$ also for the representative in the quotient)

$$\begin{aligned} \det\nolimits_D DF^{\mu}(z,w)[h_1, h_2] &= \begin{vmatrix} h_1 & -\mu h_2 \\ \widehat{\partial}_z f(z - \mu w)[h_1] & h_2 - \mu \widehat{\partial}_z f(z - \mu w)[h_2] \end{vmatrix} \\ &= h_1 h_2 - \mu h_1 \widehat{\partial}_z f(z - \mu w)[h_2] + \mu h_1 \widehat{\partial}_z f(z - \mu w)[h_1] \big(h_1\big)^{-1} h_2. \end{aligned}$$

The last two terms do not cancel out in general, but they do if $h_1 = h_2$. Therefore we could say that for $|h| = 1$ the determinant $\det_D DF^j(z,w)[h,h] = 1$, which means, that shears could be considered in a way as volume preserving maps. However, this property is no longer preserved if we compose two shears or if μ is not real. For instance, let $f(u) = u^2$ and consider F^{μ} as above. Recall that $\det_D A = 1$ precisely when its representative has modulus 1. Even if we simplify the calculation by inserting $h = 1$, we get

$$\begin{aligned} \det\nolimits_D DF^{\mu}(z,w)[1,1] &= \begin{vmatrix} 1 & -\mu \\ 2(z - \mu w) & 1 - (\mu(z - \mu w) + (z - \mu w)\mu) \end{vmatrix} \\ &= 1 - (\mu(z - \mu w) - (z - \mu w)\mu). \end{aligned}$$

The number in the bracket is purely imaginary and so the only possibility for such a number to have modulus 1, is, that the term in the bracket vanishes for all z and w. This is if and only if $\mu \in \mathbf{R}$.

Assume μ is real; in order to calculate the derivative of the overshear flow

$$\Phi^Y(z,w,t) = (z,w) + (\mu,1)\big(\mu^2+1\big)^{-1}(\mu z + w)\big(e^{tf(z-\mu w)} - 1\big)$$

of the vector field

$$Y(z,w) = \big(\mu,1\big)(z\mu + w)f(z-\mu w)\big(\mu^2+1\big)^{-1}$$

we notice first that

$$\widehat{\partial}_w e^{f(z-\mu w)}[h] = -\mu \widehat{\partial}_z e^{tf(z-\mu w)}[h]$$

and then put

$$(-\mu)A := \widehat{\partial}_w e^{f(z-\mu w)}[h] = -\mu\widehat{\partial}_z e^{tf(z-\mu w)}[h], \quad B := e^{tf(z-\mu w)} - 1.$$

Then,

$$D\Phi^Y(z,w,t)[h,h] := \begin{bmatrix} h + \dfrac{\mu}{\mu^2+1}(\mu hB + (z\mu+w)A) & \dfrac{\mu}{\mu^2+1}(hB - \mu(z\mu+w)A) \\ \dfrac{1}{\mu^2+1}(\mu hB + (z\mu+w)A) & h + \dfrac{1}{\mu^2+1}(hB - \mu(z\mu+w)A) \end{bmatrix}.$$

After applying Gaussian elimination on rows, we see that

$$\begin{aligned} \det_D D\Phi^Y(z,w,t)[h,h] &= h\begin{vmatrix} 1 & -\mu \\ \dfrac{\mu hB + (z\mu+w)A}{\mu^2+1} & h + \dfrac{hB - \mu(z\mu+w)A}{\mu^2+1} \end{vmatrix} \\ &= h^2 e^{tf(z-\mu w)}, \end{aligned}$$

so we can say that the Dieudonné determinant of $D\Phi^Y(z,w,t)$ is represented by the function $V(z,w;t) = e^{tf(z-\mu w)}$ and in this case the function $V(z,w;t)$ also solves the differential equation

$$\frac{d}{dt}V(z,w,t) = f(z-\mu w)V(z,w,t), \quad V(z,w,0) = 1,$$

where $\operatorname{div}Y(z,w) = f(z-\mu w)$. Therefore we can say that overshears form a class of automorphisms which resemble the property of having volume and the quantity V resembles the volume element at $\Phi^Y(z,w,t)$.

