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A Novel Fuzzy *c*-Means Clustering Algorithm Using Adaptive Norm

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Abstract The fuzzy *c*-means (FCM) clustering algorithm is an unsupervised learning method that has been widely applied to cluster unlabeled data automatically instead of artificially, but is sensitive to noisy observations due to its inappropriate treatment of noise in the data. In this paper, a novel method considering noise intelligently based on the existing FCM approach, called adaptive-FCM and its extended version (adaptive-REFCM) in combination with relative entropy, are proposed. Adaptive-FCM, relying on an inventive integration of the adaptive norm, benefits from a robust overall structure. Adaptive-REFCM further integrates the properties of the relative entropy and normalized distance to preserve the global details of the dataset. Several experiments are carried out, including noisy or noise-free University of California Irvine (UCI) clustering and image segmentation experiments. The results show that adaptive-REFCM exhibits better noise robustness and adaptive adjustment in comparison with relevant state-of-the-art FCM methods.

Keywords Fuzzy *c*-means clustering · Adaptive norm · Noise robustness · Relative entropy

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1 Introduction

Clustering is a significant and promising method to uncover the structure of a given dataset by pattern recognition. To capture the overall structure of the data, specific hypotheses including linear and nonlinear embedding constraints have been suggested, such as *k*-means [1], fuzzy logic [2, 3], etc.

Hard clustering techniques, such as *k*-means, stick to the rigid principle that an observation strictly belongs to one specific cluster, which means that an observation will not interact with other clusters at all. As a result, the potential distribution of a dataset is not well reflected by such hard clustering methods. Later, Zadeh pioneered the concept of fuzzy sets, with a tangible definition [4], to interpret the potential distribution. Further clustering methods inspired by fuzzy logic have been presented, including the class of isodata clustering algorithms which can effectively detect compact structures [5] and for unsupervised clustering of datasets into a given number of classes [2]. Among clustering algorithms, fuzzy clustering techniques have benefited from successive extensions [6–9] and have been widely used, e.g., for cloud intrusion detection [10], color image segmentation [11], brain segmentation enhancement [12], etc.

Inspired by such fuzzy logic and hard clustering methods, the FCM approach considers the relative relationship between all the observations in order to make fuzzier judgements. The FCM approach allows an observation to belong to different clusters, offering greater flexibility to handle the uncertainties found in real-word datasets. Specifically, the FCM approach provides a reasonable representation of clustering probabilities, known as the “membership degree.” In comparison with hard clustering algorithms, the FCM approach removes the “all or none” restriction, making the boundaries much fuzzier, which is more in line with real-world situations and facilitates outlier detection. Researchers have advanced the study of FCM and related approaches in recent decades using approaches such as kernel fuzzy clustering, algorithms based of weighted methods [e.g., new weighted fuzzy *c*-means (NWFCM) [13] and fuzzy clustering with the entropy of attribute weights (EWFCM) [14]], sparse representation-based methods [e.g., fuzzy double *c*-means (FDCM-SSR) [6]], etc.

Although the FCM approach has been refined in many ways, data processing remains a thorny issue. To the best of the authors' knowledge, noise can vary between different events. For clustering tasks, how to define noise and how to avoid the impact of noise are two tough problems, and increased noise can make the prediction process difficult. Therefore, the subject of noise rejection is still of great importance.

To further improve the robustness of such methods to noise, great efforts have been made by researchers from many directions; For instance, spectral subtraction can extract the interesting features of the implied error resulting from noise estimation in the power-spectral domain. However, in FCM, noise is treated in an intuitive way. In ordinary fuzzy logic, a point is identified as noise with an extremely low membership degree. However, this approach does not offer perfect noise robustness, so different forms of regularization have been adopted to improve this aspect of its performance. This problem is discussed below based on various approaches to noise robustness, after the discussion of the following two essential questions:

- There is no reasonable standard for the evaluation of noise, so how can one define noise universally?
- How can one address noise compatibly when applying the FCM approach?

In fact, noise is indeed undefinable in real-world datasets because noise can behave in different ways, resulting in unmeasurable uncertainty. Mathematically, of all types of noise, the outlier is the most typical and definable form. Outliers generally have abnormal features in contrast to some cluster and fall into the category that one would like to describe as UNCLASSIFIABLE [3]. Practically speaking, to obviate the undesirable effects of such outliers, outlier detection methods are applied to try to reveal their contrasting features based on statistical methods [15].

Various methods have been applied in this regard for FCM in recent years. To eliminate noisy features in high-dimensional data, Chang et al. proposed $L_{1/2}$-CM, which introduces L_q-norm $(0<q\leq 1)$ sparse regularization into FCM to shrink the weights of irrelevant features in an analytic form when $q = 1/2$, although its performance may be limited by the fact that such sparse feature selection cannot deal with outliers that are hidden in relevant features [16]. Brayda addressed this problem from the standpoint of the sensitivity to noise estimation errors and proposed the TeFCM (L_2) L_2R and TeFCM (L_2) L_1R based on the use of tolerance vectors [17, 18], although how to determine suitable parameters for the upper bound of the tolerance vectors and regularization parameters remains a problem [17]. To deal with the uncertainty of fuzzy coefficients and limit the impact of outliers, the setting of empirical intervals to design and manage the uncertainty of fuzzy coefficients is another idea. Rubio et al. combined a pattern set with interval type-2 fuzzy sets using more than one fuzzification to handle uncertainty and susceptibility to noise [19], with applications such as website hotel selection [20]. To reduce the computational complexity of this approach, all the secondary memberships are weighted uniformly for each primary membership, hence limiting its generalizability. To improve the performance of interval type-2 FCM, Minh et al. applied multiple kernels [21], although this requires the introduction of more fuzzifications, which greatly increases the parameter complexity. Instead of assigning a single possible interval to each element in a given reference set, with detailed analysis of fuzzy multisets [22] and intuitionistic fuzzy sets [23], the concept of hesitant fuzzy sets based on the application of a set of membership functions to each element to deal with uncertainty has also been introduced [24–26].

To avoid high computational complexity, we add relevant regularization to remove as much uncertainty as possible. We take the noise robustness and retention capacity of fuzzy clustering as the starting point and attempt to disperse the impact of outliers within the overall range. Besides, it is vital to promote the divisibility between different clusters as well as the similarity within each cluster.

To achieve this goal, the distance from an observation to a cluster, denoted as $\|d\|$, could be a good criterion to judge the reliability of an observation. The less reliable the observation, the more serious the penalty it produces. In other words, the penalty guides how the loss function behaves. Besides, the loss caused by unreliable observations should not be so severe that the the impact of noise is overemphasize and unstable clustering results are induced. It is well validated that norm normalization techniques can efficiently inhibit the undesirable impact of noisy data [16, 27, 28]. Applying norm regularization with respect to $\|d\|$ in FCM helps a lot to control the overall effect in theory [6, 16, 17].

Ding [29] and Nie [30] proposed the adaptive-loss concept, which serves as an assembling type of norm regularization as an adaptive embedding for semisupervised learning. The adaptive norm [30] smoothly interpolates between L_1 and L_2 error functions, consistent with the expected effects of unsupervised fuzzy clustering tasks. To advance such research, the application of adaptive loss in unsupervised cluster algorithms represents a great innovation. Stimulated by Ding's and Nie's work, we first study the adaptive loss function in fuzzy clustering algorithms in this paper. Note that the outlier sensitivity of FCM originates from the fact that constrained memberships cannot distinguish between EQUAL EVIDENCE and IGNORANCE when two points are quite far away from the centroid of a cluster, resulting in heavy tails on the membership assignment [31]. Based on this observation, we explore a membership-assignment-based strategy, which differs from previous distance-based ones, with the aid of an adaptive norm.

In terms of fuzzy logic, a troublesome obstacle to the application of the adaptive norm is that the L_2-norm and L_1-norm restrain the fuzzy level of FCM, hence limiting its clustering performance. Taking the adaptive norm as a prototype, we extend it to a general norm called the adaptive $L_{1,m}$-norm to fit FCM. The adaptive $L_{1,m}$-norm is expressed as

$$\|u\|_{1,m} = \sum_{i=1}^{n} \frac{(1+\delta)|u_i|^m}{\delta + |u_i|^{m-1}}, \tag{1}$$

where u denotes the membership degree of a point and δ is a positive coefficient controlling the adjustment of the L_m-norm and L_1-norm. We bridge the $L_{1,m}$-norm over the standard FCM objective function, resulting in a novel model called adaptive-FCM, with the aim of achieving noise robustness, divisibility, and similarity. Sects. 2.2 and 5 analyze the characteristics of adaptive-FCM and different types of norm regularization.

In addition, entropy information contained in the data should be fully utilized to achieve noise robustness. Researchers have used the maximum-entropy model as a regularization to make the clusters much fuzzier or more dissimilar via its maximizing strategy [14, 32]. Li et al. proposed a maximum approach to fuzzy clustering, and Zhou et al. proposed fuzzy clustering with the entropy of attribute weights (EWFCM) by combining attribute-weighted information with the theory of entropy [14]. EWFCM provides a good criterion for attribute weight assignment and works well for nonspherically shaped clusters. Zarinbal et al. introduced the distribution metric characteristic of relative entropy, the general case of entropy, into FCM (REFCM) to measure the distance between two distributions [32]. This combination combines the objective loss with a Gaussian distribution, making FCM more robust to noise to some degree.

Recall that, for real-world datasets, erratic noise occurs by chance in any distribution. However, in addition to the problem of how to treat noise compatibly, another issue is the treatment of the imbalance which exists between different dimensions. To deal with high-dimensional datasets, Donald et al. embedded a fuzzy covariance matrix as a nature metric into the FCM model and obtained more accurate clustering [33].

Ultimately, inspired by recent entropy-based FCM methods, we further include the relative entropy function and propose a novel FCM model (adaptive-REFCM) to address these problems and capture the overall structure of the dataset.

