Università degli Studi di Udine
Dipartimento di Scienze Matematiche, Informatiche e Fisiche
Corso di Dottorato in Informatica e Scienze Matematiche e Fisiche

Filling cages

Reverse mathematics and combinatorial principles

Tesi di dottorato di
Marta Fiori Carones
Supervisore
Prof. Alberto Marcone
Ciclo XXXII

UNIVERSITÀ DEGLI STUDI DI UDINE
hic sunt futura

PhD candidate:
Marta Fiori Carones
Università di Udine, DMIF
via delle Scienze 206,
33100 Udine,
Italia

Supervisor:
Prof. Alberto Marcone
Università di Udine, DMIF
via delle Scienze 206,
33100 Udine,
Italia

Thesis submitted on October, 31st 2019
Thesis defended on March, 12th 2020

Sommario

Nella tesi sono analizzati alcuni principi di combinatorica dal punto di vista della reverse mathematics. La reverse mathematics è un programma di ricerca avviato negli anni settanta e interessato a individuare l'esatta forza, intesa come assiomi riguardanti l'esistenza di insiemi, di teoremi della matematica ordinaria. — Dopo una concisa introduzione al tema, è presentato un algoritmo incrementale per reorientare transitivamente grafi orientati infiniti e pseudo-transitivi. L'esistenza di tale algoritmo implica che un teorema di Ghouila-Houri è dimostrabile in RCA_0. — Grafi e ordini a intervalli sono la comune tematica della seconda parte della tesi. Un primo capitolo è dedicato all'analisi di diverse caratterizzazioni di grafi numerabili a intervalli e allo studio della relazione tra grafi numerabili a intervalli e ordini numerabili a intervalli. In questo contesto emerge il tema dell'ordinabilità unica di grafi a intervalli, a cui è dedicato il capitolo successivo. L'ultimo capitolo di questa parte riguarda invece enunciati relativi alla dimensione degli ordini numerabili a intervalli. — La terza parte ruota attorno due enunciati dimostrati da Rival e Sands in un articolo del 1980. Il primo teorema afferma che ogni grafo infinito contiene un sottografo infinito tale che ogni vertice del grafo è adiacente ad al più uno o a infiniti vertici del sottografo. Si dimostra che questo enunciato è equivalente ad ACA_0, dunque più forte rispetto al teorema di Ramsey per coppie, nonostante la somiglianza dei due principi. Il secondo teorema dimostrato da Rival e Sands asserisce che ogni ordine parziale infinito con larghezza finita contiene una catena infinita tale che ogni punto dell'ordine è comparabile con nessuno o con infiniti elementi della catena. Quest'ultimo enunciato ristretto a ordini di larghezza k, per ogni $k \geq 3$, è dimostrato equivalente ad ADS. Ulteriori enunciati sono studiati nella tesi.—

Abstract

— In the thesis some combinatorial statements are analysed from the reverse mathematics point of view. Reverse mathematics is a research program, which dates back to the Seventies, interested in finding the exact strength, measured in terms of set-existence axioms, of theorems from ordinary non set-theoretic mathematics. — After a brief introduction to the subject, an on-line (incremental) algorithm to transitively reorient infinite pseudo-transitive oriented graphs is defined. This implies that a theorem of Ghouila-Houri is provable in RCA_0 and hence is computably true. — Interval graphs and interval orders are the common theme of the second part of the thesis. A chapter is devoted to analyse the relative strength of different characterisations of countable interval graphs and to study the interplay between countable interval graphs and countable interval orders. In this context the theme of unique orderability of interval graphs arises, which is studied in the following chapter. The last chapter about interval orders inspects the strength of some statements involving the dimension of countable interval orders. — The third part is devoted to the analysis of two theorems proved by Rival and Sands in 1980. The first principle states that each infinite graph contains an infinite subgraph such that each vertex of the graph is adjacent either to none, or to one or to infinitely many vertices of the subgraph. This statement, restricted to countable graphs, is proved to be equivalent to ACA_0 and hence to be stronger than Ramsey's theorem for pairs, despite the similarity of the two principles. The second theorem proved by Rival and Sands states that each infinite partial order with finite width contains an infinite chain such that each point of the poset is comparable either to none or to infinitely many points of the chain. For each $k \geq 3$, the latter principle restricted to countable poset of width k is proved to be equivalent to ADS. Some complementary results are presented in the thesis.

ALBER T O MARCONE Paul Sha fer Giovann i k s oldà

to anyone who taught me

to anyone who made me wonder

to anyone who encouraged me

Contents

Filling cages

«Considerate la vostra semenza:
fatti non foste a viver come bruti,
ma per seguir virtute e canoscenza»

Reverse mathematics indulges an essential inclination of humans, the one towards simplification. Once one knows that something can be done, then one tries to do it with the least possible effort. In mathematics this corresponds to say that once it is known that a theorem can be proved, then one would like to know how much is needed to prove it. Reverse mathematics gives a precise framework to study the previous question. The Main Question of reverse mathematics, as phrased by Stephen Simpson in [Simpson 2009] is

> Which set existence axioms are *needed* to prove the theorems of ordinary, non-set-theoretic mathematics?

The stress in the previous quotation goes on the necessity, and not only the sufficiency, of the axioms, since only necessity and sufficiency altogether corresponds to minimality. In other words, only once some axioms are proved to be equivalent to a certain theorem, then the question about the exact amount of effort needed to prove a certain theorem is definitively settled.

The imprecise presentation of the previous paragraph leaves open several questions. First of all, how can necessity be proved? Which axioms, and which theorems, are we referring to? What does 'effort' mean? Why should one be seriously interested in minimality in mathematics if not for personal tastes?

In order to answer the first question let us get a look at how sufficiency is proved first. When a mathematician claims that a theorem is true, then he proves that if a set of axioms holds, the theorem holds too. The mathematician can also *reverse* the implication, namely he can prove that if the theorem holds, then the axioms hold. If he does so, he proves the necessity of some axioms to prove a theorem. Necessity and sufficiency altogether give the equivalence between axioms and theorem. To prove the equivalence the mathematician needs some base theory, or some tools, to work with, and actually proves the equivalence of axioms and theorems over the base theory. If the axioms belong to second order arithmetic and the base theory is RCA_0, Recursive Comprehension Axiom, then the mathematician does reverse mathematics. Harvey Friedman [Friedman 1975] summarises the answer to the first question as follows

> When the theorem is proved from the right axioms, the axioms can be proved from the theorem.

We come closer to an answer to the second question. The formal theory in which reverse mathematics moves its steps is second order arithmetic Z_2, which is a theory about arithmetic formulated in a two sorted language, so that it is possible to speak about both natural numbers and collections of natural numbers. The reasons of the choice of Z_2 are technical, historical and philosophical as well. Notice that the Main Question focuses on ordinary, or not set-theoretical, mathematics, namely the mathematics which studies objects which are either countable or that can be coded by countable objects (for example real numbers follow in the second category). Hence a theory about natural numbers is able to capture them. Moreover, from an historical point of view, a formal theory of arithmetic was identified by David Hilbert a century ago as a (or better 'the' in Hilbert's dreams) theory with foundational import, since it should have grounded infinitary mathematics. Fifty years ago, Harvey Friedman and Stephen Simpson revived Hilbert's heritage noticing that a large part of statements from ordinary mathematics are not only provable from Z_2, but are provable in a (often very) weak subsystem of Z_2. And now the historical perspective matches with the philosophical one, because results from reverse mathematics let Simpson argue that a big portion of mathematics is finitistically reducible (see [Simpson 1988; Simpson 2014a] and, for a detailed historical perspective, [Dean and Walsh 2017]).

So far we concentrated on one of the key words of the Main Question, namely 'necessity', but we do not have to overlook the other one: *set-existence* axioms. As explained in the following section the relevant axioms of Z_2 are the comprehension axioms. Thus showing that a theorem is equivalent to a certain subset of axioms of Z_2 actually allows identification of which sets are needed to be assumed existent to claim that the theorem holds. The ontological import of reverse mathematics is thus quite clear and it is definitely a declination of another question Simpson asked in the preface of [Simpson 2009], namely

> What are the appropriate axioms for mathematics?

One can also intend 'appropriate' as epistemologically justifiable. In the last decade the epistemological import of reverse mathematics received more attention, thanks to recent development in the area and to the introduction of complementary approaches as computable reducibility and Weihrauch reducibility. These two research projects let slightly shift the focus of reverse mathematics as well towards an understanding of the computational content of the theorems, which is complementary to the traditional proof theoretic approach (see [Shore 2010; Hirschfeldt 2015]). This confluence of ideas and approaches allow to interpret the relationships among the theorems as carrying information about the computational core of some theorems and about the methodological core to which each subsystem of second order arithmetic can be associated. These reveal the typical kinds of reasoning underneath mathematical statements, for example if a theorem is equivalent to WKL_0, then compactness is indispensable in the proof of the theorem. For a more extensive discussion on this topic see [Simpson 2014b; Eastaugh 2019; Eastaugh 2018].

We believe that we already answered the fourth question. To add another reason we emphasise that the main subsystems of second order arithmetic studied in reverse mathematics correspond to programs in foundations of mathematics as constructivism, finitistic reductionism, predicativism, predicative reductionism, and impredicativity.

Reverse Mathematics

The axiom systems employed in reverse mathematics are subsystems of the theory of second order arithmetic Z_2, whose intended interpretation are the natural numbers and their subsets. The theory Z_2 is formulated in the language $\mathcal{L}_2$, a two sorted language with first order variables, $x, y, \dots$, intended to range over numbers, and second order variables $X, Y, \dots$ intended to vary over sets of numbers. The non logical symbols are the constant symbols 0 and 1, the functional symbols $+$ and $\times$, the relational symbols $<$ and $\in$ for membership. The latter symbol links first order with second order terms, while the remaining symbols concern only first order terms.

The basic axioms of Z_2 deal with the interpretation of $0, 1, +, \times, <$, so as to require the interpretation to be an ordered commutative semiring, and include a form of induction that applies only to sets:

$$\forall X\,((0 \in X \wedge \forall n\,(n \in X \rightarrow n+1 \in X)) \rightarrow \forall n\,(n \in X))$$

The most important axiom, for our purposes, is the axiom schema of comprehension:

$$\exists X \forall n (n \in X \leftrightarrow \varphi(n))$$

for each formula φ in $\mathcal{L}_2$ such that X is not free in φ.

A structure for Z_2 consists of a domain M and a set S_M of subsets of M, besides the interpretation of the non logical symbols. In the intended interpretation M is ω and S_M corresponds to $\mathcal{P}(\omega)$.

The subsystems of second order arithmetic are obtained limiting the comprehension axiom and the induction to specific classes of formulae. An *arithmetical formula* contains quantification only over number variables (it may contain free set variables). The number of alternations of quantifiers arranges the arithmetical formulae into an hierarchy. Δ^0_0-formulae, which contain only bounded quantifiers, form the base level. Σ^0_n-formulae, for each $n \in \mathbb{N}$, are formulae of the form $\exists x_0 \, \forall x_1 \ldots Q x_{n-1} \, (\psi(x_0, \ldots, x_{n-1}))$, such that ψ is Δ^0_0, $Q = \forall$ if n is odd and $Q = \exists$ if n is even. A Π^0_n-formula is the negation of a Σ^0_n-formula. A formula φ is Δ^0_n if it is Σ^0_n (or Π^0_n) and there exist a formula ψ which is Π^0_n (respectively Σ^0_n) and such that $\forall n(\varphi(n) \leftrightarrow \psi(n))$.

A Σ^1_n-formula is of the form $\exists X_0 \, \forall X_1 \ldots Q X_{n-1} \, (\psi(X_0, \ldots, X_{n-1}))$, such that ψ is arithmetical, $Q = \forall$ if n is odd and $Q = \exists$ if n is even. A Π^1_n-formula is the negation of a Σ^1_n-formula.

Definition 1. For each n, Σ^0_n*-induction*, denoted as $\mathsf{I}\Sigma^0_n$, is the schema of formulae

$$\varphi(0) \land \forall n \, (\varphi(n) \to \varphi(n+1)) \to \forall n(\varphi(n))$$

where φ is a Σ^0_n-formula.

For each n, Σ^0_n*-bounding*, denoted as $\mathsf{B}\Sigma^0_n$, is the schema of formulae

$$\forall m \, (\forall n < m \, \exists i \, (\varphi(n,i)) \to \exists b \, \forall n < m \, \exists i < b \, (\varphi(n,i)))$$

where φ is a Σ^0_n-formula.

For each n, *bounded* Σ^0_n*-comprehension* is the schema of formulae

$$\forall m \, \exists X \, \forall i \, (i \in X \leftrightarrow i < m \land \varphi(i))$$

where φ is a Σ^0_n-formula in which X does not occur freely.

For each n, $\mathsf{B}\Sigma^0_n$ lays strictly in between $\mathsf{I}\Sigma^0_n$ and $\mathsf{I}\Sigma^0_{n-1}$ (see [Paris and Kirby 1978]). Moreover, for each n, $\mathsf{I}\Sigma^0_n$ is equivalent to bounded-Σ^0_n-comprehension over RCA_0 (see [Simpson 2009, Exercise II.3.13]). Thus induction is comprehension in disguise for finite sets. Since reverse mathematics is interested in controlling the existence of sets, the restriction on induction seems quite natural to fulfil this purpose.

The base theory. RCA_0, Recursive Comprehension Axiom, is the subsystem of second order arithmetic obtained by limiting comprehension to Δ^0_1-formulae and induction to Σ^0_1-formulae. Since, Δ^0_1-definable sets coincide with computable sets by Post's theorem, it is often said that RCA_0 corresponds to computable mathematics (indeed computable sets form the minimal model for RCA_0).

RCA_0 suffices to prove certain, even non trivial, theorems, but more importantly offers a base theory to prove implications among statements. There are some criteria a good base theory T should satisfy as Hirschfeldt pointed out in [Hirschfeldt 2015].

> From a foundational point of view, we would like provability over T to have some philosophical meaning. From a combinatorial one, when we say that P and Q are equivalent over T, we are saying that P and Q have the same "fundamental combinatorics" up to the combinatorial procedures that can be performed in T, so we would like this class of procedures to be one we can understand and think of as natural in some sense.

The choice of RCA_0 as base theory is considered to respect the two requirements. Essentially, it guarantees that two principles are equivalent up to 'effective transformation'. This description of RCA_0 may lead to confusion. In fact, the mere fact that a statement is provable in RCA_0 does not guarantee that there exists an algorithm to solve it, because RCA_0 allows arguments by cases, whose case distinction is often non computably recognisable. Interestingly, the choice of RCA_0 as base theory is probably due to the recognition of some conceptual priority to the computable tools and methods.

The Big Five. There are five main subsystems in reverse mathematics. We already mentioned RCA_0, which is one of them. The others, which form a spine through Z_2, are the following:

WKL_0 RCA_0 *plus Weak König's Lemma.*

ACA_0 *comprehension and induction are limited to arithmetical formulae.*

ATR_0 *arithmetical comprehension can be iterated along any well-order.*

Π^1_1-CA_0 *comprehension is limited to Π^1_1-formulae.*

Weak König's Lemma states that each infinite binary tree has an infinite path. Those subsystems are commonly called the 'Big Five' because the majority of theorems analysed so far are equivalent to one of these subsystems. We refer to [Simpson 2009] for a detail exposition of the basic facts and for results in the area. We only recall the following well known equivalences (see [Simpson 2009, Theorem III.1.3, Lemma IV.4.4, Lemma VI.1.1] and [Marcone 1996, Theorem 6.5]), which are used in the thesis to prove reversals.

Theorem 2 (RCA_0)**.** ACA_0 *is equivalent to the following statement: for each injective function $f\colon \mathbb{N} \to \mathbb{N}$ there exists its range* $\operatorname{ran}(f)$.

Theorem 3 (RCA_0)**.** WKL_0 *is equivalent to the following statement: for each injective functions $f, g\colon \mathbb{N} \to \mathbb{N}$ such that* $\operatorname{ran}(f) \cap \operatorname{ran}(g) = \emptyset$ *there exists a set X such that* $\operatorname{ran}(f) \subseteq X$ *and* $\operatorname{ran}(g) \cap X = \emptyset$.

Theorem 4 (RCA_0)**.** Π^1_1-CA_0 *is equivalent to the following statements:*

1. *for each sequence of trees $\langle T_n \subseteq \mathbb{N}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$ there exists a set $X \subseteq \mathbb{N}$ such that $\forall n\,(n \in X \leftrightarrow T_n$ has a path$)$;*
2. LPP_0*: each non well founded tree has a leftmost path.*

The zoo of reverse mathematics. The Big Five do not exhaust the interesting subsystems of Z_2. In recent years a constellation of principles, mainly from combinatorics and more specifically from Ramsey's theory, have been studied in reverse mathematics. They revealed to form a zoo of principles, since they are hardly equivalent to other, even apparently similar, statements.

Ramsey's theorem for pairs RT^2_2 plays a prominent role in this picture. It was one of the first principles proved to lay between RCA_0 and ACA_0 and to be incomparable with WKL_0. After it many principles, often easy consequences of RT^2_2, were analysed and proved to be not only incomparable with WKL_0, but also not equivalent one to the other. [Hirschfeldt 2015] provides a very nice introduction to the zoo of reverse mathematics and it is also a precious source of references. We recall here only those principles mentioned in the thesis starting from Ramsey's theorem.

Definition 5. Let $c\colon [\mathbb{N}]^n \to k$ be a colouring, for some $n, k \in \mathbb{N}$. A set $H \subseteq \mathbb{N}$ is homogeneous for $i < k$ if for each $h_0, \dots, h_{n-1} \in H$ it holds that $c(h_0, \dots, h_{n-1}) = i$.

A colouring $c\colon [\mathbb{N}]^2 \to 2$ is stable if for each $x \in \mathbb{N}$ there exists $y \in \mathbb{N}$ and $i < 2$ such that $c(x, z) = i$ for each $z > y$.

Definition 6. Let $\langle S_n \mid n \in \mathbb{N}\rangle$ be a sequence of subsets of $\mathbb{N}$. A set $C \subseteq \mathbb{N}$ is cohesive if, for each $n \in \mathbb{N}$, either C is a subset of S_n, up to finitely many elements, or C is a subset of $\mathbb{N} \setminus S_n$, up to finitely many elements.

RT^n_k *For each colouring* $c\colon [\mathbb{N}]^n \to k$ *there exists an infinite homogeneous set.*

SRT^2_k *For each stable colouring* $c\colon [\mathbb{N}]^2 \to k$ *there exists an infinite homogeneous set.*

COH *For each sequence of subsets of* $\mathbb{N}$ *there exists an infinite cohesive set.*

Notoriously, RT^2_2 is equivalent to SRT^2_2 plus COH [Cholak, Jockusch, and Slaman 2001]. We denote as $\mathsf{RT}^n_{<\omega}$ the statement 'for each k RT^n_k holds' and we recall that $\mathsf{RT}^1_{<\omega}$ is equivalent to $\mathsf{B}\Sigma^0_2$ [Hirst 1987].

Among the consequences of RT^2_2 we mainly focus on ADS and its stable variant.

Definition 7. A linear order $(L, <_L)$ is stable if every element has either finitely many predecessors or finitely many successors.

ADS *For each linear order there exists an infinite ascending or an infinite descending chain.*

SADS *For each stable linear order there exists an infinite ascending or an infinite descending chain.*

The previous statements are Π^1_2-statements, so of the form $\forall X\, \exists Y\, (\varphi(X) \to \psi(X, Y))$, for some arithmetical formulae φ and ψ. For principles of this form we often refer, following a well establish terminology, to X as an instance of the principle and to Y as a solution.

The thesis mentions basic notions from computability theory. We refer to [Soare 2016] as a monograph on the topic.

And combinatorial principles

In the thesis some combinatorial principles are analysed from the reverse mathematics point of view. The literature on this topic is rather vast, as we already pointed out in the previous section and as witnessed by [Hirst 1987; Hirst 1990; Hirst 1992; Hirst and Hughes 2015; Hirst and Hughes 2016; Schmerl 2005; Cenzer and Remmel 2005], which are only a sample of the articles in this area. For a complementary approach to the topic of combinatorics and computability theory see the surveys [Gasarch 1998; Downey 1998], where it is also possible find many references.

In this section we recall some basic definitions from order theory and graph theory used extensively in the thesis. More definitions and notations are introduced along the chapters when needed. For a deeper introduction to graph theory and order theory see [Diestel 2017] and [Harzheim 2005]. When not specified differently we intend that the objects we consider are countable.

A *graph* is a pair (V, E) where $V \subseteq \mathbb{N}$ and $E \subseteq \mathbb{N} \times \mathbb{N}$ is a symmetric relation. We write $v\, E\, u$ to mean that $\{v, u\} \in E$. Notice that 'graph' denotes an undirected graph, if not specified differently. If $V' \subseteq V$, then (V', E) denotes the induced subgraph, i.e. E stands for $E \cap V' \times V'$.

The graph $(V, \overline{E})$ is the *complementary graph* of a graph (V, E) if the two graphs have dual edges, namely for each vertices $v, u \in V$ it holds that $v\, \overline{E}\, u$ if and only if $\neg v\, E\, u$.

Definition 8. A graph (V, E) is a *comparability graph* if there exists a partial order $(V, \prec)$ such that for each vertices $v, u \in V$ it holds that $v\, E\, u$ if and only if either $v \prec u$ or $u \prec v$.

Comparability graphs, as the name suggests, represent the comparability relation of orders. We often call such orders the *orders associated* to a comparability graph. Building on this idea, we name a graph an *incomparability graph* if edges represent the incomparability relation of an order. Notice that an incomparability graph is the complementary graph of a comparability graph.

Definition 9. Let (V, E) be a graph. A *path* in (V, E) is a sequence $v_0, \ldots, v_n$, for some $n \in \mathbb{N}$, of elements of V such that $v_i \, E \, v_{i+1}$ for each $i < n$. The length of the path $v_0, \ldots, v_n$ is n.

A *cycle* in (V, E) is a path with $v_n = v_0$. A *simple cycle* $v_0, \ldots, v_n$ is a cycle such that each vertex in $v_0, \ldots, v_{n-1}$ does not occur more than once. A *chord* of a cycle $v_0, v_1, \ldots, v_n$ is a pair $\langle v_i, v_j \rangle$ for $i < j \leq n$ such that $v_i \, E \, v_j$ and $2 \leq |i - j| < n - 1$. The chord is *triangular* if either $j = i + 2$ or $i = 1$ and $j = n - 1$.

A *partial order*, or poset for short, is a pair $(P, <_P)$ where $P \subseteq \mathbb{N}$ and $<_P \subseteq \mathbb{N} \times \mathbb{N}$ is a irreflexive, asymmetric and transitive relation. We generally refer to $(P, <_P)$ simply as P. If $(P, <_P)$ is an order, and $p \not<_P q$ and $q \not<_P p$ hold, then we write $p \mid_P q$. In this case the p and q are called 'incomparable'.

The symbol $<$ denotes the standard order on $\mathbb{N}$.

A set $C \subseteq P$ is an *antichain* if $c \mid_P d$ for each $c, d \in C$. A set $C \subseteq P$ is a *chain* if $c \nmid_P d$ for all distinct $c, d \in C$. A *linear order* is a poset which is also a chain. We often deal with chains of a specific order type.

Definition 10. Let $(P, <_P)$ be a poset. A sequence $A = \langle a_n \in P \mid n \in \mathbb{N} \rangle$ is an *ascending* chain if $n < m$ implies $a_n <_P a_m$, for each integers n and m. A is *descending* if for each n and m, $n < m$ implies $a_m <_P a_n$. Occasionally we use ω and ω^* chain to indicate ascending and descending chains respectively.

Notice that if $(P, <_P)$ is a poset and $A \subseteq P$ is such that, for each $a, a' \in A$, $a < a'$ implies $a <_P a'$ (i.e. the $<$-order of the elements of A correspond to the $<_P$-order), then it is possible to define computably an ascending sequence of elements of A simply enumerating them in $<$-increasing order. For this reason we sometimes tacitly oscillate between the two notions. An analogous observation holds for descending sequences.

Definition 11. Let $(P, <_P)$ be a poset and A and D be subsets of P (in many cases A and D will be chains in P). We say that D *is above* A, or $A <_P D$, if $a <_P d$ for each $a \in A$ and each $d \in D$. When $A = \{a\}$ (or $D = \{d\}$) we write $a <_P D$ (resp. $A <_P d$).

Analogously, we say that D *is incomparable with* A, or $A \mid_P D$, if $a \mid_P d$ for each $a \in A$ and each $d \in D$. When $A = \{a\}$ (or $D = \{d\}$) we write $a \mid_P D$ (resp. $A \mid_P d$).

The *width* of a partial order $(P, <_P)$ is the supremum of the sizes of its antichains. The *chain-decomposition-number* of P is the least number k such that P is union of k chains. The *height* of P is the supremum of the sizes of its chains.

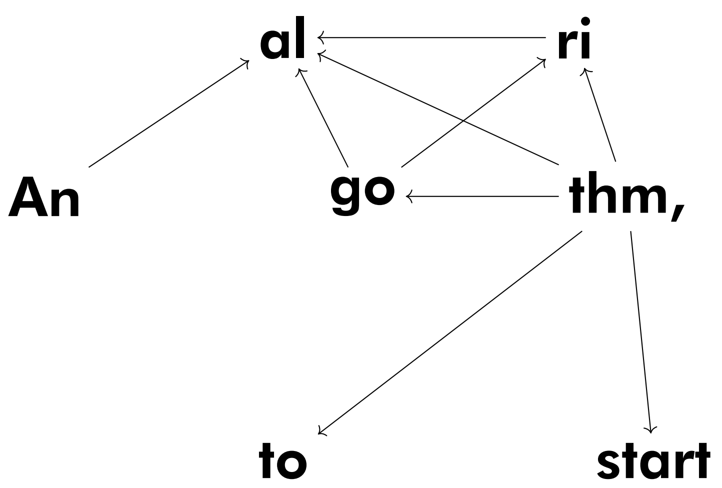
The symbol $\mathbb{N}^{<\mathbb{N}}$ denotes the set of finite sequences of natural numbers, while $2^{<\mathbb{N}}$ stands for the set of finite sequences from $\{0, 1\}$. A *tree* T is a subset of $\mathbb{N}^{<\mathbb{N}}$ such that if $\sigma \in T$ then every initial segment of σ is in T. T is *finitely branching* if each node in T has only finitely many successors in T, and it is *binary* if it is a subset of $2^{<\mathbb{N}}$. A *path* in T is a function $f \colon \mathbb{N} \to \mathbb{N}$ such that $\langle f(0), \ldots, f(n) \rangle \in T$ for each n.

If $\sigma \in \mathbb{N}^{<\mathbb{N}}$, then $|\sigma|$ denotes the length of σ. If $\sigma, \tau \in \mathbb{N}^{<\mathbb{N}}$ we write $\tau \sqsubseteq \sigma$ whenever $\forall n < |\tau| \, (\tau(n) = \sigma(n))$.

OVERVIEW OF THE THESIS The thesis is divided into three main parts. In the first one an on-line algorithm to transitivelly reorient infinite pseudo-transitive oriented graphs is defined. Interval graphs and interval orders are the common theme of the second part of the thesis. The third part is devoted to the analysis of two theorems proved by Rival and Sands.

Each part begins with an introduction to the topic and with an overview of the contents.

PART I



REORIENTATIONS

CHAPTER 1

Content

An asymmetric and irreflexive relation $\to$ is an *orientation* of a graph (V, E) if for every $a, b \in V$ we have $a\,E\,b$ if and only if $a \to b$ or $b \to a$. An orientation $\to$ is *transitive* if for every $a, b, c \in V$ such that $a \to b$ and $b \to c$ we have also $a \to c$. Graphs having a transitive orientation are also known as *comparability graphs*: in fact E is the comparability relation of the strict partial order $\to$.

A characterization of comparability graphs was given by Alain Ghouila-Houri [Ghouila-Houri 1962; Ghouila-Houri 1964] (using a different terminology and dealing only with finite graphs) and reproved by Paul Gilmore and Alan Hoffman [Gilmore and Hoffman 1964]. Further results were obtained by Tibor Gallai [Gallai 1967], who provided another characterisation of comparability graphs listing all the forbidden subgraphs.

Theorem 1.1. *An undirected graph has a transitive orientation if and only if every cycle of odd length has a triangular chord.*

In Figure 1.1 the left graph has a cycle of length nine with no triangular chord, while the right one has no cycles of odd length without triangular chords.

Figure 1.1: A graph which is not a comparability graph, to the left, and a comparability graph, to the right.

This chapter is the outcome of a joint research with Alberto Marcone.
We thank Nicola Gigante and Paul Shafer for useful discussions about the topic of the paper.

The forward direction of Theorem 1.1 is easily proved. The backward direction was proved directly by Gilmore and Hoffman, while the original proof by Ghouila-Houri uses an intermediate step. The latter approach is also taken in several expositions of the theorem ([Berge 1976, Theorem 16.8], [Fishburn 1985, Theorem 1.7], [Harzheim 2005, Theorem 11.2.5]) and hinges on the following notion.

An orientation $\rightarrow$ is *pseudo-transitive* if for every $a, b, c \in V$ such that $a \rightarrow b$ and $b \rightarrow c$ we have also either $a \rightarrow c$ or $c \rightarrow a$.

Ghouila-Houri proves the backward direction of Theorem 1.1 by first showing that if every cycle of odd length has a triangular chord then there exists a pseudo-transitive orientation, and then that any pseudo-transitive orientation can be further reoriented to obtain a transitive one.

The effectiveness of Theorem 1.1 has already been studied, in particular using the framework of reverse mathematics ([Simpson 2009] is the basic reference in this area), by Jeffry Hirst in his PhD thesis [Hirst 1987, Theorem 3.20]. Hirst indeed showed that a compactness argument (disguised as an application of Zorn's lemma in [Gilmore and Hoffman 1964] and of Rado's theorem in [Fishburn 1985; Harzheim 2005]) is necessary for countable graphs and hence the theorem is not computably true. The following lemma includes Hirst's theorem and provides a direct proof for it.

Lemma 1.2. *The following are equivalent over the base system* RCA_0*:*

1. WKL_0*;*
2. *every countable graph such that every cycle of odd length has a triangular chord has a transitive orientation;*
3. *every countable graph such that every cycle of odd length has a triangular chord has a pseudo-transitive orientation.*

Proof. To prove that (1) implies (2) assume the statement is true for finite graphs (any of the proofs mentioned above can be formalised in RCA_0). Let (V, E) be a countable graph such that every cycle of odd length has a triangular chord. We define a binary tree $T \subseteq 2^{<\mathbb{N}}$, whose paths code transitive orientations of E. Let $\sigma \in 2^{<\mathbb{N}}$ be in T if for each $\langle i, j\rangle, \langle j, i\rangle, \langle j, k\rangle, \langle i, k\rangle < |\sigma|$ the following conditions are satisfied.

- if $\sigma(\langle i, j\rangle) = 1$, then $i \mathrel{E} j$,
- if $i \mathrel{E} j$, then exactly one between $\sigma(\langle i, j\rangle) = 1$ and $\sigma(\langle j, i\rangle) = 1$ holds,
- $\sigma(\langle i, j\rangle) = 1$ and $\sigma(\langle j, k\rangle) = 1$ imply $\sigma(\langle i, k\rangle) = 1$.

The tree T is infinite because for each finite subgraph $(\{0, \ldots, n-1\}, E \restriction \{0, \ldots, n-1\})$ there exists a transitive orientation $\rightarrow_n$ by assumption. Therefore, each $\sigma \in 2^{<\mathbb{N}}$ such that $\forall i, j < n\, (\sigma(\langle i, j\rangle) = 1 \leftrightarrow i \rightarrow_n j)$ belongs to T. Let g be a path and set $i \rightarrow j$ if and only if $g(\langle i, j\rangle) = 1$. By construction $\rightarrow$ transitively orients (V, E).

The implication from (2) to (3) is trivial.

To check that (3) implies (1) let $f, g \colon \mathbb{N} \rightarrow \mathbb{N}$ be injective functions such that $\forall m\, \forall n\, (f(m) \neq g(n))$. Define a graph (V, E) as follows: $V = \{a_n, b_n, c_n, d_n \mid n \in \mathbb{N}\} \cup \{x_m, y_m \mid m \in \mathbb{N}\}$ and E is defined by the following clauses for each n and m:

$$\begin{cases} c_n \mathrel{E} a_n \mathrel{E} b_n \mathrel{E} c_n \mathrel{E} d_n; & \\ x_m \mathrel{E} a_n & \text{if } f(m) = n; \\ y_m \mathrel{E} b_n & \text{if } g(m) = n. \end{cases}$$

Every cycle of odd length has a triangular chord because every connected component of (V, E) is isomorphic to a subgraph of the right graph in Figure 1.1. Let $\rightarrow$ be a pseudo-transitive orientation of E. It is easy to check that the set

$$X = \{n \in \mathbb{N} \mid b_n \rightarrow a_n \leftarrow c_n \vee b_n \leftarrow a_n \rightarrow c_n\}$$

contains the range of f and is disjoint from the range of g. $\square$

The proof of Lemma 1.2 yields the following results in the framework of computability theory and of the Weihrauch lattice (see [Brattka, Gherardi, and Pauly 2017] for an introduction to this research program).

Lemma 1.3. *There exists a computable graph such that every cycle of odd length has a triangular chord which has no computable pseudo-transitive orientation.*

Every computable graph such that every cycle of odd length has a triangular chord has a low transitive orientation.

Lemma 1.4. *Consider the multi-valued functions that map every countable graph such that every cycle of odd length has a triangular chord to the set of its transitive (resp. pseudo-transitive) orientations. Each of these two multi-valued functions is Weihrauch equivalent to choice on Cantor space.*

Starting with [Gilmore and Hoffman 1964] there has been an interest in algorithms providing transitive orientations for finite comparability graphs. For example, the influential textbook [Golumbic 2004] devotes a whole chapter to algorithmic aspects of comparability graphs, including complexity issues. However, the first part of Lemma 1.3 shows that there is no algorithm to (pseudo-)transitively orient countable computability graphs. In particular, an algorithm which computes a (pseudo-)transitive orientation of finite comparability graphs cannot work in an incremental way (i.e. extending the previous orientation as new vertices are added to the graph), and thus is not on-line. Here we understand the notion of on-line algorithm as defined in [Bazhenov et al. 2019], which is a recent survey of the theoretical study of on-line algorithms for computable structures. To be more precise, we assume the input of an *on-line (incremental) algorithm* to consist of vertices coming one at a time together with all information about the edges connecting them to previous vertices. (So at step s the size of the input increases of at most s.) When the algorithm sees a new vertex, it must reorient all the edges connecting it to previous vertices while preserving the reorientations already set at previous stages.

Lemmas 1.2, 1.3, and 1.4 provide an analysis of the first step in Ghouila-Houri's proof of Theorem 1.1. Our main interest is the analysis of the complexity of the second step of this proof, which is best stated using *oriented graphs*, i.e. directed graphs such that at most one of the edges between two vertices exist. In this paper we abbreviate 'oriented graph' as *ograph*. The notions of pseudo-transitivity and transitivity are readily extended to ographs, and a reorientation of an ograph is an ograph obtained by reversing some of the edges. Then the second step of Ghouila-Houri's proof is the following result.

Theorem 1.5. *Every pseudo-transitive ograph has a transitive reorientation.*

This is the main lemma in [Ghouila-Houri 1962], the lemma on page 329 in [Ghouila-Houri 1964], Theorem 16.7 in [Berge 1976], Theorem 1.5 in [Fishburn 1985], and Theorem 11.2.2 in [Harzheim 2005]. Ghouila-Houri's proof deals only with finite graphs and uses induction on the number of vertices. The same proof is presented in [Berge 1976; Fishburn 1985; Harzheim 2005] and extended to the infinite case by some compactness argument. From this proof it is easy to extract an algorithm to transitively reorient finite pseudo-transitive ographs. However, the induction step requires, in a nutshell, partitioning the set of vertices into two subsets with specific properties, to reorient each of the induced subographs by induction hypothesis, and then to set the reorientation between them (see Algorithm 1 for the pseudocode of the algorithm). Thus this algorithm is not incremental and does not apply to infinite ographs. This analysis led us to conjecture that we could obtain results similar to Lemmas 1.2, 1.3 and 1.4 for Theorem 1.5. We were actually wrong.

Overview of the main results. We state the main result of this chapter in various different ways (the first three items of the theorem correspond to the approaches of Lemmas 1.2, 1.3 and 1.4, respectively).

Main Theorem.

1. RCA_0 *proves that every countable pseudo-transitive ograph has a transitive reorientation;*
2. *every computable pseudo-transitive ograph has a computable transitive reorientation;*

3. *the multi-valued function that maps a countable pseudo-transitive ograph to the set of its transitive reorientations is computable;*

4. *there exists an on-line (incremental) algorithm to transitively reorient pseudo-transitive ographs;*

5. *Player II has a winning strategy for the following game: starting from the empty graph, at step $s+1$ player I plays a pseudo-transitive extension $(V_s \cup \{x_s\}, \to_{s+1})$ of the pseudo-transitive ograph $(V_s, \to_s)$ he played at step s. Player II replies with a transitive reorientation $\prec_{s+1}$ of $\to_{s+1}$ such that $\prec_{s+1}$ extends $\prec_s$ she defined at step s. Player II wins if and only if she is always able to play.*

We concentrate on proving (4) of the Main Theorem, as this easily implies (2) and (3), while (5) is just a restatement in a different language of (4) for countable ographs. More attention has to be paid in order to derive (1) from (4), since in this case one also needs to check that the proof of the correctedness of the algorithm can be caried out in RCA_0, namely that the amount of comprehension used is limited to Δ^0_1-formulae and the amount of induction used is limited to Σ^0_1-formulae. The unique subtle passage on this respect concerns the double induction used in the proof of Property 1.38, which however is fine since the matrix of the formula is Δ^0_0.

We deal explicitly only with countable ographs; however it is easily seen that our algorithm applies to ographs of any cardinality, as long as the set of vertices can be well-ordered.

An upper bound for the complexity of the algorithm we define (when applied to finite pseudo-transitive ographs) is $O(|V|^3)$. The problem of orienting comparability graphs can be solved by an algorithm with complexity $O(\delta \cdot |E|)$, where δ is the maximum degree of a vertex ([Golumbic 2004, Theorem 5.33]), and further fine-tuning has been subsequently made.

Overview of the chapter. Section 1.1 contains the preliminary definitions and a presentation of two pseudo-transitive ographs with transitive reorientations which are the main obstacles in designing the algorithm. Sections 1.2 and 1.3 analyse in detail these two configurations. In Section 1.4 we present the on-line algorithm and prove its correctness. We also sketch the ideas needed to obtain the upper bound for the complexity mentioned above.

Algorithm 1 Ghuilà-Houri algorithm

```
Require: (V, →) is a pseudo-transitive digraph
Require: V is an initial segment of (ℕ, <)
  procedure REORIENT(V, →)
    if (V, →) is transitive then
      for a ∈ V do
        for b ∈ V do
          if a | b then
            a ⊀ b, b ⊀ a
          else if a → b then
            a ≺ b
          else
            b ≺ a
          end if
        end for
      end for
    else
      let i, j, k be such that i → j → k → i                ▷ ijk is non transitive
      n ← 0
      while n ∈ V do
        if n ∈ N(i) then
          if n ∈ N(j) then
            if n ∈ N(k) then
              case=1                    ▷ Case=1: if n ∈ N(i), then n ∈ N(j) ∩ N(k)
              n ← n + 1
            else
              case=2
              witness=[i,j,k]           ▷ Case=2 is witnessed by n ∈ N(i) ∩ N(j) \ N(k)
            end if
          else
            case=2
            witness=[i,k,j]             ▷ Case=2 is witnessed by n ∈ N(i) ∩ N(k) \ N(j)
          end if
        else if n ∈ N(j) then    ▷ To check if case=1, check if each N(j) ∩ N(k) ⊆ N(i)
          if n ∈ N(i) then
            if n ∈ N(k) then
              case=1
              n ← n + 1
            else
              case=2
              witness=[i,j,k]
            end if
          else
            case=2
            witness=[j,k,i]
          end if
        end if
      end while
```

```
        if case=1 then                                          ▷ If case 1 holds
            REORIENT(V \ {j, k}, →)                              ▷ Call REORIENT: it outputs ≺′
            for n ∈ V do
                for m ∈ V do
                    if m = j ∨ m = k then
                        if n ≺′ i then
                            n ≺ j, n ≺ k
                        else if i ≺′ n then
                            j ≺ n, k ≺ n
                        else
                            n ⊀ j, j ⊀ n, n ⊀ k, k ⊀ n
                        end if
                    else
                        n ≺ m according to ≺′
                    end if
                end for
            end for
        else                                                    ▷ If case 2 holds
            A = ∅
            for m ∈ V do
                if m → witness[0] → witness[1] → m then
                    m ∈ A
                else
                    m ∈ Ā
                end if
            end for
            REORIENT(A, →)                                       ▷ Call REORIENT: it outputs ≺_A
            REORIENT(V \ A ∪ {witness[2]}, →)                    ▷ Call REORIENT: it outputs ≺_Ā
            for u ∈ V do
                if u ∈ V \ A then
                    for v ∈ V do
                        if v ∈ V \ A then
                            u ≺ v according to ≺_Ā
                        else
                            if u ≺_Ā witness[2] then
                                u ≺ v
                            else if witness[2] ≺_Ā u then
                                u ≺ v
                            else
                                v ⊀ u, v ⊀ u
                            end if
                        end if
                    end for
                else
                    for v ∈ V do
                        if v ∈ A then
                            u ≺ v according to ≺_A
                        end if
                    end for
                end if
            end for
        end if
    end if
end procedure
```

1.1 Preliminaries

We have already introduced our terminology in the previous pages; we now give the formal definitions of the central notions.

Definition 1.6. An ograph $(V, \to)$ is *transitive* if for each $a, b, c \in V$, if $a \to b \to c$, then $a \to c$. $(V, \to)$ is *pseudo-transitive* if for each $a, b, c \in V$, if $a \to b \to c$, then $a \to c$ or $c \to a$.

A relation R on V is a *reorientation* of $\to$, if for each $a, b \in V$, if $a \to b$ then either $a\, R\, b$ or $b\, R\, a$ and if $a\, R\, b$ then either $a \to b$ or $b \to a$.

A *transitive reorientation* of $(V, \to)$ is a reorientation of $(V, \to)$ which is also transitive. In this case we often use $\prec$ in place of R.

A triple $(V, \to, \prec)$ is a *Ghouila-Houri triple* (*GH-triple* for short) if $(V, \to)$ is a pseudo-transitive ograph and $\prec$ a transitive reorientaion of $\to$.

Notice that each reorientation R of $(V, \to)$ preserves both $\to$-comparability and $\to$-incomparability. In other words, the undirected graphs associated with $(V, \to)$ and with (V, R) coincide.

Notation 1.7. Let $(V, \to)$ be an ograph and $a, b, c \in V$.

- $a - b$ means that either $a \to b$ or $b \to a$;
- $N(a) = \{b \in V \mid a - b\}$ is the *neighbourhood of* a;
- $a \mid b$ means that neither $a \to b$ nor $b \to a$;
- when we write '$a - b$ by $\mathrm{pt}(c)$' we mean that we know that $\to$ is pseudo-transitive and we are deducing $a - b$ because we have either $a \to c \to b$ or $b \to c \to a$.

Definition 1.8. Let $(V, \to)$ be an ograph. If $V' \supseteq V$ we say that $(V', \to')$ is an *extension* of $(V, \to)$ if $(V', \to')$ is an ograph such that for every $a, b \in V$ we have $a \to b$ if and only if $a \to' b$.

An on-line algorithm computing a transitive reorientation of a pseudo-transitive ograph must produce at each step a reorientation which can further be extended, in the sense made precise by the following definition.

Definition 1.9. A GH-triple $(V, \to, \prec)$ is *extendible* if for every $(V \cup \{x\}, \to')$, pseudo-transitive extension of $(V, \to)$, there exists $\prec'$ which extends $\prec$ and is such that $(V \cup \{x\}, \to', \prec')$ is a GH-triple.

Some simple cases of GH-triples which are not extendible are depicted in Figures 1.2 and 1.3.

Example 1.10. In Figure 1.2 we have the *transitive triangle examples*: $\to$ is transitive on $\{a, b, c\}$ and the transitive reorientation is defined by $a \prec c \prec b$. Notice that in the left ograph we have $a \to c \leftarrow b$, while in the right one we have $a \leftarrow c \to b$: in both cases all edges involving the vertex c have the same direction. We can add a vertex x connected to c by an edge going in the same direction and connected with neither a nor b. Then $(\{a, b, c, x\}, \to')$ is pseudo-transitive and if $\prec'$ is a reorientation of $\to'$ extending $\prec$ we must have either $x \prec' c$ or $c \prec' x$: both choices lead to the failure of transitivity of $\prec'$.

Example 1.11. In Figure 1.3 we have the $2 \oplus 2$ *example*: there are two edges $a \to c$ and $b \to d$ (with no other edges between these four vertices) and the transitive reorientation defined by $a \prec c$ and $d \prec b$. In the left ograph we add a vertex x such that $a \to' x$, $b \to' x$, $x \mid' c$ and $x \mid' d$. Then $(\{a, b, c, d, x\}, \to')$ is pseudo-transitive. Suppose $\prec'$ were a transitive reorientation of $\to'$ extending $\prec$: since $a -' x$ and $x \mid' c$, then $a \prec c$ implies $a \prec' x$; since $b -' x$ and $x \mid' d$, then $d \prec b$ implies $x \prec' b$. But $a \prec' x \prec' b$ is not compatible with $a \mid b$. The situation in the right ograph is the same as the previous one as far as the first four vertices are concerned, but the new vertex x is now such that $x \to' c$, $x \to' d$, $x \mid' a$ and $x \mid' b$. We can argue analogously to show that $(\{a, b, c, d, x\}, \to')$ is a pseudo-transitive ograph with no transitive reorientation extending $\prec$.

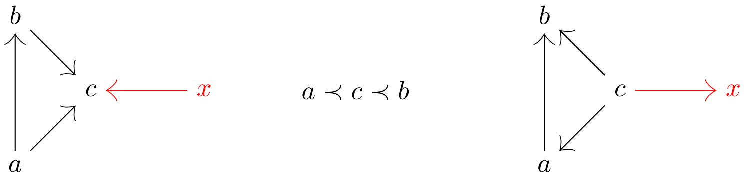


Figure 1.2: The transitive triangle examples.

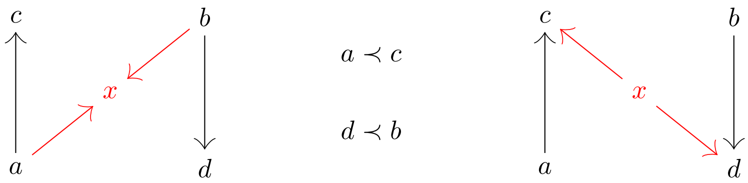


Figure 1.3: The $2 \oplus 2$ example.

We eventually show that the examples above are the only obstructions to extendibility of a GH-triple. To do this we analyse in detail Examples 1.10 and 1.11 using the following notions.

Definition 1.12. Let $(V, \rightarrow, \prec)$ be a GH-triple. If $(V \cup \{x\}, \rightarrow')$ is a pseudo-transitive extension of $(V, \rightarrow)$ define

$$N^+(x) = \{a \in N(x) \mid \forall b\,(a \prec b \Rightarrow b \in N(x))\};$$
$$N^-(x) = \{a \in N(x) \mid \forall b\,(b \prec a \Rightarrow b \in N(x))\}.$$

(Here $N(x)$ is the neighbourhood of x in $(V \cup \{x\}, \rightarrow')$.)

Remark 1.13. Under the hypothesis of the previous definition we have that if $a \in N(x) \backslash N^+(x)$, $b \in N^+(x)$ and $a - b$, then $a \prec b$. In fact, since $a \notin N^+(x)$ there is $d \succ a$ with $d \mid x$. If $b \prec a$, then $b \prec d$ against $b \in N^+(x)$. Thus $a \prec b$.