5 | ANDERSEN–LEMPERT THEOREM FOR AUTOMORPHISMS WITH VOLUME

As shown in the previous section any notion of volume and of volume-preserving maps are not well-defined in general if one uses a definition which involves the notion of the determinant. Therefore we prefer to use another approach and, as for the case of automorphisms of $\mathbf{C}^n$, we consider the volume-preserving automorphisms to be those which are perturbations of the identity by vector fields with divergence.

Definition 5.1. The space of *automorphisms with volume* is defined as

$$\mathrm{Aut}_V\big(\mathbf{H}^2\big) = \big\{\Phi^X(z,w,1), \quad \mathrm{Div}X(z,w)[h] = h\,\mathrm{div}X(z,w)\big\}$$

where X is a vector field with the corresponding flow Φ^X. The space of *automorphisms with volume 1* is defined as

$$\mathrm{Aut}_1\left(\mathbf{H}^2\right) = \left\{\Phi^X(z,w,1), \quad \mathrm{div}X(z,w) = 0\right\}.$$

Examples in the previous sections show the remarkable fact that

$$\mathcal{S}_{\mathbf{R}} \subset \mathrm{Aut}_1\left(\mathbf{H}^2\right) \qquad \text{but} \qquad \mathcal{S}_{\mathbf{H}} \not\subset \mathrm{Aut}_V\left(\mathbf{H}^2\right).$$

Similar conclusions hold for overshears and generalized overshears.

Example 5.2. In the complex case for every automorphism $F(z,w) = (z,w) + h.o.t.$, there is vector field X defined by the flow $\Phi(z,w,t) = F(tz,tw)/t$. If F is volume preserving, then $\mathrm{div}X = 0$. The same holds for a composition of two automorphisms F and G and a corresponding associated flow. This no longer holds true in the quaternionic case. After composing the shears $F(z,w) = \left(z, w+z^2\right)$ and $G(z,w) = \left(z+w^2, w\right)$, one can define as corresponding flow the mapping

$$\Phi(z,w,t) = F \circ G(tz,tw)/t = (z,w) + t\left(w^2,z^2\right) + t^2\left(0, zw^2 + w^2z\right) + t^3\left(0,w^4\right).$$

The equation $d/dt(\Phi(z,w,t)) = X(\Phi(z,w,t),t)$ defines the time-dependent vector field

$$X(z,w,t) = \sum_0^\infty X_n(z,w)t^n.$$

If the vector field X is supposed to have divergence 0, then all the vector fields X_n should have divergence 0, in particular, they should be bidegree full. The defining equation in our case is then

$$\begin{aligned}&\left(w^2,z^2\right) + 2t\left(0, zw^2 + w^2z\right) + 3t^2\left(0,w^4\right)\\ &\quad = \sum_0^\infty X_n\left(z + tw^2, w + tz^2 + t^2\left(zw^2 + w^2z\right) + t^3w^4\right)t^n\end{aligned}$$

which by identity principle on t implies

$$X_0(z,w) = \left(w^2,z^2\right), \quad X_1(z,w) + \left(wz^2 + z^2w, zw^2 + w^2z\right) = 2\left(0, zw^2 + w^2z\right).$$

The vector field $X_1(z,w) = \left(wz^2 + z^2w, -zw^2 - w^2z\right)$ is not BF. Notice, that we do not claim that there does not exist another divergence zero vector field Y with the flow Φ_t^Y such that $F \circ G = \Phi_1^Y$. Therefore, we remark that in general a finite composition of shears is an automorphism but not necessarily a map with volume 1. In other words, it is possible that a sufficiently small neighborhood of a finite composition of shears does not contain any time one map with volume 1.

Having said that, the following theorem follows from an adaptation of the Andersen–Lempert theory as developed in [2].