In summary, the contributions of this paper are threefold:

- To study the use of the adaptive-loss function in the FCM domain to achieve superior noise robustness.
- To provide a membership-assignment-based viewpoint to address the impact of outliers in FCM, in contrast to traditional distance-based approaches.
- To propose two complete FCM-based models (adaptive-FCM and its extension adaptive-REFCM) to handle noisy and dimensionally imbalanced situations, which outperform related state-of-the-art FCM methods according to experiments on real-world (noise-free or noisy UCI repository and image segmentation) and artificial datasets.

2 Related Work

Definition Given $\{x_1, x_2, \ldots, x_n\}$ as n unsupervised data points of the same dimensionality, denote the data matrix as $X = \{x_1, x_2, \ldots, x_n\}, X \in R^{s*n}$, where s is the dimensionality. Define c as the expected number of clusters and $C = \{v_1, v_2, \ldots, v_c\}$ as the vectors of all the clusters.

2.1 FCM

In fuzzy c-means clustering algorithms, m is defined and given as the fuzzy coefficient. The goal is to achieve the best assignment for u_{ij} $\forall i, j$. The Euclidean distance from the j_{th} observation to the centroid of the i_{th} cluster is defined as $d_{ij} = \|x_j - v_i\|_{2,1}$. The whole loss function is defined as

$$\begin{cases} \arg\min_{u,d} \mathrm{J}(u,d) = \arg\min_{u,d} \sum_{j=1}^{n} \sum_{i=1}^{c} u_{ij}^m d_{ij}^2 \\ s.t. \sum_{i=1}^{c} u_{ij} = 1, 0 \le u_{ij} \le 1. \end{cases} \tag{2}$$

Applying the Lagrange multiplier method to complete the whole iterative optimization computation with respect to u_{ij}, the optimization can be expressed as

$$\begin{cases} u_{ij} = \left(\sum_{k=1}^{c} \left(\left(\frac{d_{ij}^2}{d_{kj}^2} \right)^{\frac{1}{m-1}} \right) \right)^{-1} \\ i = 1, 2 \ldots c, j = 1, 2 \ldots n. \end{cases} \tag{3}$$

As m increases, all the u_{ij} tend to become closer, making the assignment fuzzier, which increases the impact on the whole dataset and individual observations. In comparison with hard clustering algorithms, such fuzzy assignment extracts the information about clusters more reasonably, because the interaction among the observations is taken into consideration.

It is natural to identify points with extremely low membership degree as noise. However, this definition of u_{ij} suffers from the disadvantage that one cannot automatically assign observations with quite low membership degree to noise, resulting in serious noise sensitivity.

2.2 Norm Regularization

To handle outliers in optimization problems, norm regularization techniques such as the well-known L_q-norm family mainly concentrate on the reasonable applications of distance-based loss function strategies.

Typically, $\|d\|_2$ (L_2-norm) and $\|d\|_1$ (L_1-norm) are taken as the two major forms for such regularization methods, among the L_q-norm family. From the perspective of numerical analysis, $\|d\|_n (n > 1)$ is smaller than $\|d\|_1$ when $0 < d < 1$, and $\|d\|_1$ grows more slowly than $\|d\|_n$ $(n > 1)$ when $d > 1$. In comparison with $\|d\|_2$, the use of $\|d\|_1$ as a penalty softens the treatment of unreliable observations but penalizes reliable ones too much. In conclusion, use of $\|d\|_2$ results in central representations while use of $\|d\|_1$ aids regularization to achieve noise robustness.

Moreover, the L_1-norm has also been proven to be effective for variable selection. The least absolute shrinkage and selection operator (LASSO) method uses the L_1-norm penalty function, viz. $\|x\|_1 = \sum_{i=1}^{n}(|x_i|)$, to achieve complete variable selection while also reducing the computational complexity [34]. However, in the case of variables that are highly correlated or when $p \gg n$, LASSO tends to select parts of the variables while ignoring others because it is not strictly convex and does not have a unique solution [35]. The use of an elastic net [35] overcomes the overfitting problem of LASSO by bridging the L_2-norm penalty function into the estimation of $\hat{\beta}$, which is defined as $\hat{\beta} = \arg\min_{\beta}(\|Y - X\beta\|^2 + \lambda_1\|\beta\|_1 + \lambda_2\|\beta\|_2)$. By adjusting λ_1 and λ_2 separately, the elastic net controls the behavior of the L_2-norm and L_1-norm to achieve the desired benefits of group effects and noise robustness.

For noise rejection, the above-mentioned regularization theories can be introduced into the knowledge system of FCM; For instance, L_1-norm regularization provides a rigid constraint on the positive membership degree, resulting in sparse assignments during the iterations of the optimization. This operation extracts the principal characters adaptively. In comparison with the L_1-norm, application of the L_2-norm for regularization in FCM promotes the compactness within each class but does not favor the divisibility between different classes. Nie et al. studied an adaptive norm as an elastic embedding constraint for linear models, ultimately simplifying the adjustment of the L_2-norm and L_1-norm functions [30] to enhance the robustness of noise during semisupervised learning.

The adaptive loss function, first studied by Ding [29], serves as an assembling type of regularization which smoothly interpolates between the L_1 and L_2 error functions.

From the perspective of optimization, we extend the adaptive norm to fit the FCM approach and design adaptive-FCM as shown in Eq. 4 to serve as a fuzzy clustering model that combines the L_1-norm ($\|x\|_1 = \sum_i |x_i|$) and the L_m-norm ($\|x\|_m = \sum_i |x_i|^m$).

$$\begin{cases} \arg\min_{u,d} \mathrm{J}(u, d) = \arg\min_{u,d} \sum_{j=1}^{n} \sum_{i=1}^{c} \frac{(1+\delta)u_{ij}^m}{u_{ij}{}^{m-1} + \delta} d_{ij}{}^2 \\ \text{s.t.} \ \sum_{i=1}^{c} u_{ij} = 1, \quad 0 \le u_{ij} \le 1, \delta > 0. \end{cases} \tag{4}$$

In Sect. 5, we discuss and compare the properties of the original adaptive-norm and the proposed adaptive-FCM in detail.

2.3 Relative Entropy

The relative entropy, also called the Kullback–Leibler (KL) divergence, of two distributions Q and P is defined as

$$\mathrm{D_{KL}}(\mathrm{Q}\|\mathrm{P}) = \sum_{i=1}^{c} \ln \frac{\mathrm{Q}(i)}{\mathrm{P}(i)},$$

which has the nonnegative property that $\mathrm{D_{KL}}(\mathrm{Q}\|\mathrm{P}) \ge 0$, while $\mathrm{D_{KL}}(\mathrm{Q}\|\mathrm{P})$ equals zero if and only if $\forall i, \mathrm{P}(i) = \mathrm{Q}(i)$. These properties of cooperation and nonnegativity make it suitable for convex optimization.

Considering relative entropy as the general case of entropy that measures the distance between two distributions, it can be applied for noise robustness in both hard and soft clustering.

REFCM adds the relative entropy to the objective function of FCM by considering the degree of fuzziness [32]. The objective function of REFCM can be expressed as

$$\begin{cases} \arg\min_{u,d} \mathrm{J}(u,d) = \arg\min_{u,d} \sum_{j=1}^{n}\sum_{i=1}^{c} u_{ij}^{m} d_{ij}^{2} \\ \qquad - \beta \sum_{j=1}^{n}\sum_{i=1}^{c}\sum_{k=1,k\neq i}^{c} u_{ij} \ln \frac{u_{ij}}{u_{kj}} \\ s.t. \sum_{i=1}^{c} u_{ij} = 1, \quad 0 \le u_{ij} \le 1, \end{cases} \tag{5}$$

where β is the tradeoff coefficient to adjust the impact of the relative entropy term. Based on the summation of the relative entropy of u_{ij} and u_{kj} $(k \neq i)$, REFCM achieves discriminable assignment to the membership degrees, thereby maximizing the dissimilarity between clusters and more effectively detecting true negative data points to improve the robustness to noise. However, REFCM does not handle dimension normalization and thus is not suitable for nonspherical datasets.

2.4 Dimension Normalization

In practical applications such as geographic information integration or text classification, some elements always behave according to Gaussian distributions with different variances. It is thus advisable to normalize the feature dimensions. EWFCM applies different weights to each dimension in order to achieve the desired improvement [14]. Later, the Mahalanobis distance was introduced into FCM for dimensional regularization. In terms of statistical analysis, the Mahalanobis distance is used to establish a unified measurement standard for each feature dimension and to achieve better adjustment for existing membership estimation. Liu et al. improved FCM by using the standard Mahalanobis distance [36]. The Mahalanobis distance has been proved to be effective for complex clustering tasks. Zhao et al. introduced the Mahalanobis distance based on a fuzzy clustering algorithm for image segmentation [37].

In this paper, we apply the normalized negative exponential of the Mahalanobis distance as a form for the membership possibility. We then apply this possibility for the relative entropy regularization in combination with the membership degrees of the observations.

In this paper, the Mahalanobis distance in the dataset is defined as $\mathrm{D_M} = (\mathrm{X} - C)^{\mathrm{T}} \Sigma_{x}^{-1} (\mathrm{X} - C)$, where C and X are defined as

Definition Under this full-rank linear transformation of the data space, hidden information is fully preserved.

In summary, adaptive-REFCM introduces the adaptive norm, relative entropy regularization, and Gaussian cooperation with the Mahalanobis distance into FCM. A complete analysis of the modeling process is presented in Sect. 5.