Similarly, if $c \in N(x) \setminus N^-(x)$, $b \in N^-(x)$, and $b - c$, then $b \prec c$.

The next lemma states some properties of extendible GH-triples.

Lemma 1.14. *Let $(V, \rightarrow, \prec)$ be an extendible GH-triple. Then for any $(V \cup \{x\}, \rightarrow')$ pseudo-transitive extension of $(V, \rightarrow)$ we have:*

1. *$N(x) = N^+(x) \cup N^-(x)$;*
2. *$N^-(x) \setminus N^+(x) \prec N^+(x) \setminus N^-(x)$.*

Proof. If condition (1) does not hold for some pseudo-transitive extension $(V \cup \{x\}, \rightarrow')$, then there exist $c \in N(x)$ and $a, b \notin N(x)$ such that $a \prec c \prec b$. This impedes both $x \prec' c$ and $c \prec' x$ for any transitive reorientation of $\rightarrow'$ with $\prec' \supseteq \prec$. (Notice that we found in $(V, \rightarrow, \prec)$ a copy of one of the transitive triangle examples.)

If condition (2) does not hold for some pseudo-transitive extension $(V \cup \{x\}, \rightarrow')$, then there exist $a \in N^-(x) \setminus N^+(x)$ and $b \in N^+(x) \setminus N^-(x)$ such that $a \not\prec b$. Since $a \in N^-(x) \setminus N^+(x)$ there exists c such that $a \prec c$ and $c \mid' x$. Since $b \in N^+(x) \setminus N^-(x)$, there exists d such that $d \prec b$ and $d \mid' x$. If $\prec'$ were a transitive reorientation of $\rightarrow'$ with $\prec' \supseteq \prec$ then these conditions imply respectively $a \prec' x$ and $x \prec' b$; it would follow $a \prec' b$, contrary to $a \not\prec b$. (Notice that in this case we found in $(V, \rightarrow, \prec)$ a copy of the $2 \oplus 2$ example.) $\square$

1.2 Avoiding the transitive triangle examples

This section is devoted to a careful study of the first condition of Lemma 1.14. The next lemma shows that this condition captures precisely the lack of the transitive triangle examples. Recall that in that situation $(V, \to, \prec)$ is a GH-triple. Moreover, there exist $a, b, c \in V$ such that $a \prec c \prec b$ and the new vertex x is connected with c, but not with a and b. Notice that this might happen only if a, b, c form a transitive triangle and either $a \to c \leftarrow b$ or $a \leftarrow c \to b$.

Lemma 1.15. *Let* $(V, \to, \prec)$ *be a GH-triple and* $(V \cup \{x\}, \to')$ *be a pseudo-transitive extension of* $(V, \to)$*, then* $N(x) = N^+(x) \cup N^-(x)$ *is equivalent to* $\forall a, b, c \in V\ (a \prec c \prec b \land x -' c \Rightarrow x -' a \lor x -' b)$.

Proof. Notice that $c \in N(x) \setminus N^+(x) \cup N^-(x)$ means that there exist a and b such that $a \prec c \prec b$ and $a, b \notin N(x)$. From this observation the equivalence is immediate. $\square$

Lemma 1.15 involves all possible pseudo-transitive extensions of $(V, \to)$ by one vertex. It is convenient to have a characterization of the GH-triples such that $N(x) = N^+(x) \cup N^-(x)$ for every pseudo-transitive extension, which involves only the GH-triple itself. To this end we introduce two formulae, Φ and Ψ. In order to do this, we define formulae $\varphi(a, b, c)$ and $\psi(a, b, c)$ which do not mention the reorientation $\prec$. Notice that Lemmas 1.14 and 1.15 imply that the non extendibility of $\prec$ may be caused by only three vertices. With this in mind, it is not hard to understand the rationale for $\varphi(a, b, c)$, $\psi(a, b, c)$, Φ, and Ψ.

Definition 1.16. Let $(V, \to, \prec)$ be a GH-triple. Let $\varphi(a, b, c)$ assert the existence of $e_0, \ldots, e_n \in V$ such that:

(φ_1) $c \to e_0$;

(φ_2) $\forall i < n\, ((a \to e_i \land b \to e_i \to e_{i+1}) \lor (e_{i+1} \to e_i \to b \land e_i \to a))$;

(φ_3) $a \to e_n \to b \lor b \to e_n \to a$.

Then Φ is

$$\forall a, b, c \in V\ (a \to c \leftarrow b \land a \prec c \prec b \Rightarrow \varphi(a, b, c)).$$

Symmetrically, let $\psi(a, b, c)$ assert the existence of $e_0, \ldots, e_n \in V$ such that:

(ψ_1) $e_0 \to c$;

(ψ_2) $\forall i < n\, ((a \to e_i \land b \to e_i \to e_{i+1}) \lor (e_{i+1} \to e_i \to b \land e_i \to a))$;

(ψ_3) $a \to e_n \to b \lor b \to e_n \to a$.

Then Ψ is

$$\forall a, b, c \in V\ (a \leftarrow c \to b \land a \prec c \prec b \Rightarrow \psi(a, b, c)).$$

Notice that the only difference between φ and ψ occurs in conditions (φ_1) and (ψ_1), where the direction of the edge is reversed. Φ and Ψ further differ in applying to triples such that $a \to c \leftarrow b$ and $a \leftarrow c \to b$ respectively.

Remark 1.17. Let $(V, \to, \prec)$ be a GH-triple. Fix $a, b, c \in V$. If $e_0, \ldots, e_n$ witness $\varphi(a, b, c)$ (or $\psi(a, b, c)$) then they witness $\varphi(b, a, c)$ (resp. $\psi(b, a, c)$) as well.

The following duality principle is useful to avoid checking Φ and Ψ separately.

Remark 1.18. Using Remark 1.17 it is immediate to notice that $(V, \to, \prec)$ satisfies Φ if and only if $(V, \leftarrow, \succ)$ (i.e. the ograph and the reorientation where all edges are reversed) satisfies Ψ.

We start with some properties concerning basic facts about φ and ψ.

Property 1.19. *Let* $(V, \to)$ *be a pseudo-transitive ograph. Suppose that* $a \to c \leftarrow b$ *and* $\varphi(a, b, c)$ *is witnessed by* $e_0, \ldots, e_n$*. Then there exists* $k \leq n$ *such that* $e_k, \ldots, e_n$ *witness* $\varphi(a, b, d)$ *for each* $d \in V$ *such that* $d \mid c$ *and* $a - d - b$*.*

The same holds starting from $a \leftarrow c \to b$ *and* $\psi(a, b, c)$*, and concluding that* $e_k, \ldots, e_n$ *witness* $\psi(a, b, d)$*.*

Proof. Suppose we are in the first case, i.e. $a \to c \leftarrow b$ and $e_0, \ldots, e_n$ witness $\varphi(a, b, c)$. Let $k \leq n$ be largest such that $c \to e_k$, and notice that $e_k, \ldots, e_n$ witness $\varphi(a, b, c)$ as well.

We claim that $e_i \to c$ for all i such that $k < i \leq n$. The claim is proved by a 'backward' induction. We obtain $e_n - c$ by (φ_3) and pt(b) or pt(a). Hence $e_n \to c$ by our assumption (unless $n = k$). Suppose now that $e_{i+1} \to c$. If $e_i \to a$, then $e_i - c$ by pt(a). Otherwise, $e_i \to e_{i+1}$ by (φ_2) and so $e_i - c$ by pt(e_{i+1}). Hence, if $i > k$ we have $e_i \to c$.

Let now d be such that $d \mid c$ and $a - d - b$. In particular we have $a \to d \leftarrow b$. Notice that to check that $e_k, \ldots, e_n$ witness $\varphi(a, b, d)$ conditions (φ_2) and (φ_3) are identical to conditions (φ_2) and (φ_3) of $\varphi(a, b, c)$, since they concern only the vertices a and b. We are left to prove that condition (φ_1) is satisfied, namely that $d \to e_k$. Since $d \mid c$ and $c \to e_k$ it suffices to show that $d - e_k$.

To this end we prove that indeed we have $e_i - d$ for all i such that $k \leq i \leq n$, again by a 'backward' induction. Since $a \to d \leftarrow b$ and either $e_n \to a$ or $e_n \to b$ by (φ_3), we have $e_n - d$ by either pt(a) or pt(b). Now, assuming $i \geq k$ and $e_{i+1} - d$ so that $d - e_{i+1} - c$, we must have $e_{i+1} \to d$ because $e_{i+1} \to c$ by the choice of k. If $a \to e_i$ condition (φ_2) of $\varphi(a, b, c)$ implies $e_i \to e_{i+1}$ and hence $e_i - d$ by pt(e_{i+1}). If $e_i \to a$, then $e_i - d$ by pt(a), since $a \to d$.

If $a \leftarrow c \to b$ and $e_0, \ldots, e_n$ witness $\psi(a, b, c)$ the argument is specular with obvious changes. □

Property 1.20. *Let* $(V, \to)$ *be a pseudo-transitive ograph and let* $v, u, e_0, \ldots, e_n \in V$*. Suppose* $u \mid v$*,* $u - e_0$ *and* $\forall i < n\, (v \to e_i \to e_{i+1} \vee e_{i+1} \to e_i \to v)$*. Then* $u - e_i$ *for each* $i \leq n$*.*

Proof. The proof is by induction on i. The base case holds by assumption, so assume $u - e_i$ for $i < n$. If $u \to e_i$, then $v \to e_i$ because $u \mid v$. Thus $e_i \to e_{i+1}$ and $u - e_{i+1}$ by pt(e_i). If $e_i \to u$ the argument is specular inverting the arrows. □

We can now show that Φ and Ψ are sufficient for the first condition of Lemma 1.14.

Lemma 1.21. *Let* $(V, \to, \prec)$ *be a GH-triple. If* Φ *and* Ψ *are satisfied, then for each* $(V \cup \{x\}, \to')$ *pseudo-transitive extension of* $(V, \to)$ *we have* $N(x) = N^+(x) \cup N^-(x)$*.*

Proof. Fix $(V \cup \{x\}, \to')$. By Lemma 1.15 it suffices to show that for any $a, b, c \in V$ such that $a \prec c \prec b$ and $x -' c$ either $x -' a$ or $x -' b$.

If $b \to c \to a$ then $x \to' c$ implies $x -' a$, while $c \to' x$ implies $x -' b$. If $a \to c \to b$ the situation is similar.

If $a \to c \leftarrow b$ then Φ implies that $\phi(a, b, c)$ holds. Let $e_0, \ldots, e_n$ witness $\varphi(a, b, c)$. Assume $x -' c$. If $c \to' x$, then both $a -' x$ and $b -' x$ follow immediately by pt(c). Otherwise we have $x \to' c$, and suppose towards a contradiction that $x \mid' a$ and $x \mid' b$. Notice that $x \to' c \to e_0$ implies $x -' e_0$. Hence by condition (φ_2) and Property 1.20 it holds that $\forall i \leq n\, (x -' e_i)$. In particular we have $x -' e_n$, and then one of $x -' a$ and $x -' b$ by pt(e_n) follows by (φ_3).

If $a \leftarrow c \to b$ we argue similarly, using Ψ. □

We now prove that Φ and Ψ are necessary conditions for $N(x) = N^+(x) \cup N^-(x)$.

Lemma 1.22. *Let* $(V, \to, \prec)$ *be a GH-triple such that one of* Φ *and* Ψ *fails. Then there is a pseudo-transitive extension* $(V \cup \{x\}, \to')$ *of* $(V, \to)$ *such that* $N(x) \neq N^+(x) \cup N^-(x)$ *and hence* $(V, \to, \prec)$ *is not extendible by Lemma 1.14.*

Proof. We assume the failure of Φ: if Ψ fails the argument is symmetric.

Let $a, b, c \in V$ be such that $a \to c \leftarrow b$, $a \prec c \prec b$ and $\neg\varphi(a, b, c)$. We fix $x \notin V$ and define an extension $\to'$ of $(V, \to)$ to $V \cup \{x\}$ in stages, as an increasing union $\to' = \bigcup_{n\in\mathbb{N}} \to_n$. For each stage n, $\to_n$ is defined as follows:

- $\to_0$ extends $\to$ by adding the single edge $x \to_0 c$;
- $\to_{n+1}$ extends $\to_n$ by adding edges

$$\begin{cases} x \to_{n+1} u & \text{if } \exists v((x \to_n v \to u) \vee (u \to v \to_n x)) \text{ and } a \to u \leftarrow b; \\ u \to_{n+1} x & \text{if } \exists v((x \to_n v \to u) \vee (u \to v \to_n x)) \text{ and } a \leftarrow u \to b. \end{cases}$$

Notice that $x -' c$ but $x \mid' a$ and $x \mid' b$ and hence $c \in N(x)$ but $c \notin N^+(x) \cup N^-(x)$. Therefore to complete the proof it suffices to check the pseudo-transitivity of $\to'$. We first make a couple of preliminary observations.

Claim 1.22.1. *For all $u \in V$ such that there exists $v \in V$ satisfying either $x \to' v \to u$ or $u \to v \to' x$ we have $a - u - b$.*

Proof. Let us first suppose that $x \to' v \to u$ holds. By definition of $\to'$ (or by hypothesis when $v = c$) it holds that $a \to v \leftarrow b$. Hence $a - u - b$ by pt(v). If $u \to v \to' x$ the argument is similar. □

Claim 1.22.2. *If $u \in V$ is such that $u \neq c$ and $u -_1 x$ then $c \to u$.*

Proof. Let us suppose that $u \neq c$ and $u -_1 x$, so that $u -_0 x$ does not hold. The definition of $\to_1$ implies that for some v we have either $x \to_0 v \to u$ or $u \to v \to_0 x$. Since the only v such that $v -_0 x$ is c and $x \to_0 c$ we must have the first possibility with $v = c$, so that $c \to u$ holds. □

In order to show that $\to'$ is pseudo-transitive, we have to consider the following three cases for $v, u \in V$:

a $v \to' x \to' u$. Then $v - u$ because $v \to a \to u$ by definition of $\to'$;

b $x \to' v \to u$. Then Claim 1.22.1 guarantees that $a - u - b$. Let n be the least stage such that $x \to_n v$. If $a \to u \leftarrow b$ or $a \leftarrow u \to b$, then $x -_{n+1} u$ by definition of $\to_{n+1}$. Thus we assume that either $a \to u \to b$ or $b \to u \to a$. Since n is the minimum stage such that $x \to_n v$, there exists e_{n-2} such that $x -_{n-1} e_{n-2} - v$ and $x \to_{n-1} e_{n-2} \Leftrightarrow e_{n-2} \to v$. Notice that $x -_{n-2} e_{n-2}$ does not hold, otherwise we would have $x \to_{n-1} v$. Analogously, there must be an e_{n-3} such that $x -_{n-2} e_{n-3} - e_{n-2}$ and $x \to_{n-2} e_{n-3} \Leftrightarrow e_{n-3} \to e_{n-2}$. For each step $i < n$, we can repeat this search of e_{i-2} witnessing that $x -_i e_{i-1}$. After $n - 1$ steps we get to $x -_1 e_0$ and, since $x -_0 e_0$ does not hold, $e_0 \neq c$. This means, by Claim 1.22.2, that $c \to e_0$. Let j be maximum such that $c \to e_j$ and set $e_{n-1} = v$ and $e_n = u$. We claim that $e_j, \dots, e_n$ witness $\varphi(a, b, c)$. To this end we need to check the three clauses in the definition of $\varphi(a, b, c)$:

(φ_1) $c \to e_j$ by hypothesis.

(φ_2) Fix $i < n$: $e_i - e_{i+1}$ holds by our choice of the sequence of the e_i's and we have either $a \to e_i \leftarrow b$ or $a \leftarrow e_i \to b$ by definition of $\to_i$. Moreover, if $x \to_{i+1} e_i$, then $b \to e_i$, by definition of $\to_{i+1}$, and $e_i \to e_{i+1}$, by choice of e_i. If $e_i \to_{i+1} x$ the argument is similar.

(φ_3) $a \to e_n \to b$ or $b \to e_n \to a$ by hypothesis.

c $u \to v \to' x$. This is similar to the previous case. □

Summarizing the results obtained in Lemma 1.21 and Lemma 1.22 we obtain:

Corollary 1.23. *Let $(V, \to, \prec)$ be a GH-triple. The following are equivalent:*

1. *for each pseudo-transitive extension $(V \cup \{x\}, \to')$ of $(V, \to)$, it holds that $N(x) = N^+(x) \cup N^-(x)$;*
2. *Φ and Ψ are satisfied.*

1.3 Avoiding the $2 \oplus 2$ example

This section is devoted to study more carefully the second condition of Lemma 1.14. The next lemma shows that, assuming that $N(x) = N^+(x) \cup N^-(x)$, this condition captures precisely the lack of the $2 \oplus 2$ example. Recall that in that example $(V, \to, \prec)$ is a GH-triple and there exist $a, b, c, d \in V$ such that $a \prec c$, $d \prec b$, $a \mid b$, $a \mid d$, $c \mid b$, and $c \mid d$. Then, a new vertex x is connected with a and b but not with c and d, or vice versa. Notice that this is possible only if either $a \to c$ and $b \to d$, or $c \to a$ and $d \to b$.

Lemma 1.24. *Let $(V, \to, \prec)$ be a GH-triple and $(V \cup \{x\}, \to')$ a pseudo-transitive extension of $(V, \to)$. We use Λ to denote the following property of $(V \cup \{x\}, \to')$ and $\prec$:*

$$\forall a, b, c, d \in V\ (a \mid b \wedge c \mid d \wedge a \prec c \wedge d \prec b \wedge x -' a \wedge x -' b \Rightarrow x -' d \vee x -' c)$$

Then:

1 if Λ holds then $N^-(x) \setminus N^+(x) \prec N^+(x) \setminus N^-(x)$;

2 if $N(x) = N^+(x) \cup N^-(x)$ and $N^-(x) \setminus N^+(x) \prec N^+(x) \setminus N^-(x)$ then Λ holds.

Proof. (1) Assume $N^-(x) \setminus N^+(x) \not\prec N^+(x) \setminus N^-(x)$, i.e. there exist $a \in N^-(x) \setminus N^+(x)$ and $b \in N^+(x) \setminus N^-(x)$ such that $a \not\prec b$. Since $a \notin N^+(x)$ there is $c \succ a$ with $c \mid x$. Since $b \notin N^-(x)$ there is $d \prec b$ with $d \mid x$. If $b \prec a$, then $b \prec c$ but this is impossible since $c \mid x$ and $b \in N^+(x)$. Since we are assuming $a \not\prec b$ we have $a \mid b$.

We claim that $c \mid d$ also holds. Since $a - x - b$, but $a \mid b$, then either $a \to x \leftarrow b$ or $a \leftarrow x \to b$. The argument for the two cases is similar, so let us assume that $a \to x \leftarrow b$. This implies $a \to c$ and $b \to d$ because $c \mid x$ and $x \mid d$. Hence if $c \to d$, then $a - d$ by pt(c). Since $a \mid b$ and $d \prec b$, it must be $d \prec a$ but this contradicts $a \in N^-(x)$. If $d \to c$, then $c - b$ by pt(d). Since $a \mid b$ and $a \prec c$, it must be $b \prec c$ which contradicts $b \in N^+(x)$. We have thus shown that $c \mid d$ as claimed.

Now a, b, c and d witness the failure of Λ.

(2) Assume that $a, b, c, d \in V$ witness the failure of Λ. Then $a \notin N^+(x)$, $b \notin N^-(x)$ and $a \not\prec b$. If $N(x) = N^+(x) \cup N^-(x)$ holds then $a \in N^-(x)$ and $b \in N^+(x)$, showing that $N^-(x) \setminus N^+(x) \prec N^+(x) \setminus N^-(x)$ fails. □

Observation 1.25. *Notice that the first four conjuncts of the antecedent of the implication appearing in Λ imply that a, b, c and d form a $2 \oplus 2$ because $c \mid b$ and $a \mid d$ follow from these. In fact, if $c - b$, then $a \prec c$ and $a \mid b$ imply that $b \prec c$, but then $d - c$ contrary to the assumption. A similar argument shows that $a \mid d$.*

We now define two formulae Θ and Σ characterizing the reorientations such that the condition Λ of Lemma 1.24 is satisfied whenever $N(x) = N^+(x) \cup N^-(x)$. As for Φ and Ψ, the main feature of Θ and Σ is that they mention only $(V, \to)$ and $\prec$. In order to define Θ and Σ it is necessary to define $\theta(a, b, c, d)$ and $\sigma(a, b, c, d)$ (which do not mention $\prec$).

Definition 1.26. Let $(V, \to, \prec)$ be a GH-triple. Let $\theta(a, b, c, d)$ assert the existence of $e_0, \dots, e_n \in V$ such that:

(θ_1) $e_0 \to b$;

(θ_2) $\forall i < n\, (e_{i+1} \to e_i \to d)$;

(θ_3) $d \to e_n$;

(θ_4) $e_n \mid a$.

Then Θ is

$$\forall a, b, c, d \in V\ (a \to c \wedge b \to d \wedge a \mid b \wedge c \mid d \wedge a \prec c \wedge d \prec b \Rightarrow \theta(a, b, c, d) \vee \theta(b, a, d, c)).$$

Symmetrically, let $\sigma(a, b, c, d)$ assert the existence of $e_0, \dots, e_n \in V$ such that:

(σ_1) $d \to e_0$;

(σ_2) $\forall i < n\,(b \to e_i \to e_{i+1})$;

(σ_3) $e_n \to b$;

(σ_4) $e_n \mid c$.

Then Σ is

$$\forall a, b, c, d \in V\ (a \to c \wedge b \to d \wedge a \mid b \wedge c \mid d \wedge a \prec c \wedge d \prec b \Rightarrow \sigma(a, b, c, d) \vee \sigma(b, a, d, c)).$$

Example 1.27. Suppose $(\{a, b, c, d, e\}, \to)$ is the pseudo-transitive graph whose only edges are $a \to c$, $b \to d$ and $d \to e \to b$. Then $\theta(a, b, c, d)$ and $\sigma(a, b, c, d)$ hold with $n = 0$ and $e_0 = e$. Thus a $2 \oplus 2$ such as the one obtained restricting $\to$ to $\{a, b, c, d\}$ can satisfy θ and σ simply because one of its edges belongs to a non transitive triangle. See the first paragraph of the proof of lemma 1.47 below for more on this.

Remark 1.28. Let $(V, \to, \prec)$ be a GH-triple. Suppose $e_0, \dots, e_n$ witness $\theta(a, b, c, d)$ for some $a, b, c, d \in V$. Clearly, if there is an $i > 0$ such that $e_i \to b$, then $e_i, \dots, e_n$ witness $\theta(a, b, c, d)$ as well. Thus we can assume that for every $i \leq n$ with $i > 0$ we have $b \to e_i$ whenever $b - e_i$. Under this assumption it actually holds that $b \to e_i$ holds for every $i \leq n$ with $i > 0$. In fact, $b - e_n$ by pt(d) and if $b \to e_{i+1}$, then $b - e_i$ by pt(e_{i+1}).

Before proving the usefulness of Θ and Σ, we would like to comment on their mutual relationship and on the difference between the connection between Θ and Σ and the connection between Φ and Ψ. Let $(V, \to, \prec)$ be a GH-triple and suppose $a, b, c, d \in V$ satisfy the antecedent of Θ and Σ (which is the same). Consider a pseudo-transitive extension $(V \cup \{x, y\}, \to')$ such that $a \to' x \leftarrow' b$ and $c \leftarrow' y \to' d$. The two extensions correspond respectively to the left and right ograph of Figure 1.2. As explained at the beginning of this section, if either x is incomparable with both c and d or if y is incomparable with both a and b, then $(V, \to, \prec)$ is not extendible. We emphasize that under these hypotheses we could have both x and y witnessing the non extendibility of $(V, \to, \prec)$. To compare this situation with the one Φ and Ψ take care of, suppose $a \to b \to c \leftarrow a$ and add x and y such that $a \to x$ and $y \to c$. Since $c \prec a \prec b$ and $a \prec c \prec b$ cannot occur simultaneously, only one of x and y can witness (if $\varphi(a, b, c)$, resp. $\psi(b, c, a)$, fails) the non extendibility of $(V, \to, \prec)$.

Despite the previous considerations the next lemma shows that x witnesses the non extendibility of $(V, \to, \prec)$ if and only if y does.

Lemma 1.29. *Let $(V, \to)$ be a pseudo-transitive ograph and suppose $a, b, c, d \in V$ are such that $a \to c$, $b \to d$, $a \mid b$ and $d \mid c$. Then $\theta(a, b, c, d)$ holds if and only if $\sigma(a, b, c, d)$ does.*

Therefore, if $(V, \to, \prec)$ is a GH-triple then Θ holds if and only if Σ does.

Proof. Since the antecedents of Σ and Θ coincide and imply the hypothesis of the first statement, it is clear that the second statement follows from the first.

For the forward direction of the first statement, let $e_0, \dots, e_n$ witness $\theta(a, b, c, d)$. By Remark 1.28 we can assume that $b \to e_i$ whenever $i > 0$. We claim that $e_n, \dots, e_0$ witness $\sigma(a, b, c, d)$. In fact, conditions (σ_1) and (σ_3) of $\sigma(a, b, c, d)$ are exactly conditions (θ_3) and (θ_1) of $\theta(a, b, c, d)$. Condition (σ_2) of $\sigma(a, b, c, d)$ is now $\forall i < n\,(b \to e_{i+1} \to e_i)$ and follows easily from our assumption on the e_i's and from condition (θ_2) of $\theta(a, b, c, d)$. We are left with showing condition (σ_4) of $\sigma(a, b, c, d)$, i.e. $e_0 \mid c$. Suppose on the contrary that $e_0 - c$. Since $c \mid d$, by Property 1.20 it follows that $\forall i \leq n\,(c - e_i)$. In particular, $c - e_n$ and so $e_n \to c$ because $a \mid e_n$ by (θ_4) of $\theta(a, b, c, d)$. But then $c - d$ by pt(e_n), contrary to the assumptions.

The proof of the backward direction is analogous. $\square$

Thanks to the previous lemma it suffices to concentrate on Θ.

The following duality principle is analogous to Remark 1.18. It is not needed elsewhere and we include it here for completeness without proof.

Remark 1.30. Notice that the GH-triple $(V, \to, \prec)$ satisfies Θ if and only if the GH-triple $(V, \leftarrow, \succ)$ satisfies Θ.

Lemma 1.31. *Let $(V, \to, \prec)$ be a GH-triple. Let $a, b, c, d \in V$ be such that $a \to c$, $b \to d$, $a \mid b$, and $d \mid c$ and assume that $\theta(a, b, c, d)$ holds. Then for each $(V \cup \{x\}, \to')$ pseudo-transitive extension of $(V, \to)$ if $a -' x -' b$ holds we have $x -' d$, and if $c -' x -' d$ holds we have $x -' b$.*

Proof. Let $(V \cup \{x\}, \to')$ be a pseudo-transitive extension of $(V, \to)$ with $a -' x -' b$. Notice that, since $a \mid b$, either $a \leftarrow' x \to' b$ or $a \to' x \leftarrow' b$. In the first case $\mathrm{pt}(a)$ and $\mathrm{pt}(b)$ guarantee that $c -' x -' d$, so we concentrate on the other case.

Suppose that $a \to' x \leftarrow' b$ and let $e_0, \dots, e_n$ witness $\theta(a, b, c, d)$. Towards a contradiction, assume $x \mid' d$. Notice that $x -' e_0$ by $\mathrm{pt}(b)$ (we use condition (θ_1)). Hence by condition (θ_2) and Property 1.20 it holds that $\forall i \leq n\, (x -' e_i)$, so that in particular $x -' e_n$. It cannot hold that $x \to e_n$, otherwise $a - e_n$ by $\mathrm{pt}(x)$ contrary to (θ_4). Hence $e_n \to x$ holds. Moreover, $d \to e_n$ by condition (θ_3) and so $\mathrm{pt}(e_n)$ implies $d -' x$.

A similar argument shows that $c -' x -' d$ implies $x -' b$. The only change is due to the fact that when $c \leftarrow' x \to' d$ then we use $\sigma(a, b, c, d)$, which holds by Lemma 1.29. □

We can now show that Θ is sufficient for the second condition of Lemma 1.14.

Lemma 1.32. *Let $(V, \to, \prec)$ be a GH-triple satisfying Θ. For each $(V \cup \{x\}, \to')$ pseudo-transitive extension of $(V, \to)$ we have $N^-(x) \setminus N^+(x) \prec N^+(x) \setminus N^-(x)$.*

Proof. By Lemma 1.24.1 it suffices to prove condition Λ. Fix $a, b, c, d \in V$ such that $a \prec c$, $d \prec b$, $a \mid b$, and $c \mid d$ and assume that $a -' x -' b$. We need to prove that either $x -' c$ or $x -' d$.

Since $a - c$ and $d - b$ there are four possible situations. If $a \to c$ and $d \to b$, but $x \mid' c$, then $a \to' x \leftarrow' b$ and $x -' d$ follows by $\mathrm{pt}(b)$. If $c \to a$ and $b \to d$ the argument is similar. If instead $a \to c$ and $b \to d$ notice that Θ implies $\theta(a, b, c, d)$ or $\theta(b, a, d, c)$: then Lemma 1.31 yields the conclusion. The last possibility is $c \to a$ and $d \to b$, where we use the second part of Lemma 1.31 (in this case a, b, c, d play roles which are opposite to those of the Lemma). □

We now prove that Θ is necessary for $N^-(x) \setminus N^+(x) \prec N^+(x) \setminus N^-(x)$ if Φ and Ψ hold.

Lemma 1.33. *Let $(V, \to, \prec)$ be a GH-triple such that Φ and Ψ hold and Θ fails. Then there is a pseudo-transitive extension $(V \cup \{x\}, \to')$ of $(V, \to)$ such that $N^-(x) \setminus N^+(x) \not\prec N^+(x) \setminus N^-(x)$ and hence $(V, \to, \prec)$ is not extendible by Lemma 1.14.*

Proof. Let $a, b, c, d \in V$ be such that $a \to c$, $b \to d$, $a \mid b$, $c \mid d$, $a \prec c$, $d \prec b$ and $\neg\theta(a, b, c, d)$. We fix $x \notin V$ and define an extension $\to'$ of $(V, \to)$ to $V \cup \{x\}$ in stages, as an increasing union $\to' = \bigcup_{n \in \mathbb{N}} \to_n$. For each stage n, $\to_n$ is defined as follows:

- $\to_0$ extends $\to$ by adding the edges $a \to x$ e $b \to x$;
- $\to_{n+1}$ extends $\to_n$ by adding edges

$$\begin{cases} x \to_{n+1} u & \text{if } \exists v((x \to_n v \to u) \vee (u \to v \to_n x)) \text{ and either } c \to u \text{ or } d \to u; \\ u \to_{n+1} x & \text{if } \exists v((x \to_n v \to u) \vee (u \to v \to_n x)) \text{ and either } u \to c \text{ or } u \to d. \end{cases}$$

Notice that $x \to' u$ and $u \to' x$ are incompatible, since if $c \to u$ or $d \to u$ then we have neither $u \to c$ nor $u \to d$.

If we assume that $\to'$ is pseudo-transitive we can complete the proof as follows. Since Φ and Ψ hold, by Lemma 1.21 we have $N(x) = N^+(x) \cup N^-(x)$. On the other hand, by definition of $\to'$, $a -' x -' b$ but

$x \mid' c$ and $x \mid' d$ (because $c \mid d$ and hence we never set $x \to_{n+1} c$ or $x \to_{n+1} d$) and condition Λ fails. Thus, by Lemma 1.24.2, $N^-(x) \setminus N^+(x) \not\leftarrow N^+(x) \setminus N^-(x)$.

Therefore to complete the proof it suffices to check the pseudo-transitivity of $\to'$. We first make a few preliminary observations.

Claim 1.33.1. *If $v \in V$ is such that $v \to' x$ then either $v \to c$ or $v \to d$. Similarly, if $u \in V$ is such that $x \to' u$ then either $c \to u$ or $d \to u$.*

Proof. Let n be least such that $v \to_n x$. If $n = 0$ then v is either a or b, which satisfy the conclusion. If $n > 0$ then $v \to c$ or $v \to d$ is required by definition. When dealing with u, the case $n = 0$ cannot hold. □

Claim 1.33.2. *Let us assume that for $z, w \in V$ we have either $x \to' z \to w$ or $w \to z \to' x$. Then if $w - c$ and $w \mid d$ we have also $z - c$ and $z \mid d$, and similarly if $w - d$ and $w \mid c$ we have also $z - d$ and $z \mid c$.*

Proof. Assume $w - c$ and $w \mid d$. If $x \to' z \to w$, then $c \to z$ or $d \to z$ by Claim 1.33.1. If $d \to z$, then $d - w$ by pt(z), contrary to the assumption. So $c \to z$, while $z \to d$ cannot hold because $c \mid d$. Thus we have $z - c$ and $z \mid d$. If $w \to z \to' x$ the argument is similar.

The second statement is proved analogously. □

Claim 1.33.3. $\forall e\,(e \neq a \wedge e \neq b \wedge e -_1 x \Rightarrow e \to a \vee e \to b)$

Proof. Let us suppose that $e \neq a$, $e \neq b$ and $e -_1 x$, so that $e -_0 x$ does not hold. The definition of $\to_1$ implies that for some v we have either $x \to_0 v \to e$ or $e \to v \to_0 x$. Since the only v's such that $v -_0 x$ are a and b, and $a \to_0 x$ and $b \to_0 x$, we must have the second possibility with v either a or b. □

In order to prove that $\to'$ is pseudo-transitive, there are some cases to consider.

a $v \to' x \to' u$. By Claim 1.33.1 either $v \to c$ or $v \to d$ and also $c \to u$ or $d \to u$. If either $v \to c \to u$ or $v \to d \to u$, then $u - v$ follows by pt(c) or pt(d) of $\to$.

We now concentrate on the case $v \to c$ and $d \to u$, the other being similar. Notice that $c \mid d$ implies that $u \to c$ and $d \to v$ do not hold. Moreover, we can assume that $v \to d$ and $c \to u$ both fail, else we are in one of the previous cases. Hence $u \mid c$ and $v \mid d$. If n is the minimum stage such that $x \to_{n+1} u$ (notice that $x \to_0 u$ cannot happen), there exists e_{n-1} such that $x \to_n e_{n-1} \to u$ or $u \to e_{n-1} \to_n x$. Analogously, there must be an e_{n-2} such that $x \to_{n-1} e_{n-2} \to e_{n-1}$ or $e_{n-1} \to e_{n-2} \to_{n-2} x$. Iterating this procedure, we get to $x -_1 e_0$. Set also $e_n = u$. Similarly, let k be least such that $v \to_k x$ (in this case $k = 0$ is possible) and set $h_k = v$. If $k > 0$, with a procedure similar to the one used before, we find $h_0, \ldots, h_{k-1}$ such that h_j witnesses that $x -_{j+1} h_{j+1}$ for each $j < k$.

Notice that a backward induction using Claim 1.33.2 easily entails $\forall i < n\,(e_i - d \wedge e_i \mid c)$ and $\forall j < k\,(h_j - c \wedge h_j \mid d)$. Notice also that for each $i < n$ either $d \to e_i \to e_{i+1}$ or $e_{i+1} \to e_i \to d$ holds. In fact, if $d \to e_i$, then $x \to' e_i$ by definition and so $e_i \to e_{i+1}$ by choice of e_i. If $e_i \to d$ the argument is specular. Arguing as in the previous lines it is easy to show that for each $j < k$ either $c \to h_j \to h_{j+1}$ or $h_{j+1} \to h_j \to c$ holds as well.

Let $i \leq n$ be least such that $d \to e_i$. We claim that $e_0, \ldots, e_i$ satisfy the first three conditions of $\theta(a, b, c, d)$:

(θ_1) $e_0 \to b$ by Claim 1.33.3 because $e_0 \to a$ implies $e_0 - c$ by pt(a), which contradicts the above observation;

(θ_1) $\forall j < i\,(e_{j+1} \to e_j \to d)$: this is immediate by the minimality of i and the observation in the previous paragraph;

(θ_1) $d \to e_i$ by choice of i;

Since $\theta(a,b,c,d)$ fails, condition (θ_4) must fail, i.e. we have $e_i - a$.

Since $a \mid d$ we can apply Property 1.20 to obtain that $e_j - a$ for every $j \leq n$ with $j \geq i$. Recalling that $e_n = u$, we obtained $u - a$: then $a \to u$ because $a \mid d$.

We show that $h_j - u$ for every $j \leq k$. Arguing as in the proof of (θ_1) above, we have $h_0 \to a$ so that $h_0 - u$ by pt(a). Thus, since $u \mid c$, we can apply Property 1.20 again to obtain the desired conclusion. Recalling that $h_k = v$ we have obtained $u - v$.

b $x \to' v \to u$ then $c \to v$ or $d \to v$ by Claim 1.33.1. By pt(v), either $c - u$ or $d - u$ and u satisfies one of the conditions in the definition of $\to'$. Thus $x -' u$.

c $u \to v \to' x$ is similar to the previous item.

This shows that $\to'$ is pseudo-transitive and hence that $(V, \to, \prec)$ is not extendible. □

Summarizing, we obtained a characterization of the conditions of Lemma 1.14.

Theorem 1.34. *Let $(V, \to, \prec)$ be a GH-triple. The following are equivalent:*

1. *for each pseudo-transitive extension $(V \cup \{x\}, \to')$ of $(V, \to)$ both $N(x) = N^+(x) \cup N^-(x)$ and $N^-(x) \setminus N^+(x) \prec N^+(x) \setminus N^-(x))$ hold;*
2. *Φ, Ψ and Θ are satisfied.*

Proof. The implication (1) ⇒ (2) follows from Lemmas 1.22 and 1.33. The implication (2) ⇒ (1) follows from Lemmas 1.21 and 1.32. □

Thanks to Theorem 1.34 we can now reformulate Lemma 1.14 in a way that does not refer to all possible pseudo-transitive extensions of $(V, \to)$ but mentions only structural properties of $(V, \to)$ and $\prec$.

Theorem 1.35. *Let $(V, \to, \prec)$ be an extendible GH-triple. Then Φ, Ψ and Θ are satisfied.*

It follows from Lemma 1.48 below that the reverse implication holds as well, namely that Φ, Ψ and Θ are also sufficient conditions of the extendibility of a GH-triple.

1.4 The Smart Extension Algorithm

In this section we define an on-line algorithm to transitively reorient a countable pseudo-transitive ograph. Before defining the algorithm we give some preliminary definitions.

Definition 1.36. Let $(V, \to, \prec)$ be a GH-triple. If $(V \cup \{x\}, \to')$ is a pseudo-transitive extension of $\to$, we define inductively the following subsets of $N(x)$:

$$
\begin{aligned}
S_0^-(x) &= N^-(x) \setminus N^+(x);\\
S_0^+(x) &= N^+(x) \setminus N^-(x);\\
S_i(x) &= S_i^-(x) \cup S_i^+(x);\\
S_{i+1}^-(x) &= \{a \in N(x) \setminus \textstyle\bigcup_{j\leq i} S_j(x) \mid \exists s \in S_i^-(x)(a \mid s)\};\\
S_{i+1}^+(x) &= \{a \in N(x) \setminus \textstyle\bigcup_{j\leq i} S_j(x) \mid \exists s \in S_i^+(x)(a \mid s)\}.
\end{aligned}
$$

Let $S^+(x) = \bigcup_{i\in\mathbb{N}} S_i^+(x)$, $S^-(x) = \bigcup_{i\in\mathbb{N}} S_i^-(x)$ and $S(x) = S^-(x) \cup S^+(x) = \bigcup_{i\in\mathbb{N}} S_i(x)$. Let also $T(x) = N(x) \setminus S(x)$.

If $* \in \{+, -\}$ we say that a sequence $\rho = \langle \rho(0), \rho(1), \ldots, \rho(|\rho| - 1)\rangle$ of elements of V is a $*$-sequence if $\rho(i) \in S_i^*(x)$ for every $i < |\rho|$ and $\rho(i) \mid \rho(i+1)$ for every $i < |\rho| - 1$.

Remark 1.37. If $N(x) = N^+(x) \cup N^-(x)$ then $S(x) \setminus S_0(x)$ and $T(x)$ are both included in $N^+(x) \cap N^-(x)$. Moreover $S(x) \subseteq N(t)$ for every $t \in T(x)$ (because if $t \in N(x)$ and $S_i(x) \setminus N(t) \neq \emptyset$ then $t \in S_{i+1}(x)$).

Notice that if $s \in S_i^*(x)$ then there exists a $*$-sequence ρ such that $\rho(i) = s$.

For the remainder of the section we use $S^*(x)$ as a shorthand for either $S^+(x)$ or $S^-(x)$. $S_i^*(x)$ is used similarly and s_i^* always denotes an element of $S_i^*(x)$. We now prove some properties of $S^*(x)$ and its subsets.

Property 1.38. *Let $(V, \to, \prec)$ be a GH-triple. Let $(V \cup \{x\}, \to')$ be a pseudo-transitive extension of $\to$.*

1. *Fix $v \in V \cup \{x\}$ and $* \in \{+, -\}$. If ρ is a $*$-sequence such that $\forall i < |\rho| \, (v -' \rho(i))$ then either $\forall i < |\rho| \, (\rho(i) \to' v)$ or $\forall i < |\rho| \, (v \to' \rho(i))$.*
2. *$S^-(x) \prec T(x)$ and $T(x) \prec S^+(x)$.*
3. *If ρ^* is a $*$-sequence for $* \in \{+, -\}$, $\rho^+(0) \to' x \leftarrow' \rho^-(0)$ and $e_0, \dots, e_n$ witness $\varphi(\rho^-(0), \rho^+(0), f)$ for some $f \mid' x$, then there exists $i \le n$ such that $e_i, \dots, e_n$ witness $\varphi(\rho^-(k), \rho^+(j), x)$, for each k and j. Moreover, $\rho^-(k) \to' x \leftarrow' \rho^+(j)$. The same statement holds with ψ in place of φ.*

Proof. (1) is obvious by pseudo-transitivity of $\to'$.

To prove (2) we fix $t \in T(x)$ and prove by induction on i that $t \prec S_i^+(x)$ for every i. For the base of the induction, $t \prec S_0^+(x)$ follows from $S_0^+(x) \subseteq N(t)$ (Remark 1.37), $t \in N^-(x)$ and $S_0^+(x) \cap N^-(x) = \emptyset$. For the induction step let $s_{i+1}^+ \in S_{i+1}^+(x)$ and choose $s_i^+ \in S_i^+(x)$ such that $s_{i+1}^+ \mid s_i^+$. By induction hypothesis $t \prec s_i^+$ and hence, since $s_{i+1}^+ - t$ (again by Remark 1.37), we have $t \prec s_{i+1}^+$. This shows $T(x) \prec S^+(x)$. Analogously we prove $S^-(x) \prec T(x)$.

To prove (3) fix ρ^+, ρ^-, f, $e_0, \dots, e_n$ satisfying the hypothesis. Let m^* be the length of ρ^* for $* \in \{+, -\}$. We write s_k^* in place of $\rho^*(k)$. Since $s_0^* \to' x$ and $S(x) \subseteq N(x)$, (1) implies that $s_k^* \to' x$ for each $k < m^*$.

Applying Property 1.19 to $(V \cup \{x\}, \to')$ we obtain that there exists $i \le n$ such that $e_i, \dots, e_n$ witness $\varphi(s_0^-, s_0^+, x)$. For the sake of convenience assume $i = 0$, so that $e_0, \dots, e_n$ witness $\varphi(s_0^-, s_0^+, x)$ as well.

Fix $* \in \{+, -\}$. We claim that $\forall k < m^* \, \forall i \le n \, (e_i - s_k^*)$. The proof is by double induction. Suppose $\forall i \le n \, (e_i - s_\ell^*)$ for each $\ell < k$. We prove by induction on i that $\forall i \le n \, (e_i - s_k^*)$. For the base case, $e_0 - s_k^*$ by $\mathrm{pt}(x)$ since $s_k^* \to' x$ and $x \to' e_0$ by (φ_1) of $\varphi(s_0^-, s_0^+, x)$. For the induction step suppose $e_i - s_k^*$. If $s_k^* \to e_i$, then $s_0^* \to e_i$ by (1) (that applies because $\forall \ell < k \, (e_i - s_\ell^*)$). Then $e_i \to e_{i+1}$ by $\varphi(s_0^-, s_0^+, x)$. Hence $e_{i+1} - s_k^*$ by $\mathrm{pt}(e_i)$. If $e_i \to s_k^*$, the argument is analogous.

Let $k < m^-$ and $j < m^+$. We check that the three conditions of $\varphi(s_k^-, s_j^+, x)$ are satisfied. Condition (φ_1) holds trivially since it coincides with (φ_1) of $\varphi(s_0^-, s_0^+, x)$. To check that (φ_2) holds suppose $s_k^- \to e_i$. Then $s_0^- \to e_i$ by (1) and thus $s_0^+ \to e_i \to e_{i+1}$ by (φ_2) of $\varphi(s_0^-, s_0^+, x)$. By (1) again it holds that $s_j^+ \to e_i$ holds as well. An analogous argument shows that if $e_i \to s_k^-$, then $e_{i+1} \to e_i \to s_j^+$. These establish that (φ_2) of $\varphi(s_k^-, s_j^+, x)$ holds. Condition (φ_3) is checked in a similar way. ☐

Notice that Property 1.38.2 implies $S^-(x) \prec S^+(x)$ whenever $T(x) \neq \emptyset$. To see that this holds in general we need to strengthen the hypothesis on the reorientation of $(V, \to)$.

Lemma 1.39. *Let $(V, \to, \prec)$ be a GH-triple such that Ψ, Φ and Θ are satisfied. Let $(V \cup \{x\}, \to')$ be a pseudo-transitive extension of $\to$. Then $S^-(x) \prec S^+(x)$ and hence $S^-(x) \cap S^+(x) = \emptyset$.*

Proof. Let $s^- \in S^-(x)$. We first claim that $s^- - s^+$ for every $s^+ \in S^+(x)$, which is obviously necessary for $S^-(x) \prec S^+(x)$. Since $s^-, s^+ \in N(x)$, there are four possibilities.

If $s^+ \to' x \to' s^-$ or $s^- \to' x \to' s^+$, then by $\mathrm{pt}(x)$ we have $s^- - s^+$.

Otherwise, $s^- \to' x \leftarrow' s^+$ or $s^+ \leftarrow' x \to' s^-$. Suppose the former holds. For $* \in \{+, -\}$ choose a $*$-sequence $\langle s_0^*, \dots, s_{m^*}^* \rangle$ such that $s_{m^*}^* = s^*$. Recall that, by definition of $*$-sequence, $s_i^* \in S_i^*(x)$ for each $i \leq m^*$ and $s_i^* \mid s_{i+1}^*$ for each $i < m^*$.

Since $s^- \to' x \leftarrow' s^+$, Property 1.38.1 implies that $s_0^+ \to' x \leftarrow' s_0^-$. Since $s_0^+ \notin N^-(x)$, there exists f such that $f \prec s_0^+$ and $f \mid' x$. Analogously, there exists e such that $s_0^- \prec e$ and $e \mid' x$. Given that $f \mid' x \mid' e$ and $s_0^+ \to' x \leftarrow' s_0^-$, then $s_0^+ \to f$ and $s_0^- \to e$. Moreover, since $s_0^- \prec s_0^+$ by Theorem 1.34, it holds $s_0^+ \to s_0^-$ or $s_0^- \to s_0^+$. Suppose the latter, the other case being similar using e in place of f. We have $s_0^- - f$ by pt(s_0^+), and thus $s_0^- \to f$ since $s_0^- \to' x$ and $x \mid' f$. Since $s_0^- \in N^-(x)$, then $s_0^- \prec f$. Summarizing, we have just shown that $s_0^- \prec f \prec s_0^+$ and $s_0^+ \to f \leftarrow s_0^-$. Since we are assuming Φ holds, there are $e_0, \dots, e_n$ witnessing $\varphi(s_0^-, s_0^+, f)$. Applying Property 1.38.3 we obtain that there exists an $i \leq n$ such that $e_i, \dots, e_n$ witness $\varphi(s_k^-, s_j^+, x)$ for each $k \leq m^+$ and $j \leq m^-$. In particular $\varphi(s^-, s^+, x)$ is satisfied and so either $s^- \to e_n \to s^+$ or $s^+ \to e_n \to s^-$ holds. In both cases, by pt(e_n), $s^+ - s^-$ as we wanted to show.

