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# Structured generalized eigenvalue condition numbers for parameterized quasiseparable matrices

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## Abstract

In this paper, when $A$ and $B$ are $\{1;1\}$-quasiseparable matrices, we consider the structured generalized relative eigenvalue condition numbers of the pair $(A, B)$ with respect to relative perturbations of the parameters defining $A$ and $B$ in the quasiseparable and the Givens-vector representations of these matrices. A general expression is derived for the condition number of the generalized eigenvalue problems of the pair $(A, B)$, where $A$ and $B$ are any differentiable function of a vector of parameters with respect to perturbations of such parameters. Moreover, the explicit expressions of the corresponding structured condition numbers with respect to the quasiseparable and Givens-vector representation via tangents for $\{1; 1\}$-quasiseparable matrices are derived. Our proposed condition numbers can be computed efficiently by utilizing the recursive structure of quasiseparable matrices. We investigate relationships between various condition numbers of structured generalized eigenvalue problem when $A$ and $B$ are $\{1;1\}$-quasiseparable matrices. Numerical results show that there are situations in which the unstructured condition number can be much larger than the structured ones.

**Keywords** Condition numbers · Simple generalized eigenvalue · Low-rank structured matrices · $\{1;1\}$-quasiseparable matrices · Quasiseparable representation · Givens-vector representation

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## 1 Introduction

The generalized eigenvalue problem of the pair $(A,\ B)$ is defined as follows:

$$Ax = \lambda Bx, \quad A,\ B \in \mathbb{C}^{n\times n}. \tag{1.1}$$

If $x \neq 0$, we say that $\lambda$ is an eigenvalue of the pair $(A,\ B)$ and $x$ is the corresponding right eigenvector. A nonzero vector $y$ is called the left eigenvector corresponding to the generalized eigenvalue $\lambda$ of the pair $(A,\ B)$, when it satisfies $y^*A = \lambda y^*B$, where $y^*$ is the conjugate transpose of $y$. The generalized eigenvalue problem (1.1) is an extension of the classical eigenvalue problem

$$Ax = \lambda x, \quad 0 \neq x \in \mathbb{C}^n,\ A \in \mathbb{C}^{n\times n}, \tag{1.2}$$

when $B = I_n$ in (1.1). Many researchers have defined and derived the condition number for the classical eigenvalue problem (1.2); see [7,31,37] and references therein. However, condition numbers mentioned above are normwise type, which measure the errors for input and output data by means of norms. When the data is sparse or badly scaled, it is reasonable to consider condition numbers by measuring the componentwise perturbation for the input data, which are named *componentwise* condition numbers for eigenvalue problems. We refer the reader to look through the papers [8,19,22,27,29] for detailed explanations. For structured eigenvalue problems, it is suitable to investigate structured perturbations on the input data, because structure preserving algorithms that preserve the underlying matrix structure can enhance the accuracy and efficiency of the eigenvalue computation. There have been many papers on structured condition numbers for structured eigenvalue problems; see references [6,8,9,20,24,25,28,29].

Theories and algorithms of the generalized eigenvalue problem (1.1) can be found in the books [30,31]. Throughout this paper, we always assume that the pair $(A,\ B)$ is *regular*, that is $\det(A - \lambda B)$ is not identically zero in $\lambda$. The structured backward error and condition number for the generalized eigenvalue problem has been studied in [21]. Higham and Higham concentrated on the case that the generalized eigenvalue $\lambda$ of the pair $(A,\ B)$ is simple and finite. For existing perturbation results on deflating subspaces of a regular matrix pair $(A,\ B)$, we refer readers to [23] and [31, Section VI.2.4]. Also the normwise perturbation theory for the regular matrix pair $(A,\ B)$ can be found in [16].