3 Algorithm Process

Adaptive-REFCM is designed based on standard FCM. The viable objective function of this model is defined as

$$\begin{cases} \arg\min_{u,d} \mathrm{J}(u,d) = & \arg\min_{u,d} \sum_{j=1}^{n}\sum_{i=1}^{c} \frac{(1+\delta)u_{ij}^{m}}{u_{ij}{}^{m-1}+\delta} d_{ij}{}^{2} \\ & +\beta \sum_{j=1}^{n}\sum_{i=1}^{c} u_{ij} \ln \frac{u_{ij}}{\gamma_{ij}} \\ s.t. \sum_{i=1}^{c} u_{ij} = 1, & 0 \le u_{ij} \le 1, \delta, \beta > 0, \end{cases} \tag{6}$$

which satisfies

$$\begin{cases} \gamma_{ij} = \frac{\exp^{-(x_j - C_j)^T \Sigma_i{}^{-1} (x_j - C_i)}}{\sum_{k=1}^{c} \exp^{-(x_j - C_k)^T \Sigma_k{}^{-1} (x_j - C_k)}} \\ \Sigma_i = \frac{1}{n} \sum_{j=1}^{n} \pi_{ij} (x_i - C_j)^T (x_i - C_j) \\ \pi_{ij} = \frac{u_{ij}^{m}}{\sum_{k=1}^{n} u_{kj}{}^{m}}, \end{cases} \tag{7}$$

where d_{ij} denotes the distance from the j_{th} observation to the centroid of the i_{th} cluster, u_{ij} denotes the membership degree of the j_{th} datum with respect to the i_{th} cluster, γ_{ij} denotes the prior approximate evaluation of u_{ij}, and δ and β are positive coefficients for the adaptive norm and relative entropy terms, respectively.

The algorithm is completed via the following steps:

Step 1: Simplify the model

We discuss the adaptive norm term separately. The objective function can be modified to

$$\frac{\partial J(u,d)}{\partial u_{ij}} = f'(u) + 2(1+\delta) \frac{m\delta + g_{ij}(u)}{2(g_{ij}(u)+\delta)^2} d_{ij}^{2}\, g_{ij}(u) g_{ij}'(u), \tag{8}$$

where $f(u)$ is the relative entropy term. To minimize Eq. 8 w.r.t. u_{ij}, let

$$\begin{cases} g_{ij}(u) = u_{ij}^{m-1} \\ g_{ij}{}'(u) = (m-1) u_{ij}^{m-2} \\ M_{ij} = (1+\delta)\, \frac{m\delta + g_{ij}(u)}{(g_{ij}(u)+\delta)^2}. \end{cases} \tag{9}$$

Equation 8 is equivalent to

$$\frac{\partial J(u,d)}{\partial u_{ij}} = f'(u) + M_{ij}\, d_{ij}^{2}\, g_{ij}(u)\, g_{ij}'(u) = 0, \tag{10}$$

M_{ij} is considered as a variable uncorrelated with u_{ij}. This upfront operation reduces the computational complexity.

Thus, the optimization of the model is simplified to the solution of Eq. 11.

$$\min_u f(u) + \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{c} M_{ij} g_{ij}(u)^2 \, d_{ij}^2. \tag{11}$$

Step 2: Apply the Lagrangian multiplier method

Following step 1, $\lambda_j = 1, 2, 3, \ldots, N$, the following function is minimized by the application of Lagrangian multipliers:

$$\begin{aligned} J(u,\lambda,d) = &\min_u \beta \sum_{j=1}^{n}\sum_{i=1}^{c} u_{ij}\left(\ln u_{ij} - \ln \gamma_{ij}\right) \\ &+ \frac{1}{2}\sum_{j=1}^{n}\sum_{i=1}^{c} M_{ij} g_{ij}(u)^2 \, d_{ij}^2 \\ &- \sum_{j=1}^{n} \lambda_j \left(\sum_{i=1}^{c} u_{ij} - 1\right) \end{aligned} \tag{12}$$

Minimization of Eq. 11 w.r.t. u_{ij} while satisfying Eq. 10 yields

$$\beta\left(\ln u_{ij} - \ln \gamma_{ij} + 1\right) - \lambda_j = -M_{ij} d_{ij}^2 u_{ij}^{2m-3}(m-1). \tag{13}$$

Both sides of the equation are multiplied by $\frac{-2m+3}{\beta}$ to give

$$\begin{aligned} &(-2m+3)\left[\left(\ln u_{ij} - \ln \gamma_{ij} + 1\right) - \frac{\lambda_j}{\beta}\right] \\ &= \frac{(2m-3)(m-1)}{\beta} M_{ij} d_{ij}^2 u_{ij}^{2m-3}. \end{aligned} \tag{14}$$

In Sect. 4, it is proved that the solution u_{ij} of Eq. 14 is the optimum solution of Eq. 11.

Step 3: Determine u_{ij}

Letting $Y_{ij} = -\ln u_{ij}$ and $u_{ij} = e^{(-Y_{ij})}$, Eq. 14 can be converted to

$$\begin{aligned} &(2m-3)\left[\left(Y_{ij} + \ln \gamma_{ij} - 1\right) + \frac{\lambda_j}{\beta}\right] \\ &= \frac{(2m-3)(m-1)}{\beta} M_{ij} d_{ij}^2 e^{-(2m-3)Y_{ij}}. \end{aligned} \tag{15}$$

Let

$$\begin{cases} C = (2m-3)\left[\left(Y_{ij} + \ln \gamma_{ij} - 1\right) + \frac{\lambda_j}{\beta}\right] \\ D = \frac{(2m-3)(m-1)}{\beta} M_{ij} d_{ij}^2 e^{-(2m-3)Y_{ij}} \\ E = De^C \end{cases} \tag{16}$$

Then, based on the computation method of the Lambert W function, we get

$$E = \frac{(2m-3)(m-1)}{\beta} M_{ij} d_{ij}^2 e^{(2m-3)\left[\ln \gamma_{ij} - 1 + \frac{\lambda_j}{\beta}\right]}. \tag{17}$$

Note that $ce^C = E$ is expressed as a transcendental equation, viz. the Lambert W function, which is also called the omega function, suggested to calculate the solution $C = W_0(E)$ [38]. With the aid of the auxiliary function, Y_{ij} can be determined.

$$Y_{ij} = \frac{1}{2m-3} W_0(E) + 1 - \ln \gamma_{ij} - \frac{\lambda_j}{\beta}, \tag{18}$$

then

$$\begin{cases} u_{ij} = e^{-Y_{ij}} = e^{-\frac{1}{2m-3}W_0(E) + 1 - \ln\gamma_{ij} - \frac{\lambda_j}{\beta}} \\ E = \frac{(2m-3)(m-1)(1+\delta)\left[m\delta + g_{ij}(u)\right]}{2\beta\left(g_{ij}(u)+\delta\right)^2} d_{ij}^2 e^{(2m-3)\left[\ln\gamma_{ij} - 1 + \frac{\lambda_j}{\beta}\right]} \end{cases} \tag{19}$$

Step 4: Determine λ_j

Because of the complex form of λ_j, it is hard to obtain an analytical solution λ_j. However, this situation can be handled by supposing a range for λ_j. According to the equation $u_{ij} = e^{-Y_{ij}} = e^{\frac{-1}{2m-3}W_0(E)} e^{-1+\ln\gamma_{ij}+\frac{\lambda_j}{\beta}}$, we get

$$\begin{cases} W_0(E)\left(\frac{u_{ij}}{e^{-1+\ln\gamma_{ij}+\frac{\lambda_j}{\beta}}}\right)^{-(2m-3)} = W_0(E)e^{W_0(E)}, \\ W_0(E)e^{W_0(E)} = E. \end{cases} \tag{20}$$

$$\begin{cases} E = \frac{(2m-3)(m-1)(1+\delta)\left[m\delta + g_{ij}(u)\right]}{2\beta\left(g_{ij}(u)+\delta\right)^2} d_{ij}^2 e^{(2m-3)\left[\ln\gamma_{ij} - 1 + \frac{\lambda_j}{\beta}\right]} \\ u_{ij} = \left(\frac{E\left(e^{-1+\ln\gamma_{ij}+\frac{\lambda_j}{\beta}}\right)^{-(2m-3)}}{W_0(E)}\right)^{-\frac{1}{2m-3}} \end{cases} \tag{21}$$

It is obvious that $E \geq 0$ when $m \geq 1$. Thus, we explore how to determine λ_j to allow u_{ij} to satisfy $0 \leq u_{ij} \leq 1$ as follows:

- According to the mapping relationship in Eq. 20, it is inevitable that Eq. 22 makes sense.

  $$\mathrm{sgn}(E) = \mathrm{sgn}(W_0(E)) \tag{22}$$

  Furthermore, u_{ij} satisfies Eq. 23,

  $$sgn\left(\left(e^{-1+\ln\gamma_{ij}+\frac{\lambda_j}{\beta}}\right)^{-(2m-3)}\right) = 1, \tag{23}$$

  so u_{ij} is positive and $u_{ij} \geq 0$ is a necessary inequality, which proves the first condition of $u_{ij} \geq 0$.
- We now explore the other condition, viz. $u_{ij} \leq 1$, which is equivalent to

$$\left(\frac{E\left(\mathrm{e}^{-1+\ln\gamma_{ij}+\frac{\lambda_j}{\beta}}\right)^{-(2m-3)}}{W_0(E)}\right)^{-\frac{1}{2m-3}} \leq 1 \tag{24}$$

Applying simple operations, Eq. 24 is equivalent to

$$\lambda_j \leq \beta\left(\frac{-1}{2m-3}\ln\frac{W_0(E)}{E} + 1 - \ln\gamma_{ij}\right) \tag{25}$$

Finally, the upper bound on λ_j is determined as

$$\lambda_j \leq \beta\left(\frac{1}{2m-3}W_0(E) + 1 - \ln\gamma_{ij}\right). \tag{26}$$

In conclusion, the range of λ_j is determined to be $(-\infty, \beta(\frac{1}{2m-3}W_0(E) + 1 - \ln\gamma_{ij})]$.