If instead $s^- \leftarrow' x \to' s^+$ the argument is similar, reversing all arrows and using Ψ.

We have thus established our claim that $s^- - s^+$ for every $s^+ \in S^+(x)$. Now we prove by induction on i that $s^- \prec S_i^+(x)$ for every i. For the base of the induction, $s^- \prec s_0^+$ for every $s_0^+ \in S_0^+(x)$ because $s^- - s_0^+$, $s^- \in N^-(x)$ and $s_0^+(x) \notin N^-(x)$. For the induction step let $s_{i+1}^+ \in S_{i+1}^+(x)$ and choose $s_i^+ \in S_i^+(x)$ such that $s_{i+1}^+ \mid s_i^+$. By induction hypothesis $s^- \prec s_i^+$ and hence, since $s_{i+1}^+ - s^-$, we have $s^- \prec s_{i+1}^+$. □

These relations between subsets of $N(x)$ explain the choices for the reorientation of $V \cup \{x\}$ made in the following definition.

Definition 1.40. Let $(V, \to, \prec)$ be a GH-triple satisfying Φ, Ψ and Θ and such that $V \subseteq \mathbb{N}$. Let $(V \cup \{x\}, \to')$ be a pseudo-transitive extension of $\to$.

We define $\prec'$, the *smart extension of* $\prec$ *to* $(V \cup \{x\}, \to')$, as the binary relation that extends $\prec$ to $V \cup \{x\}$ by establishing the relationship between x and each $v \in V$ recursively as follows:

(1) if $v \notin N(x)$ let $v \not\prec' x$ and $x \not\prec' v$;

(2) if $v \in S(x)$ then

  (a) if $v \in S^-(x)$ let $v \prec' x$,

  (b) if $v \in S^+(x)$ let $x \prec' v$;

(3) if $v \in T(x)$ then

  (a) if there exists $u < v$ such that $v \prec u \prec' x$ let $v \prec' x$,

  (b) if there exists $u < v$ such that $x \prec' u \prec v$ let $x \prec' v$,

  (c) otherwise let $v \prec' x$ if $v \to' x$ and $x \prec' v$ if $x \to' v$.

Notice that $\prec'$ depends on the order $<$ on $\mathbb{N}$, is always a reorientation of $(V \cup \{x\}, \to')$, and extends $\prec$.

For a visual understanding of $\prec'$ see Figure 1.4. Here we denote by $T^-(x)$, resp. $T^+(x)$, the subset of $T(x)$ consisting of the vertices which are below, resp. above, x. Moreover the picture shows that $S_i^-(x) \prec S_{i+2}^-(x)$ and $S_{i+2}^+(x) \prec S_i^+(x)$: we leave to the reader to prove these relations, since we do not need them. The picture may suggest that $(N(x), \prec)$ has width two, but this is not the case because there may be nontrivial antichains within some $S_i^*(x)$ and/or $T^*(x)$.

The hypothesis that $(V, \to, \prec)$ satisfies Φ, Ψ and Θ makes sure that Conditions (2a) and (2b) of Definition 1.40 are mutually exclusive, by Lemma 1.39. Some of the clauses of Definition 1.40 are necessary for $\prec'$ to be a transitive reorientation of $\to'$. Condition (1) is obviously necessary for $\prec'$ to be a reorientation. The choice $S^-(x) \prec x$ made by Condition (2a) is explained by an inductive argument: $S_0^-(x) \prec' x$ is required because $S_0^-(x) \cap N^+(x) = \emptyset$, and if $S_i^-(x) \prec x$ then the members of $S_{i+1}^-(x)$ (each incomparable with some

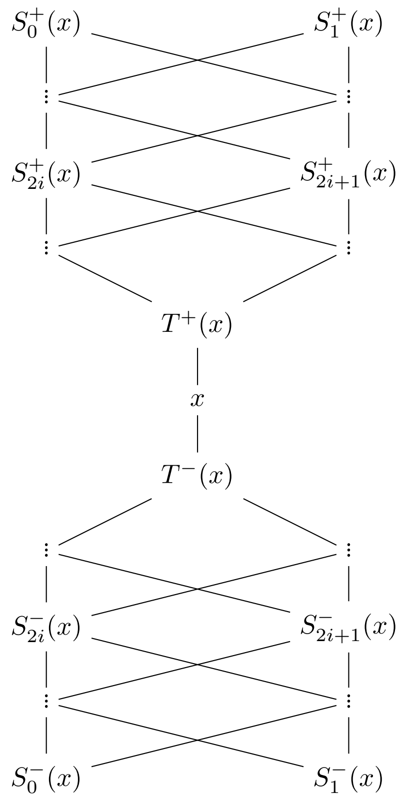


Figure 1.4: A smart extension.

element of $S_i^-(x)$) cannot lie above x. The same argument applies to $S^+(x)$ and justifies Condition (2b). Conditions (3a) and (3b) are clearly necessary for transitivity. Condition (3c) is applied when the relationship between x and v_i is not decided by the previous conditions and in this case $\prec'$ simply preserves the direction of $\rightarrow'$.

From a complexity point of view, defining the sets $S^+(x)$ and $S^-(x)$ requires more resources than setting the relation between x and $v \in V$ according to Definition 1.40. The sets $S_0^+(x)$ and $S_0^-(x)$ are computed in at most $|V^2|$ steps, since one needs to consider each $v \in N(x)$ and for each such v to go through each $u \in N(v) \setminus N(x)$. The remaining members of $S^+(x)$ and $S^-(x)$ can be found by a depth-first search algorithm applied to the non-adjacency graph $(V \cup \{x\}, E')$ (the complexity of depth-first search algorithm is $O(|V| + |E|)$, see [Cormen et al. 2009, Section 22.3]). To this end notice that from each $s_0 \in S_0^+(x)$ start sequences[1] $v_0 = s_0, v_1, \ldots, v_n$, for some $n < |V|$, such that $v_i \, E' v_{i+1}$ and $v_i \in N(x)$, for each $i \leq n$. Then each $v_i \in S^+(x)$. The same obviously applies also to $S^-(x)$.

Therefore an upper bound for the complexity of the smart extension is $O(|V|^2)$.

Definition 1.41. Let $(V, \rightarrow)$ be a pseudo-transitive ograph with V an initial interval of $\mathbb{N}$. The relation $\prec$ is the *smart reorientation* of $\rightarrow$ if it at each step s the reorientation $\prec_{s+1} = \prec\restriction \{0, \ldots, s\}$ is obtained as the smart extension of $\prec_s$.

For the pseudocode of the smart reorientation see Algorithm 2.

[1]Such sequences may not be $+$-sequences, because it may be the case that $v_i \in S_j^+(x)$, for some $j < i$ due to the incomparability chain caused by some other element of $S_0^+(x)$.

Algorithm 2 Smart reorientation

Require: $(V, \to)$ is a pseudo-transitive digraph
Require: V is an initial segment of $(\mathbb{N}, <)$
  $i \leftarrow 1$
  while $i \in V$ **do**
    for $j < i$ and $j - i$ **do** ▷ Define S_0
      for $k < i$ and $k \mid i$ **do**
        if $k \prec j$ **then**
          $j \in S_0^+$
        else if $j \prec k$ **then**
          $j \in S_0^-$
        end if
      end for
    end for
    for $n < i$ **do**
      for $j < i$ and $j - i$ and $j \notin S_0 \cup \cdots \cup S_n$ **do** ▷ Define S_n
        for $k \in S_n$ **do**
          if $j \mid k$ and $k \in S_n^-$ **then**
            $j \in S_{n+1}^-$
          else if $j \mid k$ and $k \in S_n^+$ **then**
            $j \in S_{n+1}^+$
          end if
        end for
      end for
    end for
    for $j < i$ **do**
      if $j \mid i$ **then** ▷ Smart extension
        $j \not\prec i$ and $i \not\prec j$
      else
        if $j \in S^-(i)$ **then**
          $j \prec i$
        else if $j \in S^+(i)$ **then**
          $i \prec j$
        else
          $k \leftarrow 0$
          while $k < j$ **do** ▷ $j < i$
            if $j \prec k \prec i$ **then**
              $j \prec i$
            else if $i \prec k \prec j$ **then**
              $i \prec j$
            else
              $k \leftarrow k + 1$
            end if
          end while
          if $j \to i$ **then**
            $j \prec i$
          else
            $i \prec j$
          end if
        end if
      end if
    end for
    $i \leftarrow i + 1$
  end while

Remark 1.42. Notice that the smart reorientation of $(V, \leftarrow)$ is the reversal of the smart reorientation of $(V, \rightarrow)$.

Theorem 1.49 proves that the smart reorientation algorithm is correct. To obtain this result we prove some properties of smart reorientations. In particular we introduce the notion of 'lazy reorientation' in Definition 1.44. The intuitive idea behind it is the following one: an edge $a \rightarrow b$ is reversed only when this is really needed to obtain a transitive reorientation, because $a \rightarrow b \rightarrow c \rightarrow a$ and the edges $b \rightarrow c$ and $c \rightarrow a$ are not reversed.

Property 1.43. *Let* $(V, \rightarrow, \prec)$ *be a GH-triple with* $V \subseteq \mathbb{N}$. *Let* $(V \cup \{x\}, \rightarrow')$ *be a pseudo-transitive extension of* $\rightarrow$. *Let* $\prec'$ *be the smart extension of* $\prec$.

If $a \prec' x$ *because we applied condition (3a) with witness* b *then* $b \in T(x)$. *Moreover we can choose* b *so that* $b \rightarrow' x$. *Similarly, if* $x \prec' a$ *because we applied condition (3b) with witness* b *then* $b \in T(x)$ *and we can assume* $x \rightarrow' b$.

Proof. Let $a \in T(x)$ and b with $b < a$ be such that $a \prec b \prec' x$. Since $b \prec' x$ then $b \in S^-(x) \cup T(x)$. But $b \notin S^-(x)$ by Property 1.38.2 and hence $b \in T(x)$. Let b be least (as a natural number) such that $a \prec b \prec' x$. If $x \rightarrow' b$, then we used condition (3a) when dealing with b and so there exists $c < b$ such that $b \prec c \prec' x$, contrary to the minimality of b. Hence, $b \rightarrow' x$.

The proof of the second statement is analogous. □

Definition 1.44. Let $(V, \rightarrow)$ be a pseudo-transitive ograph. The reorientation $\prec$ of $\rightarrow$ is a *lazy reorientation* if it satisfies the following property: for each $a, b \in V$ such that $a \rightarrow b$ and $b \prec a$ there exists $c \in V$ such that $b \rightarrow c \rightarrow a$ (i.e. abc is a non transitive triangle), $b \prec c \prec a$, and $c < \min(a, b)$.

$(V, \rightarrow, \prec)$ is a *lazy triple* if $(V, \rightarrow)$ is a pseudo-transitive ograph and the reorientation $\prec$ of $\rightarrow$ is a lazy reorientation.

Notice that a lazy triple is not necessarily a GH-triple, because we are not requiring $\prec$ to be transitive.

Remark 1.45. $(V, \rightarrow, \prec)$ is a lazy triple if only if $(V, \leftarrow, \succ)$ is a lazy triple, where $(V, \leftarrow)$ is the reverse ograph of $(V, \rightarrow)$.

Property 1.46. *Let* $(V, \rightarrow)$ *be a pseudo-transitive ograph with* $V \subseteq \mathbb{N}$ *and let* $\prec$ *be the smart reorientation of* $(V, \rightarrow)$. *Assume that* $\prec$ *is transitive, so that* $(V, \rightarrow, \prec)$ *is a GH-triple. Then* $\prec$ *is lazy, i.e.* $(V, \rightarrow, \prec)$ *is a lazy triple.*

Proof. The proof of the laziness condition for every $a, b \in V$ is by induction on the lexicographic order of the pair of natural numbers $(\max(a, b), \min(a, b))$. Suppose $a \rightarrow b$ and $b \prec a$ and assume that for each a' and b' such that $a' \rightarrow b'$, $b' \prec a'$ and either $\max(a', b') < \max(a, b)$ or $\max(a', b') = \max(a, b)$ and $\min(a', b') < \min(a, b)$ there exists c' such that $b' \rightarrow c' \rightarrow a'$, $b' \prec c' \prec a'$, and $c' < \min(a', b')$.

By remark 1.42 we can assume without loss of generality that $a < b$. According to Definition 1.40 either $a \in S^+(b)$ or $a \in T(b)$.

If $a \in S^+(b)$ let i be such that $a \in S^+_i(b)$. We first show that $i > 0$ is impossible. If $a \in S^+_i(b)$ with $i > 0$ let ρ be a $+$-sequence of length at least 2 such that $\rho(i) = a$. By Property 1.38.1 we have $\rho(1) \rightarrow b$. Since $\rho(1) \in S^+_1(b)$, there exists $d < b$ such that $d \in S^+_0(b)$ and $d \mid \rho(1)$. Then $b \prec d$ and $d \rightarrow b$. Since $d \in S^+_0(b)$ there exists $f < b$, $f \mid b$, $f \prec d$. Thus $d \rightarrow f$. As $\max(d, f) < b = \max(a, b)$ we can apply the induction hypothesis and there exists c such that $f \rightarrow c \rightarrow d$ and $f \prec c \prec d$. We have $c - b$ by $\mathrm{pt}(d)$, and hence $b \rightarrow c$ because $b \mid f$. But now $\rho(1) - c$ by $\mathrm{pt}(b)$, and hence $c \rightarrow \rho(1)$ since $\rho(1) \mid d$. Using again $\mathrm{pt}(c)$ we have $\rho(1) - f$, a contradiction with $\rho(1) \in S^+_1(b) \subseteq N^-(b) \cap N^+(b)$ as $f \notin N(b)$.

Thus $i = 0$ and $a \notin N^-(b)$. In particular there exists $f < b$, $f \mid b$, $f \prec a$ and so $a \rightarrow f$. As $\max(a, f) < \max(a, b)$ we can apply the induction hypothesis and there exists $c < \min(a, f)$ such that $f \rightarrow c \rightarrow a$ and $f \prec c \prec a$. We have $c - b$ by $\mathrm{pt}(a)$, and hence $b \rightarrow c$ because $b \mid f$. Hence $b \rightarrow c \rightarrow a$, $b \prec c \prec a$ (because $\prec$ is transitive and $b \mid f$), and $c < \min(a, b)$, as required.

If $a \in T(b)$ we applied condition (3b) of Definition 1.40 to set $b \prec a$. Hence, by Property 1.43 there exists c such that $c < a$, $b \prec c \prec a$ and $b \to c$. We can assume that c is least (as a natural number) with these properties. If $c \to a$ we have our conclusion. We now rule out the possibility that $a \to c$. If this was the case, by induction hypothesis (as $\max(a, c) < \max(a, b)$) there exists $d < \min(a, c)$ such that $c \to d \to a$ and $c \prec d \prec a$. By transitivity of $\prec$ we have $b \prec d \prec a$ and $b - d$. If $b \to d$ then $d < c$ violates the minimality of c. If $d \to b$ then by induction hypothesis (as $\max(d, b) = \max(a, b)$ and $\min(d, b) < \min(a, b)$) there exists $e < d$ such that $b \prec e \prec d$ and $b \to e \to d$. But then $e < c$, $b \prec e \prec a$ (by transitivity of $\prec$) and $b \to e$ contradict the minimality of c. □

Lemma 1.47. *Let $(V, \to, \prec)$ be a GH-triple which is also a lazy triple and such that $V \subseteq \mathbb{N}$. Then Φ, Ψ and Θ are satisfied.*

Proof. Thanks to laziness checking that Θ holds is straightforward. In fact, suppose $a \to c$, $b \to d$, $a \prec c$ and $d \prec b$ for some $a, b, c, d \in V$. Since $b \to d$ but $d \prec b$, there exists an e_0 such that $d \to e_0 \to b$. It is immediate to check that e_0 witnesses $\theta(a, b, c, d)$[2].

To check that Φ holds let $a, b, c \in V$ be such that $a \to c \leftarrow b$ and $a \prec c \prec b$. Since $b \to c$, $c \prec b$ and $\prec$ is lazy, there exists $e_0 \in V$ such that $c \to e_0 \to b$, $c \prec e_0 \prec b$ and $e_0 < \min(b, c)$. By transitivity of $\prec$ it holds that $a \prec e_0$ and thus $a - e_0$, since $\prec$ is a reorientation. If $a \to e_0$, it is immediate to check that e_0 witnesses $\varphi(a, b, c)$.

Otherwise $e_0 \to a$ and, since $a \prec e_0$, by laziness there exists $e_1 \in V$ such that $a \to e_1 \to e_0$, $a \prec e_1 \prec e_0$ and $e_1 < \min(e_0, a)$. Notice that even if $a \to c \to e_0$ and $a \prec c \prec e_0$ it must be $c \neq e_1$ because $e_1 < e_0 < c$ by construction. By transitivity we get that $e_1 \prec b$ and so either $e_1 \to b$ or $b \to e_1$. If the former holds then e_0, e_1 witness $\varphi(a, b, c)$.

We have now to analyse the case when $b \to e_1$. Since $e_1 \prec b$ by laziness there exists e_2 such that $e_1 \to e_2 \to b$, $e_1 \prec e_2 \prec b$ and $e_2 < \min(b, e_1)$. By transitivity it holds that $a \prec e_2$. If $a \to e_2$, it is easy to check that e_0, e_1, e_2 witness $\varphi(a, b, c)$. Otherwise $e_2 \to a$ and we can apply laziness again to obtain e_3.

This procedure provides a $<$-decreasing sequence (e_i) such that $a \to e_{i+1} \to e_i$ when i is even, and $e_i \to e_{i+1} \to b$ when i is odd. The sequence stops with e_n such that $a \to e_n \to b$. We claim that $e_0, \dots, e_n$ witness $\varphi(a, b, c)$. In fact (φ_1) is guaranteed by $c \to e_0$. Moreover, for each $i < n$ either $a \to e_i \leftarrow b$ or $a \leftarrow e_i \to b$ by assumption. If the former is the case then $e_i \to e_{i+1}$, while if the latter holds $e_{i+1} \to e_i$ by construction. These two facts guarantee that (φ_2) is satisfied as well. The vertex e_n satisfies condition (φ_3) by construction.

It is now easy to check that Ψ is satisfied as well applying the duality principle of Remark 1.18. Consider the graph $(V, \leftarrow)$ and the transitive reorientation $\succ$. Remark 1.45 guarantees that $\succ$ is lazy as well. Hence, Φ holds by what we have just shown. Then, by Remark 1.18, Ψ holds in $(V, \to)$ and $\prec$. □

Lemma 1.48. *Let $(V, \to, \prec)$ be a GH-triple such that $V \subseteq \mathbb{N}$. Assume Φ, Ψ and Θ are satisfied. Let $(V \cup \{x\}, \to')$ be a pseudo-transitive extension of $(V, \to)$. Then the smart extension $\prec'$ to $\to'$ is transitive.*

Proof. To check that $\prec$ is transitive, we have to consider the following cases, where $a, b \in V$:

1. $a \prec' x \prec' b$. Obviously $a, b \in N(x)$ and, if $a \in S(x)$ then $a \in S^-(x)$ while if $b \in S(x)$ then $b \in S^+(x)$. We consider four possibilities:

   (a) $a \in S^-(x), b \in S^+(x)$: then $a \prec b$ follows from Lemma 1.39.

   (b) $a, b \in T(x)$: if $a \to' x \to' b$ or $b \to' x \to' a$, then $a - b$ by pseudo-transitivity. So we are left to check that $b \not\prec a$. Suppose $b \prec a$. Then, according to the definition of $\prec'$, if $b < a$, then $x \prec' b$ entails $x \prec' a$, while if $a < b$, then $a \prec' x$ entails $b \prec' x$.

[2] Notice that laziness implies a strong form of Θ, in fact the sequence witnessing the formulae have always length one.

Otherwise, $a \to' x \leftarrow' b$ or $a \leftarrow' x \to' b$. Suppose the latter holds, the former being similar. Since $x \to' a$, but $a \prec' x$ by assumption, there is, by Property 1.43, $c < a$ such that $c \in T(x)$, $c \to' x$ and $a \prec c \prec' x$. Notice that $c - b$ by $\mathrm{pt}(x)$. We claim that $b \not\prec c$. Suppose $b \prec c$. If $c < b$, then, since $b \prec c \prec' x$, then $b \prec' x$ by definition, contrary to the assumption. Otherwise, $b < c$; then, since $x \prec' b \prec c$, then $x \prec' c$, contrary to the assumption. Thus it must be $c \prec b$ and so $a \prec b$ because $\prec$ is transitive by hypothesis.

(c) $a \in S^-(x), b \in T(x)$: $a \prec b$ follows by Property 1.38.2.

(d) $a \in T(x), b \in S^+(x)$: $a \prec b$ follows by Property 1.38.2.

2. $a \prec b \prec' x$. Since $b \in S^-(x) \cup T(x)$ we have $b \in N^-(x)$ and thus $a \in N(x)$. If $a \in S(x)$, Property 1.38.2 or Lemma 1.39 imply $a \in S^-(x)$, and thus $a \prec' x$.

   If instead $a \in T(x)$ then $b \in T(x)$ by Property 1.38.2. If $b < a$ then $a \prec b \prec' x$ implies $a \prec' x$. If $a < b$ then $x \prec' a$ would imply $x \prec' b$; hence $a \prec' x$ holds also in this case.

3. $x \prec' a \prec b$. The argument is similar to the previous case. $\square$

The following theorem proves that Definition 1.40 provides an algorithm to transitively reorient pseudo-transitive graphs.

Theorem 1.49. *Let $(V, \to)$ be a pseudo-transitive ograph with V an initial interval of $\mathbb{N}$ and let $\prec$ be the smart reorientation of $(V, \to)$. Then $\prec$ is transitive.*

Proof. For each $s \in \mathbb{N}$, let $\prec_s$ be the restriction of $\prec$ to $\{0, \ldots, s-1\}$. Notice that $\prec_s$ is the smart reorientation of the restriction of $\to$ to $\{0, \ldots, s-1\}$. To prove that $\prec$ is transitive it is enough to check that $\prec_s$ is transitive for each s. We do so by induction on s. For the base case there is nothing to check. Suppose $\prec_s$ is transitive. Then by Property 1.46 $\prec_s$ is lazy. Moreover, by Lemma 1.47 Φ, Ψ and Θ are satisfied. Hence, by Lemma 1.48 the smart extension $\prec_{s+1}$ is transitive. $\square$

PART II

INTERVAL ORDERS GRAPHS

I
N
T
O R D E R S
R
V
G R A P H S
L

An interval graph is a graph (V, E) whose vertices can be mapped into intervals of a linear order $(L, <_L)$ in such a way that two vertices are adjacent if and only if the intervals associated to them overlap. Consequently, if two vertices are incomparable in the graph, their intervals are placed one before the other in the linear order $(L, <_L)$. The definition itself of interval graphs lead to imagine that there is an analogous concept for orders, namely interval orders, which are defined similarly. In fact, an order $(P, <_P)$ is an interval order if its points can be mapped to intervals of a linear order $(L, <_L)$ in such a way that $x <_P y$ if and only if the interval associated to x completely precedes the interval associated to y. Given these definitions, it is easy to imagine that an interval order gives rise to an interval graph and vice versa.

Norbert Wiener was probably the first to pay attention to interval orders, disguised under the less familiar name 'relations of complete sequence', in [Wiener 1914]. Fifty years later interval graphs and interval orders were rediscovered and named with the current name. Many articles were published about this topic. [Trotter 1997] provides a survey for many result in this area, focusing primarily on finite structures. We refer mainly to the monograph [Fishburn 1985], where it is possible to find more results, examples and references about interval graphs and interval orders.

Interval graphs are, as interval orders, extensively employed in very different fields like psychology, archaeology and physics, just to mention some of them. Wiener himself noticed that interval orders are useful for the analysis of temporal events and in the representation of measures subject to a margin of error. For example it might be the case that two appointments in our agenda overlap and precede a third appointment. The representation of this situation adopted by some digital calendars is actually a (bi-dimensional) interval representation, where hours and days form a linear order and a rectangle covers the time assigned to an appointment. If two rectangles intersect, we had better choose which event we will miss. Intervals are also suitable for representations of measurements of physical properties which are subject to error, since they can take into account the accuracy of the measuring device much better than a representation with points. In psychology and economics the overlap between two intervals often indicates that the corresponding stimuli or preferences are indistinguishable. To this end indifference graphs and proper interval orders were introduced. They are subclasses of interval graphs and interval orders respectively where no vertices are mapped to intervals such that one is a proper subinterval of the other.

Interval graphs are the incomparability graphs of interval orders. The notion of comparability graphs, as well as a structural characterisation, has already been introduced in the previous chapters. Indeed comparability graphs, and so the interplay between graphs and orders, is a common thread of the thesis (in the next part a result concerning graphs can be improved when considering a subclass of comparability graphs). As far as this part of the thesis in concerned, the relationship between interval graphs and interval orders is considered from three viewpoints. First, we clarify in which subsystems of second order arithmetic interval graphs are actually related to interval orders, taking into account different characterisations of both of them. Unique orderability is the second point of view: generally an interval (or broadly a comparability) graph gives rise to many orders, but not always. It is interesting to delimit the class of interval graphs which are associated to a unique order, up to duality. The third aspect under which the interplay between graphs and orders is analysed concerns the dimension of posets, one of the comparability invariants (all orders with the same comparability graph have the same dimension). For more results in this direction and more references see [Rival 1985], especially [Kelly 1985].

Interval graphs have already been analysed from the computability, and more specifically from the reverse mathematics, point of view. The main reference for the analysis of interval orders in reverse mathematics is [Marcone 2007], which contains a number of results about the strength of different characterisations of interval orders. Chapters 2 and 3 follow the line of that article. Damir Dzhafarov in [Dzhafarov 2011] studied the notion of saturated orders, a generalisation of interval orders introduced by Reinhard Suck only for finite posets. Dzhafarov extended this notion to infinite saturated orders and studied it within subsystems of second order arithmetic.

More attention has been paid to interval graphs and orders from the on-line combinatorics point of view. The following theorem, proved in [Kierstead and Trotter 1981], is an example.

Theorem. *Each computable interval order of width k can be covered by $3k-2$ computable chains.*

Section 7.2 covers the material about partial orders with finite width and their decomposition into chains. We just anticipate here that each poset of width k can be covered by k chains, even if sometimes the chains are not computable from the poset. However, Kierstead in [Kierstead 1981] proved that not all is lost when one deals with computable posets. In fact, it is always possible to decompose a poset of width k into at most $(5^k-1)/4$ chains. The previous theorem improves the result just mentioned in case one takes into account only interval orders[3]. Since interval orders are a subclass of orders with quite peculiar properties it turns out that results stated for orders, or graphs, can be improved when restricting only to interval orders, or interval graphs. The statements about the dimension of interval orders, presented in Chapter 5, are examples of this phenomenon.

A colouring of a graph (V,E) is a function $c\colon V\to\mathbb{N}$ such that if $u\,E\,v$ then $c(v)\neq c(u)$ for each $u,v\in V$. A graph is k colourable if $\operatorname{ran}(c)\subseteq\{0,\dots,k\}$. A simple corollary of the previous theorem is the following.

Corollary. *If (V,E) is a computable interval graph without complete subgraphs of size $k+1$, there is a computable $3k-2$ colouring of (V,E).*

Proof. If (V,E) has no complete subgraph of size $k+1$, then every interval posets $(V,\prec)$ associated to (V,E) (i.e. $\prec$-incomparability corresponds to E-adjacency) has no antichain of size k. Then $\prec$ is decomposable into $3k-2$ computable chains $C_0,\dots,C_{3k-1}$. Define $c\colon V\to\mathbb{N}$ such that $c(v)=i$ if and only if $v\in C_i$. Notice that c is computable. Moreover, it is immediate to check that the function c colours (V,E) with at most $3k-2$ colours. □

The previous statement is a computable version of Rado's theorem (see [Rado 1948]), which states the following: an interval graph is k-colourable if and only if each subgraph of size $k+1$ is k-colourable (i.e. it does not contain any complete subgraph of size $k+1$).

In the same article Keirstead and Trotter proved that there is a computable interval graph with no complete subgraphs of size $k+1$ without computable $3k-3$ colouring. James Schmerl in [Schmerl 2005, Corollary 3.1] shows that this statement is provable in $\mathsf{RCA}_0+\neg\mathsf{WKL}_0$. Consequently, Rado's theorem is equivalent to WKL_0.

Theorem (RCA_0)**.** WKL_0 *is equivalent to the following statement: an interval graph is k-colourable if and only if each subgraph of size $k+1$ is k-colourable.*

Proof. It is easy to see that the theorem is provable by compactness. For the reverse implication notice that if each subgraph of size $k+1$ is k-colourable, it is $3k-3$ colourable. The conclusion follows from Schmerl's remark. □

[3]Both papers give lower bounds, namely examples of partial orders which cannot be covered by a certain amounts of chains. For interval orders there is no gap between the upper and lower bound.

Overview of the main results. We prove that WKL_0 is equivalent to the statement that a graph is an interval graph if and only if is triangulated and has no asteroidal triples. This result contrasts with Theorem 2.1 in [Marcone 2007], which proves that an analogous structural characterisation for interval orders is provable in RCA_0. An analogous result is proved for indifference graphs, again contrasting the specular result for proper interval orders.

We prove that in RCA_0 there are three non equivalent definitions of interval graphs (besides the structural one), as happens for interval orders, and that they collapse into one in WKL_0. On the other hand there are two non equivalent definitions of indifference graphs in RCA_0 (besides the structural one), as it happens for proper interval orders, but they still collapse into one in WKL_0.

We generalise to the infinite case a sufficient condition and a necessary condition for unique orderability of connected interval graphs. We show that both of them are computably true. A satisfactory characterisation for unique orderability of infinite connected interval graphs is still missing.

We prove some basic equivalence between WKL_0 and statements about upper and lower bounds of the dimension of interval orders.

Overview of the chapters. The first chapter analyses the strength of various characterisations of interval graphs and the interplay between interval orders and interval graphs. After stating the preliminary definitions the first section is devoted to the analysis of the structural characterisations for interval graphs, while the second concerns different possible definitions of interval graphs. It is split into a subsection about the relationship between interval graphs and interval orders and into a subsection where the relative strength of the various definitions presented is examined.

The second chapter is devoted to the strength of various characterisations of indifference graphs and the interplay between proper interval orders and indifference graphs. It is organised as the previous chapter.

The third chapter discusses the theme of unique orderability of interval graphs. It is noticed that the characterisation of unique orderability of non connected interval graphs and of indifference graphs can be proved in RCA_0. A characterisation for connected finite interval graphs is presented. Some preliminary results towards its generalisation to infinite connected interval graphs are shown.

The fourth chapter contains some notes on the dimension of interval orders. The strength of some basic fact, as the existence of linear extensions whose intersection is a given poset or the existence of posets with arbitrary dimension, is analysed. The second section focuses on interval orders showing that some tight bounds on their dimension are provable in WKL_0.

The results mentioned would not have been obtained without the help and the suggestions of my supervisor.

INTERVAL GRAPHS

CHAPTER 2

Content

The following definition formalises the intuitive idea of interval graph given in the previous pages. Definitions 2.1 and 3.10 are adaptations to interval graphs of the corresponding definitions of interval orders given in [Marcone 2007].

Definition 2.1. A graph (V, E) is an *interval graph* if it is reflexive and there exist a linear order $(L, <_L)$ and a relation $F \subseteq V \times L$ such that, abbreviating $\{x \in L \mid (p, x) \in F\}$ by $F(p)$, for all $p, q \in V$ the following hold

(i1) $F(p) \neq \emptyset$ and $\forall x, y \in F(p) \, \forall z \in L \, (x <_L z <_L y \rightarrow z \in F(p))$,

(i2) $p \, E \, q \Leftrightarrow F(p) \cap F(q) \neq \emptyset$.

The requirement of reflexivity for interval graphs corresponds to the requirement that comparability graphs are irreflexive. The link between the two will become clear thanks to Theorems 2.5 and 2.6. Figure 2.1 provides an example of interval graph, while the graph in figure 2.2 does not have an interval representation.

Since interval graphs are strictly linked with interval orders we report here the definition of interval order (see [Marcone 2007]).

Figure 2.1: An example of interval graph with its representation

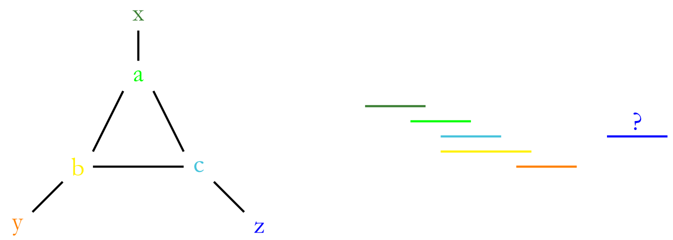


Figure 2.2: A graph which is not an interval graph, with a partial representation

Definition 2.2. A order $(V, \prec)$ is an *interval order* if there exist a linear order $(L, <_L)$ and a relation $F \subseteq V \times L$ such that, abbreviating $\{x \in L \mid (p, x) \in F\}$ by $F(p)$, for all $p, q \in V$ the following hold

(i1) $F(p) \neq \emptyset$ and $\forall x, y \in F(p)\, \forall z \in L\, (x <_L z <_L y \to z \in F(p))$,

(i2) $p \prec q \Leftrightarrow \forall x \in F(p)\, \forall y \in F(q)\, (x <_L y)$.

We are going to use the following result proved in [Marcone 2007]. Following Marcone we say that $(V, \prec)$ "does not contain a $2 \oplus 2$" if there is no $P \subseteq V$ such that $\prec \restriction P$ is the partial order with Hasse diagram . In other words $(V, \prec)$ "does not contain a $2 \oplus 2$" if for each $p, q, r, s \in V$ such that $p \prec q$ and $r \prec s$ it holds that either $p \prec s$ or $r \prec q$.

Theorem 2.3 (RCA_0). *An order $(V, \prec)$ is an interval order if and only if it does not contain a $2 \oplus 2$.*

Proposition 2.4 (RCA_0). *Let $(V, \bar{E})$ be a comparability graph. The orders associated to the comparability graph contain a $2 \oplus 2$ if and only if the complementary graph of $(V, \bar{E})$ contains a four cycle without chords.*

Proof. Let $(V, \bar{E})$ be a comparability graph, $(V, \prec)$ an associated order and (V, E) the complementary graph of $(V, \bar{E})$. Suppose $(V, \prec)$ contains a $2 \oplus 2$. This means that there are a, b, c, d such that $a \prec b, c \prec d$ and $a \mid d, c \mid b$, which entail $a \mid c, b \mid d$. Hence $a\, E\, d\, E\, b\, E\, c\, E\, a$ is a four cycle without chords. For the reverse implication the reasoning is analogous. □

We now turn to the announced results about the relationship between interval graphs and interval orders mentioned at the beginning. The first one claims that a graph (V, E) is an interval graph if and only if the orders associated to its complementary graph are interval orders. The second goes the other way around: if a poset $(V, \prec)$ is fixed, $(V, \prec)$ is an interval order if and only if the complementary graph of its comparability graph is an interval graph. In particular, we show that the proofs of these basic facts given in the literature go through in RCA_0.

Theorem 2.5 (RCA_0). *Let (V, E) be a graph. Then (V, E) is an interval graph if and only if there is an interval order $(V, \prec)$ such that $p\, E\, q \Leftrightarrow p \mid q$ for all $p, q \in V$.*

Proof. ($\Rightarrow$) Let $(L, <_L)$ be a linear order and F be a representing relation witnessing that (V, E) is an interval graph. Let $\prec$ be the order defined by $p \prec q \Leftrightarrow \forall x \in F(p)\, \forall y \in F(q)\, (x <_L y)$. Definition 2.1 guarantees that the Π^0_1-definition of $\prec$ is equivalent to the following Σ^0_1-definition

$$p \prec q \Leftrightarrow \neg pEq \wedge \exists x \in F(p)\, \exists y \in F(q)\, (x <_L y)$$

Hence, $\prec$ is definable in RCA_0. Moreover, $\prec$ is clearly an order on $(V, \bar{E})$, the complementary graph of (V, E), since $<_L$ is a linear order, and it satisfies $p\, E\, q \Leftrightarrow p \mid q$ for all $p, q \in V$. It is easy to check that L and F themselves witness that $(V, \prec)$ is an interval order.

($\Leftarrow$) Let (V, E) be a graph and $(V, \prec)$ be an interval order such that $p\, E\, q \Leftrightarrow p \mid q$ for all $p, q \in V$. Let also $(L, <_L)$ be a linear order and F be a representation for $(V, \prec)$. We claim that $(L, <_L)$ and F witness also that (V, E) is an interval graph. In fact, suppose $p\, E\, q$ for $p, q \in V$. Since $p \not\prec q \wedge q \not\prec p$, by assumption, it holds that $\exists x \in F(p) \exists y \in F(q)(y \leq_L x)$ and $\exists x' \in F(p) \exists y' \in F(q)(x' \leq_L y')$. Suppose, without loss of generality, that $y \leq_L y'$. Then $y \leq_L x'$ or $x' \leq_L y$, being L a linear order. In the first case, $x' \in F(q)$. In the second case, $y \in F(p)$. Hence $F(p) \cap F(q) \neq \emptyset$ as required by Definition 2.1. □

For sake of completeness, we prove here the subsequent theorem, which provides a conclusion to the previous one, even if it uses a theorem we have not proven yet.

Theorem 2.6 (RCA$_0$)**.** *Let $(V, \prec)$ be an order. $(V, \prec)$ is an interval order if and only if (V, E), where $p\, E\, q \Leftrightarrow p \mid q$ for all $p, q \in V$, is an interval graph.*

Proof. ($\Rightarrow$) This implication follows easily from the definitions.

($\Leftarrow$) Let $(V, \prec)$ be an order and let (V, E) be such that $p\, E\, q \Leftrightarrow p \mid q$ for all $p, q \in V$. Assume (V, E) is an interval graph, then, by Theorem 2.8(a), each cycle of length four has a chord. Proposition 2.4 entails that $(V, \prec)$ does not contain a $2 \oplus 2$. Therefore $(V, \prec)$ is an interval order by Theorem 2.3. □

Even if Theorems 2.5 and 2.6 look very similar their proofs reveal an asymmetry between the passage from an interval order to an interval graph and the reverse passage. In fact, in the proof of the former theorem the very same $(L, <_L)$ and F witness both that (V, E) and $(V, \prec)$ are an interval graph and an interval order respectively, while this is not the case in the later statement. To explain the reason of this suppose $(V, \prec)$ is a chain. Clearly there are infinitely many possible interval representations F witnessing that (V, E), with $E = \mid$, is an interval graph, but among them only one representation is suitable for $(V, \prec)$ as well. We stress here this point because this difference between interval graphs and interval orders will appear again later on.

2.1 Structural characterisations of interval graphs

Definition 2.1 explains clearly the reason of the name of this class of graphs. At the same time it is not a very nice definition to work with, in fact if a graph is given and one wants to check whether it is an interval graph or not, he has to find a suitable linear order and a suitable representation which witness that the graph can be represented as requested by Definition 2.1. It would be useful to understand what structural properties a graph must have in order to be representable by a linear order. Theorems 1.2 and 2.3 give examples of the properties we are looking for. Combinatorialists provided two of these structural properties which identify interval graphs. We prove that the first one holds in RCA$_0$, while the second one is equivalent to WKL$_0$.

The following two theorems deal with necessary conditions to be an interval graph. We underline that they both go through in RCA$_0$.

Definition 2.7. A graph (V, E) is *triangulated* if every simple cycle of length four or more has a chord.

An *asteroidal triple* in (V, E) is an independent set of three vertices (i.e. set of pairwise non adjacent vertices) of V such that each pair is connected by a path that avoids the neighbourhood of the third.

Theorem 2.8 (RCA$_0$)**.** *If a graph (V, E) is an interval graph, then a) every simple cycle of length four has a chord and b) the complementary graph $(V, \bar{E})$ is a comparability graph.*

Proof. This follows immediately from Theorems 2.5 and 2.3 and Proposition 2.4. □

Theorem 2.9 (RCA$_0$)**.** *If a graph (V, E) is an interval graph, then it is triangulated and has no asteroidal triples.*

Proof. Let (V, E) be an interval graph, $(L, <_L)$ be a linear order and F be its representing relation. Suppose (V, E) is not triangulated, i.e. there is a simple cycle $a_0\, E\, \ldots\, E\, a_n = a_0$, for some $n \geq 4$, without chords. It is easy to check that both $F(a_0) <_L F(a_{n-1})$, by transitivity of $<_L$, and $F(a_0) \mid_L F(a_{n-1})$ since $a_0\, E\, a_{n-1}$.

Suppose (V, E) contains an asteroidal triple $\{x, y, z\}$. Without loss of generality, we assume that the interval associated to y is between the intervals associated to x, z. Since x, y, z form an asteroidal triple, there exists a path P from x to z, which avoids the neighbourhood of y. This means that all the points in P must be represented by intervals between $F(x)$ and $F(z)$ overlapping one to the other. These intervals must not overlap $F(y)$: the two requests are incompatible. □

The previous two theorems give two necessary conditions for a graph to be an interval graph. These are actually sufficient conditions, thus they provide the two characterisations of interval graphs we were looking for. Before turning to the proof of sufficiency, we would like to highlight a difference between theorems 2.8 and 2.9. The former mentions comparability graphs, which are defined in 8 as graphs whose adjacency relation is the comparability relation of an order. Thus to verify if a graph is a comparability graph or not, one has to build a suitable order for it. Instead, the later theorem presents pure structural conditions, namely conditions which have only to do with the adjacency relation of (V, E) and that can be read off from (V, E) itself, so to speak, without constructing any other entity. Keeping this observation in mind and thinking about the strength of Theorem 1.2, it is possible to understand the reason of the different strength between the two following theorems.

Theorem 2.10 (RCA_0)**.** *If a graph (V, E) is such that every simple cycle of length four has a chord and the complementary graph $(V, \bar{E})$ is a comparability graph, then it is an interval graph.*

Proof. Let (V, E) be a graph. Assume every simple cycle of length four has a chord and the complementary graph $(V, \bar{E})$ is a comparability graph. Let $\prec$ be the order associated with $\bar{E}$. Given that $p \, E \, q \Leftrightarrow \neg p \, \bar{E} \, q \Leftrightarrow p \mid q$ for all $p, q \in V$, (V, E) is an interval graph if $(V, \prec)$ is an interval order by Theorem 2.5. So suppose that $(V, \prec)$ is not a interval order. Then $(V, \prec)$ contains a $2 \oplus 2$ by Theorem 2.3. Hence, by Proposition 2.4, (V, E) contains a cycle of length four with no chords, contrary to the assumption. □

Theorem 2.11 (RCA_0)**.** *The following are equivalent:*

1. WKL_0
2. *If a graph (V, E) is triangulated and has no asteroidal triples, then is an interval graph.*

Proof. ($1 \Rightarrow 2$) By Theorem 2.10 it is enough to prove that if a graph (V, E) is triangulated and has no asteroidal triples, then every simple cycle of length four has a chord and the complementary graph $(V, \bar{E})$ is a comparability graph. The first requirement follows from the definition of triangulated graph. For the second one, suppose $(V, \bar{E})$ is not a comparability graph. Then, by Lemma 1.2 it contains an odd cycle without triangular chords. Let $a_0 \, \bar{E} \, \dots \, \bar{E} \, a_n$ be its minimal sub-cycle without chords. Then $a_0 \, E \, a_2 \, E \, a_4 \, E \, a_1 \, E \, a_3 \, E \, a_0$ is a cycle in (V, E) of length five without chords, contrary to the assumption.
($2 \Rightarrow 1$) Let $f, g \colon \mathbb{N} \to \mathbb{N}$ be two injective functions such that there is no $x \in \operatorname{ran}(f) \cap \operatorname{ran}(g)$. We want to define a graph satisfying the hypotheses in (2) and such that its interval representations give enough information to define a set X which separates the range of f, g in RCA_0.

Let $V = \{a_k, b_k, c_k, x_k, y_k \mid k \in \mathbb{N}\}$ be the set of vertices. Define E such that for each $k \in \mathbb{N}$ the following are satisfied:

$$\begin{cases} b_k \, E \, y_k \, E \, c_k & \\ a_k \, E \, x_n \, E \, b_k & \text{if } f(n) = k \\ a_k \, E \, x_n \, E \, c_k & \text{if } g(n) = k \end{cases}$$

(V, E) is triangulated and has no asteroidal triples, so there are a linear order $(L, <_L)$ and a relation F which witness that (V, E) is an interval graph.

It is very easy to realise that if $f(n) = k$, then $F(a_k) <_L F(b_k) <_L F(c_k) \vee F(c_k) <_L F(b_k) <_L F(a_k)$. While if $g(n) = k$, then $F(a_k) <_L F(c_k) <_L F(b_k) \vee F(b_k) <_L F(c_k) <_L F(a_k)$. The following figure represents the k-segment of $(L, <_L)$ in case $f(n) = k$, to the left, and in case $g(n) = k$, to the right.

By Π^0_1-separation (which is provable in RCA_0 [Simpson 2009, Exercise IV.4.8.]) there exists a set X such that

$$\forall n \forall k((\forall x \in F(a_k) \forall y \in F(b_k) \forall z \in F(c_k)(x \prec y \prec z \lor z \prec y \prec x) \to k \in X) \land$$
$$(\forall x \in F(a_k) \forall y \in F(b_k) \forall z \in F(c_k)(x \prec z \prec y \lor x \prec z \prec y) \to k \notin X))$$

From what we said before, we obtain that $\forall n(f(n) \in X \land g(n) \notin X)$ as wanted. □

Theorems 2.8 and 2.10 are not surprisingly at all given the ties between interval graphs and interval orders as Proposition 2.4 expresses. On the other hand, Theorems 2.9 and 2.11 are the real analogues of Theorem 2.3, because they provide a full structural characterisation of interval graphs and orders respectively. Their different strengths shows the difference between interval orders and interval graphs. One one hand, $a \prec b$ carries the information about the order of the intervals associated to the two points, while $a \mathrel{E} b$ does not do the same. Moreover, if $\neg a \mathrel{E} b$ it is possible to choose between the representations $F(a) <_L F(b)$ and $F(b) <_L F(a)$. This freedom in the representations of an interval graph implies a greater difficulty in the construction of the relations F. This difficulty amounts to find a transitive orientation for the complementary graph (which has to be a comparability graph), which requires WKL_0 because of Theorem 1.2.

James Schmerl in [Schmerl 2005] noticed that the claim "A graph is an interval graph if and only if each subgraphs is representable by intervals" is equivalent to WKL_0. The previous theorem confirms his claim. Notice also that the corresponding claim for interval orders, i.e. an order is an interval order if and only if each suborders is representable by intervals, is provable in RCA_0 because of Theorem 2.3.

[Lekkerkerker and Boland 1962] provides another characterisation of interval graphs listing all the forbidden subgraphs. It is routine to check that those graphs are a complete list of graphs whose cycles of length greater than four do not have chords or which contain an asteroidal triple.

2.2 More definitions of intervals and representations

As mentioned in the first lines, interval graphs and interval orders take their name from their representation on intervals of a linear ordered set $(L, <_L)$. In the literature it is possible to find slightly different definitions of them, which depend on the notion of interval employed. For example intervals may be required to be closed or not, it is possible to allow that intervals associated to different points share two extremities, one or none. The combinations of these notions give rise to five conceptually distinct definitions of interval graphs, which are stated in the following definition.

Definition 2.12. Let (V, E) be an interval graph and let $(L, <_L)$ be a linear order and $F \subseteq V \times L$ be a relation for it. (V, E) is a *1-1 interval graph* if it also satisfies

(i3) $F(p) \neq F(q)$ whenever $p \neq q$.