In this paper, we are concerned with the structured componentwise condition number of (1.1) when the matrices $A$ and $B$ have *parameterized representations* [35, Ch. 2]. Especially, we concentrate on the case that $A$ and $B$ are $\{1;1\}$-quasiseparable matrices, which belong to the subcategory of the low-rank structured matrix and first appeared in [13]. Low-rank structured matrices, which satisfy that submatrices of them have ranks much smaller than the size of the matrices, have been studied extensively in numerical linear algebra and appear in many applications; see the recent books [14,15,35,36] and references therein. The rounding error analyses of fast algorithms for computations with low-rank structured matrices can be found in [1,2,10,32,38]. Dopico and Pomés introduced and investigated structured condition numbers with respect to perturbations of the parameters to the eigenvalue problem (1.2) and the solutions of linear systems of equations with low-rank structured coefficient matrices in [11,12], respectively. The results of [11,12] reveal that the simple eigenvalue of {1;1}-quasiseparable matrices and the solution to linear systems with {1;1}-quasiseparable matrices may be much less sensitive to perturbations of the parameters describing the matrices than to perturbations of the entries of the matrix. Moreover, the structured condition number analysis for linear systems with multiple right-hand sides when the coefficient matrix is low-rank structured matrix is investigated in [26].

The generalized eigenvalue problem (1.1) with the low-rank structured pair $(A,\ B)$ comes, for instance, from problems of polynomial root-finding where polynomials are expressed in a certain basis for example the monomial or Chebyshev-likes bases; see [3–5,17,18,33] for more details. For example, a modified QZ algorithm for computing the generalized eigenvalues of certain $n \times n$ rank structured pairs $(A,\ B)$ using $\mathcal{O}(n^2)$ flops and $\mathcal{O}(n)$ memory storage is proposed in [3], where $A$ is a rank-one perturbation of a Hermitian matrix and $B$ is a rank-one correction of the identity matrix. This kind of the generalized eigenvalue problem (1.1) arises from zero-finding problems for polynomials expressed in Chebyshev-likes bases. A structured pair $(A,\ B)$ can be represented by some parameters denoted by $(A,\ B) \in \mathcal{P}_n$; see [3, p. 94] for more details. The QZ iteration then perform multiplications among the representations and moreover the structure of $\mathcal{P}_n$ is preserved during the QZ iteration. There are some common parameters defining $A$ and $B$ simultaneously. In this paper, we propose the structured condition numbers for the generalized eigenvalue problem [30,31] with low-rank structured coefficient matrices. Specifically, we consider the structured perturbation analysis of (1.1) with respect to parameters defining the pair $(A,\ B)$ when there are common parameters representing $A$ and $B$ simultaneously. The situation that the representing parameters of $A$ are independent of the ones of $B$ will be a special case of the general setting.

Throughout this paper, we assume that the generalized eigenvalue $\lambda$ of a regular matrix pair $(A,\ B)$ is nonzero, simple and finite. The structured componentwise condition numbers for the generalized eigenvalue of a regular matrix pair $(A,\ B)$ with respect to the *general quasiseparable* and *Givens-vector representation via tangents* [11,12] are introduced and investigated. We prove that the condition number with respect to any general quasiseparable representation is identical, because it relies on the entries of the matrices instead of on the parameters of the representation. Also, we prove that the condition number with respect to the Givens-vector representation via tangents is smaller than the corresponding one with respect to any quasiseparable representation. Furthermore, our proposed condition numbers can be computed efficiently by utilizing the recursive structure of quasiseparable matrices. Numerical experiments show that there exist some particular pairs $(A,\ B)$ such that structured generalized eigenvalue condition numbers can be much smaller than the unstructured condition numbers.

The rest of the paper is organized as follows. In Sect. 2, some previous classical results about the perturbation theory for the generalized eigenvalue problem are reviewed. Also we will give brief introductions of some previous results on {1;1}-

quasiseparable matrices. Then, the general theory of the structured componentwise condition number for the simple and finite generalized eigenvalue of the pair $(A, B)$ with respect to the parameters that represent the pair $(A, B)$ is derived, which can be applied for studying the generalized eigenvalue condition numbers in Sect. 3. We compare the two proposed structured condition numbers with respect to the general quasiseparable representation and the Givens-vector representation via tangents in Sect. 4. Numerical experiments are done in Sect. 5 to illustrate the differences between structured and unstructured condition numbers.

**Notations.** In this paper, we adopt the following notations. For a given matrix $C \in \mathbb{R}^{m\times n}$, the symbols $C(:, i)$ and $C(j, :)$ are $i$-th and $j$-th column and row of $C$ respectively. For a given matrix $C = (C_{ij}) \in \mathbb{R}^{n\times n}$, $|C| = (|C_{ij}|)$, at the same time we adopt a similar notation for vectors. The notation $C(i_1 : i_2, j_1 : j_2)$ is a submatrix of $C \in \mathbb{C}^{n\times n}$ consisting of rows $i_1$ up to and including $i_2$ and columns $j_1$ up to and including $j_2$ of $C$ with $1 \le i_1 \le i_2 \le n$ and $1 \le j_1 \le j_2 \le n$. For two conformal dimensional matrices $C$ and $D$, the notation $C \le D$ should be understood componentwisely.