Step 5: Update the centers of the clusters

$$\begin{cases} d_{ij} = x_j - c_i \\ \frac{\partial J(u,d)}{\partial c_i} = 0 \end{cases} \tag{27}$$

The i_{th} center is updated using Eq. 28,

$$c_i = \frac{\sum_{j=0}^{n}\frac{(\delta+1)+u_{ij}^{m-1}}{\delta+u_{ij}^{m-1}}x_j}{\sum_{k=0}^{n}\frac{(\delta+1)+u_{ik}^{m-1}}{\delta+u_{ik}^{m-1}}}. \tag{28}$$

4 Convergence and Constancy

Proof The solution u_{ij} of Eq. 14 is the optimum solution of Eq. 11.

- Compute the Hessian matrix of $J(u,d)$ as

$$\frac{\partial^2 J(u,d)}{\partial u_{ij}^2} = \frac{\beta}{u_{ij}} + (m-1)(2m-3)M_{ij}d_{ij}^2u_{ij}^{2m-4}.$$

  Because $M_{ij} = (1+\delta)\,\frac{m\delta+g_{ij}(u)}{\left(g_{ij}(u)+\delta\right)^2} > 0$, all the elements of $\frac{\partial^2 J(u,d)}{\partial u_{ij}^2}$ satisfy $\frac{\partial^2 J(u,d)}{\partial u_{ij}^2} > 0$.

- Compute the first-order derivative of $J(u,d)$ as

$$\begin{aligned}\frac{\partial J(u,d)}{\partial u_{ij}} =& \beta\left(\ln u_{ij} - \ln\gamma_{ij} + 1\right) \\ &+ M_{ij}d_{ij}^2u_{ij}^{2m-3}(m-1) - \lambda_j.\end{aligned} \tag{29}$$

$\frac{\partial J(u,d)}{\partial u_{ij}}$ is a monotonically increasing function of u_{ij} and $\frac{\partial^2 J(u,d)}{\partial u_{ij}^2} > 0$, which proves the theorem. □

5 Analysis of the Algorithm

To solve the clustering task properly with multiclass datasets, adaptive-REFCM effectively integrates the adaptive norm, relative entropy term, and Gaussian mixture model (GMM), which cooperate with each other in this method. Great performance is achieved in experiments. In this section, three main properties of adaptive-REFCM are analyzed.

5.1 Property I: Noise Robustness

The adaptive loss-minimizing method [30] relaxes the rigid linear model constraint by applying an elastic constraint, such that the data structure can be better explored. The original adaptive norm can be expressed in vector form as shown in Eq. 30 or in matrix form as shown in Eq. 31, where x^i denotes the i_{th} vector of matrix X:

$$x_\delta = \sum_i \frac{(\delta+1)x_i^2}{\delta+|x_i|}, \tag{30}$$

$$X_\delta = \sum_i \frac{(\delta+1){x^i}_2^2}{\delta+{x^i}_2}. \tag{31}$$

Nie [30] pointed out that the adaptive norm is an integration of the L_1-norm and L_2-norm with the adaptive coefficient δ. In Fig. 1, the abscissa represents the Euclidean distance d_{ij} between a random point and the center of a cluster while the ordinate represents the value that the loss function gains. In comparison with the L_1-norm and L_2-norm, Fig. 1 shows the results for the adaptive norm obtained from the L_1-norm and L_2-norm based on the value of δ, here set to 0.01, 0.1, or 1, to reveal its behavior.

The core of the design for the clustering loss function is to set a proper standard for the evaluation of observations in the dataset. Generally speaking, it is more reasonably that those points which are closely gathered into a compact structure belong to the same cluster. To follow this core concept of clustering, an increase in the absolute value of $d_{ij}(i = 1,2,\ldots,c; j = 1,2,\ldots,n)$ indicates that the point will make a larger contribution to the total loss. On the other hand, qualitatively speaking, a point that deviates a lot from the centroids of all the clusters is more likely to be considered as noise. From this perspective, one tends to allocate a relatively smaller cost for small d_{ij} values but larger values for large d_{ij}.

However, this is accompanied by another problem, i.e., that outliers produce too high a cost, seriously affecting the retention of the potential structure of the data when extracted by the clustering process. Thus, it is advisable to adjust the cost into a reliable range for large d_{ij} values, to enhance the robustness of the algorithm to noise.

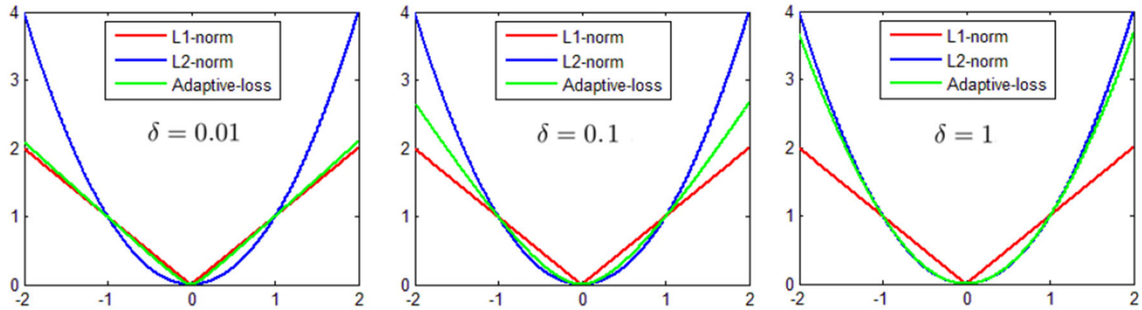


Fig. 1 Loss of L_1-norm, L_2-norm, and adaptive norm with $\delta = 0.01, 0.1, 1$

Use of the L_1-norm as the loss function offers the advantage of weakening the effect of noise, but is less favorable for cluster concentration compared with use of the L_2-norm. Nevertheless, using the squared L_2-norm as the loss function can preserve the local structure but is sensitive to outliers.

Therefore, the adaptive norm based on a mixture of the L_1-norm with the L_2-norm is a compromise to increase the degree of compactness by adjusting the cost function of smaller d_{ij} values to approach the L_2-norm while increasing the robustness to noise by decreasing the cost of higher d_{ij} values to approach the L_1-norm, as shown in Fig. 1.

Inspired by this adaptive norm, we extend it to Eq. 1 and propose the adaptive-FCM shown in Eq. 32 ($m = 2$) by adopting the adaptive norm with the cost of membership assignment. Note that u_{ij} and d_{ij} are negatively correlated, so the effects of the L_2-norm and L_1-norm in the FCM are opposite.

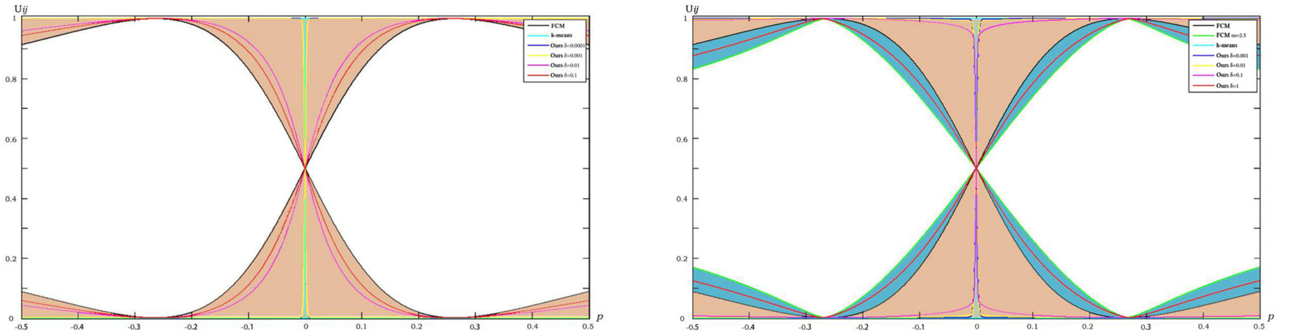
$$min\ J(u, d) = min \sum_{j=1}^{n} \sum_{i=1}^{c} \frac{(1+\delta)|u_{ij}|^m}{|u_{ij}|^{m-1} + \delta} d_{ij}^2 \tag{32}$$

Consider the situation of $c = 2$. Limited by the rigid constraint $\sum_{i=1}^{c} u_{ij} = 1, 0 \leq u_{ij} \leq 1$, in case the jth observation is likely to be outlier, one of the two membership degrees of the jth observation tends to drop from 1 while the other tends to be quite small and increase from 0 (Fig. 2). However, it is suggested that $u_{ij} (i = 1, 2)$ be allocated more uniformly for a outlier, so as to decentralize the impact of noise on all the clusters and reduce the loss. From this point of view, standard FCM (Eq. 2) allocates more uniformly for a outlier with larger m, leading to the fact that the L_2-norm offers better noise robustness than the L_1-norm. Moreover, the L_2-norm simultaneously makes the margin between clusters softer compared with the L_1-norm, providing greater potential for partition between clusters. On the other hand, adding the L_1-norm to u_{ij} tends to make the judgement explicit for observations quite close to the centroid of some cluster, resulting in better condensation within clusters.

However, a problem occurs in that FCM combined with the original adaptive norm loss suffers from an inherent limitation in the regulation of the fuzzy level, because the integration of the L_1-norm and L_2-norm limits the fuzzy ability from k-means to standard FCM. To confirm this, we consider a dataset with a uniform distribution and cluster into two subclusters using FCM and adaptive-FCM (Eq. 32) with different values of δ. The curves representing the membership degrees in the different situations are shown in Fig. 2 to prove this inference. The ordinate refers to the value of the membership degree u, while the abscissa shows the absolute one-dimensional position of an observation. For a fuzzier result, the curve tends to be closer to the middle horizontal line in Fig. 2a. It can also be seen from Fig. 2a that the curves of the original adaptive-FCM with different values of δ always lie between those of FCM and k-means (dark-red-shaded region in Fig. 2a) and cannot get fuzzier than the results of FCM.