A graph (V, E) is a *closed interval graph* if it is reflexive and there exists a linear order $(L, <_L)$ and two functions $f_0, f_1 \colon V \to L$ such that for all $p, q \in V$

(c1) $f_0(p) <_L f_1(p)$,

(c2) $p \mathrel{E} q \Leftrightarrow f_0(p) \leq_L f_0(q) \leq_L f_1(p) \lor f_0(q) \leq_L f_0(p) \leq_L f_1(q)$

A graph (V, E) is a *1-1 closed interval graph* if we also have

(c3) $f_0(p) \neq f_0(q) \lor f_1(p) \neq f_1(q)$ whenever $p \neq q$.

A graph (V, E) is a *distinguishing interval graph* if (c1) and (c2) hold and

(c4) $f_i(p) \neq f_j(q)$ whenever $p \neq q \vee i \neq j$.

The corresponding definitions for orders are the obvious modifications of Definition 2.2 following the ideas in the Definition 2.12. For example a 1-1 interval order is an interval order which satisfies condition (i3) in Definition 2.12.

2.2.1 Interval graphs and interval orders

Since we are already acquainted with the link between interval graphs and interval orders, we immediately analyse the strength of this tie for the various definitions of intervals given. The next theorem is the equivalent of Theorem 2.5 for 1-1 and closed representations. Even in these two cases if the graph is fixed, it is computably true that 1-1 (closed) interval graphs correspond to 1-1 (closed) interval orders.

Theorem 2.13 (RCA_0)**.** *Let (V, E) be a graph. Then (V, E) is 1-1 (closed) interval graph if and only if there is a 1-1 (closed) interval order $(V, \prec)$ such that $p\,E\,q \Leftrightarrow p \mid q$ for all $p, q \in V$.*

Proof. For 1-1 interval graphs the proof is the same as for interval graphs (see Theorem 2.5). Hence, we turn to closed interval graphs and orders.

($\Rightarrow$) Let (V, E) be a closed interval graph and $f_0, f_1 \colon V \to L$ its representing functions. Define the order relation $\prec$ such that $p \prec q \Leftrightarrow f_1(p) <_L f_0(q)$ for all $p, q \in V$. By definition, $p\,E\,q \Leftrightarrow p \mid q$ for all $p, q \in V$ and $(V, \prec)$ is a closed interval order.

($\Leftarrow$) Let (V, E) be a graph, $(V, \prec)$ a closed interval order, where that $p\,E\,q \Leftrightarrow p \mid q$ for all $p, q \in V$, and let $f_0, f_1 \colon V \to L$ be its representing functions. It is straightforward to verify that f_0, f_1 witness also that (V, E) is a closed interval graph. □

The previous theorems about the characterisation of interval graphs allow to deduce some straightforward corollaries. For example the following is equivalent to WKL_0: for each graph (V, E), (V, E) is triangulated and has no asteroidal triples if and only if there is an interval order $(V, \prec)$ such that $p\,E\,q \Leftrightarrow p \mid q$ for all $p, q \in V$.

The following theorems are more interesting, especially when compared with Theorem 2.6. In that case if $(V, \prec)$ is an order and (V, E) is an interval graph, for $E = |$, then one can argue in RCA_0 that $(V, \prec)$ is interval as well, thanks to the characterisation of interval order given in Theorem 2.3. While the next theorems show that, even if (V, E) is known to be a 1-1 (closed) interval graph, WKL_0 is necessary to conclude that $(V, \prec)$ is a 1-1 (closed) interval graph. The proofs of the reversals are modifications of Theorem 6.4 in [Marcone 2007].

Theorem 2.14 (RCA_0)**.** *The following are equivalent:*

1. WKL_0
2. *Let $(V, \prec)$ be an order. $(V, \prec)$ is a 1-1 interval order if and only if (V, E), where $p\,E\,q \Leftrightarrow p \mid q$ for all $p, q \in V$, is a 1-1 interval graph.*
3. *Let $(V, \prec)$ be an order. $(V, \prec)$ is a closed interval order if and only if (V, E), where $p\,E\,q \Leftrightarrow p \mid q$ for all $p, q \in V$, is a closed interval graph.*

Proof. $(1 \Rightarrow 2, 3)$ The proof proceeds as for Theorem 2.13. Notice that WKL_0 is needed to infer, from $(V, \prec)$ is not a 1-1, or not a closed, interval order, that $(V, \prec)$ contains a $2 \oplus 2$ (see Theorem 6.1 in [Marcone 2007]).

$(2 \Rightarrow 1)$ Let $f, g \colon \mathbb{N} \to \mathbb{N}$ be two injective functions with $rg(f) \cap rg(g) = \emptyset$. Define an order $(V, \prec)$ such that $V = \{a_k, b_k \mid k \in \mathbb{N}\} \cup \{c_k^n \mid n, k \in \mathbb{N}\}$. If $x_k, y_h \in V$, then $x_k \prec y_h$ if and only if $k < h$. If

$k = h$, then x_k, y_k are incomparable except in these two cases

$$\begin{cases} c_k^n \prec a_k \prec c_k^{n+1} & \text{if } f(n) = k \\ c_k^n \prec b_k \prec c_k^{n+1} & \text{if } g(n) = k \end{cases}$$

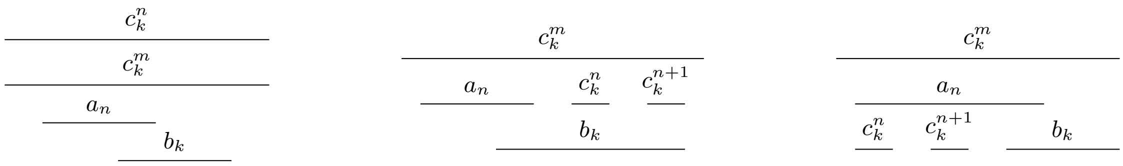
The hypotheses on f, g guarantee that at most one of the possibilities occurs for a given n and that such n is unique. $\prec$ is definable in RCA_0.

$(V, \prec)$ induces a unique comparability graph $(V, \bar{E})$. Consider its complementary graph (V, E). We claim that it is a 1-1 interval graph. It is in fact possible to show a linear order $(L, <_L)$ and a relation F which do the job. Let $L = \mathbb{Q}$ and F be the following representation for each k

$$\begin{cases} F(a_k) = (k + 2/16, k + 8/16) \\ F(b_k) = (k + 7/16, k + 14/16) \\ F(c_k^n) = (k + 1/16 - r_n, k + 15/16 + r_n) \text{ for } 0 < r_n < r_{n+1} < 1/16 & \text{if } f(n) \neq k \neq g(n) \\ F(c_k^n) = (k + 9/16, k + 11/16) \wedge F(c_{k+1}^n) = (k + 10/16, k + 12/16) & \text{if } f(n) = k \\ F(c_k^n) = (k + 3/16, k + 5/16) \wedge F(c_{k+1}^n) = (k + 4/16, k + 6/16) & \text{if } g(n) \neq k \end{cases}$$

It is not difficult to check that L and F satisfy conditions (i1)-(i3) of Definition 2.12. Hence, we conclude that $(V, \prec)$ is a 1-1 interval order.

This is a representation of a segment of V in the case, going from left to right, where $\forall n(f(n) \neq k \neq g(n)), f(n) = k, g(n) = k$.



Let $(L, <_L)$ be a linear order and F be a representing relation for $(V, \prec)$. Let $\varphi(k)$ and $\psi(k)$ be the Π_1^0-formulas $F(a_k) \subseteq F(b_k)$ and $F(b_k) \subseteq F(a_k)$ respectively. Notice that if $\exists n f(n) = k$, then $\varphi(k)$ holds. Instead if $\exists n g(n) = k$, then $\psi(k)$ holds (see [Marcone 2007], Theorem 6.4, for the proof). F being one-to-one assures that $\forall k \neg(\varphi(k) \wedge \psi(k))$. These facts together allow to infer, by Π_1^0-separation, which is provable in RCA_0, that there exists a set X such that

$$\forall k((\varphi(k) \to k \in X)) \wedge (\psi(k) \to k \notin X))$$

Thus, we proved the existence of a set X which separates the range of f from the range of g.

$(3 \Rightarrow 1)$ The proof follows the same steps as the proof before. The definition of the two representing function $f_0, f_1 \colon V \to L$ is an immediate modification of the definition of F, e.g. $f_0(a_k) = k + 3/16$ and $f_1(a_k) = k + 7/16$. While the set X is definable in RCA_0 as $X = \{k \mid f_0(b_k) <_L f_0(a_k)\}$. For the same reasons as before, X separates the range of f from the range of g. $\square$

Notice that in the above proof the graph (V, E) is shown to be a 1-1 (closed) interval graph directly giving a 1-1 (closed) representation F (f_0, f_1 respectively) for its points. What is remarkable here is the fact that F is not a suitable representation for $(V, \prec)$ since it does not maintain the correct $\prec$-relation among a_k, c_k^n, c_k^{n+1} or among b_k, c_k^n, c_k^{n+1}. This feature is actually what allows to define in RCA_0 a 1-1 representation for E but not for $\prec$.

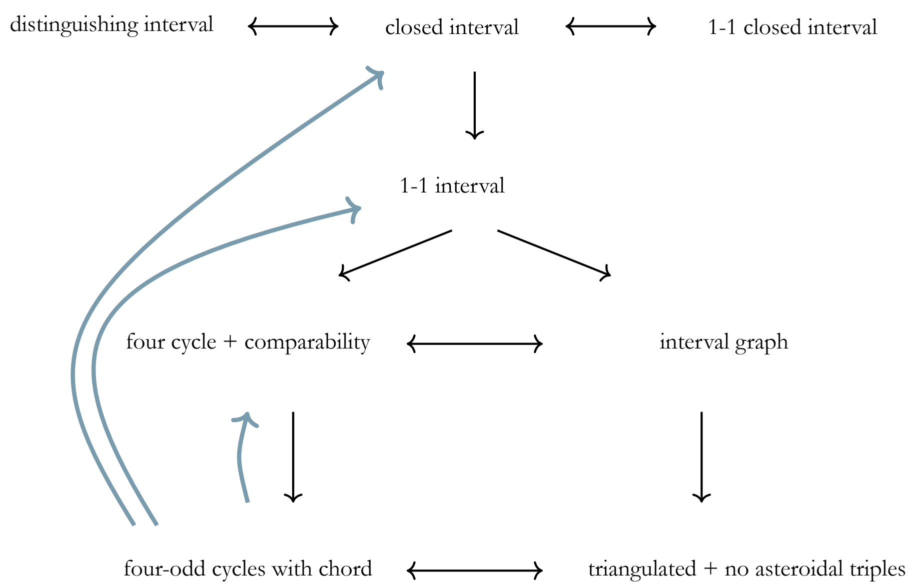


Figure 2.3: $\longrightarrow$ represents implications in RCA_0, $\longrightarrow$ represents implications in WKL_0

This theorem allows also to make a point about the link between interval orders and interval graphs. Getting back to the observation which follows Theorem 2.6, it is evident that sometimes the same (1-1, closed) representation F witnesses that a graph and an order are representable by intervals, but this is not always the case. In particular, it may not be the case when F is a representing relation for a graph (V, E) which is given as the incomparability graph of an order $(V, \prec)$. The reason of this lies on the fact that, in general, more that one order can be associated to a given interval graph (while the contrary does not hold). Said in other words (see Theorem 2.8), a comparability graph can be the graph of the comparability relation of various orders. The trick used in the reversal of Theorem 2.14 exploits exactly this fact and takes advantage of the fact that some associated orders may be defined computably, others cannot.

2.2.2 The strength of the different notions of representation

This subsection is devoted to the analysis of the relative strength of the various definitions of interval graph given in the Definition 2.12. The same investigation for interval orders has already been carried out in [Marcone 2007]. Indeed, in this respect, interval graphs and interval orders share the same behaviour.

Intuitively the Definition 2.12 enumerates increasingly strong definitions of interval graph. For example if (V, E) is a 1-1 interval graph, then the same linear order $(L, <_L)$ and the same representation F witness that it is also an interval graph. Analogously, if (V, E) is a distinguishing interval graph, then the same functions f_0, f_1 witness also that it has a closed representation. Moreover, if the functions f_0, f_1 witness that (V, E) has a (1-1) closed representation, then a relation F such that $F(p) = \{(p, x) \mid f_0(p) \leq_L x \leq_L f_1(p))\}$, for each $p \in V$, witnesses that (V, E) is a (1-1) interval graph. The following fact is perhaps more surprising.

Theorem 2.15 (RCA_0). *If a graph (V, E) is a closed interval graph, then it is a distinguishing interval graph.*

Proof. Suppose (V, E) is a closed interval graph, then there exists an order $\prec$ such that $(V, \prec)$ is a closed interval order by Theorem 2.13. Hence, $(V, \prec)$ is a distinguishing interval order by Theorem 5.1 in [Marcone 2007]. It is then possible to conclude that (V, E) is a distinguishing interval graph. $\square$

We prove that, as happens for interval orders, there are three distinct notions of interval graphs in RCA_0, namely that of interval, 1-1 interval and closed interval graph. This means that given an interval representation F it is not always possible to find computably the functions f_0 and f_1 which give information on the extreme points of the interval associated to a given vertex. But also given a representation F it is not always possible to refine it computably to a 1-1 representation. Nonetheless, these notions collapse into one in WKL_0. Figure 2.3 summaries the implications.

Theorem 2.16 (RCA_0). *The following are equivalent:*

1. WKL_0
2. *If a graph (V, E) is an interval graph, then it is a 1-1 interval graph.*

Proof. $(1 \Rightarrow 2)$ Let (V, E) be an interval graph and $(V, \prec)$ its associated interval order by Theorem 2.5. Then $(V, \prec)$ is a 1-1 interval order by lemma 6.1 in [Marcone 2007]. Then (V, E) is a 1-1 interval graph by 2.13.

$(2 \Rightarrow 1)$ This is a slight modification of proof of Theorem 6.4 in [Marcone 2007]. Suppose (2) holds and let $f, g\colon \mathbb{N} \to \mathbb{N}$ be two injective functions with $rg(f) \cap rg(g) = \emptyset$. Define a graph (V, E) such that $V = \{a_k, b_k \mid k \in \mathbb{N}\} \cup \{c_k^n, d_k^n \mid k, n \in \mathbb{N}\}$ and E is the set of pairs $\{(a_k, b_k), (c_k^n, d_k^n) \mid n, k \in \mathbb{N}\}$ plus

- $c_k^n \, E \, b_k \, E \, c_k^{n+1}$ if $f(n) = k$,
- $c_k^n \, E \, a_k \, E \, c_k^{n+1}$ if $g(n) = k$.

We claim that (V, E) does not contain cycles of length four and its complementary graph is a comparability graph $(V, \bar{E})$. To prove this claim, we define an order $\prec$ related to $(V, \bar{E})$. Let $\prec$ be the set constituted by $x_k \prec y_h$ for each $x_k, y_h \in V$ such that $k < h$ and by

1. $c_k^n \prec a_k \prec c_k^{n+1}, d_k^n \prec b_k \prec d_k^{n+1}$ if $f(n) = k$,
2. $c_k^n \prec b_k \prec c_k^{n+1}, d_k^n \prec a_k \prec d_k^{n+1}$ if $g(n) = k$.

$\prec$ is definable in RCA_0 and it is easy to check that its comparability graph is $(V, \bar{E})$. Thus (V, E) is an interval graph by Theorem 2.10 and hence a 1-1 interval graph.

It is not difficult to check that if $f(n) = k$, then $F(a_k) \subseteq F(b_k)$, while if $g(n) = k$, then $F(b_k) \subseteq F(a_k)$, because the conditions force the following representations (or its overturning) respectively

Since (V, E) is a 1-1 interval graph, $F(a_k) \neq F(b_k)$ for each $k \in \mathbb{N}$. By Π^0_1-separation we can define, in RCA_0, a set X such that

$$\forall k((F(a_k) \subseteq F(b_k) \to k \in X) \land (F(b_k) \subseteq F(a_k) \to k \notin X))$$

By the previous claim, we thus obtain that $\forall n(f(n) \in X \land g(n) \notin X)$. $\square$

Property 2.17 (RCA_0). *If (V, E) is an interval graph such that $\forall p, q \in V \exists r \in V(pEr \leftrightarrow \neg qEr)$, then (V, E) is a 1-1 interval graph.*

Theorem 2.18 (RCA_0). *The following are equivalent:*

1. WKL_0
2. *If a graph (V, E) is a 1-1 interval graph, then it is a closed interval graph.*

Proof. $(1 \Rightarrow 2)$ If a graph (V, E) is a 1-1 interval graph, then there exists an order $\prec$ such that $(V, \prec)$ is a 1-1 interval order, by Theorem 2.13. Then $(V, \prec)$ is a closed interval order, by Theorem 6.5 in [Marcone 2007]. Hence (V, E) is a closed interval graph.

$(2 \Rightarrow 1)$[1]. Suppose (2) holds and let $f, g \colon \mathbb{N} \to \mathbb{N}$ be two injective functions with $rg(f) \cap rg(g) = \emptyset$. Define a graph (V, E) such that $V = \{a_k, b_k, c_k \mid k \in \mathbb{N}\} \cup \{d^n_k \mid k, n \in \mathbb{N}\}$ and E is the set of pairs $\{(a_k, b_k), (a_k, c_k) \mid n, k \in \mathbb{N}\}$ plus the pairs

- $b_k \, E \, d^n_k, c_k \, E \, d^{n+1}_k$ if $f(n) = k$,
- $d^n_k \, E \, a_k \, E \, d^{n+1}_k$ if $g(n) = k$.

(V, E) is an interval graph because it does not contain cycles of length four and its complementary graph is a comparability graph $(V, \bar{E})$. To prove this claim, we define an order $\prec$ for $(V, \bar{E})$. Let $\prec$ be the relation defined by the following clauses

1. $x_k \prec y_h$ for each $x_k, y_h \in V$ such that $k < h$,
2. $b_k \prec c_k$ for each k,
3. $d^n_k \prec d^m_k$ for each k and for each $n < m$,
4. $a_k, b_k, c_k \prec d^n_k$ if $f(n) \neq j \neq g(m)$, for each k, n,
5. $d^n_k \prec a_k \prec d^{n+1}_k$ if $f(n) = k$,
6. $d^n_k \prec b_k \prec c_k \prec d^{n+1}_k$ if $g(n) = k$.

$\prec$ is definable in RCA_0 and it is easy to check that its comparability graph is $(V, \bar{E})$. The following is a picture of a chunk of (V, E) in case $f(n) = k$, to the left, and $g(n) = k$, to the right.

In order to fulfil the premise of statement (2) we need to check that (V, E) is also a 1-1 interval graph. It is immediate to verify that for each $v, u \in V$ there is a $w \in V$ connected only with one of v and u. Hence, by Property 2.17, (V, E) is a 1-1 interval graph.

By statement (2), we conclude that (V, E) is a closed interval graph. So there are a linear order $(L, <_L)$ and two functions $f_0, f_1 \colon V \to L$ satisfying conditions (c1)-(c2) of Definition 2.12. Let X be the set $\{k \mid f_1(a_k) <_L f_1(c_k) \vee f_1(a_k) <_L f_1(b_k)\}$. We claim that $\forall n (f(n) \in X \wedge g(n) \notin X)$. In fact, suppose that $f(n) = k$ and $f_1(b_k) <_L f_1(a_k)$. Since, according to the definition of E, $b_k \, E \, d^n_k, \neg a_k \, E \, d^n_k$, it must be that $f_0(b_k) <_L f_0(a_k)$. Given that $a_k \, E \, c_k$ and $\neg b_k \, E \, c_k$, this implies $f_0(c_k) <_L f_0(a_k) <_L f_1(c_k)$. But then $f_1(a_k) <_L f_1(c_k)$, otherwise it would be impossible that the intervals associated with c_k and with d^{n+1}_k overlap, but those of a_k, d^{n+1}_k do not. Hence, $k \in X$.

To conclude, suppose $g(n) = k$. Then, according to the definition of E, it holds that $d^n_k \, E \, a_k \, E \, d^{n+1}_k$. Suppose, without loss of generality, that $f_0(d^n_k) <_L f_0(a_k) <_L f_1(d^n_k)$. Since $\neg d^n_k \, E \, d^{n+1}_k$, this implies $f_0(a_k) <_L f_0(d^{n+1}_k) <_L f_1(a_k)$. Hence, $f_1(d^n_k) <_L f_0(d^{n+1}_k)$ by transitivity of $<_L$. Clearly, it also holds that $f_1(d^n_k) <_L f_0(b_k) <_L f_1(b_k) <_L f_0(d^{n+1}_k)$ and $f_1(d^n_k) <_L f_0(c_k) <_L f_1(c_k) <_L f_0(d^{n+1}_k)$, which implies $f_0(a_k) <_L f_0(b_k) <_L f_1(b_k) <_L f_1(a_k)$ and $f_0(a_k) <_L f_0(c_k) <_L f_1(c_k) <_L f_1(a_k)$. Therefore, $k \notin X$. $\square$

Theorem 2.19 (RCA_0)**.** *The following are equivalent:*

1. WKL_0

[1]This is a slight modification of proof of Theorem 6.5 in [Marcone 2007]

2. *If a graph (V, E) is an interval graph, then it is a closed interval graph.*

Proof. It follows from Theorems 2.16 and 2.18. □

INDIFFERENCE GRAPHS

CHAPTER 3

Content

Indifference graphs are a subclass of interval graphs. The distinguish feature of this class is the fact that there are no distinct vertices such that the interval associated to one is included in the interval associated to the other. The name 'Indifference graphs' was first introduced by Fred Roberts in [Roberts 1969] and is motivated by the theory of indifference in economics and psychology. Studies in the mid of the XX century underline the fact that the human beings have imperfect powers of discrimination, so they are able to distinguish two stimuli only when they are different enough. For example we do not mind if a bag weighs 10 or 10,4 kilos, but we do mind if it weighs 10 or 14 kilos. In other words, one is generally indifferent between a bag of 10 or 10,4 kilos, but it is more likely that one has preference between one of 10 and one of 14 kilos.

We analyse their characterisations and their link to proper interval orders in reverse mathematics following the same pattern as for interval graphs. Some of the proofs are easy consequences of the corresponding proofs for interval graphs. The unique point worth mentioning is the fact that it is provable in RCA_0 that 1-1 indifference graphs are closed indifference graphs, while for interval graphs this passage requires WKL_0.

Definition 3.1. A graph (V, E) is an *indifference graph* if it is reflexive and there exist a linear order $(L, <_L)$ and a relation $F \subseteq V \times L$ witnessing that (V, E) is an interval graph and such that

(i4) $F(p) \subseteq F(q)$ implies $F(p) = F(q)$ for all $p, q \in V$.

For completeness we give here the corresponding expected definition of proper interval orders, followed by their structural characterisation (see [Marcone 2007] for more details on proper interval orders). We say that an order $(V, \prec)$ "contains a $1 \oplus 3$" if there is a suborder $\{(a, b, c, d), \prec\}$ such that $a \prec b \prec c$ and d is incomparable with all of a, b, c.

Definition 3.2. An order $(V, \prec)$ is a *proper interval order* if there exist a linear order $(L, <_L)$ and a relation $F \subseteq V \times L$ witnessing that $(V, \prec)$ is an interval order and such that (i4) $F(p) \subseteq F(q)$ implies $F(p) = F(q)$ for all $p, q \in V$.

Theorem 3.3 (RCA$_0$)**.** *An order* $(V, \prec)$ *is a proper interval order if and only if it contains neither a* $2 \oplus 2$ *nor a* $1 \oplus 3$.

The theorems about structural characterisations for indifference graphs follow trivially from the previous analysis of interval graphs and from the subsequent observation, which link suborders to subgraphs of the incomparability graph and vice versa. $K_{1,3}$ is the complete bipartite graph with four vertices such that three of them are pairwise incomparable and connected with the fourth vertex.

Proposition 3.4. *An order* $(V, \prec)$ *contains a* $3 \oplus 1$ *if and only if its incomparability graph* (V, E) *contains a* $K_{1,3}$.

Proof. Let $(V, \prec)$ be an order. Suppose $(V, \prec)$ contains a $3 \oplus 1$. This means that there are a, b, c, d such that $a \prec b \prec c$ and $a \mid d, b \mid d, c \mid d$. Hence $\{a, b, c, d\}$ is a $K_{1,3}$ subgraph of (V, E) with $E = \mid$. For the converse the reasoning is analogous. □

Analogously to interval graphs it is immediate to see that the order associated to the complementary graph of an indifference graph is indeed a proper interval order and vice versa.

Theorem 3.5 (RCA$_0$)**.** *Let* (V, E) *be a graph. Then* (V, E) *is an indifference graph if and only if there is a proper interval order* $(V, \prec)$ *such that* $pEq \Leftrightarrow p \mid q$ *for all* $p, q \in V$.

Proof. The proof follows the same reasoning as the proof 2.5. Notice that conditions $(i4)$, which characterises indifference graphs, is the same for proper interval orders and indifference graphs. □

Theorem 3.6 (RCA$_0$)**.** *Let* $(V, \prec)$ *be an order.* $(V, \prec)$ *is a proper interval order if and only if* (V, E)*, where* $p\, E\, q \Leftrightarrow p \mid q$ *for all* $p, q \in V$*, is an indifference graph.*

Proof. $(\Rightarrow)$ By Definition 3.10 and Theorem 2.6.
$(\Leftarrow)$ Let $(V, \prec)$ be an order and define a relation E such that $p\, E\, q \Leftrightarrow p \mid q$ for all $p, q \in V$. Assume (V, E) is an indifference graph. Then (V, E) is an interval graph which does not contain a $K_{1,3}$ as subgraph, by Theorem 3.7. Theorem 2.6 assures that $(V, \prec)$ is an interval order, while Proposition 3.4 entails that $(V, \prec)$ does not contain a $3 \oplus 1$. Therefore $(V, \prec)$ is a proper interval order by Theorems 3.3 and 2.3. □

3.1 Structural characterisations for indifference graphs

The key feature of indifference graphs is the fact that if $p \neq q$, then the interval associated with p is not included in the interval associated with q. This immediately translates, in terms of forbidden subgraphs, to not having $K_{1,3}$ as subgraph.

Theorem 3.7. (V, E) *is an indifference graph if and only if it is an interval graph which does not contain a* $K_{1,3}$ *as subgraph.*

Proof. The left to right implication is immediate to check. For the contrary, let (V, E) be an interval graph which does not contain a $K_{1,3}$ as subgraph. By Theorem 2.13, there exists an interval order $(V, \prec)$ such that $pEq \Leftrightarrow p \mid q$ for all $p, q \in V$. Moreover, by Claim 3.4, $(V, \prec)$ does not contain a $3 \oplus 1$, so it is a proper interval order by Theorem 7.12 in [Marcone 2007]. Then (V, E) is an indifference graph by theorem □

Theorem 3.8 (RCA$_0$)**.** *A graph* (V, E) *is an indifference graph if and only if a) every simple cycle of length four has a chord, b) the complementary graph* $(V, \bar{E})$ *is a comparability graph and c) it does not contain a* $K_{1,3}$ *as subgraph.*

Proof. The left to right implication follows from Theorem 2.8 and Claim 3.4. The reverse implication follows from Theorem 2.10 and Claim 3.4. □

Theorem 3.9 (RCA_0)**.** *The following are equivalent:*

1. WKL_0
2. *a a graph* (V, E) *is an indifference graph if and only if it is triangulated, has no asteroidal triples and does not contain a* $K_{1,3}$.

Proof. Straightforward consequence of Theorem 2.9 and Proposition 3.4 plus Theorem 2.11 (notice that the graph used in the proof of the reversal is an indifference graph). □

Notice the different strength between the structural characterisation of proper interval orders, Theorem 3.3, and of indifference graphs. We have already commented this for interval graphs.

3.2 More definitions of intervals and representations

The same variants of the definition of interval graphs are applicable also to indifference graphs. For example it is possible to ask that the mapping is 1-1 or to have a closed representation. The following definition collects all the options and is the analogous of Definition 2.12.

Definition 3.10. A graph (V, E) is an *1-1 indifference graph* if it is reflexive and there exist a linear order L and a representation $F \subseteq V \times L$ witnessing that (V, E) is a 1-1 interval graph which also satisfy the following

(i3) $F(p) \neq F(q)$ whenever $p \neq q$.

A graph (V, E) is a *closed indifference graph* if it is reflexive and there exist a linear order L and two functions $f_0, f_1 \colon V \to L$ witnessing that (V, E) is a closed interval graph which also satisfy the following

(c5) $f_0(p) <_L f_0(q) \Leftrightarrow f_1(p) \leq_L f_1(q)$ for all $p, q \in V$.

A graph (V, E) is a *1-1 closed indifference graph* if we also have

(c3) $f_0(p) \neq f_0(q) \vee f_1(p) \neq f_1(q)$ whenever $p \neq q$.

A graph (V, E) is a *distinguishing indifference graph* if (c1) and (c2) holds and

(c4) $f_i(p) \neq f_j(q)$ whenever $p \neq q \vee i \neq j$.

3.2.1 Indifference graphs and proper interval orders

We briefly analyse the relation between 1-1 or closed indifference graphs and 1-1 or closed proper interval orders. On this respect, indifference graphs behave as interval graphs.

Theorem 3.11 (RCA_0)**.** *Let* (V, E) *be a graph. Then* (V, E) *is an 1-1 (closed) indifference graph if and only if there is a proper 1-1 (closed) interval order* $(V, \prec)$ *such that* $p \mathrel{E} q \Leftrightarrow p \mid q$ *for all* $p, q \in V$.

Proof. The proof follows the same reasoning as the proof 2.13. Notice that conditions $(i4)$ and $(c5)$, which characterize indifference graphs, are the same for proper interval orders and indifference graphs. □

Theorem 3.12 (RCA_0)**.** *The following are equivalent*

1. WKL_0
2. *Let* $(V, \prec)$ *be an order.* $(V, \prec)$ *is a proper 1-1 interval order if and only if* (V, E)*, where* $p \mathrel{E} q \Leftrightarrow p \mid q$ *for all* $p, q \in V$*, is a 1-1 indifference graph.*
3. *Let* $(V, \prec)$ *be an order.* $(V, \prec)$ *is a proper closed interval order if and only if* (V, E)*, where* $p \mathrel{E} q \Leftrightarrow p \mid q$ *for all* $p, q \in V$*, is a closed indifference graph.*

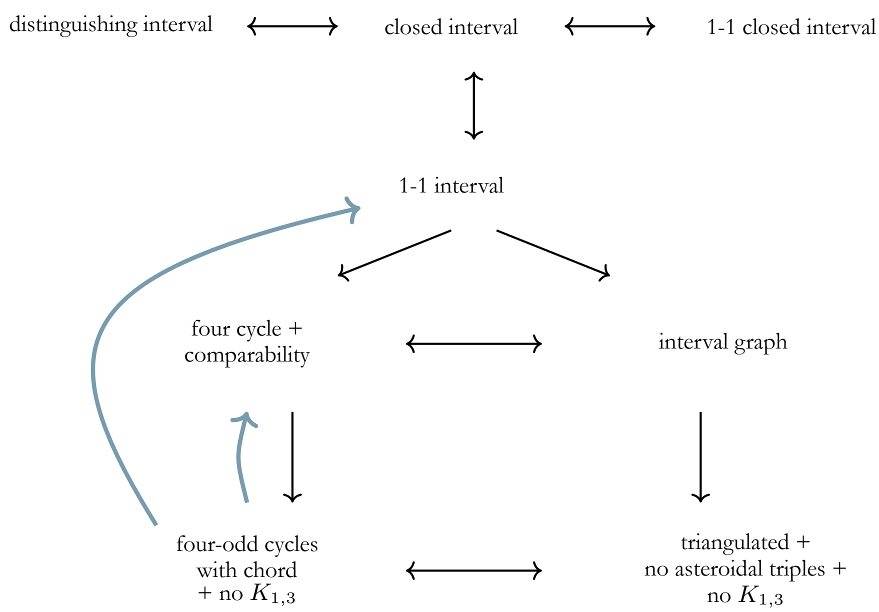


Figure 3.1: $\longrightarrow$ represents implications in RCA_0, $\rightarrow$ represents implications in WKL_0

4. *Let* $(V, \prec)$ *be an order.* $(V, \prec)$ *is a proper closed interval order if and only if* (V, E)*, where* $p\,E\,q \Leftrightarrow p \mid q$ *for all* $p, q \in V$*, is a closed interval graph without* $K_{1,3}$.

Proof. $(1 \Rightarrow 2, 3)$ The proof proceeds as for Theorem 3.6. Notice that WKL_0 is needed to infer, from (V, E) is a 1-1 (closed) interval graph that $(V, \prec)$ is a 1-1 (closed) interval order (see Theorem 2.14).

$(2 \Rightarrow 3)$ and $(3 \Rightarrow 4)$ are immediate.

$(4 \Rightarrow 1)$[1] Let $f, g \colon \mathbb{N} \to \mathbb{N}$ be injective function with $\operatorname{ran}(f) \cap \operatorname{ran}(g) = \emptyset$. We define an order $(V, \prec)$ and we argue that its incomparability graph is closed interval without $K_{1,3}$. Let $V = \{a_k, b_k \mid k \in \mathbb{N}\} \cup \{c_k^n \mid n, k \in \mathbb{N}\}$ and let $\prec$ be defined as

- $a_k \prec c_k^n$ if $f(n) = k$
- $c_k^n \prec a_k$ if $g(n) = k$

No other comparability occurs between the points of V. Let (V, E) be the graph such that $pEq \Leftrightarrow p \mid q$ for all $p, q \in V$. Clearly, (V, E) does not contain $K_{1,3}$. Moreover, the linear order $(\mathbb{N}, <)$ and the functions $f_0, f_1 \colon \mathbb{N} \to \mathbb{N}$ defined by the following items provide a closed representation for (V, E).

$$
\begin{aligned}
f_0(a_k) = f_1(a_k) &= 3k + 1 && \\
f_0(b_k) &= 3k && \\
f_1(b_k) &= 3k + 1 && \\
f_0(c_k^n) &= 3k && \text{if } f(n) \neq k \\
f_1(c_k^n) &= 3k + 2 && \text{if } g(n) \neq k \\
f_0(c_k^n) &= 3k + 2 && \text{if } f(n) = k \\
f_1(c_k^n) &= 3k && \text{if } g(n) = k
\end{aligned}
$$

[1]This is a slight modification of Theorem 7.21 of [Marcone 2007]

We can then conclude that $(V, \prec)$ is proper closed interval order. Let $(L, <_L)$ and $f_0, f_1 : \mathbb{N} \to \mathbb{N}$ be its witnesses. Let $X = \{k \in \mathbb{N} \mid f_1(a_k) <_L f_1(b_k)\}$. If $f(n) = k$, then $f_1(a_k) <_L f_0(c^n_k)$ by definition. Moreover, $f_0(c^n_k) <_L f_1(b_k)$ because $c^n_k \mid b_k$. Hence, $k \in X$. To show that if $k \notin \mathrm{ran}(g)$, then $k \notin X$ the reasoning is analogous. □

3.2.2 Characterisation of indifference graphs

We now turn to the analysis of the strength of the various definitions introduced in 3.10. As for interval graphs some implications are trivial, for example each distinguishing indifference graph is a closed indifference by definition. Moreover, the implications from closed to indifference or from closed to distinguishing is obtainable as for interval graphs (see Theorem 2.15), so we do not repeat it here. More interesting is the fact that 1-1 indifference and closed indifference graphs collapse into one notion already in RCA_0. This is the unique difference in strength between the characterisations of interval and indifference graphs (see Theorem 2.18), and reflects the difference between interval and proper interval orders. Figure 3.1 summarises the implications.

Theorem 3.13 (RCA_0)**.** *If a graph (V, E) is a 1-1 indifference graph, then it is a closed indifference graph.*

Proof. Let (V, E) be a 1-1 indifference graph. By Theorem 3.5, there is a proper 1-1 interval order $(V, \prec)$ such that $p \, E \, q \Leftrightarrow p \mid q$ for all $p, q \in V$. By Theorem 7.16 in [Marcone 2007] $(V, \prec)$ is a proper closed interval order, so (V, E) is a closed indifference graph. □

So we are left to prove the following.

Theorem 3.14 (RCA_0)**.** *The following are equivalent:*

1. WKL_0
2. *If a graph (V, E) is an indifference graph, then it is a closed indifference graph.*

Proof. (1 $\Rightarrow$ 2) Let (V, E) be a indifference graph. By Theorem 3.5, there is a proper interval order $(V, \prec)$ such that $p \, E \, q \Leftrightarrow p \mid q$ for all $p, q \in V$. By Theorem 7.19 in [Marcone 2007] $(V, \prec)$ is a proper closed interval order, so (V, E) is a closed indifference graph.
(2 $\Rightarrow$ 1) We show that if (V, E) is an indifference graph, which is also a closed interval graph, then it is a closed indifference graph. Used the same idea as in the proof of Theorem 2.11, claiming that that graph is a closed interval and an indifference graph showing an interval representation in $\mathbb{Q}$. □

UNIQUE ORDERABILITY

CHAPTER 4

Content

This chapter is devoted to the characterisation of interval graphs whose associated poset is unique up to duality. To give two extreme examples, any complete graph admits a unique order, because its associated interval order is an antichain (recall that the graph is the incomparability graph of the order), while any totally disconnected graph can be turned into infinitely many different chains. Graphs theorists are thus interested in giving structural characterisations for interval graphs which are uniquely orderable[1].

Fishburn's monography contains some characterisations for interval graphs, building on results proved in [Hanlon 1982] and in [Roberts 1971], but it considers mainly finite interval graphs. Our first aim is thus to check whether those characterisations are peculiar of finite graphs or whether they can be extended to the infinite case. The question is interested on its own, but in our opinion Theorem 2.14 increase its interest. We already mentioned that the interval graph defined in Theorem 2.14 is not uniquely orderable and some of its associated orders are definable in RCA_0, while others require WKL_0. We exploited this fact to prove the reversal.

Definition 4.1. An interval graph (V, E) is *uniquely orderable* if there exists an unique (up to duality) order $\prec$ such that for each $v, u \in V$ it holds that $\neg u\, E\, v$ if and only if $u \mid v$.

Notice that if (V, E) is uniquely orderable by $\prec$, then (V, E) is the incomparability graph of $(V, \prec)$. This choice is due the fact that we are mainly dealing with interval graphs, which, we remind, are incomparability graphs of interval orders.

Question 4.2. Is it possible to characterise those infinite interval graphs which are uniquely orderable?

The following easy theorem turns out to be useful to narrow down the classes of graphs which admit a unique orderability. Notice that no assumption on being interval is needed. Indeed the theorem holds for any incomparability graph, namely for the largest class of graphs for which it makes sense to wonder about unique orderability.

[1] The same question arises for comparability graphs in general. [Golumbic 2004; Aigner and Prins 1971; Kelly 1985, Chapter 5.2], just to cite a sample of papers, contain results about this topic for finite comparability graphs.

Lemma 4.3 (RCA_0). *Let* (V, E) *be a* not connected *incomparability graph.* (V, E) *is uniquely orderable if and only if for each choice of three points at most two are non adjacent (i.e. has exactly two connected components each of which is complete).*

Proof. Let (V, E) be a non connected incomparability graph.

($\Rightarrow$) Assume there exist $a, b, c \in V$ such that they are pairwise non adjacent. We claim that there exists $v \in V$ which is not connected with either a, b or c (notice that v may be equal to a, b or c), otherwise (V, E) would be connected. Suppose on the contrary that each vertex in V is connected with a, b and c and let $v, u \in V$. By assumption there exist $v_1, \dots, v_n$ and $u_1, \dots, u_m$, for some $n, m \in \mathbb{N}$, such that $v\,E\,v_1\,E\,\dots\,E\,v_n\,E\,a$ and $a\,E\,u_1\,E\,\dots\,E\,u_m\,E\,u$. Then $v, v_1, \dots, v_n, a, u_1, \dots, u_m, u$ is a path connecting u, v contrary to the fact that (V, E) is assumed to be non connected.

Without loss of generality assume v is not connected with a. Since (V, E) is an incomparability graph, there exists an order $\prec$ on V. Suppose $a \prec b \prec v$ and assume also that b is not connected with v. We claim that there exists an order $\prec'$ on V such that either $v \prec' a \prec' b$ or $b \prec' a \prec' v$. If such $\prec'$ does not exist, then there must be a point x such that $a \prec x$ and $v \mid x \mid b$, that is $\neg x\,E\,a$ and $b\,E\,x\,E\,v$. Thus x witnesses that b and v are connected, contrary to the assumption. Otherwise, i.e. if b is not connected with v, then it holds that b is not connected with a, otherwise a and v would be connected trough b, an analogous argument shows that there exists $\prec'$ such that either $a \prec' v \prec' b$ or $b \prec' v \prec' a$.

The orders $\prec$ and $\prec'$ witness that (V, E) is not uniquely orderable.

($\Leftarrow$) Suppose that for each $a, b, c \in V$ it holds that either $a\,E\,b$ or $b\,E\,c$ or $a\,E\,c$. We show that (V, E) has two connected components. Let $\langle v_n \mid n \in \mathbb{N}\rangle$ be an enumeration of V and set $v_0 \in U$. Let $U = \{v_n \mid v_n\,E\,v_0\}$ and $W = V \setminus U$. It is trivial to check that U is connected, since for each $i, j \in \mathbb{N}$, $v_i\,E\,v_0\,E\,v_j$. Since $W \neq \emptyset$, because (V, E) is not connected by assumption, let i minimum such that $v_i \in W$. Notice that for each m such that $v_m \in W$ it holds that $v_m\,E\,v_i$ (this follows from the fact that $\neg v_i\,E\,v_0$, $\neg v_m\,E\,v_0$, and from the assumption). Hence W is connected as well.

We claim that both U and W are complete graphs. In fact, if $v_m, v_\ell \in U$, then $\neg v_i\,E\,v_m$, $\neg v_i\,E\,v_\ell$, thus $v_m\,E\,v_\ell$ by assumption. A similar argument, with 0 in place of i, shows that W is complete too. This implies that $(U, \prec)$ and $(W, \prec)$ are antichain in any order $\prec$ associated to E. It follows that any order $\prec$ on (V, E) must choose one of the following: $u \prec w$, for each $u \in U$ and $w \in W$, or $w \prec u$, for each $u \in U$ and $w \in W$. The graph (V, E) admits a unique ordering up to duality. □

Thanks to the previous lemma we can restrict our attention to connected graphs. Moreover, we can safely assume that the graph we are dealing with are not complete, given that RCA_0 proves that any complete incomparability graph is uniquely orderable, because it gives rise to an antichain. Notice also that *universal points*, i.e. vertices connected with all the remaining vertices of a graph, are uninfluential for the uniqueness of the ordering, since they are isolated in any order. It is then more convenient to assume that a graph does not contain such vertices, once one is aware of the fact that recognising those points is not a computable task.

Roberts in [Roberts 1971] proved that connected indifference graphs have only one associated order.

Lemma 4.4 (RCA_0). *Each connected indifference graph is uniquely orderable.*

Proof. Suppose (V, E) is a connected indifference graph and $\prec$, $\prec'$ are two (non dual) orderings of E. Let $a, b, c \in V$ such that $a \prec b \prec c$ and $c \prec' a \prec' b$. By Theorem 3.5 $(V, \prec)$ and $(V, \prec')$ are proper interval orders, so let $F \subseteq V \times L$ and $F' \subseteq V \times L'$, for some linear orders $(L, <_L)$ and $(L', <_{L'})$, be interval representations of $\prec$ and $\prec'$ respectively. Since the graph is connected and $a \prec b \prec c$ there exists a path $v_0, \dots, v_n$, for some $n \in \mathbb{N}$, between b and c which does not contain a. This implies that there exists $i \leq n$ such that $v_i \neq a$ and either $F'(a) \subseteq F'(v_i)$ or $F'(v_i) \subseteq F'(a)$, contrary to the fact that (V, E) is an indifference graph. □

Notice that the last passage of the previous proof exploits Condition (i4) of Definition 3.2 which differentiate indifference graphs from interval graphs. Lemmas 4.3 and 4.4 settle the question about the uniquely orderability of indifference graphs.

4.1 Uniquely orderable finite interval graphs

The graph we mention in this section are all connected, finite and without universal points. Suppose (V, E) is an interval graph. Saying that (V, E) is not uniquely orderable amounts to checking that there are two orders $\prec$ and $\prec'$ and three vertices $a, b, c \in V$ such that $a \prec b \prec c$ and $b \prec' a \prec' c$. The vertices a and b can be reoriented without regard, so to speak, to the order of c. The connected graph with vertices $\{a, b, k, r, s\}$ and edges $a\,E\,k\,E\,b$, $k\,E\,r\,E\,s$, whose interval representation is represented in Figure 4.4, is a very basic example of a non uniquely orderable connected finite interval graph. The first characterisation of uniquely orderable interval graphs exploits the above observation to identify subgraphs which are forbidden in uniquely orderable interval graphs.

Theorem 4.5. *Let (V, E) be a finite connected interval graph. The following are equivalent:*

1. *(V, E) is uniquely orderable,*
2. *(V, E) does not contain a buried subgraph (defined below),*
3. *the graph (W, Q) has two connected components, where $W = \{(a, b) \mid a, b \in V \wedge \neg a\,E\,b\}$ and $ab\,Q\,cd \Leftrightarrow a\,E\,c \wedge b\,E\,d$.*

The previous theorem provides a definitive answer to the individuation of finite connected interval graph which are uniquely orderable. A proof can be found in [Fishburn 1985, Theorems 3.11-3.12]. We now give the definitions needed to understand the previous theorem.

Definition 4.6. Let (V, E) be an graph. For each subgraph B let $K(B) = \{v \mid \forall b \in B\,(v\,E\,b)\}$. The subgraph B is a *buried subgraph* of (V, E) if the following holds:

1. there exists $a, b \in B$ such that $\neg a\,E\,b$,
2. $K(B)$ is not empty and $K(B) \cap B = \emptyset$,
3. if $x_0, \ldots, x_n$ is a path between $b \in B$ and $v \in V \setminus B$, then there exists $i \le n$ such that $x_i \in K(B)$.

The last point of the previous enumeration encapsulates the idea that B is connected with the remaining vertices of V only via $K(B)$.

On a first sight, in the situation of the previous definition, if (V, E) is infinite $K(B)$ requires ACA_0 to be defined. The following observation shows that $K(B)$ exists in any model of RCA_0 which contains the buried subgraph B.

Property 4.7. *Let B be a buried subgraph of a graph G. Then $K(B)$ is Δ^0_1-definable in B.*

Proof. We claim that $K(B) = \{v \notin B \mid \exists b \in B\,(v\,E\,b)\}$.

If $v \in K(B)$, then there is a $b \in B$ such that $v\,E\,b$ by definition of $K(B)$. Moreover, since B is a buried subgraph $K(B) \cap B = \emptyset$, so $v \notin B$.

For the reverse inclusion let $v \notin B$ and $b \in B$ be such that $v\,E\,b$. By point (3) of Definition 4.6 $v \in K(B)$. □

Property 4.8. *Let (V, E) be an interval graph and B be a buried subgraph in (V, E). Then the following holds:*

1. *$K(B)$ is a complete subgraph,*
2. *for each $r \notin B \cup K(B)$ and for each $b \in B$, $\neg r\,E\,b$.*

Proof. To prove (1) assume that there exists $k, k' \in K(B)$ such that $\neg k\,E\,k'$. Let $a, b \in B$ be such that $\neg a\,E\,b$ (the existence of such a, b is guaranteed by Definition 4.6). Then $x\,E\,k\,E\,y\,E\,k'\,E\,x$ is a cycle of length four without chords. By Theorem 2.8 (V, E) is not an interval graph, contrary to the assumption.

Item (2) follows immediately from Definition 4.6, since $b\,E\,r$ would imply that there is a path between b and r with no element in $K(B)$. □

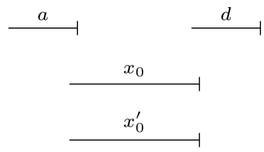


Figure 4.1: A connected uniquely orderable interval graph which contains a buried subgraph.