Theorem 5.3. *Every automorphism in* $\mathrm{Aut}_V\left(\mathbf{H}^2\right)$ *can be approximated uniformly on compacts by finite composition of shears and overshears and every automorphism with volume 1 can be approximated uniformly on compacts by a finite composition of shears.*

By Andersen–Lempert theory, any automorphism F of $\mathbf{C}^2$ tangent to the identity at 0 can be approximated by a finite composition of shears and overshears. If such automorphism F is volume preserving then it can be approximated by a finite composition of shears.

Let us present the sketch of the proof in the holomorphic case. For an automorphism F tangent to the identity define the one-parameter family $\Phi_t(z,w) = \Phi(z,w,t) = F(tz,tw)/t$. It is a flow of a time-dependent vector field $X(z,w,t)$ and $F(z,w) = \Phi(z,w,1)$. Notice that if $\det DF(z,w) = 1$, the divergence of the vector field X is 0. The rest of the proof is an application of convergence of Euler method and it requires just mild smoothness assumptions (we refer the reader to [4] for a detailed exposition). Choose a sufficiently fine division of [0,1] to N subintervals $[t_k, t_{k+1}], t_k = k/N, k = 0, \dots, N$.

Then by the definition of flow we have

$$\Phi_t(z,w) = \overbrace{\Phi_{t/N}\circ \ldots \circ\Phi_{t/N}}^{N \text{ times}}(z,w).$$

On a given compact set, for N sufficiently large and each $k \in 0, \ldots, N$, the vector field $X_k(z,w) := X\left(z,w,t_k\right)$ approximates the vector field X for $t \in \left[t_k, t_{k+1}\right]$ and its flow Φ_t^k approximates the flow Φ in the sense that $\left|\Phi_t\left(\Phi_{t_k}(z,w)\right) - \Phi_t^k\left(\Phi_{t_k}(z,w)\right)\right|$ is small for $t \in \left[0, t_{k+1}-t_k\right]$. Each vector field X_k can be further approximated by a polynomial vector field $\tilde{X}_k$. If X_k has divergence 0, then $\tilde{X}_k$ can be chosen to have divergence 0 just by appropriately cutting off the Taylor series expansion. The flow $\tilde{\Phi}^k$ of $\tilde{X}_k$ also approximates the original flow in the same manner as above. But such a polynomial vector field is a sum of shear and overshear vector fields (in the divergence 0 case it is a sum of shear vector fields) and the composition of the corresponding shears and overshears (or just shears in the volume preserving case) approximates the flow $\tilde{\Phi}^k$.

In the quaternionic case we are already in trouble with a notion of "quaternionic Jacobian," therefore we have to assume that our automorphism already is a time-one map of a (maybe time-dependent) vector field X with divergence. Then we proceed as above. On small time intervals we approximate a time-dependent vector field X with divergence by time-independent polynomial vector fields $\tilde{X}_k$ with divergence; they can be written as a sum of shear and overshear vector fields (in divergence 0 case just as a sum of shear vector fields) by Proposition 4.3. Then the compositions of their flows approximates the initial automorphism. We show in the example below that without the condition on existence of divergence, the time-one map is not necessarily approximable by shears.

Example 5.4. In this example we show that the map $F(z,w) = \left(ze^{zw}, e^{-zw}w\right)$ from Example 3.10 is not approximable by finite compositions of shears. It is, though, a time one map of a complete vector field, but this vector field does not have divergence. Its restriction to complex subspaces $C_I \times C_I$ is approximable by complex shears (but it is not a finite composition of shears as proved in [1]).