To achieve better robustness to outliers in FCM, one approach is to blur the impact of outliers by making their membership fuzzier than FCM. A larger value of m assigns fuzzier membership in FCM, at the cost of reduced compactness within clusters, which may result in unexpected uncertainty. Pal concluded based on cluster validity that the best interval for m is [1.5, 2.5] [39]. Bezdek concluded that, when $m = 2$, the most meaningful partition can be obtained by FCM [40]. Therefore, simply increasing the value of m is not a compatible strategy. Alternative ways to handle this situation include interpolative selecting of different values of m (e.g., interval-FCM [21]) or to retain the compactness of the L_1-norm in adaptive-FCM and expand the properties of the L_m-norm ($m = 2$) by using a larger value of m.

To achieve flexibility in the regulation of the fuzzy level to work better with noisy datasets, we further expand its form into a novel form by combining the L_1-norm with the L_m-norm ($m > 2$) as shown in Eq. 4. Use of a larger value for the fuzzy coefficient m enables fuzzier performance. According to Pal's range for m, we choose $m = 2.5$ for adaptive-FCM. Taking Fig. 2a as a reference, it can be seen



(a) k-means, FCM (m=2) and adaptive-FCM (m=2) **(b)** k-means, FCM (m=2, m=2.5) and adaptive-FCM (m=2.5)

Fig. 2 Comparison of membership degrees of k-means, FCM, and adaptive-FCM w.r.t. δ and m

from Fig. 2b that, in comparison with adaptive-FCM ($m = 2$), adaptive-FCM ($m = 2.5$) further expands the domain of U_{ij} (dark-blue-shaded region) w.r.t. the value of δ.

Figure 3 shows the clustering results of FCM ($m = 2$) and adaptive-FCM ($m = 2.5$, $\delta = 1$), confirming the effectiveness of adaptive-REFCM in terms of the membership degrees. Define p as the position of an observation. As shown in Fig. 3, the p of the two cluster centroids is -0.25 and 0.25. In comparison with FCM, adaptive-FCM keeps u explicit (close to 1 or 0) when p is close to -0.25 or 0.25, much fuzzier (closer to 0.5) when p is close to the edges ($p = -0.5$ and $p = 0.5$), and much fuzzier when p ranges from -0.2 to 0.2 (the margin that is fuzzy for clustering). Obviously, the expected performance in terms of noise robustness is achieved.

Summing up the results described above, the adaptive loss minimization method extracts a holistic representation of the whole dataset while showing robustness to noise.

5.2 Property II: Global Adaptive Adjustment

In this section, we focus on the global adaptive adjustment of the algorithm and explore the effect of the relative

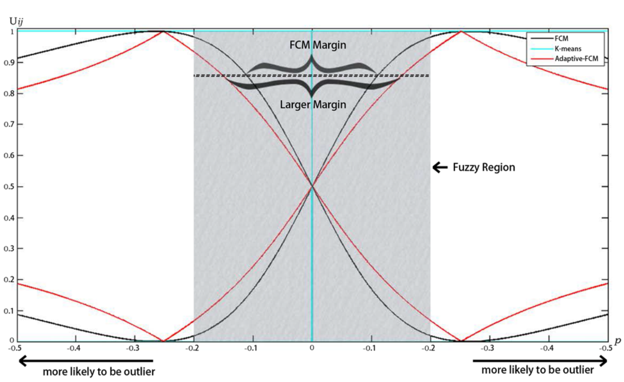


Fig. 3 Membership degrees of FCM and adaptive-FCM in terms of distances

entropy. Firstly, recall the relative entropy (RE) from Eq. 6:

$$\mathrm{RE}_{\mathrm{part}} = \min_{u} \sum_{j=1}^{n} \sum_{i=1}^{c} u_{ij} \left(\ln u_{ij} - \ln \gamma_{ij} \right). \tag{33}$$

In our method, we combine GMM in the RE for the reason that datasets behave differently in different situations and an underlying mechanism associated with such a mixture model is observable in fields such as documents, handwriting recognition, iris datasets, etc. In the RE, γ_{ij} denotes the probabilistic presence of subpopulations within the overall population to correspond to the distribution of the clusters. The concept of membership degree is similar to the probability of a subpopulation, thus GMM provides prior knowledge for the learning of degrees of belonging in our clustering method. To minimize the RE ideally, u_{ij} and γ_{ij} must satisfy $u_{ij} = \gamma_{ij}\ \forall i, j$, which indicates that γ_{ij} expresses the prior presence of all the subpopulations.

5.3 Property III: Dimensional-Wise Normalization

As mentioned in GMM, the Mahalanobis distances of all the observations to all the centroids of the clusters are calculated. The Mahalanobis distance uses the covariance matrix of Euclidean distances of observations to clusters in order to normalize high-dimensional data to a specific standardization in order to eliminate the side-effect of dimensional scale disunity, making the method more reliable for calculating the imbalance between different observations and clustering high-dimensional data such as nonspherical, ellipsoidal, or speech recognition datasets.

For standard FCM, the membership degree is obtained by Eq. 34,

$$u_{ij} = \left(\sum_{k=1}^{c} \left(d_{ij}^2 / d_{kj}^2 \right)^{1/(m-1)} \right)^{-1}, \tag{34}$$

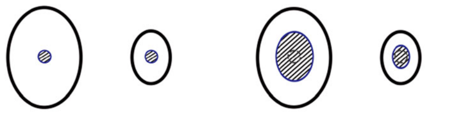


Fig. 4 Comparison of fuzzy regions of FCM and adaptive-REFCM

where d_{ij}^2/d_{kj}^2 is the decisive factor used to determine u_{ij}. As a result, the fuzzy degree is not associated with the size of the different clusters. In Fig. 4, we take the clustering of two clusters as an example to represent the shortcoming of FCM. The two thick black circles indicate the different distribution ranges of the two clusters, while the two inner blue circles filled with diagonal lines indicate the range where the membership degrees of the two clusters are more unambiguous than a given fuzzy threshold.

Figure 4a shows that FCM results in two blue circles with the same radius, while Fig. 4b shows that adaptive-REFCM results in two blue circles with different radii. In Fig. 4b, the dark-grey part retains the blue region resulting from FCM in Fig. 4a. For the same threshold α, a cluster with larger variance results in a larger blue region for the reason that the Gaussian collaborates with the Mahalanobis distance. This indicates that adaptive-REFCM is beneficial compared with FCM to tackle the problem of variance imbalance by assigning membership degrees in line with the variance of each cluster, making the model more general for fuzzy clustering tasks.

In summary, this section presents in-depth analysis of the three main properties of this model. Its experimental performance is reported in Sect. 6.

6 Experimental Analyses

This section further evaluates the classification capability of the proposed methods on noise-free and noisy datasets and nonspherical datasets. In addition, several related state-

Table 1 Comparison of average accuracy in 100 trials over 25 datasets without extra outliers

Name	FCM	AWFCM	NWFCM	EWFCM	FDCM-SSR	$L_{1/2}$-CM	REFCM	ADFCM	ADREFCM
E. coli	0.7888	0.8024	0.8219	0.7885	0.8006	0.8332	0.8076	0.8142	**0.8428**
Auto	0.7534	0.7658	0.7751	0.7894	0.7534	**0.8332**	0.7534	0.7604	0.7619
Dermatology	0.6986	**0.8694**	0.6911	0.7958	0.7057	0.6817	0.7013	0.7103	0.7038
Iris	0.8797	0.8270	0.8977	0.8797	0.8859	**0.9187**	0.8923	0.8925	0.8977
Zoo	0.8325	0.8352	0.6252	**0.9559**	0.8485	0.8586	0.8523	0.8702	0.8720
Transfusion	0.5853	0.5458	0.5799	**0.6368**	0.5929	0.5798	0.5853	0.5853	0.5992
Parkinson's	0.5929	0.5196	0.5758	**0.6218**	0.5928	0.6270	0.5929	0.6084	0.6167
Banknote	0.5236	0.5214	0.5194	0.5236	0.5245	0.5243	0.5249	0.5249	**0.5252**
Credit	0.5048	**0.6751**	0.5058	0.5182	0.5048	0.5073	0.5048	0.5153	0.5153
Breast cancer	0.9159	0.9375	0.9294	0.9159	0.9348	0.9026	0.9458	0.9267	**0.9486**
Wine	0.7105	**0.8294**	0.7269	0.6295	0.7105	0.7239	0.7105	0.7187	0.7187
Automobile	0.6882	0.6947	**0.7269**	0.6889	0.6882	0.6937	0.6882	0.6981	0.6986
Car	0.5330	0.5347	0.5387	0.5330	0.5330	0.5425	0.5456	0.5425	**0.5487**
Fertility	0.5000	0.4958	0.5083	0.5711	0.5010	0.5136	0.5056	0.5224	**0.5533**
Seeds	0.8744	0.8505	0.8621	0.8744	0.8744	0.8441	0.8744	0.8762	**0.8840**
Balance	0.5818	0.5818	0.4300	0.5918	**0.6159**	0.5807	0.5916	0.6152	0.6152
House votes	0.7752	0.7820	0.4300	0.7820	0.7752	0.7820	0.7786	0.7821	**0.7925**
Vowel	0.7290	0.5161	0.5923	0.7951	0.7590	0.7576	0.7378	0.8506	**0.8605**
Glass	0.7117	0.7180	0.6621	0.7117	0.7124	**0.7277**	0.7117	0.7160	0.7235
Mammographic	0.5683	**0.6840**	0.5738	0.6702	0.5683	0.6473	0.5729	0.5762	0.5776
Pima	0.5499	**0.5841**	0.5293	0.5458	0.5499	0.5427	0.5499	0.5516	0.5516
Bankruptcy	0.9453	0.9082	0.9010	0.9453	0.9454	0.9082	0.9453	**0.9762**	**0.9762**
Phishing	0.6614	**0.6886**	0.6460	0.6827	0.6649	0.6587	0.6676	0.6614	0.6622
Yeast	0.7148	0.7148	0.7148	0.7216	0.7148	0.6409	0.7193	**0.7498**	**0.7498**
User knowledge	0.6749	0.6630	0.6120	0.6813	0.6749	0.6672	0.6871	0.6829	**0.6939**
Average	0.6918	0.7018	0.6691	0.7140	0.6973	0.6999	0.6979	0.7091	**0.7156**

The bold numbers note the best performances of all the models listed in Table 2 of-the-art (SOTA) methods from recent years are compared under the same experimental settings and using the same initialization.