With respect to the second characterisation in Theorem 4.5 notice that if (V, E) is an interval graph and $ab\, Q\, xy$, for some $a, b, x, y \in V$, then the order of a and b determines the order of x and y. In fact, by Theorem 2.8, either $\neg a\, E\, y$ or $\neg b\, E\, x$, otherwise $a\, E\, b\, E\, x\, E\, y\, E\, a$ is a cycle of length four without chords. Without loss of generality assume $\neg a\, E\, y$ and $a \prec b$. If $y \prec x$, then $y \prec a$ because $y\, E\, a$. By transitivity it holds that $y \prec b$ contrary to the assumption that $b\, E\, y$. We emphasise that the hypothesis on (V, E) being an interval graph is crucial to settle the conclusion.

4.2 Uniquely orderable infinite interval graphs

The absence of buried subgraph is a necessary condition for the unique orderability of infinite connected interval orders as well, while (W, Q) having two connected components is a sufficient condition. The proof of these two facts is a slight modification of the proof given in [Fishburn 1985] for finite graphs.

Theorem 4.9 (RCA_0). *If a graph (V, E) is a connected interval graph without universal points and it is uniquely orderable, then it does not contain a buried subgraph.*

Proof. Assume that (V, E) is a connected interval graph without universal points and that B is a buried subgraph. Let $K(B) = \{v \mid \forall b \in B\, (v\, E\, b)\}$, by Property 4.7 $K(B)$ is definable in RCA_0 and so is the set $R(B) = V \setminus \{K(B) \cup B\}$. Since there are no universal points, it holds that $R(B) \neq \emptyset$.

Let $\prec$ be an order associated to (V, E). We claim that the order $\prec$ is such that $R'(B) \prec B \prec R''(B)$ and $R'(B) \prec K(B) \prec R''(B)$, where $R(B) = R'(B) \cup R''(B)$ (one of the two subset may be empty). To check that $R'(B) \prec K(B) \prec R''(B)$ notice that it cannot be the case that $r \prec k \prec r' \prec k'$ for some $r, r' \in R(B)$ and $k, k' \in K(B)$, otherwise $\neg k\, E\, k'$, contrary to (1) of Property 4.8. Since each $b \in B$ is $\prec$-incomparable with each $k \in K(B)$, it must hold that $R'(B) \prec B \prec R''(B)$.

We define another ordering $\prec'$ of (V, E) as follows: for each $v, u \in V$ such that $v \prec u$ if $v, u \in B$ let $u \prec' v$, otherwise let $v \prec' u$. It is easy to check that $\prec'$ is transitive. To verify that $\prec'$ is not the dual of $\prec$ let $a, b \in B$ be such that $\neg a\, E\, b$ and $r \in R(B)$ and suppose that $r \prec a \prec b$. Then it holds that $r \prec' b \prec' a$. □

Notice that the assumption on the non existence of universal points is crucial in the previous theorem. Indeed the graph in Figure 4.1 is a connected uniquely orderable interval graph and it contains a buried subgraph. However, it is easy to realise that if $B \subseteq V$ is a buried subgraph with $V = B \cup K(B)$, then (V, E) is uniquely orderable if and only if B is uniquely orderable.

Theorem 4.10 (RCA_0). *Let (V, E) be a connected interval graph. If for all three pairs of vertices there exists a Q-path between two of them (i.e. (W, Q) has two connected component), then (V, E) is uniquely orderable.*

Proof. We claim that ab and ba, for each $ab \in W$, are not Q-connected. Assume by contradiction that there exists $ab \in W$ such that there exists a path $ab\, Q\, x_0y_0\, Q\, \ldots\, Q\, x_ny_n\, Q\, ba$ connecting ab with ba. By definition of the relation Q it holds that $a\, E\, x_0\, E\, \ldots\, E\, x_n\, E\, b$ and that $a\, E\, y_n\, E\, \ldots\, E\, y_0\, E\, b$. Let $(L, <_L)$ be a linear order and F be an interval representation of (V, E) on $(L, <_L)$. We claim that this entails the

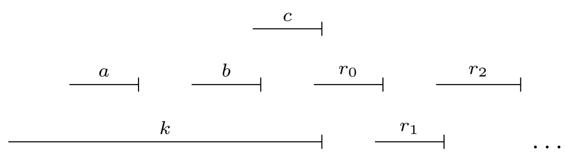


Figure 4.2: A graph without buried subgraphs and with a finite subgraph with buried subgraph.

existence of $i \leq n$ such that $F(x_i) \cap F(y_i) \neq \emptyset$, namely $x_i \, E \, y_i$ contrary to the definition of (W, Q). Assume without loss of generality that $F(a) <_L F(b)$. Firstly, we claim that if $F(x_j) <_L F(y_j)$ for all $j \leq n$, then $F(a) <_L F(y_n)$. In fact, since $F(a) <_L F(b)$ and $F(b) \cap F(x_n) \neq \emptyset$, then either $F(a) <_L F(x_n)$ or $F(b) \cap F(x_n) \neq \emptyset$. Both cases imply that $F(a) <_L F(y_n)$ contrary to the fact that $a \, E \, y_n$. Thus let $i+1$ be minimum such that $F(y_{i+1}) <_L F(x_{i+1})$. Since $x_i \, E \, x_{i+1}$, there exists $x \in F(x_i) \cap F(x_{i+1})$. Let also $y \in F(y_{i+1})$ and $y' \in F(y_i)$. By minimality of $i+1$ and by the assumptions it must hold that $y <_L x <_L y'$ and thus that $F(y_{i+1}) <_L F(y_i)$ contrary to the fact that $y_i \, E \, y_{i+1}$. Since we are assuming that (W, Q) has two connected components, thanks to the previous claim, for each $ab, cd \in W$ it holds that either $cd \, Q \, ab$ or $cd \, Q \, ba$.

Let $\prec$ be an order associated to (V, E) and $ab, cd \in W$ be such that $ab \, Q \, cd$. We claim that $a \prec b$ holds if and only if $c \prec d$ holds as well. Notice that it holds that either $\neg b \, E \, c$ or $\neg a \, E \, d$, otherwise $a \, E \, c \, E \, b \, E \, d \, E \, a$ would be a cycle of length four without chords, contrary to the assumption that (V, E) is an interval graph. Suppose now that $a \prec b$. If $\neg b \, E \, c$, then $c \prec b$ because $\neg a E c$. Hence, $c \prec d$ holds otherwise $d \prec b$ contrary to the assumption. If instead $b \, E \, c$, then it must be that $\neg a \, E \, d$ holds and so that $a \prec d$, otherwise $d \prec b$. It follows that $c \prec d$ otherwise $a \prec c$, contrary to the assumption.

Since the order of a single pair determines the order of the entire component and given that the the two components are duals, it is possible to conclude that (V, E) is uniquely orderable. □

To complete the generalisation of Theorem 4.5 to infinite graphs, it would be enough to prove that if (V, E) does not contain a buried subgraph, then (W, Q) has two connected components. This would establish, in particular, that the absence of buried sugraph is a necessary and sufficient condition for the unique orderability of infinite connected interval graphs, thus getting a structural characterisation for the later class of graphs.

The missing implication is proved in [Fishburn 1985, Theorem 3.12] for finite interval graphs showing that the contrapositive statement holds, namely building a buried subgraph from suitable $ab, xy \in W$ living in different connected components. The existence of pairs with the properties required by the proof in [Fishburn 1985] heavily depends on the fact that the graphs are finite.

Question 4.11. Is the absence of buried subgraphs a sufficient condition for the unique orderability of infinite connected interval graphs?

One would be tempted to generalise Theorem 4.5 to the infinite case arguing by compactness, but this strategy fails. In favour of this strategy one may argue that if (V, E) is a connected interval graph such that each finite subgraph is uniquely orderable, then (V, E) is itself uniquely orderable (this is due to the fact that the non unique orderability is witnessed by three points). Nonetheless, there exists a connected interval graph with no buried subgraph which contains a finite connected subgraph with a buried subgraph. In fact, let $V = \{a, b, c, k\} \cup \{r_n \mid n \in \mathbb{N}\}$ and $k \, E \, a, k \, E \, b, k \, E \, c, k \, E \, r_0, a \, E \, c, r_0 \, E \, c$ and $r_n \, E \, r_{n+1}$ for each $n \in \mathbb{N}$ as represented in Figure 4.2. It is easy to check that (V, E) does not contain a buried subgraph, but that $\{a, b\}$ is a buried subgraph in $(\{a, b, k, r_0\}, E)$.

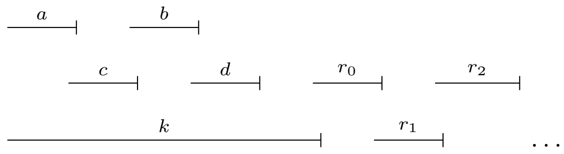


Figure 4.3: A graph with a buried subgraphs and with a finite subgraph without buried subgraph.

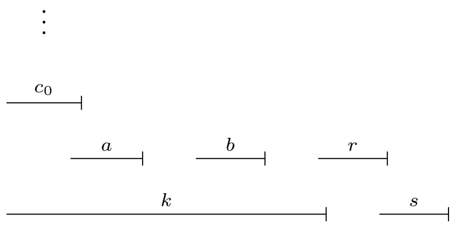


Figure 4.4: A graph which has a buried subgraph, but no finite subgraph has a buried subgraph.

Notice also that there exists a connected interval graph with a buried subgraph which contains a finite connected subgraph without a buried subgraph. In fact, let $V = \{a, b, c, d, k\} \cup \{r_n \mid n \in \mathbb{N}\}$ and $k\,E\,a, k\,E\,b, k\,E\,c, k\,E\,d, k\,E\,r_0, a\,E\,c, c\,E\,b, b\,E\,d$ and $r_n\,E\,r_{n+1}$ for each $n \in \mathbb{N}$ as represented in Figure 4.3. It is easy to check that $\{a, b, c, d\}$ forms a buried subgraph in (V, E), but that $(\{a, b, c, d\}, E)$ does not contain a buried subgraph.

To complete the inspection of the relationship between infinite graphs with buried subgraphs and their finite subgraphs, we mention the fact that there exists a connected interval graph which contains an infinite buried subgraph, but no finite buried subgraph. Let (V, E) be as in Figure 4.4, where the dots stand for infinitely many c_n such that $c_n\,E\,c_{n+1}$ and $c_n\,E\,k$ for each $n \in \mathbb{N}$. The set $B = \{x \mid x = a \vee x = b \vee xEa\}$ is a buried subgraph for (V, E), but no $B' \subseteq B$ is buried. Another example of this kind is depicted in Figure 4.5.

For sake of completeness notice also that there exists a connected uniquely orderable graph with a non uniquely orderable finite subgraph (see Figure 4.6 for an example). In contrast with this observation, one may notice that if a not connected interval graph is uniquely orderable, then each finite subgraph is uniquely

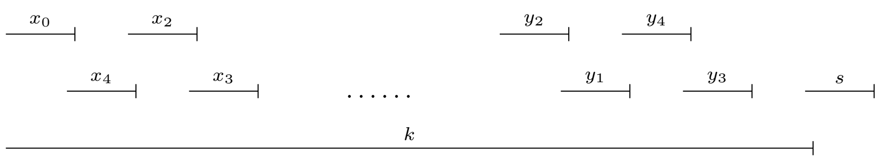


Figure 4.5: A graph which has a buried subgraph, but no finite subgraph has a buried subgraph.

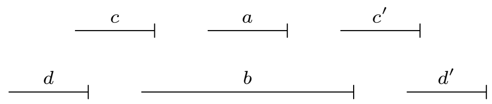


Figure 4.6: The unique interval representation of graph with a non uniquely orderable subgraph.

orderable. In fact, by Lemma 4.3, each finite subgraph is either complete or has two connected components each of which complete and it is then uniquely orderable.

DIMENSION THEORY

CHAPTER 5

Content

The dimension of a poset is one of the parameters, like width and height, which characterise a poset. Moreover, it is a comparability invariant, so that each poset associated to a comparability graph has the same dimension. This parameter was introduced by Ben Dushnik and Edwin Miller in 1941 [Dushnik and Miller 1941] and it links posets with linear orders. In fact, the notion of dimension builds upon a theorem proved by Edward Szpilrajn in 1930 which shows that each partial order can be extended to a linear one (for a proof see [Harzheim 2005, Theorem 2.3.2]).

Definition 5.1. Let $(P, <_P)$ be a poset. An order $\prec$ is *partial extension* of $<_P$ if it holds that $<_P \subseteq \prec$. A *linear extension* is a partial extension which is also a linear order.

Definition 5.2. Let $(P, <_P)$ be a poset. The *dimension* of P is the minimum number $n \leq \omega$ such that there are linear extensions $\prec_0, \ldots, \prec_{n-1}$ of $<_P$ which satisfy the following condition for each $p, q \in P$: $p <_P q$ if and only if $p \prec_i q$ for each $i < n$.

Usually the dimension of $(P, <_P)$ is defined as the minimal cardinality of a realiser, which is nothing else than a family of linear extensions whose intersection is exactly $<_P$. The previous definition is more easily formalised in arithmetic.

Notice that the dimension of a poset does not measure how far away a poset is from being a linear order. Indeed, an antichain has dimension two, as witnessed by any linear order and its dual, despite the fact that it is as far as possible from being a linear order.

To the best of our knowledge not much attention has been devoted to dimension theory from the reverse mathematics point of view. The following theorem, proved by Douglas Cenzer and Jeffrey Remmel in [Cenzer and Remmel 2005, Theorem 3.5], is an exception[1].

Theorem 5.3. WKL_0 *is equivalent to the following statement: if* $(P, <_P)$ *is a poset such that each finite subposet has dimension at most* n*, then* P *has dimension at most* n.

[1] Quite interestingly Francois Dorais in 2012 suggested using dimension to understand if ADS is equivalent to CAC (see `http://logic.dorais.org/archives/656`).

On the other hand, there are interesting results about dimension theory and computability theory. [Downey 1998, Section 6] and [Kierstead 1986] survey those results.

After its introduction in 1941 several researchers studied properties and bounds for the dimension of posets. [Harzheim 2005, Chapter 7] is a nice introduction to the topic and [Trotter 2001] is entirely devoted to dimension theory. We mainly refer to [Fishburn 1985, Chapter 5] which focuses especially on the dimension of interval orders. We notice that some statements mentioned in that chapter, in particular Theorems 5.11, 5.13 and 5.14, are easily provable in WKL_0 thanks to Theorem 5.3. The original proofs of those statements generally go through in ACA_0 and it gives a more explicit construction of the linear extensions which witness that the dimension is bounded. Moreover, we establish the equivalence of Theorems 5.10, 5.11 and 5.13 with WKL_0.

5.1 Some basic facts

The first basic fact we highlight, following results in [Harzheim 2005, Chapter 2], is that RCA_0 proves that for each poset there exist linear extensions whose intersection is the partial order.

Lemma 5.4 (RCA_0)**.** *Let $(P, <_P)$ be a poset and $p, q \in P$ be incomparable elements. Then there exists a partial (hence linear) extension $\prec$ such that $p \prec q$.*

Sketch of the proof. For each $x, y \in P$ set $x \prec y$ if and only if either $x <_P y$ or $x \leq_P p$ and $q \leq_P y$. By definition $\prec$ is an extension of $<_P$. It is routine to verify that $\prec$ is a partial order. □

Corollary 5.5 (RCA_0)**.** *Let $(P, <_P)$ be a partial order. If $p, q \in P$ are incomparable, then there exist linear extensions $\prec_1$ and $\prec_2$ such that $p \prec_1 q$ and $q \prec_2 p$.*

Lemma 5.6 (RCA_0)**.** *For each poset $(P, <_P)$ there exists $i \leq \omega$ and a sequence $\langle \prec_n | \, n \leq i \rangle$ of linear extensions which satisfy the following condition for each $p, q \in P$: $p <_P q$ if and only if $p \prec_n q$ for each $n \leq i$.*

Proof. For each $p, q \in P$ such that $p \mid_P q$, let $\prec_{pq} \supseteq \prec_P$ be a linear extension such that $p \prec_{pq} q$. The existence of such extension is guaranteed by Corollary 5.5. It is immediate to check that, for each $r, s \in P$, it holds that $r <_P s$ if and only if $r \prec_{pq} s$ in each linear extension $\prec_{pq}$. □

Notice that, by the previous lemma, the dimension of P does not exceed the cardinality of P. However, the previous lemma does not establish that RCA_0 proves that each poset has a dimension, in fact the linear extensions defined in the lemma may not be the optimal ones. $\mathsf{I}\Sigma^1_1$ surely guarantees that each poset $(P, <_P)$ has a dimension, since it allows to find the minimal i such that there exist linear extensions witnessing that the dimension of P is i. It would be interesting to understand what amount of induction is really necessary.

Lemma 5.4 may be compared with the following lemma, which is just a particular case of the following statement, proved to be equivalent to WKL_0 in [Cholak, Marcone, and Solomon 2004, Lemma 3.16]: for each acyclic relation there exists its transitive closure.

Lemma 5.7 (RCA_0)**.** *The following are equivalent:*

1. WKL_0,
2. *let $(P, <_P)$ be a partial order and $R \subseteq P$. If $(R, \prec_R)$ is a linear extension of $<_P \subseteq R \times R$, then there exists a linear extension $\prec$ of $<_P$ such that $\prec_R \subseteq \prec$.*

Proof. We have only to check that $<_P \cup \prec_R$ is an acyclic relation, so that there exists its transitive closure, and therefore its linearisation. The fact that $<_P \cup \prec_R$ is acyclic follows from the observation that if it holds that $p <_P q \prec_R r$, then it holds either that $q <_P r$ and so that $p <_P r$, or that $q \mid_P r$ and so that $r \not<_P p$ by transitivity of $<_P$. A similar consideration holds for $p \prec_R q <_P r$.

For the reversal let $f, g\colon \mathbb{N} \to \mathbb{N}$ be injective functions with $\operatorname{ran}(f) \cap \operatorname{ran}(g) = \emptyset$. Let $P = \{a_n, b_n, x_n, y_n \mid n \in \mathbb{N}\}$. The elements of P are pairwise incomparable with the following exceptions: $a_n <_P x_k$ and $y_k <_P b_n$ if $f(k) = n$; furthermore, $b_n <_P x_k$ and $y_k <_P a_n$ if $g(k) = n$. Let R be the antichain $\{x_k, y_k \mid k \in \mathbb{N}\}$ and $\prec_R$ be such that $x_k \prec_R x_{k+1}$, $y_j \prec_R y_{j+1}$ and $x_k \prec_R y_j$ for each $k, j \in \mathbb{N}$. By definition $\prec_R$ is a linear extension of $<_P \subseteq R \times R$, so let $\prec$ be a linear extension of $<_P \cup \prec_R$. It is straightforward to verify that the set $X = \{n \mid a_n \prec b_n\}$ separates $\operatorname{ran}(f)$ from $\operatorname{ran}(g)$. ☐

Since antichains have dimension two one may wonder if there are posets of arbitrary high dimension. This question was answered by Dushnik and Miller themselves. The proof of the following theorem is based on the proof in ([Fishburn 1985, Theorem 5.1]). We notice that it goes through in RCA_0.

Theorem 5.8 (RCA_0)**.** *Let $A \subseteq \mathbb{N}$ be a set and $P = \{\{a\} \mid a \in A\} \cup \{A \setminus \{a\} \mid a \in A\}$. Then $(P, \subseteq)$ has dimension $|A|$.*

Proof. Let $a^* = \{a\}$ and $a_* = A \setminus \{a\}$ for each $a \in A$. Assume without loss of generality that $|A| \geq 3$. Notice that $\{a^* \mid a \in A\}$ and $\{a_* \mid a \in A\}$ form two antichains, while $a^* \subseteq b_*$ for each $a \neq b$ in A.

Notice that if $a \neq b$ it cannot exists a linear extension $\prec$ such that it hold both that $a_* \prec a^*$ and $b_* \prec b^*$, otherwise, by transitivity, it would hold that $a_* \prec b^*$ contrary to $b^* \subseteq a_*$. This implies that the dimension of P is at least $|A|$.

For each $a \in A$ let $\prec_a$ be a linear extension of $\subseteq$ such that $b^* \prec_a c^* \prec_a a_* \prec_a a^* \prec_a b_* \prec_a c_*$ for each $b < c$. By definition, for each $b \neq c \in A$, it holds that $b^* <_a c_*$ for each $a \in A$. Moreover, it holds that $b_* <_b b^*$ and $b^* <_c b_*$, that $b_* <_b c_*$ and $c_* <_c b_*$ and finally that $b^* <_c c^*$ and $c^* <_b b^*$. Hence, $\langle <_a \mid a \in A \rangle$ witnesses that the dimension of P is at most $|A|$. ☐

The previous theorem gives a receipt to generate posets of dimension n for each $n \leq \omega$. Despite this fact it is quite hard to define posets with dimension greater than four. Notice that the order used in the proof is not an interval order, because it hold that $a^* \subseteq b_*$ and $b^* \subseteq a_*$ but it holds neither that $b^* \subseteq b_*$ nor that $a^* \subseteq a_*$ for each $a, b \in A$.

5.2 Dimension and interval orders

As already mentioned, sometimes it is possible to set sharp bounds on the dimension of interval orders. The next theorem provides another characterisation of interval orders based on the existence of a certain kind of linear extensions.

Definition 5.9. Let $(P, <_P)$ be a poset and A, B two disjoint subsets of P. Then $[A|B]$ denotes a linear extension $\prec$ of $(A \cup B, <_P)$ such that it holds that $a \prec b$ whenever $a \in A$, $b \in B$, and $a \mid_P b$.

Theorem 5.10 (RCA_0)**.** *The following are equivalent:*

1. WKL_0,
2. *a poset $(P, <_P)$ is an interval order if and only if $[A|B]$ exists for each disjoint $A, B \subseteq P$.*

Proof. It is sufficient to check that the relation $<' = <_P \cup \{\langle a, b\rangle \mid a \in A, b \in B, a \mid_P b\}$ is acyclic if and only if $<_P$ is an interval order. In fact, if $<'$ is acyclic it can be extended to a linear order $\prec$ with the desired properties.

If $<_P$ is not an interval order, then let $a <_P b$ and $c <_P d$ witness the fact that P contains a $2 \oplus 2$ (see Theorem 2.3). Let $A = \{b, d\}$ and $B = \{a, c\}$. Since it holds that $d <' a$ and $b <' c$, by definition of $<'$, it follows that $d <' a <' b <' c' <' d$, so $<'$ is cyclic.

Suppose now $p_0 <' \cdots <' p_n <' p_0$ is a cycle. We show that it contains a $2 \oplus 2$. There are three cases to consider: $p_0 \in A$, $p_0 \in B$ or $p_0 \notin A \cup B$.

Suppose the former holds. It follows from the definition of $<'$ that $p_n <_P p_0$.

Claim 5.10.1. *There exists* $i \leq n$ *such that* $p_i \in B$ *and* $p_i <_P p_0$.

Proof. If there is no $p_i \in B$, then $p_0 <_P \cdots <_P p_n <_P p_0$ would be a cycle in $<_P$. Let $i \leq n$ be maximum such that $p_i \in B$. Then it holds that $p_i <_P p_{i+1} \cdots <_P p_0$ by definition of $<'$ and thus $p_i <_P p_0$ holds by transitivity. □

Let $i \leq n$ satisfy the previous claim. Let also $j < i$ be such that $p_j \in A$ and $p_j \mid_P p_i$. Notice that such j exists, otherwise $p_0 <_P \cdots <_P p_n <_P p_0$ would be a cycle in $<_P$. Notice that it holds that $p_0 \not<_P p_j$, otherwise it would hold that $p_i <_P p_j$, contrary to the assumption.

Claim 5.10.2. *There exists* $k < j$ *such that* $p_k \in B$ *and* $p_k <_P p_j$.

Proof. Firstly, we claim that there exists $k < j$ such that $p_k \in B$. If this does not hold, it follows from the definition of $<'$ that $p_m <_P p_{m+1}$ for each $m < j$. This implies that $p_0 <_P p_j$, contrary to what we just proved.

Let $k < j$ be maximum such that $p_k \in B$. Then it holds that $p_k <_P p_j$ by transitivity of $<_P$ and by definition of $<'$. □

Notice that it holds that $p_0 \not<_P p_k$ and $p_i \not<_P p_k$, otherwise transitivity implies $p_i <_P p_j$. Suppose that $p_0 \mid_P p_k$. It follows that $p_k \mid_P p_i$, because $p_k <_P p_i$ implies $p_k <_P p_0$ by transitivity. Moreover, it holds that $p_0 \mid_P p_j$, because we proved that $p_0 \not<_P p_j$ and because $p_j <_P p_0$ implies $p_k <_P p_0$, contrary to the assumption. Thus $p_k <_P p_j$ and $p_i <_P p_0$ form the desired $2 \oplus 2$.

Otherwise, it holds that $p_k <_P p_0$. Since we are assuming that $p_0 <' \cdots <' p_k$, by definition of $<'$ there exist $a < b \leq n$ such that $p_a \in A$, $p_b \in B$, $p_0 <' p_b <_P p_a <' p_k$, $p_0 \mid_P p_b$ and $p_k \mid_P p_a$. Notice that $p_0 \mid_P p_a$ also holds, because $p_0 <_P p_a$ implies that $p_k <_P p_a$, while $p_a <_P p_0$ implies that $p_b <_P p_0$. Finally, it holds that $p_k \mid_P p_b$, because $p_k <_P p_b$ implies that $p_k <_P p_a$, and $p_b <_P p_k$ implies that $p_b <_P p_0$. In this case $p_k <_P p_0$ and $p_b <_P p_a$ form the desired $2 \oplus 2$.

If $p_0 \in B$, then it holds that $p_0 <_P p_1$ by definition of $<'$. It is possible to reason analogously to the previous case, swapping A with B and $<'$ with $>'$, to find a $2 \oplus 2$.

Lastly, suppose that $p_0 \notin A \cup B$. Let $a < b \leq n$ be such that $p_a \in A$, $p_b \in B$ and $p_a \mid_P p_b$ (their existence is guaranteed by definition of $<'$ and by the fact that $<_P$ does not have cycles). Notice that we can assume a is minimum and b is maximum having the desired properties, since $p_a <' p_b$ holds by definition of $<'$ and $p_0 <' \cdots <' p_a <' p_b <' \cdots <' p_0$ is still a cycle. The choice of a and b implies that $p_0 <_P p_a$ and that $p_b <_P p_0$. It follows by transitivity that $p_b <_P p_a$ contrary to the choice of a and b.

For the reversal let $f, g \colon \mathbb{N} \to \mathbb{N}$ be injective functions with $\operatorname{ran}(f) \cap \operatorname{ran}(g) = \emptyset$. Let $P = A \cup B$ where $A = \{a_n, b_n \mid n \in \mathbb{N}\}$ and $B = \{x_n^k \mid k, n \in \mathbb{N}\}$. The elements of P are pairwise incomparable with the following exceptions: $x_n^k <_P a_n$ if $f(k) = n$ and $x_n^k <_P b_n$ if $g(k) = n$.

$(P, <_P)$ is an interval order, so let $\prec$ be a linear extension of $<_P$ witnessing that $[A|B]$ exists. The set $X = \{n \mid b_n \prec a_n\}$ separates the ranges of f and g. In fact, if $f(k) = n$, for some $k \in \mathbb{N}$, then $b_n \prec x_n^k \prec a_n$, by definition of $\prec$. If $g(k) = n$, for some $k \in \mathbb{N}$, then $a_n \prec x_n^k \prec b_n$. □

The previous theorem turns out to be useful to prove the next one, which contrasts with Theorem 5.8. Remember that the *height* of a poset is the maximum size of the chains. Notice that the family of posets used in Theorem 5.8 has height two.

Theorem 5.11 (RCA_0). *The following are equivalent:*

1. WKL_0,
2. *every interval order* $(P, <_P)$ *with height two has dimension two.*

Proof. Suppose the statement holds for finite orders (see [Fishburn 1985, p. 86], [Rabinovitch 1978a] or [Trotter 2001, theorem 8.3.4] for proofs). Since each finite subposet of P is an interval order with height two, then each finite subposet has dimension two. It follows from Theorem 5.3 that P has dimension two as well.

For the reverse implication let $f, g\colon \mathbb{N} \to \mathbb{N}$ be injective functions with $\operatorname{ran}(f) \cap \operatorname{ran}(g) = \emptyset$. We define an interval order $(P, <_P)$ of height two. Let $P = \{a_n, b_n, c_n, d_n \mid n \in \mathbb{N}\} \cup \{x_k, y_k \mid k \in \mathbb{N}\}$. The unique comparabilities in $<_P$ are the following: $a_n <_P x_k$ and $b_n <_P x_k$ if $f(k) = n$, and $c_n <_P y_k$ and $d_n <_P y_k$ if $g(k) = k$.

It is trivial to check that $(P, <_P)$ satisfies the hypotheses. Hence, let $\prec_1$ and $\prec_2$ be linear extensions such that $<_P = \prec_1 \cap \prec_2$. Let $-_1$ denote the betweens relation, namely $a_n -_1 c_n -_1 d_n$ if and only if either $a_n \prec_1 c_n \prec_1 d_n$ or $d_n -_1 c_n -_1 a_n$. We claim that the set $X = \{n \mid \neg(a_n -_1 c_n -_1 b_n) \wedge \neg(a_n -_1 d_n -_1 b_n)\}$ separates $\operatorname{ran}(f)$ from $\operatorname{ran}(g)$.

Assume $f(k) = n$, for some $k, n \in \mathbb{N}$. We rule out the possibilities that $a_n -_1 c_n -_1 b_n$ and that $a_n -_1 d_n -_1 b_n$. Assume without loss of generality that $a_n \prec_1 c_n \prec_1 b_n$. Since $\{a_n, b_n, c_n\}$ is an antichain, then it holds that $b_n \prec_2 c_n \prec_2 a_n$. It follows from the definition of $<_P$ and from the fact that $<_P \subseteq \prec_1$ and $<_P \subseteq \prec_2$, hat $c_n \prec_1 x_k$ and $c_n \prec_2 x_k$, contrary to $c_n \mid_P x_k$. An analogous reasoning shows that d_n is not $\prec_1$-between a_n and b_n. Thus we conclude that $n \in X$.

A similar argument shows that if $g(k) = n$, for some $k, n \in \mathbb{N}$, then it cannot hold that a_n and b_n are in between c_n and d_n, otherwise they would be $\prec_1$ and $\prec_2$-below y_k, contrary to the $<_P$-incomparability. It follows that $n \notin X$. □

For strong interval orders it is possible to obtain the same conclusion on the dimension, even dropping the assumption on the height.

Definition 5.12. A poset $(P, <_P)$ is a strong interval order if $p <_P q$, $s <_P t$, $p \mid_P s$ and $q \mid_P t$ imply $p <_P t$ and $s <_P q$ for each $p, q, s, t \in P$.

Theorem 5.13 (RCA_0)**.** *The following are equivalent:*

1. WKL_0,
2. *each strong interval order it is either a linear order or it has dimension two.*

Proof. The proof follows the same line as the proof of Theorem 5.11. For the reversal notice that the order $(P, <_P)$ defined from the functions f, g is also a strong interval order. □

For proper interval orders the bound on the dimension is even tighter. Indeed Rabinovitch proved in [Rabinovitch 1978b] that each proper interval order has dimension at most three. In [Bosek et al. 2012] it is defined an on-line algorithm which, for each proper interval order, outputs four linear extensions witnessing that the dimension is at most four. It is also presented a proper interval order such that the intersection of each triple of linear extensions is not the poset itself. As a corollary we get that Rabinovitch's theorem is not computably true. On the other hand, WKL_0 proves this statements thanks to Theorem 5.3.

Theorem 5.14 (WKL_0)**.** *Each proper interval order has dimension at most three.*

Question 5.15. Is Theorem 5.14 equivalent to WKL_0?

PART III

$$_{(s\text{-})}\mathbf{RS}^{(W/CD)}_{g/po_{(k)}}$$

Ivan Rival and Bill Sands proved the following two theorems in [Rival and Sands 1980].

Theorem. *For every infinite graph G there exists an infinite subgraph H such that every vertex of G is adjacent to at most one vertex of H or to infinitely many vertices of H.*

Theorem. *Let $(P, <_P)$ be an infinite poset of finite width. Then there exists an infinite chain C such that every point of P is comparable to none or to infinitely many elements of C. Moreover, if P is countable, C may be chosen so that each $p \in P$ is comparable to none or to cofinitely many elements of C.*

The first theorem concerns graphs, of any cardinality, and the existence of certain subgraphs with peculiar adjacency relation with the other vertices of the graph. To some extent it resembles Ramsey's theorem for pairs RT^2_2. The latter statement asserts that for each infinite graph there exists a subgraph which is either complete or totally disconnected. In other words, for each graph G there exists a subgraph H such that either every vertex of H is adjacent to all vertices of H, or no vertex of H is adjacent to any vertex of H.

Rival and Sands noticed that while Ramsey's theorem fully describes the adjacency structure inside the subgraph H, it gives no information about the adjacency structure between H and the rest of the graph. Their first theorem is, in some sense, an attempt to fill this lacking information. On the other hand, to attain this result, Rival and Sands have to drop the complete information about the adjacency relation inside H. In fact, they exhibited a graph whose complete and totally disconnected subgraphs are not solutions to the Rival-Sands statement.

The second theorem improves the first one for a specific class of graphs, namely comparability graphs whose associated partial orders have finite width (i.e. the size of the antichains is bounded by some $k \in \mathbb{N}$). On one hand, the solution to Rival-Sands second theorem is always a complete subgraph, so that we have complete information about the internal structure of the solution. On the other hand, there are no points of the poset comparable with only one element of the solution[2].

This part of the thesis is the outcome of a joint project with Alberto Marcone, Paul Shafer and Giovanni Soldà. The material presented here is the result of our joint efforts.

Overview of the main results. We are interested in analysing these theorems (restricted to countable graphs and to countable posets) from the point of view of reverse mathematics, in order to understand their strengths within subsystems of second order arithmetic and to compare them with the strength of RT^2_2. The first theorem turned out to be equivalent to ACA_0, hence stronger than RT^2_2, despite the similarity between the two statements. On the other hand, a variant of RSg, suggested by Steffen Lempp and Jeffry Hirst and apparently weaker than RT^2_2, is equivalent to RT^2_2.

The analysis of the second theorem turned out to be more complicated than the first one, but more interesting and results in new and simpler proofs of Rival-Sands second principle. The proof given in [Rival and Sands 1980] goes through in Π^1_1-CA_0. We give proofs which are very different from the original one and which substantially improve the axiomatic upper bound of the statement. More specifically, we prove that, for each $k \geq 3$, ADS is equivalent to the following statement: for each poset $(P, <_P)$ of width k there exists an infinite chain C such that every point of P is comparable to none or to infinitely many elements of C.

[2] [Rival and Sands 1980] contains some open questions about possible improvement of these theorems and let rise some questions about possible further results. To the best of our knowledge the unique follow up is [Gavalec and Vojtáš 1980], concerning generalisations of Rival-Sands theorems to solutions of all cardinalities.

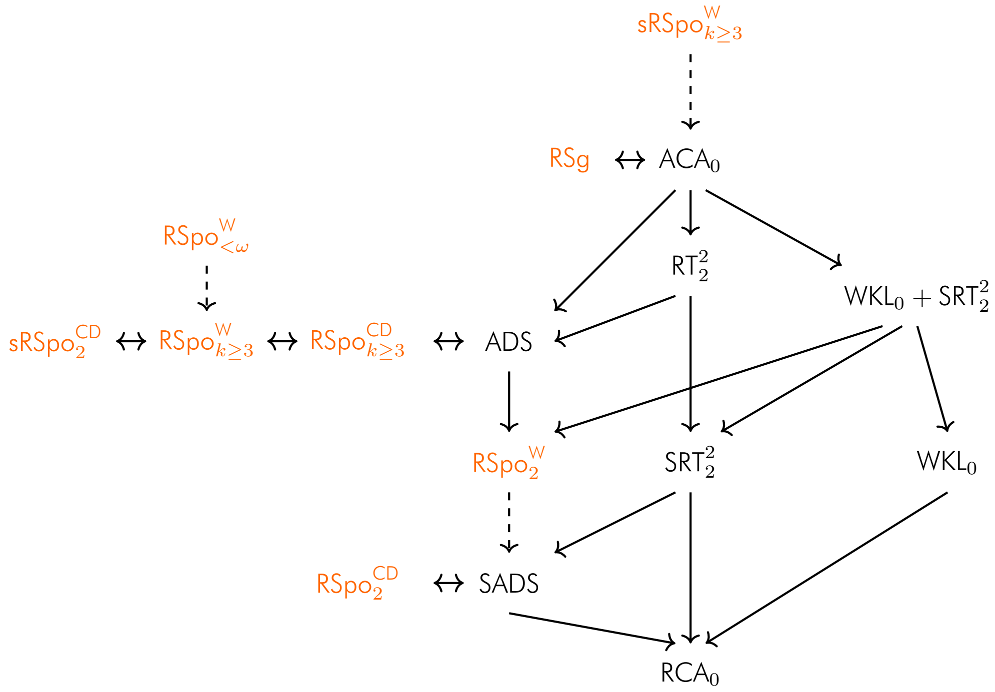


Figure 5.1: A summery of the implications generated by Rival-Sands theorems.

Moreover, we prove that the same statement restricted to posets of width two, is equivalent to SADS over WKL_0.

We also prove that ADS is equivalent to the following stronger version (limited to posets of width two) of Rival-Sands second theorem: for each poset $(P, <_P)$ of width two there exists an infinite chain C such that every point of P is comparable to none or to co-finitely many elements of C.

The two principles mentioned are, to the best of our knowledge, the first statements of ordinary mathematics known to be equivalent to ADS and SADS. In reverse mathematics ADS received attention as an easy consequence of Ramsey's theorem, which is nonetheless strictly weaker than Ramsey's theorem for pairs, but neither computably true nor comparable with WKL_0. ADS shares this behaviour with many other statements, which are quite close, yet non equivalent, to each other. This behaviour contrasts with that of the so called Big Five of reverse mathematics, which are characterised by a sort of robustness and by the equivalence to numerous theorems from different areas of mathematics.

Overview of the chapters. Chapter one covers the analysis of Rival-Sands theorem for graphs. After stating some preliminary definitions we present the forward proof in ACA_0 and the reversal. Finally, we comment on the relationship between Rival-Sands theorem and Ramsey's theorem. The second chapter presents miscellaneous results used in the final chapter, including some principles about the existence of maximal chains, some statements about the composition of posets into nicer posets and a bounded version of SRT^2_2. The third chapter is devoted to Rival-Sands theorem for orders. In section one we present the original proof of the latter theorem, while we give another proof in ACA_0 in section two. Sections three, four and five are devoted to the equivalence with ADS, and section six to the equivalence with SADS. Some considerations about a stronger form of Rival-Sands theorem end the chapter.

RIVAL-SANDS FOR GRAPHS

CHAPTER 6

Content

By analogy with the notion of homogeneous set adopted for Ramsey's theorem we introduce the notion of $(0, 1, \omega)$-homogeneous subgraph.

Definition 6.1. Let (V, E) be a graph. Then an induced subgraph (V', E)[1] is $(0, 1, \omega)$-*homogeneous* if each $v \in V$ is adjacent to at most one vertex of V' or to infinitely many of them.

Similarly, (V', E) is called $(0, 1)$-*homogeneous* if each $v \in V$ is adjacent to at most one vertex of V'. (V', E) is called $(0, \omega)$-*homogeneous* if each $v \in V$ is adjacent to none or to to infinitely many vertices of V'.

Definition 6.2. A graph (V, E) is *locally-finite* if each $v \in V$ has only finitely many neighbours.

A graph (V, E) is *highly recursive* if there exists a computable function $f : V \to \mathbb{N}$ which maps each $v \in V$ to the maximum, with respect to $<$, of its neighbours.

The first theorem in [Rival and Sands 1980] can thus be reformulated as follows:

RSg *For each graph* (V, E) *there exists a* $(0, 1, \omega)$-*homogeneous subgraph* (V', E).

RSlfg *For each locally-finite graph* (V, E) *there exists a* $(0, 1)$-*homogeneous subgraph* (V', E).

The main result of this chapter is the following.

Theorem 6.3 (RCA_0)**.** ACA_0 *and* RSg *are equivalent.*

Reading carefully the proof of RSg given in [Rival and Sands 1980] we noticed that it essentially goes through in ACA_0. For sake of completeness we rewrite the proof here (see Theorem 6.9). The reverse direction is instead proved in Theorem 6.13.

To prove RSg it is useful to introduce some notations and a definition.

[1] Abusing notation we write (V', E) in place of $(V', E \restriction V')$.

Notation 6.4. Let (V, E) be a graph. Then

- $N(a) = \{v \in V \mid v\,E\,a\} \cup \{a\}$ is the neighbourhood of $a \in V$. Notice that $a \in N(a)$,
- $N(A) = \bigcup_{a \in A} N(a)$ is the neighbourhood of $A \subseteq V$,
- $F = \{a \in V \mid N(a) \text{ is finite}\}$ is the set of elements with finitely many neighbours,
- $T = \{a \in V \mid N(a) \cap F \text{ is finite}\}$. Notice that $F \subseteq T$,
- $N^*(a) = N(N(a) \cap T) \cap F$ for each $a \in V$,
- $N^*(A) = \bigcup_{a \in A} N^*(a)$ for $A \subseteq V$.

Remark 6.5. From a computability theoretic viewpoint, the set F defined above is Σ_2-definable in (V, E), so $(V, E)^{(1)}$ enumerates it. On the other hand, the set T is Δ_3-definable in (V, E), so $(V, E)^{(2)}$ computes it. It is not difficult to see that neither set is computable in (V, E); thus RCA_0 does not suffice to prove their existence. Furthermore, $N^*(a)$ is c.e. in T and Σ_3-definable in (V, E) for each $a \in V$, so $(V, E)^{(2)}$ enumerates it.

Proposition 6.6. *Let (V, E) be a graph and $a_0, \ldots, a_{n-1}$, for some $n \in \mathbb{N}$, be a sequence of vertices of F. Then RT^1_n proves that $\bigcup_{i<n} N^*(a_i)$ is finite.*

Proof. Since $a_i \in F$, for each $i < n$, by RT^1_n it holds that $\bigcup_{i<n} N(a_i)$, and so $\bigcup_{i<n} N(a_i) \cap T$, is finite. Moreover, each $v \in \bigcup_{i<n} N(a_i) \cap T$ has finitely many neighbours in F, by definition of T. Thus $\bigcup_{i<n} N^*(a_i)$ is finite. □

Definition 6.7. Let (V, E) be a graph and $A, B \subseteq V$. We say that A is *disjoint relative to* B if $N(a) \cap N(a') \cap B = \emptyset$ for each $a, a' \in A$.

6.1 A proof for RSg

The proof of the main theorem exploits the following lemma as an essential step towards the conclusion. The sets F, T and $N^*(A)$ mentioned in the lemma are as in Notation 6.4. Since the proof of the theorem goes through in ACA_0 we just notice that the lemma can be proved in ACA_0 as well.

Lemma 6.8 (ACA_0)**.** *Let (V, E) be a graph such that $F \subseteq V$ is infinite. Let $I \subseteq F$ be infinite and $A \subseteq F$ be finite and disjoint relative to T. Then there exists a vertex $b \in I \setminus A$ such that $A \cup \{b\}$ is disjoint relative to T.*

Proof. We claim that $A \cup \{b\}$ is disjoint relative to T, for each $b \in I \setminus N^*(A)$. Suppose on the contrary that there exists $a \in A$, $b \in I \setminus N^*(A)$ and $c \in N(a) \cap N(b) \cap T$. In other words, it holds that $c \in N(a) \cap T$ and $c\,E\,b$. Thus $b \in N(N(a) \cap T)$ and $b \in I \subseteq F$ imply that $b \in N^*(A)$, contrary to the choice of b.

Let $b \in I \setminus N^*(A)$ (Proposition 6.6 guarantees that such a b exists). Thus b is the vertex sought after by the previous claim and because it holds that $A \subseteq N^*(A)$. □

Rival and Sands proved a slightly general version of the previous lemma. Namely they argued that for each $n \in \mathbb{N}$ if $I_0, \ldots, I_n$ are infinite subsets of F and $A \subseteq F$ is finite and disjoint relative to T, then there exist distinct $b_0, \ldots, b_n$ such that $b_i \in I_i \setminus A$, for each $i \leq n$, and $A \cup \{b_0, \ldots, b_n\}$ is disjoint relative to T. The proof is by induction and the base case corresponds to the proof of our restatement of the lemma. For the induction step one assumes that $b_0, \ldots, b_m$, for some $m < n$, have already being chosen and argues, very much as in base case, that there exists b_{m+1} with the desired properties. To do so RT^1_{k+m}, where $k = |A|$, is used. In the Rival-Sands variant of the lemma the pigeonhole principle is thus used n times to define the sequence $b_0, \ldots, b_n$.

Our restatement of the lemma is a particular case of Rival-Sands one, namely we prove it for $n = 1$. This is sufficient to carry out the proof of RSg and avoids using the lemma for arbitrary n inside the main proof[2].

Theorem 6.9 (RCA_0)**.** ACA_0 *proves* RSg.

Proof. Let (V, E) be a graph. If F is finite, then let $V' = V \setminus N(F)$. Because F is finite, V' is infinite. Then (V', E) is $(0, \omega)$-homogeneous. In fact, if $x \in F$, then x is connected to none of V', otherwise x is connected to infinitely many vertices in V'. Notice that in this case every $x \in V'$ has infinitely many neighbours in V'.

We can now assume that F is infinite. This case is split into two subcases depending whether $N(a) \subseteq T$ for each $a \in F$ or not.

Suppose the former is the case. We define a sequence $\langle a_n \mid n \in \mathbb{N}\rangle$ of vertices of F as follows

$$\begin{aligned} a_0 &= \min a \in F \\ a_n &= \min a \in F \setminus \bigcup_{i<n} N^*(a_i) \end{aligned}$$

Given that F is infinite and by Proposition 6.6 each $\bigcup_{i<n} N^*(a_i)$ is finite, the sequence $V' = \langle a_n \mid n \in \mathbb{N}\rangle$ is infinite.

Claim 6.9.1. (V', E) *is* $(0, 1)$*-homogeneous.*

Proof. Suppose $v \in V$ is such that $|N(v) \cap V'| > 1$ and let $m < n$ be such that $v \in N(a_m) \cap N(a_n)$. Then $v \in N(a_m) \cap T$, because $N(a_m) \subseteq T$ by assumption, and so $a_n \in N(N(a_m) \cap T) \cap F = N^*(a_m)$ contrary to the choice of a_n. (V', E) is thus $(0, 1)$-homogeneous. □

Notice also that (V', E) is totally disconnected.

Suppose now that there exists $a \in F$ such that $N(a) \nsubseteq T$. Let $a_0 \in F$ be such that $N(a_0) \setminus T \neq \emptyset$, namely such that there exists $b \in N(a_0)$ such that $N(b) \cap F$ is infinite. We define V' as an increasing union of sets $V_i \subseteq F$. To do so we also define an auxiliary sequence σ_i, for each $i \in \mathbb{N}$.

To start let $V_0 = \{a_0\}$ and $\sigma_0 = \langle a \mid a \in N(V_0) \setminus T\rangle$. The assumptions on a_0 guarantee that the sequence σ_0 is finite but not empty, and that $N(\sigma_0(i)) \cap F$ is infinite for each $i < |\sigma|$. Thanks to Lemma 6.8, applied on $N(\sigma_0(0)) \cap F$ and V_0, we obtain $a_1 \in (N(\sigma_0(0)) \cap F) \setminus V_0$ such that $\{a_0, a_1\}$ is disjoint relative to T. Thus we set

$$\begin{aligned} V_1 &= \{a_0, a_1\} \\ \sigma_1 &= \langle \sigma_0(1) \dots, \sigma_0(|\sigma_0| - 1)\rangle^\frown N(V_1) \setminus T \end{aligned}$$

We assume the elements of $N(V_1) \setminus T$ are listed in increasing order. Assume now that the sequences $V_n \subseteq F$ and $\sigma_n \subseteq N(V_n) \setminus T$ have been defined and that V_n is disjoint relative to T. The sets $N(\sigma_n(0)) \cap F$ and V_n satisfies the hypothesis of Lemma 6.8, thus let $a_{n+1} \in (N(\sigma_n(0)) \cap F) \setminus V_n$ such that $V_n \cup \{a_{n+1}\}$ is disjoint relative to T. We thus set

$$\begin{aligned} V_{n+1} &= V_n \cup \{a_{n+1}\} \\ \sigma_{n+1} &= \langle \sigma_n(1) \dots, \sigma_n(|\sigma_n| - 1)\rangle^\frown N(V_{n+1}) \setminus T \end{aligned}$$

We assume the elements of $N(V_{n+1}) \setminus T$ are listed in increasing order. Finally let $V' = \bigcup_{n\in\mathbb{N}} V_n$. By construction V' is an infinite subset of F disjoint with respect to T.