The Taylor expansion of the mapping F is of the form

$$F(z,w) = \left(z + z^2 w + \cdots, w - zw^2 + \cdots\right),$$

where the dots indicate higher order terms. Consider a generic composition of shears $S = S^d \circ \ldots \circ S^1$ with

$$S^m(z,w) = (z,w) + \left(\mu_m, 1\right)\left(\left(z-\mu_m w\right)^2 a_{m,2} + \left(z-\mu_m w\right)^3 a_{m,3} + \cdots\right)$$

and let S_n^m denote the term of order n in its expansion. Then the composition of shears S up to the third order is of the form

$$id + \sum_{m=1}^{k} S_2^m + \sum_{m=1}^{k} S_3^m + \tilde{S}_3,$$

where $\tilde{S}_3$ are the rest of the terms of order 3. If S is supposed to be approximating F, the terms of order 3 of S should approximate the term of order 3 in the expansion of F - the term $\left(z^2w, -zw^2\right)$. Since the terms S_n^m are all BF and the latter is not, the only possibility for approximating F is that the missing terms come from $\tilde{S}_3$. However, terms of order 3 arise if and only if we compose some S_2^m with a term of the form id $+ T_2$ where T_2 are terms of order 2 which are all BF. So we have

$$\begin{aligned}
&\left(z-\mu_m w\right)^2 \circ \left((z,w) + \sum_n \left(z-\mu_n w\right)^2 \left(\mu_n, 1\right) a_n\right) \\
&= \left(\left(z + \sum_n \left(z-\mu_n w\right)^2 \mu_n a_n\right) - \mu_m\left(w + \sum_n \left(z-\mu_n w\right)^2 a_n\right)\right)^2
\end{aligned}$$

$$= \left((z - \mu_m w) + \sum_n (z - \mu_n w)^2 (\mu_n - \mu_m) a_n \right)^2$$

$$= (z - \mu_m w)^2 + (z - \mu_m w) \sum_n (z - \mu_n w)^2 (\mu_n - \mu_m) a_n$$

$$+ \sum_n (z - \mu_n w)^2 (\mu_n - \mu_m) a_n (z - \mu_m w) + \ldots$$

$$= (z - \mu_m w)^2 + \sum_n (\mu_n - \mu_m) \left[(z - \mu_m w)(z - \mu_n w)^2 a_n + (z - \mu_n w)^2 a_n (z - \mu_m w) \right] + \cdots.$$

We are interested in the terms in the square brackets with bidegree (2,1). Those are

$$\begin{aligned}&(- \mu_m w z^2 - \mu_n z(zw + wz))a_n + (- \mu_m z^2 a_n w - \mu_n (zw + wz) a_n z)\\&\quad = -(wz^2 \mu_m a_n + z^2 w \mu_n a_n + zwz\mu_n a_n) - (z^2 \mu_m a_n w + zw\mu_n a_n z + wz\mu_n a_n z).\end{aligned}$$

With no loss of generality we may absorb the factor $(\mu_n - \mu_m)$ in the coefficient a_n. After summing up all possible choices we get

$$-wz^2 \left(\sum_n \mu_m a_n \right) - (z^2 w + zwz) \left(\sum_n \mu_n a_n \right) - z^2 \left(\sum_n \mu_m a_n \right) w - zw \left(\sum_n \mu_n a_n \right) z - wz \left(\sum_n \mu_n a_n \right) z.$$

The bidegree full part can cancel out only terms with coefficients on the right. So if the above-given sums are not real, we can not get rid of the terms $zw \left(\sum_n \mu_n a_n \right) z$ and $z^2 \left(\sum_n \mu_m a_n \right) w$. On the other hand, if the sums are real, we can rewrite the above expression as

$$\left(\left(\sum_n \mu_m a_n \right) - \left(\sum_n \mu_n a_n \right) \right) (wz^2 + z^2 w) - 2 \left(\sum_n \mu_m a_n \right) zwz.$$

We observe that bidegree polynomials with degree $d = 3$ can not cancel out the term wz^2 in the first component of the mapping without cancelling also the term $z^2 w$. So, the conclusion is, that F cannot be approximated by a composition of shears. Finally, we remark that in the above considerations, the monomials wz^2, zwz and $z^2 w$ are not just formally linearly independent, but also linearly independent as functions.

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