It should be emphasized that the details of the experiments are deliberately chosen for comparison of the performance of the proposed model with RE (adaptive-REFCM) and without RE (adaptive-FCM) to confirm whether the global adaptive adjustment of RE works or not.

Note that adaptive-FCM is referred to as ADFCM while adaptive-REFCM is referred to as ADREFCM for short in this section.

6.1 General Performance Comparison on UCI

Twenty-five real-world datasets are selected randomly from the UCI repository [41]. We implement several related state-of-the-art FCM methods, and Table 1 collects the clustering accuracy (also called the Rand index [42]) of the 25 UCI datasets without extra noise. It turns out that, generally, ADFCM and ADREFCM achieve better performances in comparison with FCM and REFCM. Note that the clustering accuracy of ADREFCM is higher by 2.38% on average compared with FCM and that it outperforms other enhanced fuzzy clustering algorithms in these trials.

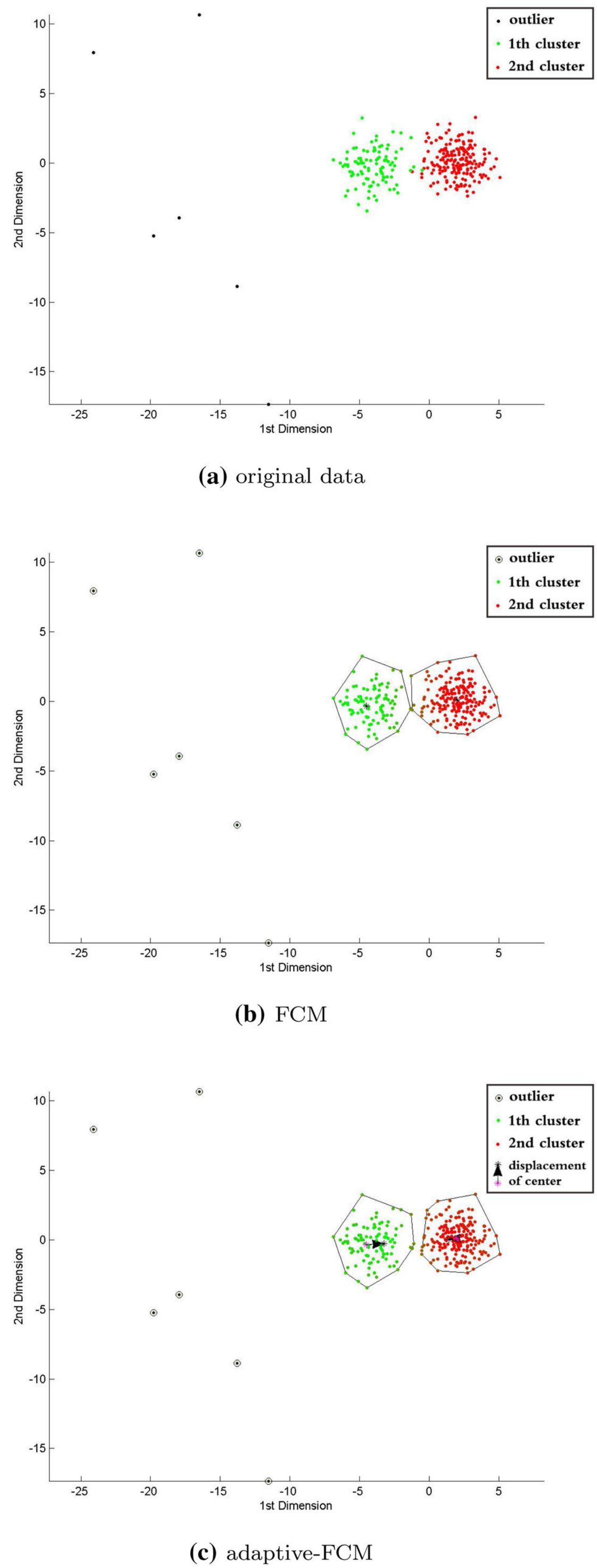
6.2 Robustness to Noise

6.2.1 Clustering of UCI Datasets with Outliers

This section compares the average accuracy in 100 trials over 25 datasets with outliers. There is no explicit mathematical definition for an outlier, so we apply the definition of a small group of observations whose size is $\frac{1}{100}$ of the size of the original dataset and whose centroid is two times the maximum distance of the original observations away from the centroid of all the points in the dataset. The performance results indicate that ADFCM and ADREFCM are superior to standard FCM and REFCM in dealing with outliers in real-world situations. The average clustering accuracy of ADFCM and ADREFCM is 2.70% and 5.75% higher compared with FCM, respectively, and both outperform other enhanced fuzzy clustering algorithms among these trials.

6.2.2 Artificial Dataset Clustering

Figure 5 shows the noise robustness in a more intuitive way by pointing out the displacements of the clusters' centers with arrows, as shown in Fig. 5c, relative to the clustering result of FCM. Moreover, the clustering convex hulls are drawn for visualization. In this experiment, the



(a) original data

(b) FCM

(c) adaptive-FCM

Fig. 5 Noise robustness of adaptive-FCM on two partially overlapping noisy clusters in comparison with FCM

Table 2 Comparison of average accuracy in 100 trials over 25 datasets with outliers

Name	FCM	AWFCM	NWFCM	EWFCM	FDCM-SSR	$L_{1/2}$-CM	REFCM	ADFCM	ADREFCM
E. coli	0.8068	0.8171	0.8169	0.8077	0.8185	**0.8628**	0.8097	0.8306	0.8527
Auto	0.7654	0.7942	0.7769	0.6363	0.7690	0.7790	0.7654	0.7931	**0.8257**
Dermatology	0.6764	0.6808	0.6728	**0.8900**	0.6782	0.6808	0.7004	0.6879	0.6879
Iris	0.7637	0.7599	0.8580	0.7934	0.7709	0.8107	0.8180	0.8629	**0.8684**
Zoo	0.8103	0.8856	0.8636	**0.9743**	0.8693	0.8310	0.8182	0.8874	0.8850
Transfusion	0.6368	**0.6369**	0.6368	0.6368	0.6368	0.6368	0.6368	0.6368	0.6368
Parkinson's	0.6270	0.6270	0.6287	0.6270	0.6270	0.6270	0.6270	0.6270	**0.6846**
Banknote	0.5205	0.5165	0.5196	0.5373	0.5205	0.5229	0.5205	0.5290	**0.7549**
Credit	0.5036	0.5036	**0.5194**	0.5036	0.5412	0.5036	0.5036	0.5036	0.5048
Breast cancer	0.9000	0.8922	0.9080	0.9000	0.9026	0.9186	0.9000	0.9000	**0.9431**
Wine	0.6688	0.6882	0.6923	0.3451	0.6689	0.6928	0.6697	0.6879	**0.7296**
Automobile	0.6578	0.6536	0.6742	0.2450	0.6688	0.6697	0.6583	0.6759	**0.6895**
Car	0.5330	0.5430	0.5516	0.5364	0.5335	0.5515	0.5425	0.5514	**0.5569**
Fertility	0.5014	0.5080	0.5033	0.7286	0.5085	0.5190	0.5216	0.5392	**0.7867**
Seeds	0.8076	0.7695	0.7827	**0.8147**	**0.8147**	0.7875	0.8102	0.8147	0.8108
Balance	0.5160	0.5323	0.5181	0.5512	0.5182	0.5270	0.5604	**0.5833**	0.5800
House votes	0.7752	0.7718	0.7821	0.7855	0.7820	0.7821	0.7810	**0.7881**	**0.7881**
Vowel	0.6514	0.6482	0.6537	0.7509	0.6601	0.6514	0.6608	**0.8543**	0.8518
Glass	0.6791	**0.7061**	0.6574	0.6797	0.6799	0.6730	0.6842	0.6962	0.6929
Mammographic	0.5757	0.5757	0.5757	0.5785	0.5776	0.5725	0.5757	0.5757	**0.5925**
Pima	0.5450	0.5451	0.5451	0.5450	0.5451	0.5451	0.5450	0.5450	**0.5668**
Bankruptcy	0.9762	0.9762	0.9762	0.9762	0.9762	0.9762	0.9762	0.9762	0.9762
Phishing	0.6490	0.6629	0.6536	0.6608	0.6673	0.6598	0.6919	0.6492	**0.6928**
Yeast	0.7115	0.6488	0.6928	0.7204	0.7127	0.7115	0.7322	**0.7376**	0.7292
User knowledge	0.6782	0.6833	0.6666	0.6807	0.6782	0.6782	0.6791	0.6804	**0.6876**
Average	0.6775	0.6811	0.6850	0.6762	0.6850	0.6868	0.6875	0.7045	**0.7350**

The bold numbers note the best performances of all the models listed in Table 2

dataset consists of two partially overlapped clusters subjected to Gaussian distributions with equal μ and equal δ. In addition, extra stochastic outliers are added. Theoretically, in this case, the overall dataset can be clustered into two clusters with balanced distribution using FCM. However, the small amount of outlying observations whose values in the first dimension are much smaller than the normal data points serve as interference in the data structure when using FCM, resulting from the fact that the computation of the membership degrees considers all the clusters. This leads to a global shift toward the location of the outliers during the clustering process. ADFCM weakens this uncertain influence of the outliers and adjusts the centers of the clusters to better positions (e.g., the centroid of the first cluster toward the right in this case), ultimately resulting in better clustering performance compared with FCM (Table 2).