Claim 6.9.2. (V', E) *is* $(0, 1, \omega)$*-homogeneous.*

[2] The lemma can actually be stated and proved in RCA_0 plus arithmetical induction. The key observation here is the fact that $N^*(A)$, for any $A \subseteq F$, can be defined by bounded Σ_4-comprehension, once it has been proved, by $\mathsf{B}\Sigma_4$ that it is finite.

Proof. Let $v \in V$ and $m < n$ be such that $v \in N(a_m) \cap N(a_n)$. Since V_n is disjoint relative to T, then $v \notin T$, so $v \in N(V_n) \setminus T$. We claim that for each $s > n$, there exists $t \geq s$ such that $a_t \in N(x) \cap V'$. Let $s > m$. Then it holds that $v \in N(V_s) \setminus T$ because $V_n \subseteq V_s$ by construction. Hence, there is an $i < |\sigma|$ such that $\sigma_s(i) = v$ by definition of σ_s. The construction guarantees that, at stage $t = s + i + 1$, V_t contains a vertex $v_t \in (N(x) \cap F) \setminus V_{s+i}$. Hence, $|N(v) \cap V'| = \omega$ as required. □

Notice also that (V', E) is totally disconnected. Assume, on the contrary, that $a_i E a_j$ for $i < j$ and $a_i, a_j \in V'$. Then $a_j \in N(a_i) \cap N(a_j) \cap T$ (since $F \subseteq T$), so A_j would not be disjoint relative to T, contrary to the construction. □

Remark 6.10. The proof of the theorem guarantees that each vertex of V' is adjacent to none or to infinitely many vertices of V'.

Moreover, the proof gives more information about the adjacency structure of (V', E), namely

- if F is finite, there exists (V', E) not totally disconnected and $(0, \omega)$-homogeneous,
- if F is infinite, then there exists (V', E) totally disconnected. Moreover, if $N(a) \subseteq T$ for each $a \in F$, then (V', E) may also be found $(0, 1)$-homogeneous.

The proof of the previous theorem is highly non uniform, since $(V, E)^{(2)}$ is needed to recognise whether F is finite or not, namely if (V, E) follows in the first or the second case of the proof.

We also emphasise that $(V, E)^{(4)}$ is required in the proof to calculate a $(0, 1, \omega)$-homogeneous subgraph.

Sharper results, concerning the computable strength of Theorem 6.9, can be obtain for locally finite graphs. Notice that if (V, E) is locally finite, then $F = T = V$ and (V, E) falls under case 2.1 of the previous theorem.

We give here a simpler, but suitable only for locally-finite graphs, proof of RSlfg.

Theorem 6.11 (RCA_0)**.** ACA_0 *proves* RSlfg.

Proof. Let (V, E) be a locally-finite graph. We define an increasing sequence $\langle a_n \mid n \in \mathbb{N} \rangle$ of vertices as follows:

$$a_0 = \min \{v \in V\}$$
$$a_n = \min \{v \in V \mid \forall y \in \bigcup_{i<n} N(a_i)(a \notin N(y))\}$$

Finally, let $V' = \langle a_n \mid n \in \mathbb{N} \rangle$. Since each $a_n \in V'$ is at distance at least two from any other $a_m \in V'$, then (V', E) is trivially $(0, 1)$-homogeneous. □

The previous proof allows to notice immediately that if (V, E) is highly recursive, then RSlfg is computably true. In fact, since at step n one can computably enumerate the neighbours of $a_0, \ldots, a_{n-1}$, then the previous proof gives an effective procedure to build (V, E'). If a computable function $f \colon \mathbb{N} \to \mathbb{N}$ bounds the neighbourhood of the vertices of C, then (V', E) is Δ_1^V-definable since $x \in V'$ if and only if $\forall y < x \forall z < f(y)(z \in N(y) \to z \notin N(x))$.

Theorem 6.12 (RCA_0)**.** *For each highly recursive graph there exists a* $(0, 1)$*-homogeneous subgraph.*

6.2 A reversal for RSg

Theorem 6.13 (RCA_0)**.** RSlfg *implies* ACA_0.

Proof. Let $f \colon \mathbb{N} \to \mathbb{N}$ be an injection. We show that the range of f exists. To do this, we build a locally-finite graph $(\mathbb{N}, E)$ such that we can decode the range of f from any $(0, 1, \omega)$-homogeneous $H \subseteq \mathbb{N}$.

The relation E is the symmetric closure of the following relation

$$(v, s) \in E \Leftrightarrow v < s \wedge f(s) < s \wedge f(s) < f(v)$$

The graph is locally finite. To see this, consider $v \in \mathbb{N}$. As f is injective, there are only finitely many $s > v$ such that $f(s) < f(v)$, and therefore there are only finitely many $s > v$ that are adjacent to v.

Let $H \subseteq \mathbb{N}$ be an infinite $(0, 1)$-homogeneous set. Enumerate H in increasing order as $u_0 < u_1 < u_2 < \cdots$. We show that, for any $n \in \mathbb{N}$, if $\exists s(f(s) = n)$, then $(\exists s \leq u_{n+1})(f(s) = n)$. So suppose that $f(s) = n$. If $n \geq s$, then $s \leq n \leq u_{n+1}$. If $n < s$, then s is adjacent to all but at most n of the vertices $v < s$. This is because f is an injection, so there are at most $n = f(s)$ many vertices $v < s$ with $f(v) < f(s)$. By the (0,1)-homogeneity of H, at most one neighbour of s is in H. Therefore there can be at most $n + 1$ many vertices in H that are $< s$. Thus $u_{n+1} \geq s$. Thus n is in the range of f if and only if $(\exists s \leq u_{n+1})(f(s) = n)$, so the range of f exists by Δ^0_1 comprehension. ☐

As a consequence of the previous theorem we get

Theorem 6.14 (RCA_0). RSg *implies* ACA_0.

Notice that it is also possible to modify the graph built in the reversal to get a non locally-finite graph. For example let $V = \mathbb{N} \cup \{a\}$ and set a adjacent to all the vertices of V. Notice that this graph follows under case 2.2 in the proof of Theorem 6.9, since each $n \in \mathbb{N}$ belongs to F and has a neighbour, namely a, not in T.

The question about the strength of RSg in the hierarchy of subsystems of second order arithmetic is thus settled. Nonetheless, it would be interesting to investigate further this principle in order to understand its computational content more deeply. Theorem 6.13 witnesses that there exists a locally-finite graph whose $(0, 1, \omega)$-homogeneous sets code $0'$. Notice that this is the best possible for locally-finite graphs, since $0'$ is enough to prove RSlfg, as underlined in Theorem 6.11. On the other hand, the proof of Theorem 6.9 for arbitrary graphs uses several jumps, since it requires to define the sets F, T and $N^*(a)$ for some vertices $a \in V$ (see Remark 6.5). Hence, we suspect that there are graphs, non locally-finite, whose coding power is higher. Some progresses have been very recently made towards answering to this problem; we now believe that any degree that is PA relative to $(V, E)''$ can compute a $(0, 1, \omega)$-homogeneous to (V, E) and that this bound is strict.

6.3 RSg and RT^2_2

Introducing Rival-Sands theorem we emphasised the link between RSg and Ramsey's theorem for graphs, RT^2_2, which inspired Rival and Sands themselves. The authors exhibited a graph such that each $(0, 1, \omega)$-homogeneous subgraph is neither complete not totally disconnected. Reverse mathematics provides additional insight into their result because it gives formal methods to prove that there exists a graph such that each solution for Ramsey's theorem does not code a $(0, 1, \omega)$-homogeneous subgraph.

Another measure of the similarity between a certain principle and Ramsey's theorem is the notion of 'Ramsey-like statements'. A statement of the form $\forall G(\varphi(G) \Rightarrow \exists H \psi(G, H))$ is said to be of Ramsey-type when it has the following properties:

- if $\varphi(G)$ and $\psi(G, H)$, then H must be infinite,
- if $\varphi(G)$, $\psi(G, H)$, and $H' \subseteq H$ is infinite, then also $\psi(G, H')$.

In other words the feature of Ramsey-like statements is the fact that each infinite subset of a solution is itself a solution.

RSg is not in general of Ramsey's type. For example let $V = \{a_n \mid n \in \mathbb{N}\} \cup \{b_n \mid n \in \mathbb{N}\} \cup \{c\}$ such that $c\,E\,a_n$ for each $n \in \mathbb{N}$. Then $H = \{a_n \mid n \in \mathbb{N}\} \cup \{b_n \mid n \in \mathbb{N}\}$ is $(0, 1, \omega)$-homogeneous, but $H' = \{b_n \mid n \in \mathbb{N}\} \cup \{a_0, a_1\}$ is not $(0, 1, \omega)$-homogeneous. More interestingly, there is a graph whose $(0, 1, \omega)$-homogeneous subgraphs are not of Ramsey type. Just take the graph mentioned in the first paragraph, whose $(0, 1, \omega)$-homogeneous subgraphs are neither complete nor totally disconnected. If it had a $(0, 1, \omega)$-homogeneous subgraph H whose infinite subgraph were still $(0, 1, \omega)$-homogeneous, then applying Ramsey's theorem to H one gets a subgraph which is either complete or totally disconnected and which is $(0, 1, \omega)$-homogeneous contrary to the assumption. However, solutions are always preserved when only finitely many elements from H are removed. This implies that if H is $(0, 1)$-homogeneous, then each infinite subgraph is still $(0, 1)$-homogeneous, and so RSlfg is a Ramsey-type principle.

To better calibrate the different strengths of RSg and RT^2_2 we introduced two weakenings of the former statement.

wRSg *For each graph (V, E) there exists an infinite subgraph H such that either for each $x \in H$ there exists at most one $y \in H$ such that xEy, or for each $x \in H$ there exist infinitely many $y \in H$ such that xEy.*

wwRSg *For each graph (V, E) there exists an infinite subgraph H such that for each $x \in H$ either there exists at most one $y \in H$ such that xEy or there exist infinitely many $y \in H$ such that xEy.*

A solution to wRSg or to wwRSg depends only about the adjacency relationship internal to the solution H, rather than to the adjacency relationship between H and the remaining points. In this respect the two statements are closer to RT^2_2 than RSg.

The two principles above were suggested by Steffen Lempp and Jeffry Hirst at the workshop 'Ramsey theory and Computability' in 2018. The following theorem was proved in the same occasion with their substantial help.

Theorem 6.15 (RCA_0)**.** *The following are equivalent:* RT^2_2*,* wRSg*, and* wwRSg.

Proof. The implication from RT^2_2 to wRSg and from wRSg to wwRSg are trivial. To prove that wwRSg implies RT^2_2 we exploit the fact that RT^2_2 is equivalent to SRT^2_2 plus COH.

To prove SRT^2_2 let $c\colon \mathbb{N} \to \mathbb{N}$ be a stable colouring and define as usual a graph (V, E) such that $x\,E\,y$ if and only if $c(x, y) = 1$. Let H be a solution to wwRSg. Notice that if $x \in H$ has two neighbours in H, then it has infinitely many of them.

Suppose there are infinitely many $x \in H$ with infinitely many neighbours in H (i.e. there are infinitely many $y \in H$ such that $c(x, y) = 1$). Notice that for those x the colouring stabilises at 1. We define an infinite homogeneous set $H' = \langle x_n \mid n \in \mathbb{N}\rangle$ which is 1-homogeneous. Let $x_0 \in H$ be such that there are $y, z \in H$ such that $y\,E\,x\,E\,z$. Suppose x_n has been defined and search for the minimum triple x, y, z such that $x, y, z \in H$, $y\,E\,x\,E\,z$ and $c(x_i, x) = 1$ for each $i \leq n$. Set $x_{n+1} = x$. It is easy to check that the procedure enumerates an infinite set $H' \subseteq H$, which is 1-homogeneous by construction.

Otherwise there exists $x \in H$ such that for each $y > x$ it holds that y is incomparable with all but at most one elements in H (i.e. such that $c(y, z) = 0$ for all $z \in H$ except at most one). Enumerate an infinite 0-homogeneous set H' as follows: if $x_0, \dots, x_n$ have been defined, let x_{n+1} be the minimum $x \in H$ such that $c(x_i, x) = 0$ for each $i \leq n$. Since there are at most n elements $y \in H$ such that $c(x_i, y) = 1$, the procedure does not halt. We have thus proved that wwRSg implies SRT^2_2.

We now prove that wwRSg implies ADS, which implies COH. Since we are dealing with linear orders it is more convenient to rephrase wwRSg, in the standard way, using colouring rather than graphs. Let $(L, <_L)$ be a linear order and define a colouring $c\colon [\mathbb{N}]^2 \to 2$ such that $c(x, y) = 1$ whenever $x <_L y$ and $x < y$.

Let H be a solution to wwRSg, namely for each $x \in H$ either there exists at most one $y \in H$ such that $c(x, y) = 1$ or there are infinitely many such y's.

Suppose there are infinitely many $x \in H$ such that $c(x, y) = 1$ for infinitely many $y \in H$ (i.e. such that there are infinitely many $y \in H$ such that $x <_L y$). We define an ascending chain $A = \langle x_n \mid n \in \mathbb{N} \rangle$ as follows: let x_0 be such that there are $y, z \in H$ such that $c(x, y) = c(x, z) = 1$; suppose $x_0, \ldots, x_n \in A$ and that $x_0 <_L \cdots <_L x_n$, then search for the minimum triple x, y, z such that $x, y, z \in H$, $c(x, y) = c(x, z) = 1 = c(x_n, x) = 1$. Set $x_{n+1} = x$. By assumption the procedure does not stop and enumerates an ascending chain.

Otherwise there is $x \in H$ such that for each $y > x$ it holds that $c(y, z) = 0$ for all but one $z \in H$. We claim that there exists a descending chain in H. Let x_0 be the minimum vertex greater that x. If $x_0 >_L \cdots >_L x_n$ have been defined, let x_{n+1} be minimum $x \in H$ such that $c(x_n, x) = 0$. Since x_n has only finitely elements in H above it by assumption, the procedure does not halt and enumerates a descending chain. We have thus proved that wwRSg implies ADS. $\square$

Notice that RT^2_2 implies a slightly stronger modification of both wRSg and wwRSg where one substitutes the condition on the existence of at most one $y \in H$ adjacent to $x \in H$ with the condition of $x \in H$ being incomparable with all $y \in H$. Since the following theorem proves that RT^2_2, wRSg, and wwRSg are equivalent over RCA_0, the previous variation is equivalent to RT^2_2 as well.

TOOLKIT FOR RIVAL-SANDS

CHAPTER 7

Content

This chapter contains quite varied material. The first section is devoted to the analysis of some principles about the existence of maximal chains. As emphasised in the next chapter Rival-Sands second theorem mentions one of these principles; we named it MMLC. What is more, the strength of MMLC determines the axiomatic system needed to carry out the original proof of Rival-Sands theorem. For the sake of completeness, analogous principles for antichains are mentioned.

The remainder of the chapter is instead related to new proofs of Rival-Sands theorem. We gather some lemmas and observations essential for the proof of Theorems 8.7, 8.8 and 8.16.

7.1 Some principles about the existence of maximal chains

Several of the following lemmas exploit the tool of true and false stages of an injective function $f\colon \mathbb{N} \to \mathbb{N}$.

Definition 7.1. Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injective function. The number $n \in \mathbb{N}$ is a *true stage* if $f(k) > f(n)$ for each $k > n$. Otherwise, n is a *false stage*.

True stages are interesting from a computability theoretic point of view because the range of f is computable in any infinite subset of true stages. In fact, if n is a true stage, one can determine $\operatorname{ran}(f)$ up to $f(n)$ simply running f on inputs $0, \dots, n$.

True stages revealed to be useful also to prove reversals to ACA_0. In fact, to prove that the range of some function $f\colon \mathbb{N} \to \mathbb{N}$ exists, it suffices to prove that an infinite subset of the true stages of f exists, because the two are computably equivalent.

We are especially interested in coding the true stages of f, and so the range of f, for some function $f\colon \mathbb{N} \to \mathbb{N}$, into chains which are solutions to various principles regarding posets. To this end we find useful a particular ordering on the true and false stages introduced in [Frittaion and Marcone 2014] (see for example Theorem 4.5) and used later in [Frittaion et al. 2016]. Since we are going to use it several times in Lemmas 7.4, 7.7, 7.10 and 7.10, we present the common construction here separately.

Construction 7.2. Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injective function. Define a linear order $(L, <_L)$ such that $L = \{\ell_n \mid n \in \mathbb{N}\}$ and for each $n < m$ the following hold:

1. $\ell_n <_P \ell_m$ if $f(k) < f(n)$ for some k such that $n < k \leq m$ (i.e. at stage m, k witnesses that n is a false stage),
2. $\ell_m <_P \ell_n$ if $f(n) < f(k)$ for all k such that $n < k \leq m$ (i.e. at stage m, we have no reason to think that n is a false stage)

It is easy to see that $(L, <_L)$ is a chain of order type $\omega + \omega^*$ (the false and true stages respectively).

7.1.1 Maximal (anti)chains

Thinking about principles on the existence of maximal chains or antichains, the first natural statement which blossomed in our mind is the following one.

Lemma 7.3 (RCA_0)**.** *Let $(P, <_P)$ be a poset. Then there exist a maximal chain D and a maximal antichain E in P.*

Proof. Define D inductively adding, at stage n, the minimum element of P comparable with points selected in the previous stages. It is immediate to check that D is the desired chain.

Analogously, build E choosing at each step a point incomparable with the points selected in the previous stages. □

Notice that a solution to the previous statement may be finite. For example D (or E) may coincide with an isolated point or with a maximum, if P contains one of those. Indeed this statement is very weak. It is enough to require that the maximal chain (or antichain) extends a given chain (resp. antichain) to jump up to the level of ACA_0.

Lemma 7.4 (RCA_0)**.** ACA_0 *is equivalent to the following statements: let $(P, <_P)$ be a poset and $C \subseteq P$ be a chain. Then there exists a maximal chain $D \supseteq C$.*

Proof. Let $(P, <_P)$ be a poset and $C \subseteq P$ be a chain. Define a sequence of points as follows:

$$\begin{aligned} d_0 &= < \text{-min} \,\{p \in P \mid \forall c \in C \,(p <_P c \vee c <_P p)\} \\ d_{n+1} &= < \text{-min} \,\{p \in P \mid \forall c \in C \cup \{d_0, \dots, d_n\} \,(p <_P c \vee c <_P p))\} \end{aligned}$$

Let $D = \langle d_n \mid n \in \mathbb{N}\rangle$. By construction D is a chain and $C \subseteq D$. In order to check the maximality, let $p \in P \setminus D$ and $d_n \in D$ be the greatest such that $d_n < p$. Then, if $d_{n+1} \neq p$, it means that p is incomparable with some $c \in C \cup \{d_0, \dots, d_n\}$ and hence with some $d \in D$.

For the reverse implication let $f\colon \mathbb{N} \to \mathbb{N}$ be an injective function. We define a poset $(P, <_P)$ and a chain $C \subseteq P$ in such a way that each maximal chain $D \supseteq C$ codes the range of f.

Let $P = \{a_n, c_n \mid n \in \mathbb{N}\}$ and let $<_P$ on $\{a_n \mid n \in \mathbb{N}\}$ be as defined in Construction 7.2. Moreover, for each m and n let

1. $c_m <_P a_n$ if $\forall k\,(n < k \leq m \to f(n) < f(k))$ (i.e. n is a true stage at stage m),
2. $c_m <_P c_n$ if $m < n$.

In order to check that $<_P$ is an order, we need to verify that it holds that $c_\ell <_P a_n$, for each $\ell, n \in \mathbb{N}$, whenever there exists m such that either $c_\ell <_P c_m$ and $c_m <_P a_n$, or $c_\ell <_P a_m$ and $a_m <_P a_n$. Firstly, suppose $c_\ell <_P c_m$ and $c_m <_P a_n$, then a_n is a true stage at stage m by definition of $<_P$. It follows that a_n is a true stage at all the previous stages and in particular at stage ℓ, so $c_\ell <_P a_n$ according to the definition.

Suppose now that $c_\ell <_P a_m$ and $a_m <_P a_n$. If $\ell \leq n$, then it follows immediately that $c_\ell <_P a_n$. So suppose that $n < \ell$. We claim that $n < m$ holds. Suppose on the contrary that $m < n$, so that m is false at stage n by definition of $<_P$. Hence, m is false at each later stage and in particular at stage ℓ, contrary to the fact that $c_\ell <_P a_m$ and $m < \ell$.

Notice that, since $n < m$ and $a_m <_P a_n$, then n is true at stage m by definition of $<_P$. If $\ell < m$, then n is also true at stage ℓ and so it holds that $c_\ell <_P a_n$ as wanted. Otherwise, it holds that $n < m < \ell$. Suppose that there exists k such that $n < k \leq \ell$ and $f(k) < f(n)$, i.e. n is false at stage ℓ. Since n is true at stage m, it holds that $m < k$. Moreover, since m is true at stage ℓ by definition of $<_P$, it holds that $f(m) < f(k)$, which entails $f(m) < f(n)$ by transitivity. Thus m itself witnesses that n is false at stage m, contrary to the assumption.

Since $C = \langle c_n \mid n \in \mathbb{N} \rangle$ forms a chain in P by definition, we apply the statement to obtain a maximal chain D extending C. Notice that a_n is comparable with all elements in C if and only if n is a true stage. Thus, thanks to the maximality of $D \supseteq C$, it holds that n is a true stage if and only if $a_n \in D$. This gives a Δ^0_1-definition of the true stages of f, from which one can recover computably the range of f. □

Lemma 7.5 (RCA_0)**.** ACA_0 *is equivalent to the following statements: let* $(P, <_P)$ *be a poset and* $C \subseteq P$ *be an antichain. Then there exists a maximal antichain* $D \supseteq C$.

Proof. To prove the statement one can reason as in the proof of Lemma 7.4 just substituting the request of comparability between d_{n+1} and $C \cup \{d_0, \dots, d_n\}$ with the requirement of incomparability.

For the reversal let $P = \{a_n, b_n \mid n \in \mathbb{N}\}$ and let $a_m <_P b_n$ if and only if $f(m) = n$, for each $m, n \in \mathbb{N}$. No other comparability holds. Let $C = \{a_n \mid n \in \mathbb{N}\}$ and $D \supseteq C$ be a maximal antichain. We claim that $P \setminus D = \operatorname{ran}(f)$. In fact, if $b_n \notin D$, for some $n \in \mathbb{N}$, then there exists m such that $a_m \nmid_P b_n$ by maximality of D. Hence $n \in \operatorname{ran}(f)$ by definition of $<_P$. Moreover, if $b_n \in D$, for some $n \in \mathbb{N}$, then for each m it holds that $a_m \mid_P b_n$ and so $n \notin \operatorname{ran}(f)$ by definition of $<_P$. □

Lemma 7.4 guarantees that, if C is infinite, then D does not degenerate to a maximum or to an isolated point, if P contains any of those. On the other hand, it does not guarantee that D does not contain a maximum. To attain this we analyse a further principle, the last one about maximal chains. We call a chain *max-less* (*min-less*) if it has no maximal (resp. minimal) element.

MMLC *For each poset* P *and each max-less (min-less) chain* $C \subseteq P$ *there exists a maximal and max-less (resp. min-less) chain* $D \supseteq C$.

Notice that MMLC is the statement actually used in the original proof of Rival-Sands second theorem. We prove that MMLC is equivalent to Π^1_1-CA_0, it is so a very strong principle compared to the previous two. In order to define a max-less maximal chain it is possible to adopt the same strategy as in the proof of the Lemma 7.4, except for the fact that at each step, one has also to be careful to choose elements which are not maximal or 'potentially maximal'. For example if a chain $C \supseteq P$ has only three points $a <_P b <_P c$ above it, then not only the maximum c cannot belong to a max-less maximal extension $D \supseteq C$, but neither a nor b can belong to D. In a nutshell choosing whether $p \in P$ has to be placed in Dinvolves not only testing its maximality, but rather involves testing whether there is an ω chain above p. This is the difficult part in the definition of max-less chains in terms of set-existence axioms.

Before turning to the proof of the equivalence between MMLC and Π^1_1-CA_0, we introduce a definition and prove a couple of intermediate lemmas that make precise the idea sketched in the previous lines.

Definition 7.6. Let $(P, <_P)$ be a poset. A point $p \in P$ is *well founded* if there is an ω chain above it. It is *reverse-well founded* if there is an ω^* chain below it.

Notice that p is well founded in $(P, <_P)$ if and only if it is reverse-well founded in $(P, >_P)$. We recall that if $T \subseteq \mathbb{N}^{\mathbb{N}}$ is a tree, then the Kleene-Brouwer ordering $<_{KB}$ on T is defined as follows: $\sigma <_{KB} \tau$ if and only if either $\tau \sqsubseteq \sigma$ or $\sigma(i) < \tau(i)$ for the minimum i such that $\tau(i) \neq \sigma(i)$, for each $\sigma, \tau \in T$.

Lemma 7.7 (RCA_0)**.** *The following principle implies* ACA_0*: for each poset* $(P, <_P)$ *there exists the set of reverse-well founded elements.*

Proof. Let $f: \mathbb{N} \to \mathbb{N}$ be an injective function and define $(P, <_P)$ as in Construction 7.2. Let R be the set of reverse-well founded elements. It is straightforward to check that $a_n \in R$ if and only if n is a true stage. □

Lemma 7.8 (RCA_0)**.** Π^1_1-CA_0 *is equivalent to the following statement: for each poset* $(P, <_P)$ *there exists the set of reverse-well founded elements.*

Proof. Let $(P, <_P)$ be an order and R be the set of $p \in P$ such that there exists a function $f: \mathbb{N} \to \mathbb{N}$ which enumerates a descending sequence and such that $\forall n\, (f(n) <_P p)$. Notice that R is Σ^1_1-definable and, according to Definition 7.6, is the set of reverse-well founded elements.

For the reversal, by Lemma 7.7 we may work in ACA_0. Let $\langle T_n \subseteq \mathbb{N}^{<\mathbb{N}} \mid n \in \mathbb{N}\rangle$ be a sequence of trees. We define a poset $(P, <_P)$ such that $P = \bigcup_{n\in\mathbb{N}} T_n$ and $\sigma <_P \tau$ if and only if $\sigma, \tau \in T_n$ and $\sigma <_{KB} \tau$ for some $n \in \mathbb{N}$. Let R be the set of reverse-well founded elements and define $X = \{n \in \mathbb{N} \mid \text{the root of } T_n \text{ is in } R\}$. We claim that $\forall n\, (n \in X \leftrightarrow T_n \text{ has a path})$. Notice that the root of T_n is reverse-well founded if and only if $(T_n, <_{KB})$ is not well founded. Moreover, ACA_0 proves that $(T_n, <_{KB})$ is not well founded if and only if T_n has a path[1]. By Theorem 4.1 we have thus proved Π^1_1-CA_0. □

The previous lemma is the key point of the proof of MMLC in Π^1_1-CA_0. As already noticed, the set of well founded elements of $(P, <_P)$ corresponds to the set of reverse-well founded elements of $(P, >_P)$. Therefore, the previous lemma shows also that the existence of the set of well founded elements of a poset is equivalent to Π^1_1-CA_0.

Lemma 7.9 (RCA_0)**.** Π^1_1-CA_0 *proves* MMLC.

Proof. Let $(P, <_P)$ be a poset and $C \subseteq P$ be a chain without maximal element. Let W be the set of well founded elements of P. Notice that $C \subseteq W$. Then, define D as in the proof of Lemma 7.4 just adding the requirement that $d_n \in W$, for each $n \in \mathbb{N}$. □

The next lemma provides a lower bound for MMLC useful to prove the reversal in 7.12.

Lemma 7.10 (RCA_0)**.** MMLC *implies* ACA_0.

Proof. Let $f: \mathbb{N} \to \mathbb{N}$ be an injective function and $(P, <_P)$ be defined as in Construction 7.2. Notice that, since false stages form a c.e. set F, then a subchain $C \subseteq F$ exists in RCA_0. By MMLC there exists a maxless maximal chain D extending C. Notice that $D = F$, because each true stage has only finitely many predecessors. Hence, $P \setminus D$ is the set of true stages, from which one can computably recover $\mathrm{ran}(f)$. □

Remember from Theorem 4.2 that LPP_0 asserts that for each non well founded tree T there exists the leftmost path through T, and that it is equivalent to Π^1_1-CA_0.

Lemma 7.11 (RCA_0)**.** *The following principle implies* Π^1_1-CA_0*: every linear order which is not well founded has a maximal min-less chain.*

[1] See [Simpson 2009, Lemma V.1.3]

Proof. Let $T \subseteq \mathbb{N}^{<\mathbb{N}}$ be a non well founded tree. We show that the principle we are considering implies that there is a leftmost path in T. Thanks to Lemma 7.10 we can reason in ACA_0.

Ordering the elements of T with the Kleene-Brouwer order $<_{KB}$ we obtain the linear order $(T, <_{KB})$. Since T is non well founded by assumption, there exists a min-less maximal chain D in $(T, <_{KB})$.

Let P be defined as follows:

$$P = \{\sigma \in D \mid \forall \tau \in D\, \exists u \in D\, (u \sqsupseteq \sigma \wedge u <_{KB} \tau)\}$$

Notice that P contains the elements of D with reverse-cofinitely many extensions in D. We claim that P is the leftmost path in T. Firstly, suppose that there are $\sigma, \sigma' \in P$ with $\sigma <_{KB} \sigma'$ such that $\sigma \not\sqsupseteq \sigma'$ and $\sigma' \not\sqsupseteq \sigma$. Notice that for each $u \sqsupseteq \sigma'$ it holds that $\sigma <_{KB} u$ by definition of $<_{KB}$, but this contradicts the assumption that $\sigma' \in P$.

To check that P is the leftmost path, suppose that there exists a path P' such that $\sigma'(n) < \sigma(n)$ for some $\sigma \in P, \sigma' \in P'$ and some $n \in \mathbb{N}$ (suppose n is minimum with this property). Then $\sigma' <_{KB} \sigma$ and $\sigma' <_{KB} u$ for each $u \sqsupseteq \sigma$, contrary to assumption that $\sigma \in P$.

Lastly, we verify that P is infinite. Suppose on the contrary that P is finite and so let μ be minimum in P. Since $\mu \in D$ and D is min-less, there exists $\delta \in D$ such that $\delta <_{KB} \mu$. Notice that it must be that $\mu \sqsubseteq \delta$, since $\mu \in P$ and P is the leftmost path. Thus let σ be the immediate $\sqsubseteq$-successor of μ. Since D is maximal, then $\sigma \in D$ and, by assumption, it also holds that $\sigma \notin P$. Thus let τ be such that $\forall u \in D\, (u \sqsupseteq \sigma \rightarrow \tau <_{KB} u)$. By choice of σ it holds that $u \sqsupset \sigma$ if and only if $u \sqsupset \mu$. This contradicts the fact that $\mu \in P$, since there exists τ such that $\forall u \in D\, (u \sqsupseteq \mu \rightarrow \tau <_{KB} u)$. □

Notice that the following weakening of the previous statement already implies Π^1_1-CA_0: each linear order $(L, <_L)$ which is not well founded has a min-less chain cofinal in L. To prove the reversal one can reason as in the proof of the previous lemma, just defining $P = \{\sigma \mid \exists \sigma' \in D\, (\sigma' \sqsupseteq \sigma) \wedge \forall \tau \in D\, \exists u \in D\, (u \sqsupseteq \sigma \wedge u <_{KB} \tau)\}$.

Summarising the previous results we get the following theorem.

Theorem 7.12 (RCA_0). *The following are equivalent:*

1. Π^1_1-CA_0,
2. LPP_0,
3. MMLC,
4. *each non well founded linear order has a maximal min-less chain,*
5. *each well founded linear order has a maximal max-less chain.*

Proof. 1 and 2 are equivalent by Theorem 4.2. 1 implies 3 by Lemma 7.9, while 4 is a weakening of 3. Statements 4 and 5 are duals. Finally, 4 implies 2 by Lemma 7.11. □

7.1.2 Reformulation of MMLC in terms of LPP_0

To complete the analysis of MMLC we show that it is possible to rephrase it in terms of the following principle, introduced in [Towsner 2013].

MPP_0 *For each ill founded tree T and for each well founded order $\prec$ there exists a path through T which is minimal with respect to $\prec$.*

Towsner pointed out that LPP_0 is nothing but the restriction of MPP_0 to the order $<$ of the integers.

Let $(P, \leq_P)$ be a poset containing a max-less chain. We define a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ such that its paths code the max-less chains in P and such that the minimal path in T is a max-less maximal chain. Notice that while the paths of T are Π^0_1-definable, being max-less is a Π^0_2 property, so "directly" coding max-less chains is not feasible. To overcome this difficulty, each $\sigma \in T$ is a pair $\langle p, b\rangle$ such that $p \in P$, $b \in \mathbb{N}$ and b contains the promise on the existence of $\tau \sqsupseteq \sigma$ in T such that $\tau(|\tau| - 1)$ witnesses that p is not a maximum. With this in mind let T be the following tree

$$\big\langle \langle p_i, b_i\rangle \,|\, i < k\big\rangle \in T \Leftrightarrow \begin{cases} \forall i < j < k\,((p_i \leq_P p_j \vee p_j \leq_P p_i) \wedge p_i < p_j) \quad \wedge \\ \forall i\,(b_i \leq p_{k-1} \rightarrow \exists j < k(p_i <_P p_j)) \end{cases}$$

Suppose g is a path in T and let g' be its projection on the first component. By definition of T it is immediate to observe that g' is a max-less chain in P. Moreover, if C is a max-less chain, since it is possible to associate to every point of C a satisfiable promise, there is at least one path in T which corresponds to C.

Let $\prec$ be the lexicographic sum of $<_P$ and $<$. The order $\prec$ is well-founded and so MPP_0 guarantees that there exists a path g through T which is minimal with respect to $\prec$. Let g' be its projection on the first component. We claim that g' is a maximal max-less chains of P. Suppose on the contrary that g' is not maximal. Hence, let $p \notin g'$ be $<_P$-comparable with g' and not a maximum of g'. Let also b be a suitable promise for p. It is possible to define a path h such that

$$h(\langle p_i, b_i\rangle) = \begin{cases} g(\langle p_i, b_i\rangle) & \text{if } p_i < p \\ g(\langle p_{i-1}, b_{i-1}\rangle) & \text{if } p_{i-1} > p \\ \langle p, b\rangle & \text{otherwise} \end{cases}$$

By the assumption on p, the projection on the first component of h is a max-less chain. Moreover, $h \prec g$, contrary to the minimality of g.

7.2 Width and chain decomposition

An instance of Rival-Sands second theorem is a poset of finite width. In this section we discuss the relationship between width and chain-decomposition-number of a poset.

It follows from the definitions that if a poset $(P, <_P)$ has chain-decomposition-number k, then P has width k. The reverse implication is less immediate and was proved by Dilworth in [Dilworth 1950] (for another proof of Dilworth's theorem see [Harzheim 2005, Section 2.5]). Jeffry Hirst in [Hirst 1987, Theorem 3.23] proved that Dilworth's theorem is equivalent to WKL_0.

Theorem 7.13 (RCA_0). *The following statement is equivalent to* WKL_0*: each poset* $(P, <_P)$ *of width* k *has chain-decomposition-number* k.

The proof of the previous theorem reveals that there exists a computable poset of width two which cannot be decomposed into two computable chains. Despite this Dilworth's theorem does have an 'effective analogue'.

Kierstead was interested in extending the algorithmic or constructive content typical of the finite combinatorics to countable structures, following the approach of what we would now call on-line combinatorics. His approach with respect to the non computability of solutions of Dilworth's theorem was thus to ask for a bound b such that each computable poset $(P, <_P)$ of width k can be decomposed into at most b computable chains. In [Kierstead 1981] the bound b is set to $(5^k - 1)/4$ providing an on-line algorithm to decompose each poset of width k into $(5^k - 1)/4$ chains. The bound has recently been greatly improved in [Bosek et al. 2018]. However, for our purposes it is not relevant the exact bound on the chain-decomposition-number, but instead the crucial fact is that the proof of Kierstead's theorem can be formalised in RCA_0, as reveals by an inspection of the proof that we omit[2].

[2] We thank Keita Yokoyama to help us overcoming some obstacles we encountered in this verification.

Theorem 7.14 (RCA_0)**.** *For each $k \in \mathbb{N}$, each poset $(P, <_P)$ of width k has chain-decomposition-number at most 5^k.*

For completeness we mention that Jeffry Hirst in [Hirst 1987, Theorem 3.24] proved also that WKL_0 is equivalent to the antichain version of Dilworth's theorem, namely each poset $(P, <_P)$ of height k can be decomposed into k antichains. These theorems can be compared with Theorem 5.3. Moreover Kierstead in [Kierstead 1986] provides computable analogs for the latter statement, in the spirit of Theorem 7.14.

Existence of (anti)chains in posets of finite width. The previous theorem turned out to be useful also in some unexpected way. If $(P, <_P)$ is an infinite poset of width (or height) k, then it surely contains an infinite chain (resp. antichain). One may wonder if these principles are computably true. The answer is positive and Theorem 7.14 allows to give a straightforward proof of the former. To be more precise we introduce the following variants of CAC:

CC_k *Each poset of width k has an infinite chain.*

CA_k *Each poset of height k has an infinite antichain.*

Recall that the height of a poset is the supremum of the cardinality of the chains. We denote with $\mathsf{CC}_{<\omega}$ and $\mathsf{CA}_{<\omega}$ the uniform version of CC_k and CA_k respectively (i.e. for each $k \in \mathbb{N}$, each poset of width (height) k has an infinite (anti)chain).

It is immediate to see that CC_k, for each $k \in \mathbb{N}$, is computably true. In fact, each poset of width k is effectively decomposable into 5^k many chains and $\mathsf{RT}^1_{5^k}$ finds an infinite chain in P. Moreover, the following lemma holds.

Lemma 7.15 (RCA_0)**.** $\mathsf{B}\Sigma^0_2$ *is equivalent to* $\mathsf{CC}_{<\omega}$.

Proof. Fix k and let $(P, <_P)$ be a poset and $C_0, \ldots, C_{k-1}$ be chains such that $P = \bigcup_{i<k} C_i$. $\mathsf{RT}^1_{<\omega}$ implies that there exists $i < k$ such that C_i is infinite.

To prove that the reverse implication holds as well, we fix k and show that CC_k implies RT^1_k. Let $c\colon \mathbb{N} \to k$ and define the poset $(P, <_P)$ setting $p <_P q$ whenever $f(p) = f(q)$ and $p < q$. P is constituted by k incomparable chains, hence it has trivially chain-decomposition-number k. Moreover, a chain is homogeneous for c. ☐

Lemma 7.16 (RCA_0)**.** $\mathsf{B}\Sigma^0_2$ *is equivalent to* $\mathsf{CA}_{<\omega}$.

Proof. Let $(P, <_P)$ be a poset of height k. We define a colouring $c\colon \mathbb{N} \to \mathbb{N}$ such that, for each $n \in \mathbb{N}$, $c(n) = \langle x, y \rangle$ where x is the maximum length of the chains below n and y is the maximum length of the chains above n. The two maxima exist because P has height k. Thus the colouring c is bounded and by $\mathsf{RT}^1_{<\omega}$ there exists an homogeneous set $H \subseteq \mathbb{N}$. It is immediate to check that H is an antichain.

To prove the reverse implication fix k and let $c\colon \mathbb{N} \to k$ be an instance of RT^1_k. We define a poset $(P, <_P)$ such that $p \mid_P q$ if $c(p) = c(q)$ and $p <_P q$ if $c(p) < c(q)$. It is immediate to check that $<_P$ is transitive and that P has height k. Moreover, an antichain A is homogeneous for c. ☐

Lemma 7.15 straightforwardly implies that ADS finds ascending and descending chains in posets of finite width (remember that ADS implies $\mathsf{B}\Sigma^0_2$).

Proposition 7.17 (RCA_0)**.** ADS *proves the following: for each k and for each poset $(P, <_P)$ of width k, P contains either an ascending or a descending chain.*

Proof. Let $(P, <_P)$ be a poset of width k. By CC_k there exists an infinite chain C in P and by ADS it contains either an ascending or a descending chain. ☐

7.3 A decomposition for linear orders

The proof of Theorem 8.7 uses the following lemma which allows us to find the well founded part and the reverse well founded part of a linear order with no suborders of order type ζ in ACA_0.

Lemma 7.18 (RCA_0). *The following statement in equivalent to ACA_0: each linear order $(L, <_L)$ with no suborder of order type ζ can be split into its well founded part and its reverse well founded part.*

Proof. Let $(L, <_L)$ be a linear order $(L, <_L)$ with no suborder of order type of ζ. Define $X = \{x \in L \mid \forall y <_P x \, \exists z \, (y <_P z <_P x)\}$; intuitively X is the set of limit points of the well founded part. We claim that the downward closure of X, $X\downarrow$, is well founded. Suppose on the contrary that $D = \langle d_n \mid n \in \mathbb{N}\rangle$ is a descending sequence contained in $X\downarrow$. Let $x \in X$ be such that $d_0 \leq_L x$, such x exists by definition $X\downarrow$. By transitivity it holds that $D <_L x$. We define an ascending chain $A = \langle a_n \mid n \in \mathbb{N}\rangle$ such that $D \setminus \{d_0\} <_L A <_L x$. Notice that $D \setminus \{d_0\} \cup A$ has order type ζ by construction, we thus reach the contradiction sought after. The ascending chain A is defined as follows:

$$
\begin{aligned}
a_0 &= \min \{z \mid d_1 <_L z <_L x\} \\
a_{n+1} &= \min \{z \mid a_n <_L z <_L x\}
\end{aligned}
$$

For each $n \in \mathbb{N}$ there exists a_n with the desired properties because $x \in X$.

Consider $L \setminus X\downarrow$. If it is reverse well founded, then X and $L \setminus X\downarrow$ provide the desired splitting of L. Otherwise, we claim that $L \setminus X\downarrow = O \cup R$ such that O has order type ω, R is reverse well founded and possibly empty. In this case $X \cup O$ and R provides the desired splitting. Since $L \setminus X\downarrow$ is not reverse well founded by assumption, there exists an ascending chain $A = \langle a_n \mid n \in \mathbb{N}\rangle$. Let $B = \{x \in L \setminus X\downarrow \mid x <_L a_0\}$ and suppose B is infinite. Under this hypothesis we define a descending chain $C = \langle c_n \mid n \in \mathbb{N}\rangle$ so that $C \cup A$ forms a ζ. Let

$$
\begin{aligned}
c_0 &= \min \{y \in L \setminus X\downarrow \mid y <_L a_0 \wedge \neg\exists z \, (y <_L z <_L a_0)\} \\
c_{n+1} &= \min \{y \in L \setminus X\downarrow \mid y <_L y_n \wedge \neg\exists z \, (y <_L z <_L y_n)\}
\end{aligned}
$$

Notice that for each $n \in \mathbb{N}$ there exists c_n with the desired properties because, since $B \subseteq L \setminus X\downarrow$ each $c >_P B$ cannot belong to X. We conclude that B is finite and so that $B \cup A$ is still ascending.

It is easy to verify that if A has an immediate successor x, then $x \in X$ and so $A \in X\downarrow$, contrary to the assumption. It follows that $R = L \setminus (X\downarrow \cup A \cup B)$ is reverse well founded.

For the reversal let $f\colon \mathbb{N} \to \mathbb{N}$ be an injective function and define a linear order $(L, <_L)$ as in Construction 7.2. The reverse well founded part of L correspond with the true stages of f by definition of $<_L$. $\square$

7.4 Bounded SRT^2_k

The last ingredient needed for the following chapter is the following 'bounded version' of SRT^2_k.

Definition 7.19. Let $c\colon [\mathbb{N}]^2 \to k$ be a colouring. We say that c is n-stable if for each $x \in \mathbb{N}$ there exists $m \leq n$ and $y_0, \ldots, y_m$, such that $c(x, y_i) \neq c(x, y_{i+1})$ for each $i < m$, and for each $u, v > \max\{y_1, \ldots, y_m\}$ it holds that $c(x, u) = c(x, v)$.

Loosely speaking c is n-stable if each $x \in \mathbb{N}$ changes colour at most n times. We thus introduce the following principle.

n-SRT^2_k Each n-stable colouring $c\colon [\mathbb{N}]^2 \to k$ contains an infinite homogeneous set.

To emphasise the differences and the commonalities between n-SRT^2_k and SRT^2_k, for some $n, k \in \mathbb{N}$, it is useful to restate the stability condition as follows: a colouring $c\colon [\mathbb{N}]^2 \to k$ is stable if for each $x \in \mathbb{N}$ there exists m and $y_0, \ldots, y_m$, such that $c(x, y_i) \neq c(x, y_{i+1})$ for each $i < m$, and for each $u, v > \max\{y_1, \ldots, y_m\}$ it holds that $c(x, u) = c(x, v)$.

Lemma 7.20. *For each standard* $n, k \in \mathbb{N}$*,* RCA_0 *proves* n-SRT^2_k.

Proof. We prove the statement by induction on n. For the base case let $c\colon [\mathbb{N}]^2 \to k$ be 0-stable we define $H_j = \{x \in \mathbb{N} \mid c(x, x+1) = j\}$ for each $j < k$. To check that H_j is homogeneous let $x, y \in H_j$; by definition on H_j it hold that $c(x, x+1) = j$ and $c(y, y+1) = j$, thus $c(x, y) = j$ because c is 0-stable. By RT^1_k there exists $i < k$ such that H_i is infinite.

Suppose the statement is true for n-stable colouring and let $c\colon [\mathbb{N}]^2 \to k$ be $(n+1)$-stable.

Suppose there are infinitely many x which witness that c is $n+1$-stable, but not n-stable. We define, for each $j < k$, a set H_j as the increasing union of sets H^s_j. At stage s, we set $x \in H^s_j$ if there are $y_0 <_{\mathbb{N}} \cdots <_{\mathbb{N}} y_{n+2} <_{\mathbb{N}} s$ such that $c(x, y_i) \neq c(x, y_{i+1})$ for each $i \leq n$ and $c(x, y_{n+2}) = j$. By assumption $\bigcup_{j<k} H_j$ is infinite and so there is a $j < k$ such that H_j is infinite by RT^1_k.

Otherwise, there is an x such that $c \restriction (\mathbb{N} \setminus \{0, \ldots, x\})$ is n-stable. The colouring contains an homogeneous set by induction hypothesis. ☐

Corollary 7.21 (RCA_0)**.** *For each* $n \in \mathbb{N}$*,* $\mathsf{B}\Sigma^0_2$ *is equivalent to for all* k n-SRT^2_k.

Proof. The forward direction is a straightforward consequence of the proof of the previous lemma which indeed shows that RCA_0 proves $\forall k(\mathsf{RT}^1_k \to n - \mathsf{SRT}^2_k)$, so that $\mathsf{B}\Sigma^0_2$ proves $\forall k$ n-SRT^2_k.

For the reversal let $c\colon \mathbb{N} \to k$ be an instance of RT^1_k. Define a colouring $d\colon [\mathbb{N}]^2 \to k$ such that $d(x, y) = c(x)$ for each $x, y \in \mathbb{N}$. By definition d is 0-stable so let H be an infinite homogeneous set for d. It is immediate to verify that H is homogeneous for c too. ☐

Since n-SRT^2_k is proved in RCA_0 by induction on n we do not get immediately that RCA_0 proves that for each $n \in \mathbb{N}$ n-SRT^2_k holds. However $\mathsf{I}\Sigma^0_4$-induction is sufficient to prove the uniform version of this principle. In fact, once $n, k \in \mathbb{N}$ are fixed, Lemma 7.20 actually proves that for each $X \subseteq \mathbb{N}$ and for each $e \in \mathbb{N}$ if φ^X_e codes a k-colouring of $[\mathbb{N}]^2$ such that each x changes colour at most n times, then there exist i and $j < k$ such that φ^X_i is homogeneous for j. Once formalised the previous statement one gets the following result.