6.3 Global Adaptive Adjustment and Dimensional-Wise Normalization

6.3.1 Image Segmentation

We carry out several trials on image segmentation using ADFCM and ADREFCM in comparison with other related algorithms. Taking red–green–blue (RGB) values (three dimensions) and spatial information as two-dimensional (2D) coordinate positions into consideration, Figs. 6 and 7 present the performance of the different clustering methods. Figure 6 focuses on object segmentation, while Fig. 7 focuses on computed tomography segmentation. Figure 6a shows the original image where four unique subclasses (three foreground subclasses and the background subclass) are included. FCM does not perform very well in the image segmentation because of the potential dimensional

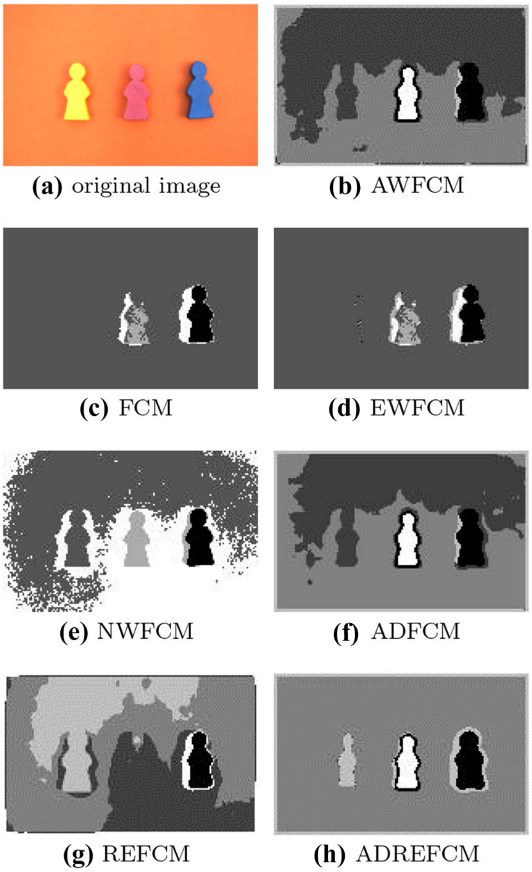


Fig. 6 Segmentation experiments on image containing three foreground subclasses and one background subclass

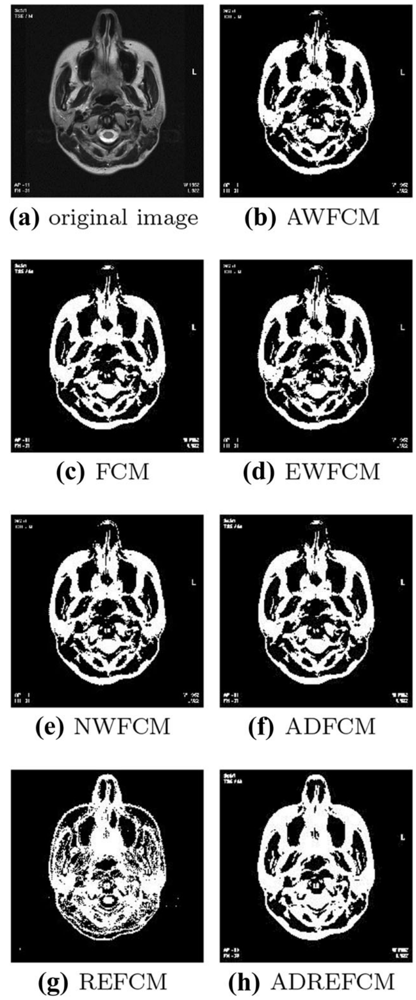


Fig. 7 Image segmentation experiments on X-ray output of a patient's skull

imbalance between the RGB and spatial information. Inheriting the result of FCM and restarting consecutive iterations using AWFCM, EWFCM, NWFCM, and ADFCM does not achieve much improvement for the same reason as mentioned above, but in combination with GMM, ADREFCM adaptively adjusts the situation. A seen in Fig. 6d, the background and three foreground objects are visibly separated when using ADREFCM.

As presented in Figs. 8, 9, 10, and 11, four more experiments are carried out on a scenery picture and three images under the condition of low illumination, where the partially overlapping objects have limited color features. Extra sparse outliers (extremely bright spots) are added to these images as shown in the "original image" in each case. The outliers pull the centers of the objects toward the negative directions and destroy their structure in the images, resulting in uncertainty in the data structure extracted by the clustering process. By global adjustment of the RE term, ADREFCM segments the components more effectively, outperforming the other related FCM algorithms as shown in Figs. 8, 9, 10, and 11.

6.3.2 Artificial Dataset Clustering

This section discusses the clustering results of two-dimensional nonspherical data belonging to two clusters.

As shown in Fig. 12a, the original dataset consists of three rectangular clusters. Figure 12b–d presents the clustering results achieved using the different algorithms. Note that points of the same color belong to the same cluster.

FCM is based on Euclidean distance, which can result in improper clustering when dealing with nonspherical datasets, as shown in Fig. 12c. ADFCM is sensitive to dimensional inconsistency, as shown in Fig. 12b. In comparison with FCM, the centroids of the clusters are adaptively pushed slightly when using ADREFCM to fit the distribution of the dataset, which can capture its latent structure well, as shown in Fig. 12d.

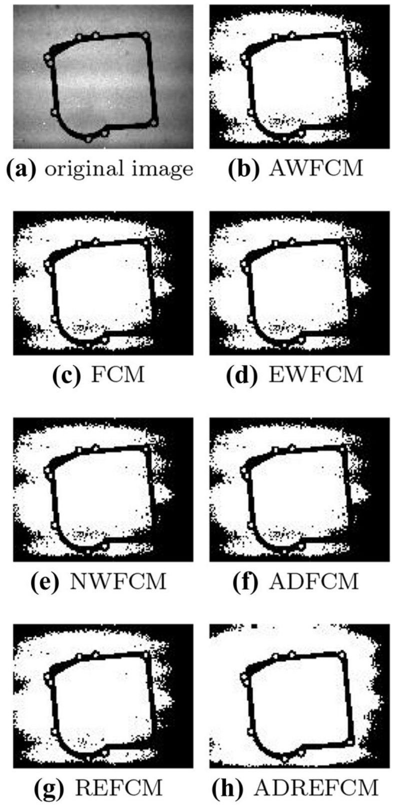


Fig. 8 Image segmentation experiments on industry image I under low-illumination condition

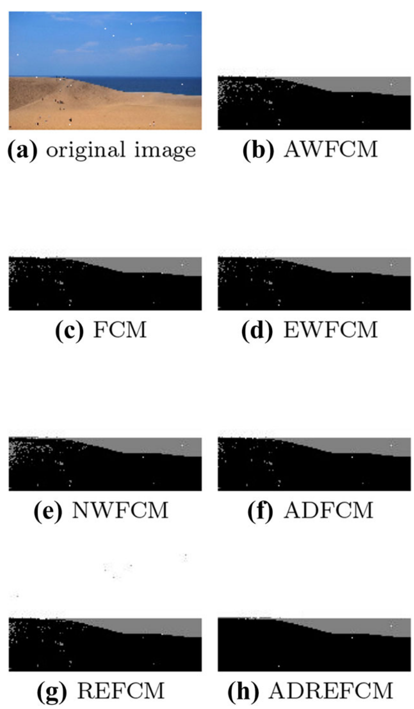


Fig. 9 Image segmentation experiments on noisy scenery image I

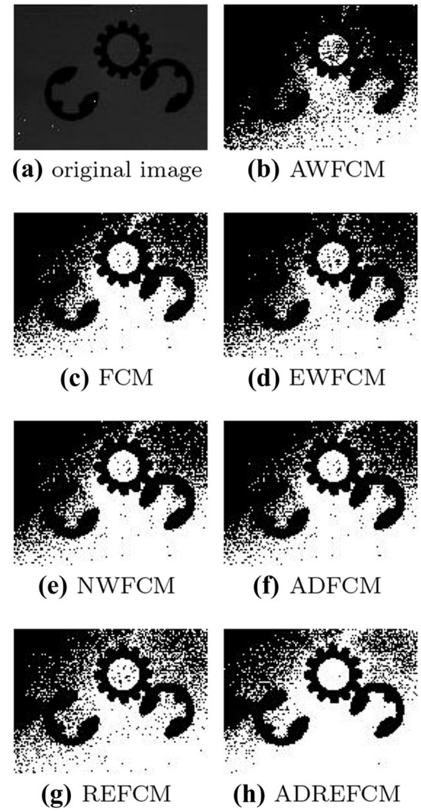


Fig. 10 Image segmentation experiments on industry image II under low-illumination condition

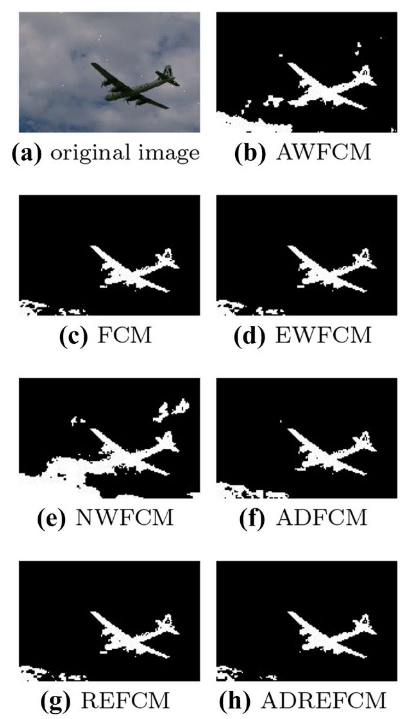


Fig. 11 Image segmentation experiments on noisy scenery image II

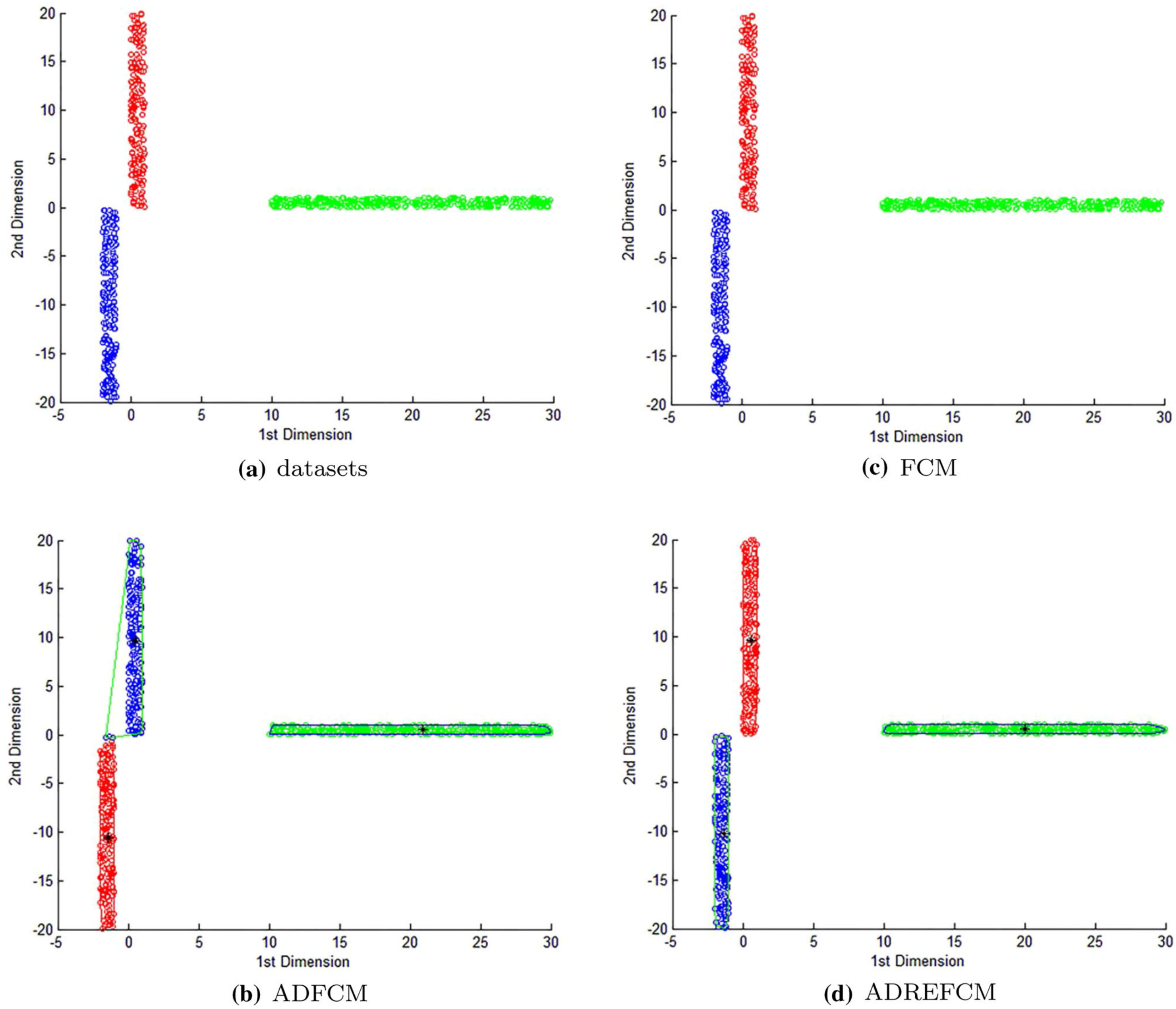
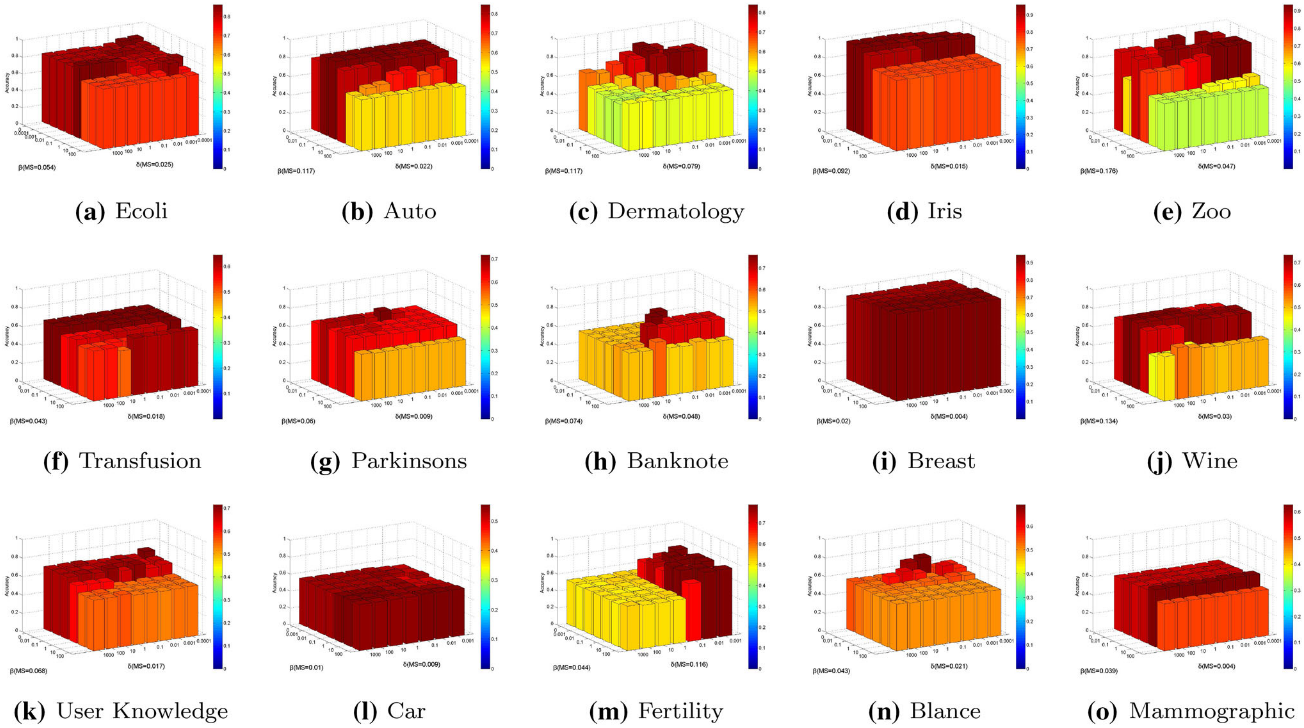


Fig. 12 Two-dimensional clustering experiments indicating the global adaptive adjustment of RE

6.4 Parameter Sensitivity Analysis

To analyze the sensitivity of the algorithm to its parameters, we arbitrarily cluster 15 splits out of the UCI repository with an extra $\frac{1}{100}$ outliers using ADREFCM while varying δ and β, as done by Luo and Wen [43, 44]. β helps consideration of the global information in the dataset. However, outliers cause data distribution deviations, thus excessive β will result in overconsideration of noisy global information, so we set the value of β within a certain range. In this experiment, δ and β are tuned within the range of $[10^{-4}, 10^{3}]$ and $[0, 10^{2}]$, respectively. Figure 13 visualizes the δ-β accuracy histogram of ADREFCM, showing the mean standard deviation (MS) of the clustering accuracy. These results indicate greater sensitivity to β than δ, although they are both important for promoting the performance of the algorithm. The optimal values of these parameters are data dependent. In most cases, we conclude that the optimal ranges of δ and β are [0.01, 1] and [0, 10], respectively.

In summary, the parameter sensitivity analysis and properties of the proposed methods discussed in detail in this section reveal that the proposed methods can achieve the targeted performance in dozens of real-world and artificial experiments.



(a) Ecoli **(b)** Auto **(c)** Dermatology **(d)** Iris **(e)** Zoo

(f) Transfusion **(g)** Parkinsons **(h)** Banknote **(i)** Breast **(j)** Wine

(k) User Knowledge **(l)** Car **(m)** Fertility **(n)** Blance **(o)** Mammographic

Fig. 13 Parameter sensitivity visualization w.r.t. δ-β-accuracy (MS, mean of standard deviation)

7 Conclusions

Adaptive-FCM is proposed as an extension of the adaptive norm to m_{th} order, to weaken the impact of noise while preserving the aggregation ability of FCM within clusters.

In addition, by combining the Gaussian mixture model and the relative entropy, adaptive-REFCM is proposed to solve the problems of both noise robustness and dimensional normalization in clustering tasks, considering not only fuzzy membership but also the distribution of clusters. Great performance of adaptive-REFCM is achieved based on its higher clustering accuracy in experiments on real-world (noise-free or noisy UCI repositories and image segmentation) and artificial datasets. With regard to future research, it is recommended to study the integration of existing noise-sensitive algorithms with the core design of adaptive-FCM and adaptive-REFCM to address their deficiency of noise sensitivity.

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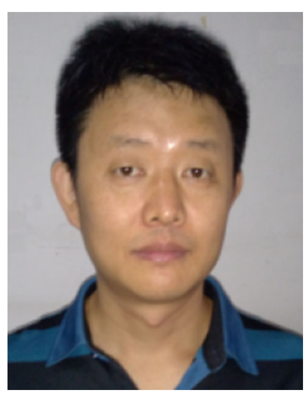

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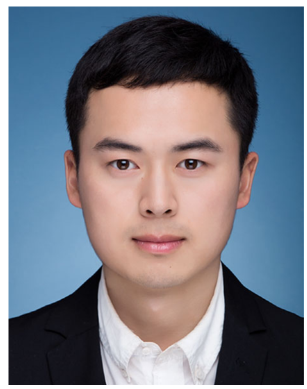

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