Corollary 7.22 (RCA_0)**.** $\mathsf{I}\Sigma^0_4$*-induction proves* n-SRT^2_k *for each* $k, n \in \mathbb{N}$.

RIVAL-SANDS FOR ORDERS

CHAPTER 8

Content

The following definition is analogous to Definition 6.1 for posets.

Definition 8.1. Let $(P, <_P)$ be a poset. Then C is a $(0, \omega)$-homogeneous chain for $(P, <_P)$ if each element of P is comparable to none of the elements of C or to infinitely many of them.

The second theorem in [Rival and Sands 1980] can thus be reformulated as follows:

$\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$ *For each k and each poset $(P, <_P)$ of width k there exists a $(0, \omega)$-homogeneous chain C.*

To better analyse the strength of $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$ it is convenient to introduce $\mathsf{RSpo}^{\mathsf{W}}_{k}$, which is simply the restriction of $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$ to posets of width k, and $\mathsf{RSpo}^{\mathsf{CD}}_{k}$, the restatement of $\mathsf{RSpo}^{\mathsf{W}}_{k}$ for posets with chain-decomposition-number k.

$\mathsf{RSpo}^{\mathsf{W}}_{k}$ *For each poset $(P, <_P)$ of width k there exists a $(0, \omega)$-homogeneous chain C.*

$\mathsf{RSpo}^{\mathsf{CD}}_{k}$ *For each k and each poset $(P, <_P)$ with chain-decomposition-number k there exists a $(0, \omega)$-homogeneous chain C.*

Since, by Dilworth's theorem, each poset of width k has chain-decomposition-number k, the second of the previous principles is a natural restatement of the first. Furthermore, the chain decomposition of a poset turned out to be a very convenient framework in any proof but the original proof of the Rival-Sands theorem (which could nonetheless be rewritten using this notion) of Rival-Sands theorem.

Each partial order of finite width $(P, <_P)$, viewed as a graph, is an instance of RSg, which thus provides a $(0, 1, \omega)$-homogeneous subgraph $(V', <_P)$ of $(P, <_P)$. Certainly such a V' contains an infinite chain C, because P has finite width, but as emphasised in Section 6.3 C may not be $(0, 1, \omega)$-homogeneous. To give a specific example let $(P, <_P)$ be composed by two ω-chains A and B which coincides for the first n points, for some $n \in \mathbb{N}$, and which are incomparable from the $n + 1$th point on. The graph P itself is $(0, 1, \omega)$-homogeneous, but it is clearly not a chain. Moreover, it contains the chain A which is not $(0, \omega)$-homogeneous, because each $b \in B$ is comparable with n elements of A.

8.1 The original proof of $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$

Unlike the case of RSg, the Rival-Sands proof for $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$ cannot be the optimal one. In fact, $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$ is a Π^1_2-statement, the original proof goes through in Π^1_1-CA_0, and a Π^1_2-statement cannot be equivalent to Π^1_1-CA_0. This implies that the complexity of $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$ is strictly weaker than Π^1_1-CA_0. Nonetheless we prefer to present in Theorem 8.3 the original proof, emphasising which set existence axioms are used, because it contains some ideas we develop and exploit in other proofs of $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$ and because, to better understand its strength, we were pushed to study some interesting principles about the existence of maximal chains we discussed in Section 7.1.

Notation 8.2. Let $(P, <_P)$ be a poset and A be an ascending chain. Then A_m denotes the ascending sequence $\langle a_n \mid n \geq m \rangle$, which forms a tail of A.

By definition a chain A is not $(0, \omega)$-homogeneous if there exists a point $p \in P$ such that p is comparable with some element of A but only with finitely many elements of them.

Suppose now that the chain $A = \langle a_n \mid n \in \mathbb{N} \rangle$ is ascending and not $(0, \omega)$-homogeneous. In this case we can be more specific about the comparability relation between such a p and the points of A. In fact, there must exists $n \in \mathbb{N}$ such that $p >_P a_n$ and $p \mid_P A_{n+1}$ (see Definition 11 for the definition of $p \mid_P A_{n+1}$). We often call such a p a counterexample to A being $(0, \omega)$-homogeneous. If each tail of A is not $(0, \omega)$-homogeneous, then there must be infinitely many of such p each of which witnesses that A_n is not $(0, \omega)$-homogeneous for cofinaly many $n \in \mathbb{N}$. Lemma 8.4 provides more details about this.

Theorem 8.3 (RCA_0). Π^1_1-CA_0 *proves* $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$.

Proof. Let $(P, <_P)$ be a poset of width k, for some $k \in \mathbb{N}$, and assume that P does not contain a $(0, \omega)$-homogeneous chain. Since P has finite width, then it contains an ascending chain or a descending chain by Proposition 7.17. Assume that P has an ascending chain (the case where P contains a descending chain may be obtained by considering $(P, >_P)$).

We define inductively triples (S_i, C_i, D_i), for each $i \leq k$, such that $S_i \subseteq P$, $C_i \subseteq S_i$ is a maximal max-less chain and D_i is cofinal in C_i and of order type ω[1]. Those triples will allow to define an antichain of length $k + 1$, contrary to the assumption.

At the base step set $S_0 = P$ and, by Lemma 7.9, let $C_0 \subseteq S_0$ be a maximal max-less chain. Let also D_0 be a cofinal in C_0 and of order type ω.

[1]It is not necessary, and not required in the original proof, that D_i is an ω chain, for each $i \leq k$. We introduce this change, which affects neither the structure of the proof nor its strength, to better emphasise the features of S_{i+1}, as defined in the inductive step, and to slightly modify Claim 8.3.1 in a way that allows us to introduce a schema of argumentation very useful in the other proofs of $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$.

Suppose $(S_0, C_0, D_0), \dots, (S_i, C_i, D_i)$ have been defined such that, for each $j \leq i$, $S_j = \{p \in P \mid \exists d \in D_{j-1} \forall e >_P d\,(p >_P d \wedge e \mid_P p)\}$, $C_j \subseteq S_j$ is a maximal chain without maximum and D_j is cofinal in C_j and of order type ω. Notice that by assumption each tail of D_i is not $(0, \omega)$-homogeneous. Let $S_{i+1} = \{p \in P \mid \exists d \in D_i \forall e >_P d\,(p >_P d \wedge e \mid_P p)\}$ (notice that, in the terminology previously introduced, S_{i+1} is the set of counterexamples of D_i). Notice that $S_{i+1} \neq \emptyset$, otherwise D_i would be $(0, \omega)$-homogeneous. The following claim proves that S_{i+1} contains a max-less chain.

Claim 8.3.1. *There exists a max-less chain in* S_{i+1}.

Proof. Suppose on the contrary that $m_0, \dots, m_n$, for $n < k$ are the maximal elements of S_{i+1}. By definition of S_{i+1} there exist $d_0, \dots, d_n$ such that $d_j \in D_i$ and $\forall e >_P d_j (e \mid_P m_j)$, for each $j \leq n$. Let $e \in D_i$ be greater than d_j for each $j \leq n$, so that $e \mid_P m_j$ for each $j \leq k$. We claim that $D = \{p \in D_i \mid p \geq_P e\}$ is a $(0, \omega)$-homogeneous chain, contrary to the assumption on P. In fact, if $p \in P \setminus S_i$, then either p is incomparable with D_i, hence with D, or it is below some $d \in D_i$, hence below infinitely many points in D. Otherwise, if $p \in S_i$, then there exists a $j \leq n$ such that $p <_P m_j$, by choice of $m_0, \dots, m_n$. This entails that $p \mid_P D$. □

The previous claim and Lemma 7.9 guarantee the existence of a maximal max-less chain $C_{i+1} \subseteq S_{i+1}$. Let D_{i+1} be cofinal in C_{i+1} and of order type ω.

Claim 8.3.2. *For each* $j \leq i$*,* $D_{i+1} \subseteq S_j$.

Proof. Fix $j < i$ and assume, as induction hypothesis, that $D_{i+1} \subseteq S_n$, for each $n < j$. To prove that $D_{i+1} \subseteq S_j$, by definition of S_j, we have to verify that for each $d_{i+1} \in D_{i+1}$ there exists a $d_{j-1} \in D_{j-1}$ such that $d_{j-1} <_P d_{i+1}$ and $e \mid_P d_{i+1}$ for each $e >_P d_{j-1}$. Fix now $d_{i+1} \in D_{i+1}$.

Since $d_{i+1} \in S_{i+1}$ there exists $d_i \in D_i$ such that $d_i <_P d_{i+1}$. For the same reason there exists $d_{i-1} \in D_{i-1}$ such that $d_{i-1} <_P d_i$. Iterating this procedure we find $d_i, \dots, d_j$ such that $d_n \in D_n$, for each $i \leq n \leq j$, and $d_{i+1} >_P d_i >_P \dots >_P d_j$. By transitivity it holds that $d_{i+1} >_P d_j$. Since $d_j \in S_j$ there exists $d_{j-1} \in D_{j-1}$ such that $d_{j-1} <_P d_j$ and $e \mid_P d_j$ for each $e >_P d_{j-1}$. By transitivity it holds that $d_{j-1} <_P d_{i+1}$.

We are left to check that $e \mid_P d_{i+1}$ for each $e >_P d_{j-1}$. Suppose on the contrary that this is not the case. Notice that if there exists $e >_P d_{j-1}$ such that $e >_P d_{i+1}$, then transitivity implies that $e >_P d_j$, contrary to the choice of d_j. Hence, it must be that $d_{i+1} >_P e$ for each $e >_P d_{j-1}$. Let $C = \{d \in D_{i+1} \mid d \geq_P d_{i+1}\}$. By induction hypothesis it holds that $C \subseteq S_{j-1}$. Moreover it holds that $C_{j-1} <_P C$, because D_{j-1} is cofinal in C_{j-1}. These two fact imply that $C_{j-1} \cup C$ is a max-less chain in S_{j-1}, contradicting the maximality of C_{j-1}. □

At step $k + 1$ it is possible to define an antichain of size $k + 1$ as follows:

$$
\begin{aligned}
c_k &= \min d \in D_k \\
c_i &= \min \{d \in D_i \mid c \mid_P c_{i+1} \wedge \dots \wedge c \mid_P c_n\}
\end{aligned}
$$

The existence of $c_n, \dots, c_0$ is guaranteed by the previous claim. In fact, if $c_n, \dots, c_i$ have been defined, then $c_j \in S_i$, for each $i \leq j \leq n$, and so it is incomparable with cofinitely many points in $d \in D_{i-1}$. The existence of this antichain contradicts the assumption on the width of P, so we conclude that there exists a $(0, \omega)$-homogeneous chain. □

Suppose that $(P, <_P)$ is a poset of width k without $(0, \omega)$-homogeneous chains and containing an ascending chain. Notice that S_i, for each $i \leq k$, is defined by a Δ^0_2 formula relative to D_{i-1}. In fact, being D_{i-1} ascending, the formula

$$\exists d \in D_{i-1} \forall e >_P d\,(s_i >_P d \wedge e \mid_P s_i)$$

is equivalent to

$$\exists d \in D_{i-1}\,(s_i >_P d) \wedge \forall e \in D_{i-1}\,\exists f >_P e\,(s_i \mid_P f)$$

The ascending chain D_i is instead computable in C_i. In fact, suppose $\langle c_n \mid n \in \mathbb{N}\rangle$ is an enumeration of C_i and $d_0, \ldots, d_n$ have been chosen to be in D_i, then, by the standard inductive construction, choose d_{n+1} to be the minimum $d > d_n$ and $d >_P d_n$. It is easy to check that D_i is cofinal and ascending.

The most delicate passage in the proof of Theorem 8.3 concerns the existence of the maximal max-less chains $C_0, \ldots, C_k$ as explained in Section 7.1.

Theorem 7.12 witnesses that the full strength of Π^1_1-CA_0 is used in the proof of Theorem 8.3 through MMLC. Nonetheless, as already mentioned, $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$ cannot be equivalent to Π^1_1-CA_0. This mismatch between the lower and upper bound reminds other Π^1_2-principles, as those analysed in [Marcone 1996] and [Shafer 2012], whose known proof goes through in Π^1_1-CA_0 because some form of maximality is used in the proof. Henry Towsner studied extensively those cases noticing that often, and in particular in the two principles previously mentioned, maximality in not used in its full strength, since one needs that certain objects are maximal with respect to certain, not all, objects. [Towsner 2013] presents a detailed framework where this idea is make explicit and defines a hierarchy of weakening of LPP_0.

To obtains a deeper comprehension of the proof of Theorem 8.3 it would be interesting to understand if one of the Towsner's principles might replace MMLC (given the fact that MMLC can be reformulated in terms of LPP_0, as shown in Section 7.1.2). More explicitly one needs to understand whether maximal max-less chains in Claim 8.3.2 can be replaced with max-less chains which are maximal with respect to a class of Σ-definable chains for some formula Σ.

Looking at the proof of $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$ under this perspective, one can notice that the chain C, defined in Claim 8.3.2 to contradict the maximality of C_{j-1}, is computable in D_{i+1} and so in C_{i+1}. Hence, the chain $C_{j-1} \cup C$ is computable in $C_{i+1} \oplus C_{j-1}$. It would be interesting to figure out what is Σ in this particular case, in order to calibrate better the strength of the maximality we are using. The difficulty in carrying on this analysis here relates to the fact that C_i is defined thanks to $C_0 \oplus \cdots \oplus C_{i-1}$, which are maximal max-less chains themselves.

8.2 A lower upper bound for $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$

Despite the great strength required by the original proof, we were able to give an entirely different proof of $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$, which goes through in ACA_0.

The key difference between the proofs of Theorem 8.3 and of Theorem 8.7 is the replacement of maximal max-less (or min-less) chains, not useful in a weaker subsystem, with well founded max-less (resp. reverse-well founded min-less) chains. To this end Lemma 7.18 comes into play.

The other two main ingredients of the proofs are: the Dilworth decomposition of the partial order into a finite number of chains (see Section 7.2) and a finer analysis of the notion of counterexamples to ascending or descending chains whose tails are not $(0,\omega)$-homogeneous. Lemma 8.4 is devoted to the latter and expands the comment on counterexamples to ascending or descending chains made at the beginning of Section 8.1.

Lemma 8.4 (ACA_0)**.** *Let $(P, <_P)$ be a poset and $C_0, \ldots, C_{k-1}$ be chains such that $P = \bigcup_{i<k} C_i$. Let $A = \langle a_n \mid n \in \mathbb{N}\rangle$ be an ascending chain included in C_i, for some $i < k$, such that each tail of A is not $(0,\omega)$-homogeneous.*

Then there exist $j < k$ and an ascending chain $B = \langle b_n \mid n \in \mathbb{N}\rangle$ included in C_j such that $\forall m\,\exists n > m\,\exists \ell\,(a_n <_P b_\ell \wedge \forall r > n\,(a_r \mid_P b_\ell))$.

Proof. Suppose $m \in \mathbb{N}$ is such that $\forall n > m\,\forall p\,(a_n \not<_P p \vee \exists r > n\,(a_r \nmid_P p))$. Then for each $p \in P$ and for each $n > m$, either p is above or incomparable with A or $p <_P a_r$ for some $r > n$. The chain A_{n+1} is thus $(0,\omega)$-homogeneous, contrary to the assumption.

Notice that the sequence of points satisfying the formula above is infinite. In fact, suppose that $a_m <_P p$ and $\forall r > m\,(a_r \mid_P p)$ for some $m \in \mathbb{N}$ and some $p \in P$. Then, there must be an $n > m$ and a p' (notice that one can choose p' such that $p < p'$) such that $a_n <_P p'$ and $\forall r > n\,(a_r \mid_P p')$. Since $a_{m+1} \leq_P a_n <_P p'$ and $a_{m+1} \nmid p$, it holds that $p \neq p'$.

Let $S \subseteq P$ be the sequence of points satisfying the formula above. RT^1_k guarantees that there exist an infinite set $B \subseteq S$ and $j < k$ such that $B \subseteq C_j$. We claim that B is ascending. In fact, let $p < p'$ as in the previous paragraph and suppose that p and p' belong to B. If $p' <_P p$, then by transitivity it holds that $a_{m+1} <_P p$, contrary to the assumption. □

In the situation of the previous lemma we often call B a *counterexample* to A or we say that B witnesses that each tail of A is not $(0,\omega)$-homogeneous. Very similar considerations hold for a descending chain A which does not contain $(0,\omega)$-homogeneous chains. Indeed an ω^* chain has order type ω in $(P, >_P)$ and it is $(0,\omega)$-homogeneous in $(P, <_P)$ if and only if it is $(0,\omega)$-homogeneous in $(P, >_P)$. The unique change to the notion of counterexample to descending chains is thus due to the fact that, in this case, the counterexample B is a descending sequence such that $\forall m\, \exists n > m\, \exists \ell\, (a_n >_P b_\ell \wedge \forall r > n\,(a_r \mid_P b_\ell))$.

Counterexamples to ascending and descending chains whose tails are not $(0,\omega)$-homogeneous have thus a very clear structure. This is the main reason why they play a central role in each of the proofs of $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$ we give, included the original one.

It is very convenient to introduce the following definition to describe some relations between A and B.

Definition 8.5. We say that D *is pointwise bounded by* A, or $A <_{\forall\exists} D$, if for each $a \in A$ there exists $d \in D$ such that $a <_P d$. If $A = \{a\}$ (or $D = \{d\}$), then we write $a <_{\forall\exists} D$ (resp. $A <_{\forall\exists} d$).

Property 8.6. *In the situation of the previous definition the following facts holds:*

- $A <_P d$ *if and only if* $A <_{\forall\exists} d$*;*
- *if* A *has no maximum and* $A \subseteq D$ *then* $A <_{\forall\exists} D$*;*
- $a <_{\forall\exists} D$ *if and only if* $a <_P D_n$ *for some* $n \in \mathbb{N}$*;*
- $<_{\forall\exists}$ *is transitive.*

Notice that, if D is a counterexample to A, it holds that $A <_{\forall\exists} D$ and it does not hold that $D <_{\forall\exists} A$.

Theorem 8.7 (RCA_0)**.** ACA_0 *proves* $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$.

Proof. Let $(P, <_P)$ be a poset of width k and let $C_0, \ldots, C_{k-1}$ be the decomposition chain according to Theorem 7.13. Assume that $(P, <_P)$ does not contain a $(0,\omega)$-homogeneous chain. Notice that any chain Z of order type ζ is $(0,\omega)$-homogeneous. In fact, if $p \in P$ is comparable with some $z \in Z$, then it is either comparable with all elements above z or with all elements below z. It follows that C_i, for each $i < k$, does not contain chains of order type ζ. By Lemma 7.18 we can split C_i, for each $i < k$, in its well founded part W_i, refined deleting maximal elements, and its reverse well founded part R_i.

Suppose that $W_0, \ldots, W_u$, for some $u < k$, are the non empty well founded chains. If all of them are empty, then some reverse well founded chain is not empty and we can reason analogously with the obvious changes. For each $i \leq u$ let $A_i = \langle a_n \mid n \in \mathbb{N}\rangle$ be a cofinal sequence in W_i of order type ω. By assumption each tail of A_i, for each $i \leq u$, is not $(0,\omega)$-homogeneous, so let B_i be a counterexample to A_i by Lemma 8.4. Let $h\colon \{0,\ldots,u\} \to \{0,\ldots,u\}$ be a bijection such that $B_i \subseteq C_{h(i)}$, for each $i \leq u$. Since B_i is ascending then it holds, for each $i \leq u$, that $B_i \subseteq W_{h(i)}$.

Notice that, for each $i \leq u$, it holds that $B_i <_{\forall\exists} A_{h(i)}$ since $A_{h(i)}$ is cofinal in $W_{h(i)}$. Since it holds that $A_{h(i)} <_{\forall\exists} B_{h(i)}$, by choice of $B_{h(i)}$, and since $<_{\forall\exists}$ is transitive, it holds that $B_i <_{\forall\exists} B_{h(i)}$, for each $i \in \mathbb{N}$. By transitivity we get that $B_{h^n(i)} <_{\forall\exists} B_{h^m(i)}$ for each $i \leq u$ and each $n \leq m \in \mathbb{N}$ ($h^m(i)$ stands for the m^{th} iteration of $h(i)$, where $h^0(i) = i$).

We claim that there exist $i, j \in \mathbb{N}$ such that $i < j$ and $h^i(0) = h^j(0)$: simply consider $h(0), h(h(0)), \ldots$. Let $i, j \in \mathbb{N}$ satisfy the previous claim. It follows from the previous paragraph that $B_{h^{i+1}(0)} <_{\forall\exists} B_{h^j(0)}$ and so that $B_{h^{i+1}(0)} <_{\forall\exists} B_{h^i(0)}$. Finally, it holds that $B_{h^{i+1}(0)} <_{\forall\exists} A_{h^{i+1}(0)}$, since $B_{h^i(0)} <_{\forall\exists} A_{h^{i+1}(0)}$ by cofinality of $A_{h^{i+1}(0)}$. This contradicts the fact that $B_{h^{i+1}(0)}$ is a counterexample to $A_{h^{i+1}(0)}$. □

8.3 An equivalence with ADS

The main theorem of this section is the following one.

Theorem 8.8. *For each* $k \geq 3$, $\mathsf{RCA}_0 \vdash \mathsf{ADS} \leftrightarrow \mathsf{RSpo}_k^{\mathsf{W}}$.

Proof. Fix $k \geq 3$ and let $(P, <_P)$ be a poset of width k. By Theorem 7.14 P has chain-decomposition-number h, for some $h \leq (5^k - 1)/4$. By Theorem 8.13 P contains a $(0, \omega)$-homogeneous chain C.

We now prove that $\mathsf{RSpo}_3^{\mathsf{W}}$, and so $\mathsf{RSpo}_k^{\mathsf{W}}$, implies ADS.

Let $(L, \leq_L)$ be a linear order and consider $(L \times 3, <_P)$ with the product partial order (with the ordering $0 <_3 1$ and $2 <_3 1$). Since $L \times 3$ has clearly width 3, let C be a $(0, \omega)$-homogeneous chain for $L \times 3$.

For each $i < 3$ set $C_i = C \cap (L \times i)$. By definition of $<_P$ it is easy to see that $C \subseteq C_0 \cup C_1$ or $C \subseteq C_1 \cup C_2$. In fact $(\ell, 0)$ and $(\ell, 2)$ are incomparable for each $\ell \in L$.

We claim that C_1 has no maximum. Suppose on the contrary that $(m, 1)$ is a maximum of C_1 and hence of C. Since $C_0 = \emptyset$ or $C_2 = \emptyset$ and both $(m, 0)$ and $(m, 2)$ are below $(m, 1)$, then at least one between $(m, 0)$ and $(m, 2)$ is comparable with some and finitely many elements of C. This contradicts the assumption that C is $(0, \omega)$-homogeneous. Hence, if $C_1 \neq \emptyset$, we can recursively define an ascending chain in it.

Otherwise, by RT_3^1 at least one between C_0 and C_2 is infinite. In this case either C_0 or C_2 has no minimum, otherwise there would be a point in $(L, 1)$ incomparable with all C but the minimum. It is thus possible to define recursively a descending chain in C_0 or C_2, which is obviously a descending chain in L. □

8.3.1 Local counterexamples to ω chains

Before getting into the proof of Theorem 8.13 we anticipate some ideas and considerations that hopefully guide the reader toward a better comprehension of the proof itself.

Let $(P, <_P)$ be a poset with chain-decomposition-number k. The main idea behind the proof is to focus almost exclusively on ω chains (as usual it would be the same focusing on ω^* chains) and to use extensively the fact that for each ascending chain whose tails are not $(0, \omega)$-homogeneous there exists a counterexample. On a first sight this may appear to be a problem since the most natural definition of counterexample requires ACA_0, as witnessed by Lemma 8.4.

The key point behind the proof of Theorem 8.13 is actually the fact that the counterexample may be found computably even if this may require trails and errors in its search. The following lemma is a computable modification of Lemma 8.4.

Lemma 8.9 (RCA_0)**.** *Let* $(P, <_P)$ *be a poset and* $C_0, \ldots, C_{k-1}$ *be chains such that* $P = \bigcup_{i<k} C_i$. *Let* $A = \langle a_n \mid n \in \mathbb{N} \rangle$ *be an ascending chain included in* C_i*, for some* $i < k$*, such that each tail of* A *is not* $(0, \omega)$*-homogeneous.*

Then there exist $j < k$ *and an ascending chain* $D = \langle d_n \mid n \in \mathbb{N} \rangle$ *included in* C_j *such that* $\forall m \, \exists n > m \, \exists \ell \, (a_n <_P d_\ell \wedge a_{n+1} \mid_P d_\ell)$.

Proof. Firstly, we prove that the formula $\forall m \, \exists n > m \, \exists p \in P \, (a_n <_P p \wedge a_{n+1} \mid_P p)$ holds. Suppose not, namely suppose that

$$\exists m \, \forall n > m \, \forall p \in P \, (a_n \not<_P p \vee a_{n+1} \nmid_P p)$$

Let m satisfy the formula. Then for each $p \in P$ and for each $n > m$, either p is above or incomparable with A or $p <_P a_r$ for some r. The chain A_{n+1} is thus $(0, \omega)$-homogeneous.

Notice that the sequence of points satisfying the formula above is infinite. In fact, suppose that $a_m <_P p$ and $a_{m+1} \mid_P p$ for some m and some $p \in P$. Then, there must be an $n > m$ and a p' such that $a_n <_P p'$ and $a_{n+1} \mid_P p'$. Since $a_{m+1} \leq_P a_n <_P p'$ and $a_{m+1} \nmid p$, it holds that $p \neq p'$.

Let $S \subseteq P$ be the sequence of points satisfying the formula above. RT_k^1 guarantees that there exist an infinite set $D \subseteq S$ and $j < k$ such that $D \subseteq C_j$. We claim that D is ascending. In fact, let p and p' as before and suppose that p and p' belong to D. If $p' <_P p$, then by transitivity it holds that $a_{m+1} <_P p$, contrary to the assumption.

Notice that the proof is identical to the proof of Lemma 8.4 except for a small change in the formula proved to be true. This change allows to enumerate S in RCA_0. $\square$

Definition 8.10. Each ascending chain D satisfying Claim 8.9 is named a *local counterexample* to A.

Notice that if D is a local counterexample to A, then $A <_{\forall\exists} D$ by definition of local counterexample.

In contrast to from the counterexamples of Lemma 8.4 there is no guarantee that a local counterexample D to A witnesses that each tail of A is not $(0, \omega)$-homogeneous. There are in fact two possibilities:

- $\forall d \in D\, \exists a \in A\, (d <_P a)$, i.e. $D <_{\forall\exists} A$
- $\exists d \in D\, \forall a \in A\, (d \not<_P a)$

In the first case D surely does not witness that each tail of A is not $(0, \omega)$-homogeneous, since each $d \in D$ is comparable with infinitely many points of A. In this case we say that the local counterexample D *interleaves* A.

In the second case it is possible to refine D in such a way that $\forall a \in A\, (d_0 \not<_P a)$. We always assume that this refinement is done and we actually call D the refined chain as well. Notice that no $d \in D$ is below any point in A. Hence, D witnesses that each tail of A is not $(0, \omega)$-homogeneous. For this reason we occasionally say that D is a real counterexample of A, besides saying that D *does not interleave* A.

Suppose $A \subseteq C_i$, for some poset $P = \bigcup_{i<k} C_i$, is an ascending chain such that its tails are not $(0, \omega)$-homogeneous. Our goal is to claim in RCA_0 that there exists a chain D which witness this. If Lemma 8.9 produces a local counterexample $D^1 \subseteq C_j$, for some $j < k$, which interleaves A, then D^1 is not useful for our purposes. However, this tell us that there must exists a local counterexample D^2 included in C_h for some $h \neq j$. Again it may be that D^2 interleaves A, but in this case we can look for another local counterexample which lives in a different chain of the decomposition, since both D^1 and D^2 do not witness that tails of A are not $(0, \omega)$-homogeneous. Iterating this procedure at most $k-1$ steps we find a local counterexample D not interleaving A.

Theorem 8.11 (RCA_0). *Let $(P, <_P)$ be a poset and $C_0, \ldots, C_{k-1}$ be chains that $P = \bigcup_{i<k} C_i$. Let $A \subseteq C_i$, for some $i < k$, be an ascending chain in P. If A and its tails are not $(0, \omega)$-homogeneous then there exists an ascending chain $D \subseteq C_j$, for some $j \neq i$, such that D is a local counterexample not interleaving A.*

Proof. Firstly we prove that if $D^1, \ldots, D^w$, for some $w < k$, are local counterexamples interleaving A and if there exists an injection $h\colon w + 1 \to k$ such that $D^j \subseteq C_{h(j)}$ for each $j \leq w$, then there exists a local counterexample D^{w+1} to A which is contained in D_e for some $e \notin \operatorname{ran}(h)$.

Fix an m and let $p \in P$ be such that $\exists r > m\, \forall s > r\, (p >_P a_r \wedge p \mid_P a_s)$. Such a p must exist because A_m is not $(0, \omega)$-homogeneous. If $p \in C_{h(j)}$ for some $j \leq n$, then either $p <_P D^j_n$, for some $n \in \mathbb{N}$, or $D^j <_P p$. In the former case, $p <_{\forall\exists} A$ since $D <_{\forall\exists} A$. In the latter case, $A <_P p$ since $A <_{\forall\exists} D$. Both cases contradict the choice of p. Thus the following formula is satisfied:

$$\forall m\, \exists n > m\, \exists p \in P\, (p >_P a_n \wedge p \mid_P a_{n+1} \wedge p \notin \textstyle\bigcup_{j \leq w} C_{h(j)})$$

Once the sequence S of points satisfying the formula is enumerated, it is possible to refine S to an ascending chain $D^{w+1} \subseteq C_e$, for some $e \notin \operatorname{ran}(h)$, as done in Lemma 8.9. By construction D^{w+1} is a local counterexample to A.

The previous lines describe a procedure to find new local counterexamples to A provided that those which have already been defined interleave A. We are left to prove that in at most $k-1$ steps the procedure finds a local counterexample not interleaving A. Suppose on the contrary that $w = k-1$ so that h is surjective onto k. We have just proved that, under these hypotheses, there exists a local counterexample D^k to A included in C_e, for some $e \in k \setminus \operatorname{ran}(h)$: this contradicts the surjectivity of h. $\square$

8.3.2 Local counterexamples to $\omega + \omega$ chains

Suppose that A and F are ω chains such that $A <_P F$. If each tail of the $\omega + \omega$ chain $A \cup F$ is not $(0, \omega)$-homogeneous, then there must be infinitely many points which are comparable with finitely many points in $A \cup F$ and incomparable with the remaining elements. Since $A \cup F$ is of order type $\omega + \omega$, such points must be above some $a_n \in A$ and incomparable with each point in $A_{n+1} \cup F$. As for ω chains, finding such D directly requires ACA_0. Hence we look for points above some $a_n \in A$ and incomparable with a_{n+1}. Thus Lemma 8.9 gives a procedure to find local counterexamples to $\omega + \omega$ chains as well. As discussed in the previous section if D is a local counterexample to $A \cup F$, then D may interleave A or not. However, that is not the notion of interleaving we are interested in for $\omega + \omega$ chains, since it gives no information about the comparability relation between elements of D and of F. The relevant possibilities in this case are the following two:

- $\forall d \in D\, (\exists a \in A\, (d <_P a) \vee \exists f \in F\, (d <_P f))$
- $\exists d \in D\, (\forall a \in A\, (d \not<_P a) \wedge \forall f \in F\, (d \not<_P f))$

If the first case holds, then D does not witness that each tail of $A \cup F$ is not a solution, because each d is comparable with infinitely many points of $A \cup F$. We say that D *interleaves* $A \cup F$. If the second case holds, then it is possible to refine D in such a way that d_0 witnesses that the formula is satisfied. We say that D *does not interleave* $A \cup F$.

Even if we use the same word 'interleaving' to indicate two slightly different properties for ω and $\omega + \omega$ chains, we believe the context always indicates which of the two notions we are mentioning.

As for ω chains, if each tail of $A \cup F$ is not $(0, \omega)$-homogeneous, we would like to find a local counterexample not interleaving $A \cup F$. The following is the analogous of Theorem 8.11.

Theorem 8.12. *Let $(P, <_P)$ be a poset and $C_0, \ldots, C_{k-1}$ be chains that $P = \bigcup_{i<k} C_i$. Let A and F be ascending chains in P such that $A <_P F$. If $A \cup F$ and its tails are not $(0, \omega)$-homogeneous, then there exists an ascending chain $D \subseteq C_j$, for some $j < k$, such that D is a local counterexample not interleaving $A \cup F$.*

Proof. The proof is a straightforward modification of Theorem 8.11. ☐

8.3.3 How to find a $(0, \omega)$-homogeneous chain

Let $(P, <_P)$ be a poset decomposable in k chains $C_0, \ldots, C_{k-1}$. Suppose that $A^0, \ldots, A^{k-1}$ are ascending chains each of which is included in a distinct decomposition chain. Then each $p \in P$ belong to C_j, for some $j < k$, and so p is comparable with all points in A^j. Moreover, since A^j is an ω chain, one of the following alternatives holds: either p is below a tail of A^j, i.e. $p <_{\forall\exists} A^j$, or it is above A^j, i.e. $A^j <_{\forall\exists} p$.

If one wants to guarantee that an $\omega + \omega$ chain $A \cup F$ is $(0, \omega)$-homogeneous, it is then enough to let $A <_{\forall\exists} A^j$ and $A^j <_{\forall\exists} F$ for each $j < k$. In fact, if $p <_{\forall\exists} A^j$, then $p <_{\forall\exists} F$ and hence $p <_{\forall\exists} A \cup F$. On the other hand, if $A^j <_{\forall\exists} p$, then $A <_{\forall\exists} p$. In both cases p is comparable with infinitely many elements of $A \cup F$ and so cannot be a counterexample to $A \cup F$ being $(0, \omega)$-homogeneous.

In a nutshell the proof of Theorem 8.13 shows that, under the hypothesis that each ω or $\omega + \omega$ chain B such that $A <_{\forall\exists} B$ is not $(0, \omega)$-homogeneous, it is possible to define F and $A^0, \ldots, A^{k-1}$ as in the previous paragraph.

8.4 Proof of $\mathsf{RSpo}^{\mathsf{CD}}_k$

Theorem 8.13. *For each k, ADS proves $\mathsf{RSpo}^{\mathsf{CD}}_k$.*

Proof. We will prove by induction on k that RCA_0 proves the following statement:

($\clubsuit_k$) *Let $(P, <_P)$ be a poset with chain-decomposition-number k. Then for each ascending (resp. descending) chain A there is a $(0, \omega)$-homogeneous chain B of order type ω or $\omega + \omega$ (resp. ω^* or $\omega^* + \omega^*$) such that $A <_{\forall\exists} B$ (resp. $\forall a \in A\, \exists b \in B\, (a \geq_P b)$).*

Using ADS it is straightforward to see that $\clubsuit_k$ implies $\mathsf{RSpo}^{\mathsf{CD}}_k$. In fact if $(P, <_P)$ is a poset with chain-decomposition-number k then by ADS there exists either an ascending chain or a descending chain in P and then $\clubsuit_k$ provides a $(0, \omega)$-homogeneous chain in P.

For $\clubsuit_1$ it suffices to let $B = A$. For the inductive step fix $k > 1$ and assume that $\clubsuit_{k-1}$ holds. Let $(P, <_P)$ be a poset with chain-decomposition-number k and fix chains $C_0, \dots, C_{k-1}$ such that $P = \bigcup_{i<k} C_i$.

Let $A = \langle a_n \mid n \in \mathbb{N} \rangle$ be an ascending chain. (The case where A is descending can be obtained by considering $(P, >_P)$.) By RT^1_k there is an $g < k$ such that infinitely many points of A are in C_g and we can assume that A itself is contained in C_g (because if $A <_{\forall\exists} B$ the same holds also for the original A).

We assume that each chain B of order type ω or $\omega + \omega$ such that $A <_{\forall\exists} B$ is not $(0, \omega)$-homogeneous in P. The proof describes a finite procedure which leads to a contradiction in at most $2k$ steps. At stage s we have a sequence $A^0, \dots, A^w$ of ascending chains included in distinct C_j's and an either empty or ascending chain F_s.

Before describing the procedure in its full generality we sketch the first few steps. At stage 0 we start with the ascending chain $A^0 = A$ and we set $F_0 = \emptyset$. At stage 1, since each tail of A is not $(0, \omega)$-homogeneous by assumption, by Theorem 8.11, let $A^1 \subseteq C_j$, for some $j \neq g$, be a local counterexample not interleaving A. Set $F_1 = F_0$. Since A^1 is an ascending chain such that $A <_{\forall\exists} A^1$ our assumption entails that each tail of A^1 is not $(0, \omega)$-homogeneous. By Theorem 8.11 there exists a local counterexample D not interleaving A^1 such that $D \subseteq C_h$, for some $h \neq j$. In Claim 8.13.2 we will show that either $h \neq g$ or D is above A. At stage 2, if $h \neq g$ we set $A^2 = D$ and $F_2 = F_1$, otherwise we set $F_2 = D$ and do not define A^2 yet. If A^2 has not been defined, at stage 3 we look for a local counterexample not interleaving $A \cup F_2$: this is A^2, while $F_3 = F_2$.

We now describe the main features of the procedure (the remainder of the proof is devoted to showing that this plan can be carried out).

Suppose that at stage s the ascending chains $A^0, \dots, A^w$, each satisfying $A <_{\forall\exists} A^j$ have been defined and that there exists an injection $h \colon w+1 \to k$ such that A^j is an ascending chain included in $C_{h(j)}$ for each $j \leq w$. Suppose also that F_s is either empty or an ascending chain above A and $F_s \subseteq C_{h(j)}$ for some $j \leq w$. Assume also that exactly one of the following facts is true: A^w has been defined at stage s, $F_s \neq F_{s-1}$.

At *stage* $s + 1$, we will make one of the following moves:

Move 1 define an ascending chain $A^{w+1} \subseteq C_i$, for some $i \notin \operatorname{ran}(h)$, local counterexample not interleaving the ω chain A^w or the $\omega + \omega$ chain $A^0 \cup F_s$; in this case $F_{s+1} = F_s$;

Move 2 define an ascending chain $F_{s+1} \subseteq C_{h(j)}$, for some $j \leq w$, such that $A^0 <_P F_{s+1}$, $F_s <_{\forall\exists} F_{s+1}$, and $A^w <_{\forall\exists} F_{s+1}$; in this case we do not define A^{w+1} yet.

To decide the move we carry out, we look at the previous move. If at stage s we made Move 1 let D be a local counterexample not interleaving A^w by Theorem 8.11 (recall that $A <_{\forall\exists} A^w$ and hence the tails of A^w are not $(0, \omega)$-homogeneous by assumption). We show in Claim 8.13.2 that D is either contained in C_i for some $i \notin \operatorname{ran}(h)$ or $A^0 <_P D$. If the former holds we apply Move 1 with $A^{w+1} = D$ (so that $h(w+1) = i$), otherwise we use D to define F_{s+1} and apply Move 2 (see Claim 8.13.3). If at stage s we made Move 2 let, by Theorem 8.12, D be a local counterexample not interleaving $A^0 \cup F_s$ (obviously $A <_{\forall\exists} A \cup F_s$ and hence the tails of $A \cup F_s$ are not $(0, \omega)$-homogeneous by assumption). We show in Claim 8.13.1 that in this case D is contained in C_i for some $i \notin \operatorname{ran}(h)$ and we apply Move 1 with $A^{w+1} = D$ (and set $h(w+1) = i$).

Notice that Move 2 is always followed by Move 1 (while the converse does not hold). This guarantees that in at most $2k - 1$ stages the function h becomes a bijection.

Claim 8.13.1. *If at stage s we make* Move 2*, i.e. $F_s \neq F_{s-1}$, then $A^j <_{\forall\exists} F_s$ for every $j \leq w$ and each local counterexample not interleaving $A \cup F_s$ is not included in $C_{h(j)}$ for every $j \leq w$, so that at stage $s+1$ we make* Move 1.

Proof. Let us suppose that the Claim holds for every stage $t < s$. Let $t < s$ be maximum such that Move 2 was played at stage t (if it exists). In this case let $i \leq w$ be such that A^i was defined at stage $t+1$, where, by induction hypothesis, we made Move 1. Otherwise let $i = 0$.

If $j \geq i$ then for every $i \leq n < w$ we have that A^{n+1} is a local counterexample to A^n and hence $A^n <_{\forall\exists} A^{n+1}$. Moreover by the properties of Move 2 we have $A^w <_{\forall\exists} F_s$. Hence, by transitivity of $<_{\forall\exists}$, $A^j <_{\forall\exists} F_s$. If $j < i$ (and hence t is defined) then by induction hypothesis $A^j <_{\forall\exists} F_t$. Since $F_t = F_{s-1} <_{\forall\exists} F_s$ by the properties of Move 2, $A^j <_{\forall\exists} F_s$.

To prove the second statement let D be a local counterexample not interleaving $A \cup F_s$. Suppose that there exists $p \in D \cap C_{h(j)}$ for some $j \leq w$. If $p <_P a^j$ for some $a^j \in A^j$, then p is below a tail of $A \cup F_s$ by the first part of the claim, which contradicts D not interleaving $A \cup F_s$. If instead $a^j <_P p$ for every $a^j \in A^j$ then p is above A because $A <_{\forall\exists} A^j$, contradicting the fact that D is a local counterexample to $A \cup F_s$. □

Claim 8.13.2. *If at stage s we made* Move 1 *defining A^w and D is a local counterexample not interleaving A^w such that $D \subseteq C_{h(j)}$ for some $j \leq w$, then $A <_P D$.*

Proof. It suffices to show that $A^j <_P D$. In fact, if $A^j <_P D$ then it is immediate to conclude that $A <_P D$ since we are assuming $A <_{\forall\exists} A^j$.

Towards a contradiction, assume that there exist $d \in D$ and $a^j \in A^j$ such that $a^j \not<_P d$. As $d, a^j \in C_{h(j)}$ they are comparable and so $d <_P a^j$. Let $a^w \in A^w$ be such that $a^w <_P d$ (such a point exists because D is a local counterexample to A^w). By transitivity it holds that $a^w <_P a^j$.

Let $t < s$ be maximum such that Move 2 was played at stage t (if it exists). In this case let $i \leq w$ be such that A^i was defined at stage $t+1$ as a local counterexample not interleaving $A \cup F_s$ (remember that $F_s = F_t$). Otherwise let $i = 0$. For every $i \leq n < w$ we have that A^{n+1} is a local counterexample to A^n and hence $A^n <_{\forall\exists} A^{n+1}$. Hence, by transitivity of $<_{\forall\exists}$, $A^n <_{\forall\exists} A^w$ for each $i \leq n < w$.

If $j \geq i$, let $a^{j+1} \in A^{j+1}$ be such that $a^{j+1} <_P a^w$. By transitivity it follows that $a^{j+1} <_P a^j$ contrary to the fact that A^{j+1} does not interleave A^j.

If $j < i$, let $a^i \in A^i$ be such that $a^i <_P a^w$. By transitivity it holds that $a^i <_P a^j$. Since $a^j \in A^j$ and $A^j <_{\forall\exists} F_s$ by Claim 8.13.1, a^i is below a tail of F_s, contrary to the assumption that A^i does not interleave $A \cup F_s$. □

Claim 8.13.3. *If at stage s we made* Move 1 *defining A^w and there exists a local counterexample D not interleaving A^w such that $D \subseteq C_{h(j)}$ for some $j \leq w$, then there exists an ascending chain $F_{s+1} \subseteq C_{h(j)}$, for some $j \leq w$, such that $A^0 <_P F_{s+1}$, $F_s <_{\forall\exists} F_{s+1}$, and $A^w <_{\forall\exists} F_{s+1}$, so that we can apply* Move 2.

Proof. Let $i \leq w$ be such that $F_s \subseteq C_{h(i)}$. Since we are assuming as induction hypothesis that $\clubsuit_{k-1}$ holds, there exists a chain S which satisfies the following properties: S is $(0, \omega)$-homogeneous in $\bigcup_{i \neq h(j)} C_i$, has order type ω or $\omega + \omega$ and $F_s <_{\forall\exists} S$.

Since $A <_P F_s$ we have $A <_{\forall\exists} S$ and so S is not $(0, \omega)$-homogeneous in P by assumption. Let E be a local counterexample to S. It must be that $E \subseteq C_{h(j)}$, because S is $(0, \omega)$-homogeneous in $\bigcup_{i \neq h(j)} C_i$.

If $D <_{\forall\exists} E$, let $e_n \in E$ be such that $d_0 <_P e_n$ and set $F_{s+1} = E_n$. Clearly $F_{s+1} \subseteq C_{h(j)}$. By Claim 8.13.2 we have $A <_P D$ and in particular d_0 is above A so that $d_0 <_P E_n$ implies $A <_P F_{s+1}$. Moreover, $A^w <_{\forall\exists} F_{s+1}$ because $A^w <_{\forall\exists} D$ (since D is a local counterexample to A^w) and by case hypothesis. Furthermore, $F_s <_{\forall\exists} F_{s+1}$ because $F_s <_{\forall\exists} S <_{\forall\exists} E$ by definition of S and because E is a local counterexample to S.

If instead $D \not<_{\forall\exists} E$, let $d_n \in D$ be such that $E <_P D_n$ (such a d_n exists because $E \cup D \subseteq C_{h(j)}$) and set $F_{s+1} = D_n$. Clearly $F_{s+1} \subseteq C_{h(j)}$ and, by Claim 8.13.2, $A <_P F_{s+1}$. In this case $A^w <_{\forall\exists} F_{s+1}$ follows immediately from $A^w <_{\forall\exists} D$. Finally, $F_s <_{\forall\exists} F_{s+1}$ because $F_s <_{\forall\exists} S <_{\forall\exists} E <_P D_n$. □

To conclude notice that at some stage $s < 2k$ the procedure defines $A^0, \ldots, A^{k-1}$ and F_s. By construction A^{k-1} is ascending and $A <_{\forall\exists} A^k$, so A^k is not $(0, \omega)$-homogeneous by hypothesis. Let D be a local counterexample not interleaving A^k. Claim 8.13.2 guarantees that $A <_P D$ since h is surjective. According to the rules at stage $s+1$ Move 2 is made and an ascending chain $F_{s+1} >_P A$ is defined. Now Claim 8.13.1 would define a chain A^k not included in $\bigcup_{j<k} C_{h(j)} = P$, which is clearly impossible. □

Notice that ADS is used only once at the very beginning of the proof. Indeed it allows to find an ascending or descending chain to start the procedure with.

Moreover, the assumption on the non existence of $(0, \omega)$-homogeneous chains is only an expositive device for the proof, which could be written more "constructively". In this case we think that the procedure stops at some stage $s \leq 2k$ finding a $(0, \omega)$-homogeneous chain B which is of order type ω or $\omega + \omega$ and such that $A <_{\forall\exists} B$. In particular B may be a tail of some A^w, if Move 1 defines A^w at stage s, or a tail of $A \cup F_s$ or even a $(0, \omega)$-homogeneous S for $(\bigcup_{i \neq j} C_i, <_P)$ for some $j < k$, found using $\clubsuit_{k-1}$ as in Claim 8.13.3.

The strategy of the previous proof is conceptually very different from the one employed in Theorems 8.3 and 8.7, which are on this respect more similar. Apart for the use of ζ in Theorem 8.7, each of the three proofs of $\mathsf{RSpo}_k^{\mathsf{CD}}$ is based on an iterative search of counterexamples, possibly local, to ascending or descending chains. The proofs consist on showing that this search eventually stops (even if each proof is written as a reductio ad absurdum we already comment on the fact that they also give a recipe to find a $(0, \omega)$-homogeneous chain). The difference between the three proofs, and in particular the difference between proofs of Theorems 8.3 and 8.7, on one hand, and of Theorem 8.8 on the other hand, stands in the reason why the procedure stops. To explain the difference let $(P, <_P)$ be a poset which does not contain ζ and such that $P = \bigcup_{i<k} C_i$ and C_i is a chain for each $i < k$. Both the proofs of Theorem 8.3 and of Theorem 8.7 cover C_i with some cofinal chain A_i, for each $i < k$, and exploit the fact that the iteration of the counterexample devise cannot provide a chain above A_i, for each $i < k$. On the contrary, in the proof Theorem 8.8 there is no guarantee that the ascending chain A^0 covers C_0 and we take advantage of this fact to define an $\omega + \omega$ chain which is $(0, \omega)$-homogeneous.

8.5 The strength of $\mathsf{RSpo}_{<\omega}^{\mathsf{CD}}$

The main effort of the previous proof is devoted to show that, for each $k \in \mathbb{N}$, RCA_0 proves that $\clubsuit_k$ holds and we do so by induction on k. Since $\clubsuit_k$ is computably true, it essentially states that for each $X \subseteq \mathbb{N}$, for each X-computable poset with chain-decomposition-number k and for each ascending chain X-computable, there exists an X-computable chain B with the desired properties.

At a first sight $\clubsuit_k$ is thus a Π^1_2-statement. However, thanks to the fact that $\clubsuit_k$ is computably true, we can rephrase it quantifying over indices of the programs enumerating the poset and the chains. Hence, $\clubsuit_k$ can be stated as: for each $X \subseteq \mathbb{N}$ and for each $e, i \in \mathbb{N}$ if φ_e^X is a poset with chain-decomposition-number k and φ_i^X is an ascending chain, then there exists $j \in \mathbb{N}$ such that φ_j^X is $(0, \omega)$-homogeneous, of order type ω or $\omega + \omega$ and $\varphi_i^X <_{\forall\exists} \varphi_j^X$.

Once each condition is carefully formalised one realises that the latter statement is a $\mathsf{I}\Sigma^0_5$-statement. This implies that $\mathsf{I}\Sigma^0_5$-induction proves that for each k $\clubsuit_k$ holds. Thanks to this observation we get the following corollary.

Corollary 8.14 (RCA_0)**.** ADS *plus arithmetical induction prove* $\mathsf{RSpo}_{<\omega}^{\mathsf{W}}$.

8.6 An equivalence with SADS

Theorem 8.8 settled the question about the strength of $\mathsf{RSpo}_k^{\mathsf{W}}$ and of $\mathsf{RSpo}_k^{\mathsf{CD}}$ for each $k \geq 3$. As happens with Ramsey theorem, $\mathsf{RSpo}_2^{\mathsf{W}}$ and $\mathsf{RSpo}_2^{\mathsf{CD}}$ are weaker principles, as shown by the following theorems.

The interesting feature of the proof of $\mathsf{RSpo}_2^{\mathsf{CD}}$ in SADS is the use of the absence of $(0,\omega)$-homogeneous chains to find an ascending or descending chain in a (non necessary stable) poset with chain-decomposition-number two.

Notation 8.15. Let $(P, <_P)$ be a poset and $p \in P$. $p\downarrow$ denotes the set $\{q \in P \mid q <_P p\}$. Similarly, $p\uparrow$ denotes the set $\{q \in P \mid q >_P p\}$.

Theorem 8.16 (RCA_0)**.** SADS *implies* $\mathsf{RSpo}_2^{\mathsf{CD}}$.

Proof. Let $(P, <_P)$ be a poset and C_0, C_1 chains such that $P = C_0 \cup C_1$. Assume that P does not contain $(0,\omega)$-homogeneous chains.

Firstly, we rule out the possibility that either C_0 or C_1 is finite. Suppose on the contrary that C_1 is finite and let $c_0 <_P \cdots <_P c_{k-1}$ be the elements of C_1. Define $L = \{x \in C_0 \mid x <_P c_{k-1}\}$. We claim that if $C_0 \setminus L$ is infinite, then it contains a $(0,\omega)$-homogeneous chain. We define the colouring $f\colon C_0 \setminus L \to k+1$ as follows[2]:

$$f(x) = \begin{cases} k & \text{if } c_{k-1} <_P x \\ i+1 & \text{if } c_i <_P x \wedge c_{i+1} \not<_P x \\ 0 & \text{otherwise} \end{cases}$$

Notice that $\operatorname{ran}(f)$ is bounded by k, so by RT^1_{k+1} there exists $j < k+1$ and $H \subseteq C_0 \setminus L$ such that H is homogeneous for j. Moreover, H is $(0,\omega)$-homogeneous because each $p \in C_0$ is comparable with H, $c_0, \dots, c_{j-1}$ are below H, and $c_j, \dots, c_{k-1}$ are incomparable with H.

Suppose now that $C_0 \setminus L$ is finite, so that L is infinite. Let us consider the colouring $g\colon L \to k$ defined as follows:

$$g(x) = \begin{cases} i & \text{if } x <_P c_i \wedge x \not<_P c_{i-1} \\ 0 & \text{otherwise} \end{cases}$$

By RT^1_k there exists $j < k$ and $H \subseteq L$ such that H is homogeneous for j. Notice that $c_j, \dots, c_{k-1}$ are above H, while c_i, for each $i < j$, can only be below some point of H and incomparable with some point of H. Lastly, we define the colouring $h\colon H \to j+1$ as follows:

$$h(x) = \begin{cases} i+1 & \text{if } c_i <_P x \wedge c_{i+1} \not<_P x \\ j & \text{if } c_{j-1} <_P x \\ 0 & \text{otherwise} \end{cases}$$

By RT^1_{j+1} there exists $i < j+1$ such that and $H' \subseteq L$ such that H' is homogeneous for i. Notice that $c_0, \dots, c_{i-1}$ are below H', $c_i, \dots, c_{j-1}$ are incomparable with H', and $c_j, \dots, c_{k-1}$ are above H' by the choice of j. H' is thus $(0,\omega)$-homogeneous contrary to the assumption[3].

[2] To be more precise $\operatorname{dom}(f)$ is an enumeration of $C_0 \setminus L$, we let it coincide with $C_0 \setminus L$ in order to simplify the notation. The same consideration holds for the functions g and h defined in the proof.

[3] Notice that in this paragraph we actually need $\mathsf{RT}^1_{<\omega}$ which is proved by SADS.

Suppose now both C_0 and C_1 are infinite and let $\langle p_n \mid n \in \mathbb{N}\rangle$ and $\langle q_n \mid n \in \mathbb{N}\rangle$ be an enumeration of C_0 and C_1 respectively. Define a colouring $c\colon [\mathbb{N}]^2 \to 4$ as follows:

$$c(n,m) = \begin{cases} 0 & \text{if } \forall i \leq m\,(p_n \mid_P q_i) \\ 1 & \text{if } \exists i\,(n < i \leq m \wedge p_n <_P q_i) \\ 2 & \text{if } \forall i\,(n < i \leq m \to p_n \not<_P q_i) \wedge \exists i\,(n < i \leq m \wedge p_n >_P q_i) \\ 3 & \text{otherwise} \end{cases}$$

Intuitively, c colours pairs $\langle p, q\rangle$, according to their comparability relation and through their indices, such that $p \in C_0$ and $q \in C_1$. Notice that, for each $m \in \mathbb{N}$, c changes colour at most twice. By 2-SRT^2_2 (see Lemma 7.20) there exists an infinite homogeneous set H for c. Thanks to H we define an ascending or descending chain in P.

We claim that H is not homogeneous for 0. Suppose on the contrary that it is and let $S = \langle p_h \mid h \in H\rangle$. Clearly each $p \in C_0$ is comparable with S, while each $q \in C_1$ is incomparable with S by the homogeneity of H. It follows that S is $(0,\omega)$-homogeneous contrary to the assumption.

Suppose now that H is 1-homogeneous and consider the set $A = \langle p_h \mid h \in H\rangle$. Let $f\colon \mathbb{N} \to \mathbb{N}$ be such that, for each $h \in H$, $f(h)$ is minimum such that $q_{f(h)}$ witnesses that $h \in H$ (i.e. $h < f(h)$ and $p_h <_P q_{f(h)}$). Notice that f is injective. In fact, suppose that $h < k$ and $h, k \in H$. Then by 1-homogeneity there exists $i < k$ such that $p_h <_P q_i$, so $f(h) < k$. Now consider $c(k, j)$, for some $j \in H$: by 1-homogeneity there exists $r > k$ such that $p_k <_P q_r$, so $f(k) > k > f(h)$.

If A is stable, then SADS finds an ascending or a descending chain in A and so in P. Otherwise, let $n \in H$ be such that $p_n{\downarrow}$ and $p_n{\uparrow}$ are both infinite. We claim that for each $h \in H$ such that $p_n \leq_P p_h$, it holds that $q_{f(h)}{\uparrow}$ is finite. Suppose this does not hold and let $q_{f(h)}{\uparrow}$ be infinite. Then $p_n{\downarrow} \cup q_{f(h)}{\uparrow}$ is a chain, it contains infinitely many elements in both C_0 and C_1 and is thus $(0,\omega)$-homogeneous, contrary to the assumption. It follows that the sequence $\langle q_{f(h)} \mid p_n <_P p_h\rangle$ can be refined to an ω^* chain.

If H is 2-homogeneous we can reason analogously, so we are left to the case of H 3-homogeneous. Notice that if $c(h,k) = 3$, for some $h, k \in H$, then there exists $i \leq h$ such that $p_h \nmid_P q_i$. There are two cases to consider: either there exists $n \in \mathbb{N}$ such that, for each $h \in H$, if $p_h \nmid_P q_i$, then $i \leq n$, or there are infinitely many $h \in H$ and infinitely many $i \in \mathbb{N}$ such that $p_h \nmid_P q_i$. If the former is the case, then for each $h \in H$ and for each $m > n$ it holds that $p_h \mid_P q_m$. So we can think as if $C_1 = \langle q_0, \dots, q_n\rangle$ and argue as in the first paragraphs to rule out this case too.

If the latter holds define $g\colon \mathbb{N} \to \mathbb{N}$ such that $g(h)$ is the minimum i such that $p_h \nmid_P q_i$, for each $h \in H$. By assumption there exists an infinite set $H' \subseteq \mathrm{dom}(g)$ such that $g \restriction H'$ is injective. Moreover, there are either infinitely many $h \in H'$ such that $p_h <_P q_{g(h)}$ or infinitely many $h \in H'$ such that $p_h >_P q_{g(h)})$. In both cases an argument analogous to the one for H 1-homogeneous provides an ascending or descending sequence.

Suppose P contains an ascending sequence. Then $\clubsuit_2$ guarantees that there exists a $(0,\omega)$-homogeneous chain. If P contains a descending sequence D, then $\clubsuit_2$ applied on $(P, >_P)$ and D guarantees that there exists a $(0,\omega)$-homogeneous chain. $\square$

It is easy to understand that the assumption on the non existence of $(0,\omega)$-homogeneous chains was only instrumental in the previous proof. Without this assumption one can either decode directly from the homogeneous set a $(0,\omega)$-homogeneous chain or argue as above to find an ascending or descending chain.

Theorem 8.17. *Over* RCA_0*,* SADS *is equivalent to* $\mathsf{RSpo}^{\mathsf{CD}}_2$.

Proof. We are left to prove the reversal. Let $(L, <_L)$ be an infinite stable poset. Consider $(L \times 2, <_P)$ with the product partial order (from $0 < 1$). Clearly, $L \times 2$ has chain-decomposition-number two. Let S be $(0,\omega)$-homogeneous and set $C_i = C \cap (L \times i)$ for each $i < 2$. By RT^1_2 at least one between C_0 and C_1 is infinite.

Suppose C_0 is infinite. If each $(c, 0) \in C_0$ has finitely many predecessors, then it is possible to enumerate computably an ω chain contained in S_0 and hence in L. Otherwise, we claim that there exists a descending chain in C_0. Let $(c, 0) \in C_0$ be such that c has finitely many successors and suppose that $(c', 0) \in C_0$ has finitely many predecessors. Notice also that C_1 must be finite, because $(c, 0)$ has only finitely many successors and $(c, 0) \mid_P (d, 1)$ for each $d <_L c$ by definition of $<_P$. Then $(c', 1)$ is above some and only finitely many elements of C, contrary to the fact that S is $(0, \omega)$-homogeneous. This proves that each element of C_0 has finitely many successors and so it is possible to enumerate computably an ω^* chain contained in C_0 and hence in L.

If C_1 is infinite and each element of C_1 has finitely many successors, then C_1 contains an ω^* chain. Otherwise, arguing as in the previous paragraph it is possible to show that C_1 contains an ω chain. □

Corollary 8.18. *Over* WKL_0*,* SADS *is equivalent to* $\mathsf{RSpo}_2^{\mathsf{W}}$.

Proof. Let $(P, <_P)$ be a poset of width two. By Dilworth's theorem let C_0 and C_1 be chains such that $P = C_0 \cup C_1$. By Theorem 8.16 P contains a $(0, \omega)$-homogeneous chain.

Since the partial order $(L \times 2, \prec)$ defined in the proof of Theorem 8.17 has width two, the same argument provides a reversal for $\mathsf{RSpo}_2^{\mathsf{W}}$ as well. □

Notice that the full strength of WKL_0 is used to prove the left to right implication of the previous corollary, since there exists a computable poset of width two which is not decomposable into two computable chains.

As a consequence of the previous theorem we get that $\mathsf{RSpo}_2^{\mathsf{W}}$ is strictly weaker than ADS, since ADS and $\mathsf{WKL}_0 + \mathsf{SADS}$ are incomparable, and not computably true. It is open whether $\mathsf{RSpo}_2^{\mathsf{W}}$ is equivalent to SADS over RCA_0 as well or whether it lies strictly in between SADS and ADS.

Question 8.19. Over RCA_0, is SADS equivalent to $\mathsf{RSpo}_2^{\mathsf{W}}$?

8.7 A reversal for the parallel version of $\mathsf{RSpo}_{<\omega}^{\mathsf{W}}$

To complete our analysis about the Rival-Sands theorem for posets we present a further result about the strength of the parallel version of the Rival-Sands theorem. In this case an instance of the problem is a sequence $\langle (P_n, <_n) \mid n \in \mathbb{N} \rangle$ of posets of finite width. The solution is thus a sequence of chains $\langle C_n \mid n \in \mathbb{N} \rangle$ such that C_n is $(0, \omega)$-homogeneous in $(P_n, <_n)$ for each $n \in \mathbb{N}$. The parallel $\mathsf{RSpo}_{<\omega}^{\mathsf{W}}$ implies ACA_0; it is in fact possible to arrange the posets in such a way that each $(0, \omega)$-homogeneous chain in P_n codes the information whether $n \in \operatorname{ran}(f)$ for some injective function $f\colon \mathbb{N} \to \mathbb{N}$.

Theorem 8.20 (RCA_0)**.** *The following statement implies* ACA_0*: let* $\{P_n \mid n \in \mathbb{N}\}$ *be a sequence of posets of width three. Then there exists a sequence* $\{D_n \mid n \in \mathbb{N}\}$ *such that, for each* $n \in \mathbb{N}$*,* D_n *a* $(0, \omega)$*-homogeneous chain in* P_n*.*

Proof. For each n we define a poset P_n of width three in such a way that each $(0, \omega)$-homogeneous chain for P_n allows to determine if n belongs to the range of an injective function $f\colon \mathbb{N} \to \mathbb{N}$. Each poset takes care about one possible value of f, so that having infinitely many of them allows to define $\operatorname{ran}(f)$.

Let $f\colon \mathbb{N} \to \mathbb{N}$ be an injective function and n an integer. For each n, P_n is the union of three ω chains $A_n = \{a_k^n \mid k \in \mathbb{N}\}, B_n = \{b_k^n \mid k \in \mathbb{N}\}, C_n = \{c_k^n \mid k \in \mathbb{N}\}$. The comparability relation $<_{P_n}$ is defined as follows:

$$\begin{cases} a_s^n <_{P_n} b_{s+1}^n \wedge c_s^n <_{P_n} b_{s+1}^n & \text{if } \forall m \leq s\,(f(m) \neq n) \\ b_s^n <_{P_n} a_{s+1}^n \wedge b_s^n <_{P_n} c_{s+1}^n & \text{if } \exists m < s\,(f(m) = n) \end{cases}$$

P_n is thus formed by three ω chains whose comparability relations is determined by n being or not in $\operatorname{ran}(f)$.

We claim that the following equivalence holds

$$\mathrm{ran}(f) = \{n \mid D_n \cap B_n = \emptyset\}$$

Notice that this equivalence, giving a Π^0_1-definition of $\mathrm{ran}(f)$, assures that the range of f is Δ^0_1 - definable in $\{D_n \mid n \in \mathbb{N}\}$.

In order to prove the claim, suppose $n \notin \mathrm{ran}(f)$. If $D_n \cap B = \emptyset$ then $D_n \subseteq A_n$ or $D_n \subseteq C_n$ because A_n and C_n are incomparable and D_n is a chain. Suppose the first possibility happens, namely $D_n \subseteq A_n$. Then each element in B_n is comparable with only finitely many elements of D_n by construction of the poset, contrary to the assumption that D_n is $(0, \omega)$-homogeneous.

For the right to left inclusion assume without loss of generality that $D_n \subseteq A_n$. If $n \notin \mathrm{ran}(f)$, then for each k there were an s such that $a^n_s <_{P_n} b^n_k \wedge \forall r > s(a^n_r \,|_{P_n}\, b^n_k)$ by definition of $<_{P_n}$. Hence for all all but finitely many k there is a t such that $d^n_t <_{P_n} b^n_k \wedge \forall v > t(a^n_v \,|_{P_n}\, b^n_k)$ because $D_n \subseteq A_n$, contrary to the assumption that D_n is $(0, \omega)$-homogeneous. □

8.8 A stronger Rival-Sands theorem

The proof of Theorem 8.3 shows that for each poset of finite width is possible to find a chain C of order type ω or ω^* which is $(0, \omega)$-homogeneous. Because of the specific order type of C one gets that actually each $p \in P$ is either incomparable with C or it is comparable with cofinitely many of elements of C. This represents a further improvement with respect to RSg, for which a solution is $(0, 1, \omega)$-homogeneous. Moreover, the latter formulation of Rival-Sands second theorem is a Ramsey-type principle, while this is not true for $\mathsf{RSpo}^{\mathsf{W}}_{<\omega}$.

Definition 8.21. Let $(P, <_P)$ be a poset. Then C is a $(0, \mathrm{cof})$-homogeneous chain for $(P, <_P)$ if each $p \in P$ is comparable to none of the elements of C or to cofinitely many of them.

$\mathsf{sRSpo}^{\mathsf{W}}_k$ *For each poset* $(P, <_P)$ *of width* k *there exists a* $(0, \mathit{cof})$*-homogeneous chain* C.

$\mathsf{sRSpo}^{\mathsf{CD}}_k$ *For each poset* $(P, <_P)$ *with chain-decomposition-number* k *there exists a* $(0, \mathit{cof})$*-homogeneous chain* C

As usual $\mathsf{sRSpo}^{\mathsf{W}}_{<\omega}$ denotes the uniform version of the former principle.

Theorem 8.7 and Theorem 8.8 do not prove $\mathsf{sRSpo}^{\mathsf{W}}_k$ nor $\mathsf{sRSpo}^{\mathsf{CD}}_k$. The proof of the former relies on the fact that a copy of $\mathbb{Z}$ is always $(0, \omega)$-homogeneous, even if not always $(0, \mathrm{cof})$-homogeneous. Despite this, if P does not contain a copy of $\mathbb{Z}$, then the proof of Theorem 8.7 provides an ascending or a descending chain and so a $(0, \mathrm{cof})$-homogeneous chain (notice that ω and ω^* chains are $(0, \omega)$-homogeneous if and only if are $(0, \mathrm{cof})$-homogeneous). On the other hand, Theorem 8.8, via Theorem 8.13, proves that for each poset of width k, for some $k \geq 3$, there always exists a $(0, \omega)$-homogeneous chain of order type ω or $\omega + \omega$ or ω^* or $\omega^* + \omega^*$. Even in this case there is no reason why a chain of order type $\omega + \omega$ or $\omega^* + \omega^*$ must also be $(0, \mathrm{cof})$-homogeneous (indeed the one exhibited in the proof of Theorem 8.8 are not $(0, \mathrm{cof})$-homogeneous).

The strength of $\mathsf{sRSpo}^{\mathsf{W}}_k$, for $k \in \mathbb{N}$, is less clear than the strength of $\mathsf{RSpo}^{\mathsf{W}}_k$. Theorem 8.3 shows that Π^1_1-CA_0 is an upper bound for $\mathsf{sRSpo}^{\mathsf{W}}_k$ as well, but it cannot be optimal. In this section proves that $\mathsf{sRSpo}^{\mathsf{CD}}_2$ is actually stronger that $\mathsf{RSpo}^{\mathsf{CD}}_2$ and settle the exact strength of $\mathsf{sRSpo}^{\mathsf{W}}_2$, contrary to the still unclear strength of $\mathsf{RSpo}^{\mathsf{W}}_2$.

Theorem 8.22 (RCA_0). ADS *is equivalent to* $\mathsf{sRSpo}^{\mathsf{W}}_2$ *and to* $\mathsf{sRSpo}^{\mathsf{CD}}_2$.

Proof. Theorem 8.24 shows that ADS implies $\mathsf{sRSpo}_2^{\mathsf{W}}$, which implies $\mathsf{sRSpo}_2^{\mathsf{CD}}$. Theorem 8.25 takes care of the reversals. □

The key step to obtain Theorem 8.24 is a refinement of Lemma 8.4 and Lemma 8.9 which exploits essentially the fact that $(P, <_P)$ has width (or chain-decomposition-number) two.

Lemma 8.23 (RCA_0)**.** *Let $(P, <_P)$ be a poset of width two and $A = \langle a_n \mid n \in \mathbb{N}\rangle$ be an ascending chain in P whose tails are not $(0, cof)$-homogeneous. Then there exists a function $g \colon \mathbb{N} \to \mathbb{N}$ which enumerates an ascending chain witnessing that each tail of A is not $(0, cof)$-homogeneous.*

Proof. Firstly, we prove that the formula $\varphi = \forall m\, \exists n > m\, \exists p \in P\, (a_n <_P p \wedge a_{n+1} \mid_P p)$ holds. Suppose it does not and so let m satisfies the following formula

$$\forall n > m\, \forall p \in P\, (a_n \not<_P p \vee a_{n+1} \nmid_P\ p)$$

Then for each $p \in P$ and for each $n > m$, either p is above A, or p is incomparable with A, or $p <_P a_r$ for some $r \in \mathbb{N}$. The chain A_{m+1} is thus $(0, \omega)$-homogeneous, contrary to the assumption.

Let $\pi_i \colon \mathbb{N}^2 \to \mathbb{N}$, for each $i < 2$, be the projection on the ith component of a pair. We define two functions $f, g \colon \mathbb{N} \to \mathbb{N}$ as follows:

$$\begin{aligned}
f(0) &= \pi_0(\min \langle n, p\rangle\, (a_n <_P p \wedge a_{n+1} \mid_P p)) \\
g(0) &= \pi_1(\min \langle n, p\rangle\, (a_n <_P p \wedge a_{n+1} \mid_P p)) \\
f(m+1) &= \pi_0(\min \langle n, p\rangle\, (n > f(m) \wedge a_n <_P p \wedge a_{n+1} \mid_P p)) \\
g(m+1) &= \pi_1(\min \langle n, p\rangle\, (n > f(m) \wedge a_n <_P p \wedge a_{n+1} \mid_P p))
\end{aligned}$$

The two functions enumerate the pairs $\langle n, p\rangle$, avoiding repetitions, satisfying φ. The previous paragraph thus guarantees that f and g are total functions and are well defined. A straightforward (Δ^0_1) induction shows that f is monotone (with respect to $\mathbb{N}$). Thus f enumerates a cofinal sequence in A. Moreover, we claim that for distinct $m, \ell \in \mathbb{N}$, it holds that $g(m) \neq g(\ell)$. Suppose on the contrary that $m < \ell$, so that $f(m) < f(\ell)$, and $g(m) = g(\ell)$. Since $a_{f(m)+1} \leq_P a_{f(\ell)} <_P g(\ell)$ holds by definition, then $a_{f(m)+1} <_P g(m)$, contrary to the definition of f and g. This claim guarantees that $\mathrm{ran}(g)$ is infinite.

The functions f and g allow to enumerate an infinite sequence of antichain $\langle \langle a_{f(m)+1}, g(m)\rangle \mid m \in \mathbb{N}\rangle$. Notice that, since $(P, <_P)$ has width two, for each $m \in \mathbb{N}$ and each $p \in P$ it holds either that $p \nmid_P a_{f(m)+1}$ or that $p \nmid_P g(m)$.

We claim that there exists $M \in \mathbb{N}$ such that for each $m > M$ it holds that $\forall i > f(m)\, (a_i \mid_P g(m))$. Suppose on the contrary that no such M exists. It follows that for each $g(m)$ there exists $i > f(m)$ such that $g(m) <_P a_i$ holds (notice that $a_i <_P g(m)$ cannot holds because it implies $a_{f(m)+1} <_P g(m)$). Let $p \in P$. If there exists m such that $p <_P a_{f(m)+1}$ or $p <_P g(m)$, then p is below a tail of A. Otherwise, p is either above $a_{f(m)}$ or above $g(m)$, for each $m \in \mathbb{N}$. In both cases p is above A. This argument shows that A is $(0, \mathrm{cof})$-homogeneous, contrary to the assumption.

We also claim that for almost all m and ℓ (namely for all $m, \ell > M$ for M satisfying the previous claim), if $m < \ell$, then $g(m) <_P g(\ell)$ holds. Let $m < \ell$, so that $f(m) < f(\ell)$. If it holds that $g(m) \mid_P g(\ell)$, then $\langle g(m), g(\ell), a_{f(\ell)+1}\rangle$ is an antichain of size three, contrary to the assumption that P has width two (remember that $g(\ell) \mid_P a_{f(\ell)+1}$ by definition and $a_{f(\ell)+1} \mid_P g(m)$ by the previous claim). If $g(\ell) <_P g(m)$ holds, then $a_{f(m)+1} \leq_P a_{f(\ell)} <_P g(\ell)$ and transitivity imply $a_{f(m)+1} <_P g(m)$ contrary to the definition of f and g. Thus, given that $g(m) \neq g(\ell)$, $g(m) <_P g(\ell)$ holds as we wanted to show.

The previous claims guarantee that g enumerates an ascending chain with the desired properties. □

Notice that the functions f and g defined in the previous proof need to inspect only an initial segment of A to give an output. Moreover, the proof of the previous lemma shows that there exists a uniform procedure to associate to each A, ascending chain whose tails are not $(0, \mathrm{cof})$-homogeneous, an ascending chain which is a counterexample to A.

It is also easy to observe that the previous lemma and related observations holds for descending chains with the obvious changes. These observations are crucial for the proof of next theorem.

Theorem 8.24 (RCA_0)**.** ADS *implies* $\mathsf{sRSpo}_2^{\mathsf{W}}$.

Proof. Let $(P, <_P)$ be a poset of width two. By Proposition 7.17 $(P, <_P)$ contains either an ascending or a descending chain A. Suppose that $A = \langle a_n \mid n \in \mathbb{N}\rangle$ is ascending (the case where A is descending can be obtained by considering $(P, >_P)$). Suppose also that P does not contain $(0, \mathrm{cof})$-homogeneous chains.

We define recursively a function $g\colon \mathbb{N}^2 \to \mathbb{N}$ which enumerates a sequence of chains, starting from A, each of which is a counterexample to the previous one. To do so we also define an auxiliary function $f\colon \mathbb{N}^2 \to \mathbb{N}$ which enumerates indices used as arguments for g. Let $\pi_i\colon \mathbb{N}^2 \to \mathbb{N}$ be the projection on the ith component of a pair. We define $f, g\colon \mathbb{N}^2 \to \mathbb{N}$ as follows:

$$\begin{aligned}
f(0, m) &= m\\
g(0, m) &= a_m\\
f(i+1, 0) &= \pi_0(\min \langle n, p\rangle\, (g(i, n) <_P p \wedge g(i, n+1) \mid_P p))\\
g(i+1, 0) &= \pi_1(\min \langle n, p\rangle\, (g(i, n) <_P p \wedge g(i, n+1) \mid_P p))\\
f(i+1, m+1) &= \pi_0(\min \langle n, p\rangle\, (n > f(i+1, m) \wedge g(i, n) <_P p \wedge g(i, n+1) \mid_P p))\\
g(i+1, m+1) &= \pi_1(\min \langle n, p\rangle\, (n > f(i+1, m) \wedge g(i, n) <_P p \wedge g(i, n+1) \mid_P p))
\end{aligned}$$

Let $g_i(m) = g(i, m)$ and $g_i = \bigcup_{m \in \mathbb{N}} g(i, m)$. Notice that to calculate $f_{i+1}(m)$ and $g_{i+1}(m)$, for some $i, m \in \mathbb{N}$, one needs to calculate f_{i+1} and g_{i+1} up to $m-1$ and f_i and g_i up to $n+1$ for some n which depends on $f(i+1, m-1)$.

By definition, g_0 enumerates the ascending chain A. We claim that, for each $i \in \mathbb{N}$, g_{i+1} is an ascending chain and witnesses that each tail of g_i is not $(0, \mathrm{cof})$-homogeneous. Suppose the claim is true for i. Since $(P, <_P)$ does not contain $(0, \mathrm{cof})$-homogeneous chains, each tail of g_i is not $(0, \mathrm{cof})$-homogeneous. Hence, by Lemma 8.23 there exists a function g which enumerates an ascending chain witnessing that each tail of g_i is not $(0, \mathrm{cof})$-homogeneous. It is immediate to see that g_{i+1} coincides with the function g defined in the proof of the lemma.

The previous claim implies also that f and g are total functions (notice that here the assumption on $(P, <_P)$ not containing $(0, \mathrm{cof})$-homogeneous chains is crucial).

We define an infinite ascending sequence $S = \langle s_n \mid n \in \mathbb{N}\rangle$ inductively. For the base case let $s_0 = g_0(0)$. Suppose that $s_0 <_P \cdots <_P s_{2i}$ have been defined and $s_{2i} = g_i(k)$ for some $k \in \mathbb{N}$. By definition of g_i there exist least n and $u \leq k$ such that $g_i(k) <_P g_i(n)$ and $n = f_{i+1}(u)$. Let $s_{2i+1} = s_{2(i+1)-1} = g_{i+1}(u)$ and $s_{2i+2} = s_{2(i+1)} = g_{i+1}(u+1)$. Notice that by definition both s_{2i+1} and s_{2i+2} are elements of g_{i+1} and that $s_{2i} <_P s_{2i+1} <_P s_{2i+2}$.

We claim that S is a $(0, \mathrm{cof})$-homogeneous chain. Let $p \in P$, assume p is not above S and so let $m \in \mathbb{N}$, be such that $p \not>_P s_i$ for each $i \geq 2m$. We argue that this entails the existence of $j \geq 2m$ such that $p <_P s_j$. Assume towards a contradiction that $p \mid_P s_j$ for each $j \geq 2m$ ($p >_P s_j$ cannot occur by assumption). Since $s_{2m+1} = g_{m+1}(k)$ for some $k \in \mathbb{N}$, it holds that $g_{m+1}(k) \mid_P g_m(f_{m+1}(k)+1)$. Since $\langle p, g_{m+1}(k), g_m(f_{m+1}(k)+1)\rangle$ cannot be an antichain it holds that $p \nmid_P g_m(f_{m+1}(k)+1)$. In particular one of the following holds:

1. $p >_P g_m(f_{m+1}(k)+1)$. Notice that $g_m(f_{m+1}(k)) >_P s_{2m}$. In fact, it holds that $g_m(f_{m+1}(k)) \nmid_P s_{2m}$ because the both live in $\mathrm{ran}(g_m)$. Moreover, it holds that $s_{2m+1} >_P s_{2m}$, since S is ascending. So $s_{2m} >_P g_m(f_{m+1}(k))$ would imply $s_{2m+1} >_P g_m(f_{m+1}(k))$, contrary to the assumption. Thus by transitivity we get that $p >_P s_{2m}$, contrary to the assumption;

2. $p <_P g_m(f_{m+1}(k)+1)$. Since $s_{2m+1} = g_{m+1}(k)$, then $s_{2m+2} = g_{m+1}(k+1)$ and so $p <_P s_{2m+2}$, contrary to the assumption.

We conclude that, for each $p \in P$, either $p >_P S$ or p is below a tail of S. Hence, S is $(0, \mathrm{cof})$-homogeneous contrary to the assumption that $(P, <_P)$ has no $(0, \mathrm{cof})$-homogeneous chains. □

Notice that ADS is used only once at the very beginning of the proof. Indeed it allows us to find an ascending or descending chain with which to start the procedure.

The assumption on the non existence of a $(0, \mathrm{cof})$-homogeneous chain in $(P, <_P)$ is only instrumental in the proof. If one drops it, it is possible that the functions f and g are no more total functions because a tail of g_i is $(0, \mathrm{cof})$-homogeneous, for some $i \in \mathbb{N}$. In this case g_{i+1} would be undefined from some $m \in \mathbb{N}$ on and g_i would be $(0, \mathrm{cof})$-homogeneous.

There are some obstacles to the generalisation of the previous proof to posets of greater width. Suppose $(P, <_P)$ is a poset such that $P = \bigcup_{i<3} C_i$ for some chain C_0, C_1, C_2 (it is more convenient to speak about chain-decomposition-number at the moment). Assume as usual that P does not contain $(0, \mathrm{cof})$-homogeneous chains and let $A \subseteq C_0$ be ascending. A first problem lay in the search of a counterexample. If one defines the functions f and g as in Lemma 8.23, then there may exist n and m such that $g(n) \mid_P g(m)$, and so g may not be a chain. It is still possible to show that if $g(n) \nmid_P g(m)$ and $n < m$, then $g(n) <_P g(m)$. Thus $\mathrm{ran}(g)$ contains an ascending chain $D = \langle d_n \mid n \in \mathbb{N} \rangle$. However, initial segment of A cannot determine any more initial segments of D, indeed one needs to get $\mathrm{ran}(g)$ in order to apply RT^1_2 and so get D. It is thus hard to imagine that there exists an uniform procedure to produce a local counterexample (notice that it no more true that every local counterexample is a counterexample) to ascending chains whose tails are not solutions. If one wants to imitate the iteration of search of local counterexamples in posets of chain-decomposition-number greater than two, then one needs to determine $\mathrm{ran}(g_0)$ and then a chain $D_0 \subseteq \mathrm{ran}(g_0)$. Once D_0 is defined, D_1 can be obtained in the same way and so on. To conclude ACA_0 seems to be needed to define the sequence $\langle D_n \mid n \in \mathbb{N} \rangle$.

The second peculiarity of posets of chain-decomposition-number two is the fact that after at most ω iterations of search of local counterexamples a $(0, \mathrm{cof})$-homogeneous chain is surely found, as proved in Theorem 8.24. It is not hard to see that this does not hold for posets with chain-decomposition-number equal or grater than three. For example suppose that the chains C_0, C_1, C_2 decompose a poset which does not contain $(0, \mathrm{cof})$-homogeneous chains and that each (local) counterexample, found with the usual iteration, lives in $C_0 \cup C_1$. Then an ω chain defined taking two points from each local counterexample, as S in the proof of Theorem 8.24, may still have a counterexample in C_2 and hence not being $(0, \mathrm{cof})$-homogeneous.

Thanks to the previous considerations we imagine that a $(0, \mathrm{cof})$-homogeneous chain can be obtained, in poset with chain-decomposition-number greater than two, iterating the search of counterexample transfinitely many steps, even probably a number of steps bounded by the chain-decomposition-number of the poset. Even if this idea would finally convert into a proof of $\mathsf{sRSpo}^{\mathsf{CD}}_k$ it would surely be a proof in a system stronger than ACA_0, while the unique lower bound for $\mathsf{sRSpo}^{\mathsf{CD}}_k$ is the following.

Theorem 8.25 (RCA_0). $\mathsf{sRSpo}^{\mathsf{W}}_2$ *and* $\mathsf{sRSpo}^{\mathsf{CD}}_2$ *imply* ADS.

Proof. Let $(L, \leq_L)$ be a linear order and let $P = (L \times 2, \leq_P)$ the order on the Cartesian product of L, so that $(\ell, i) \leq_P (m, j) \Leftrightarrow \ell \leq_L m \wedge i \leq j$. Such a poset has clearly width and chain-decomposition-number two, so let $C \subseteq P$ be $(0, \mathrm{cof})$-homogeneous. For each $i < 2$ set $C_i = C \cap (L \times i)$.

We claim that if C_0 is infinite, then C_0 has no minimum, and can thus be refined to a descending chain. Suppose on the contrary that C_0 is infinite and that $(m, 0)$ is minimum in C_0. By definition of $<_P$ it holds that $(m, 0) <_P (m, 1)$ and $(n, 0) \mid_P (m, 1)$, for each $n >_L m$. It follows that $(m, 1)$ is incomparable with infinitely many elements of C, contrary to the assumption that C is $(0, \mathrm{cof})$-homogeneous.

Similar reasoning allows us to prove that if C_1 is infinite, then C_1 has no maximum, and hence that L contains an ascending chain. □

Question 8.26. What is the strength of $\mathsf{sRSpo}^{\mathsf{W}}_k$, for each $k \in \mathbb{N}$?

It follows from Theorem 8.17 that $(0, \mathrm{cof})$-homogeneous chains of a poset of width two (or chain-decomposition-number two) codes ascending or descending chains of a linear order. There are very simple examples which witness this fact. Consider for example the linear orders L_0 or order type $\omega + \omega$ and L_1 or order type $\omega^* + \omega^*$, and let P_0 and P_1, respectively, be defined as in the previous proof. For each $i < 2$, any chain such that $|C \cap (L_i, 0)| = |C \cap (L_i, 0)| = \omega$ is $(0, \omega)$-homogeneous (this is actually a trick used in the proof of Theorem 8.16), so it does distinguish between ascending and descending chains.

Bibliography

Aigner, Martin and Geert Prins (1971), "Uniquely partially orderable graphs", *Journal of London Mathematical Society* 3, pp. 260–266.

Bazhenov, Nikolay, Rod Downey, Iskander Kalimullin, and Alexander Melnikov (2019), "Foundations of online structure theory", *The Bulletin of Symbolic Logic* 25.2, pp. 141–181.

Berge, Claude (1976), *Graphs and Hypergraphs*, second edition, North-Holland.

Bosek, Bartłomiej, Kamil Kloch, Tomasz Krawczyk, and Piotr Micek (2012), "On-line version of Rabinovitch theorem for proper intervals", *Discrete Mathematics* 312.23, pp. 3426–3436.

Bosek, Bartłomiej, Tomasz Krawczyk, Henry Kierstead, Grzegorz Matecki, and Matthew E. Smith (2018), "An easy subexponential bound for online chain partitioning", *Electronic Journal of Combinatorics* 25.2.

Brattka, Vasco, Guido Gherardi, and Arno Pauly (2017), *Weihrauch Complexity in Computable Analysis*, To appear in *Handbook of Computability and Complexity in Analysis*. Available at http://arxiv.org/abs/1707.03202.

Cenzer, Douglas and Jeffrey B. Remmel (2005), "Proof-theoretic strength of the stable marriage theorem and other problems", *Reverse Mathematics 2001*, Association for Symbolic Logic, pp. 67–103.

Cholak, Peter, Alberto Marcone, and Reed Solomon (2004), "Reverse mathematics and the equivalence of definitions for well and better quasi-orders", *Journal of Symbolic Logic* 69.3, pp. 683–712.

Cholak, Peter A., Carl G. Jockusch, and Theodore A. Slaman (2001), "On the strength of Ramsey's theorem for pairs", *The Journal of Symbolic Logic* 66.1, pp. 1–55.

Cormen, Thomas H., Charles E. Leiserson, Ronald L. Rivest, and Clifford Stein (2009), *Introduction to Algorithms*, third edition, MIT Press.

Dean, Walter and Sean Walsh (2017), "The prehistory of the subsystems of second-order arithmetic", *Review of Symbolic Logic* 10.2, 357–396.

Diestel, Reinhard (2017), *Graph Theory*, fifth edition, Springer.

Dilworth, Robert P. (1950), "A decomposition theorem for partially ordered sets", *Annals of Mathematics (2)* 51, pp. 161–166.

Downey, Rodney G. (1998), “Computability theory and linear orderings”, *Handbook of Recursive Mathematics, Vol. 2*, North-Holland, pp. 823–976.

Dushnik, Ben and Edward W. Miller (1941), “Partially ordered sets”, *American Journal of Mathematics* 63, pp. 600–610.

Dzhafarov, Damir D. (2011), “Infinite saturated orders”, *Order* 28.2, pp. 163–172.

Eastaugh, Benedict (2018), *Computational reverse mathematics and foundational analysis*, Available at http://philsci-archive.pitt.edu/14825/.

— (2019), “Set existence principles and closure conditions: unravelling the standard view of reverse mathematics”, *Philosophia Mathematica (3)* 27.2, pp. 153–176.

Fishburn, Peter C. (1985), *Interval Orders and Interval Graphs*, Wiley, pp. xi+215.

Friedman, Harvey (1975), “Some systems of second order arithmetic and their use”, *Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1*, pp. 235–242.

Frittaion, Emanuele and Alberto Marcone (2014), “Reverse mathematics and initial intervals”, *Annals of Pure and Applied Logic* 165, pp. 858–879.

Frittaion, Emanuele, Alberto Marcone, Matthew Hendtlass, Paul Shafer, and Jeroen Van der Meeren (2016), “Reverse mathematics, well-quasi-orders, and Noetherian spaces”, *Archives for Mathematical Logic* 55, pp. 431–459.

Gallai, Tibor (1967), “Transitiv orientierbare Graphen”, *Acta Mathematica. Academiae Scientiarum Hungaricae* 18, English translation in Maffray and Preissmann 2001, pp. 25–66.

Gasarch, William (1998), “A survey of recursive combinatorics”, *Handbook of recursive mathematics, Vol. 2*, North-Holland, Amsterdam.

Gavalec, Martin and Peter Vojtáš (1980), “Remarks to a modification of Ramsey-type theorems”, *Commentationes Mathematicae Universitatis Carolinae* 21.4, pp. 727–738.

Ghouila-Houri, Alain (1962), “Caractérisation des graphes non orientés dont on peut orienter les arêtes de manière à obtenir le graphe d’une relation d’ordre”, *Les Comptes Rendus de l’Académie des Sciences* 254, pp. 1370–1371.

— (1964), “Flots et tensions dans un graphe”, *Annales Scientifiques de l’École Normale Supérieure. Troisième Série* 81, pp. 267–339.

Gilmore, Paul C. and Alan J. Hoffman (1964), “A characterization of comparability graphs and of interval graphs”, *Canadian Journal of Mathematics* 16, pp. 539–548.

Golumbic, Martin C. (2004), *Algorithmic Graph Theory and Perfect Graphs*, second edition, Elsevier.

Hanlon, Philip (1982), “Counting interval graphs”, *Transactions of the American Mathematical Society* 272.2, pp. 383–426.

Harzheim, Egbert (2005), *Ordered Sets*, Springer.

Hirschfeldt, Denis R. (2015), *Slicing the Truth*, World Scientific.

Hirst, Jeffry L. (1987), "Combinatorics in Subsystems of Second Order Arithmetic", PhD thesis, The Pennsylvania State University.

— (1990), "Marriage theorems and reverse mathematics", *Logic and Computation* 106, pp. 181–196.

— (1992), "Connected components of graphs and reverse mathematics", *Archive for Mathematical Logic* 31.3, pp. 183–192.

Hirst, Jeffry L. and Noah A. Hughes (2015), "Reverse mathematics and marriage problems with unique solutions", *Archive for Mathematical Logic* 54.1-2, pp. 49–57.

— (2016), "Reverse mathematics and marriage problems with finitely many solutions", *Archive for Mathematical Logic* 55.7-8, pp. 1015–1024.

Kelly, David (1985), "Comparability Graphs", in Rival 1985, pp. 3–40.

Kierstead, Henry A. (1981), "An effective version of Dilworth's theorem", *Transactions of the American Mathematical Society* 268.1, pp. 63–77.

— (1986), "Recursive ordered sets", *Combinatorics and ordered sets (Arcata, California, 1985)*, American Mathematical Society, pp. 75–102.

Kierstead, Henry A. and William T. Trotter (1981), "An extremal problem in recursive combinatorics", *Congressus Numerantium* 33.143–153, p. 98.

Lekkerkerker, Cornelis J. and Johan C. Boland (1962), "Representation of a finite graph by a set of intervals on the real line", *Fundamenta Mathematicae* 51, pp. 45–64.

Maffray, Frédéric and Myriam Preissmann (2001), "A translation of T. Gallai's paper: "Transitiv orientierbare Graphen"", *Perfect Graphs*, Wiley, pp. 25–66.

Marcone, Alberto (1996), "On the logical strength of Nash-Williams' theorem on transfinite sequences", *Logic: from foundations to applications*, Oxford University Press, pp. 327–351.

— (2007), "Interval orders and reverse mathematics", *Notre Dame Journal of Formal Logic* 48, pp. 425–448.

Paris, Jeff B. and Laurie A. S. Kirby (1978), "Σ_n-collection schemas in arithmetic", *Logic Colloquium '77*, vol. 96, North-Holland, pp. 199–209.

Rabinovitch, Issie (1978a), "An upper bound on the "Dimension of interval orders"", *Journal of Combinatorial Theory (A)* 25.1, pp. 68–71.

— (1978b), "The dimension of semiorders", *Journal of Combinatorial Theory (A)* 25.1, pp. 50–61.

Rado, Richard (1948), “Covering theorems for ordered sets”, *Proceedings of the London Mathematical Society* 2.1, pp. 509–535.

Rival, Ivan, ed. (1985), *Graphs and Order. The role of graphs in the theory of ordered sets and its applications*, D. Reidel.

Rival, Ivan and Bill Sands (1980), “On the Adjacency of Vertices to the Vertices of an Infinite Subgraph”, *Journal of the London Mathematical Society* 2, pp. 393–400.

Roberts, Fred S. (1969), “Indifference graphs”, *Proof Techniques in Graph Theory (Proceedings Second Ann Arbor Graph Theory Conference*, ed. by F. Harary, Academic Press, pp. 139–146.

— (1971), “On the compatibility between a graph and a simple order”, *Journal of Combinatorial Theory Ser. B* 11, pp. 28–38.

Schmerl, James H. (2005), “Reverse mathematics and graph coloring: eliminating diagonalization”, *Reverse Mathematics 2001*, ed. by S. Simpson, Association of Symbolic Logic, pp. 331–348.

Shafer, Paul (2012), “Menger’s theorem in Π^1_1-CA_0”, *Archive for Mathematical Logic* 51.3-4, pp. 407–423.

Shore, Richard A. (2010), “Reverse mathematics: the playground of logic”, *Bulletin of Symbolic Logic* 16.3, pp. 378–402.

Simpson, Stephen G. (1988), “Partial realizations of Hilbert’s Program”, *Journal of Symbolic Logic* 53.2, pp. 349–363.

— (2009), *Subsystems of Second Order Arithmetic*, second edition, Cambridge University Press, pp. xvi+444.

— (2014a), “An objective justification for actual infinity?”, *Infinity and Truth*, World Scientific, pp. 157–169.

— (2014b), “Toward objectivity in mathematics”, *Infinity and Truth*, vol. 25, World Scientific, pp. 157–169.

Soare, Robert I. (2016), *Turing Computability*, Springer.

Towsner, Henry (2013), “Partial impredicativity in reverse mathematics”, *Journal of Symbolic Logic* 78, pp. 459–488.

Trotter, William T. (1997), “New perspectives on interval orders and interval graphs”, *Surveys in combinatorics, 1997 (London)*, vol. 241, Cambridge University Press, pp. 237–286.

— (2001), *Combinatorics and Partially Ordered Sets: Dimension Theory*, JHU Press.

Wiener, Norbert (1914), “A contribution to the theory of relative position”, *Proceedings of the Cambridge Philosophical Society* 17, pp. 441–449.