## 2 Preliminaries

In this section we will give brief introductions of some previous results on the $\{1;1\}$-quasiseparable $n$-by-$n$ matrix, which has the representation with $\mathcal{O}(n)$ parameters instead of its $n^2$ entries. Moreover, some previous classical results about the perturbation theory for the generalized eigenvalue problem are reviewed. Then the structured componentwise condition number for the simple and finite generalized eigenvalue of the pair $(A, B)$ with respect to the parameters that represent the pair $(A, B)$ is introduced. Moreover, we derive the general explicit formula of the proposed condition number, from which the corresponding condition number expressions for the pair $\{1;1\}$-quasiseparable matrices $(A, B)$ in the quasiseparable representation and Givens-vector representation can be deduced.

**Definition 2.1** Let $7n - 8$ real parameters be given by

$$\Omega_{QS} = \left(\{p_i\}_{i=2}^{n}, \{a_i\}_{i=2}^{n-1}, \{q_i\}_{i=1}^{n-1}, \{d_i\}_{i=1}^{n}, \{g_i\}_{i=1}^{n-1}, \{b_i\}_{i=2}^{n-1}, \{h_i\}_{i=2}^{n}\right), \tag{2.1}$$

which define a $n$-by-$n$ matrix $C$ as follows:

$$C = \begin{bmatrix} d_1 & g_1h_2 & g_1b_2h_3 & \cdots & g_1b_2\cdots b_{n-1}h_n \\ p_2q_1 & d_2 & g_2h_3 & \cdots & g_2b_3\cdots b_{n-1}h_n \\ p_3a_2q_1 & p_3q_2 & d_3 & \cdots & g_3b_4\cdots b_{n-1}h_n \\ p_4a_3a_2q_1 & p_4a_3q_2 & p_4q_3 & \cdots & g_4b_5\cdots b_{n-1}h_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ p_na_{n-1}\cdots a_2q_1 & p_na_{n-1}\cdots a_3q_2 & p_na_{n-1}\cdots a_4q_3 & \cdots & d_n \end{bmatrix}.$$

Then we say that the matrix $C \in \mathbb{R}^{n\times n}$ is a $\{1; 1\}$-quasiseparable matrix with the *quasiseparable representation* $\Omega_{QS}$, which is not unique.

In the next lemma, the derivative of the entries of a $\{1; 1\}$-quasiseparable matrix $C$ with respect to its quasiseparable representation $\Omega_{QS}$ is given, which appeared in [12, Theorem 4.4] and [11, Lemma 4.4]. Using Lemma 2.1, we will obtain explicit expressions of structured condition numbers for the generalized eigenvalue problem involving a pair of quasiseparable matrices in the quasiseparable representation in Theorem 3.1.

**Lemma 2.1** *Let $C$ be a $\{1; 1\}$-quasiseparable matrix and $C = C_L + C_D + C_U$, where $C_L$ is strictly lower triangular, $C_D$ is diagonal, and $C_U$ is strictly upper triangular. Assume that $\Omega_{QS}$ is the quasiseparable representation of $C$, where $\Omega_{QS}$ is defined in* (2.1). *Then the entries of $C$ are differentiable functions of the parameters in $\Omega_{QS}$ and their derivatives with respect to $\Omega_{QS}$ can be characterized by*

$$\begin{cases} 1)\ \ \dfrac{\partial C}{\partial d_i} = \mathbf{e}_i \mathbf{e}_i^\top, & i = 1, \ldots, n, \\ 2)\ \ p_i \dfrac{\partial C}{\partial p_i} = \mathbf{e}_i C_L(i,:),\ h_i \dfrac{\partial C}{\partial h_i} = C_U(:,i)\mathbf{e}_i^\top, & i = 2, \ldots, n, \\ 3)\ \ a_i \dfrac{\partial C}{\partial a_i} = \begin{bmatrix} 0 & 0 \\ C(i+1:n, 1:i-1) & 0 \end{bmatrix}, & i = 2, \ldots, n-1, \\ 4)\ \ q_i \dfrac{\partial C}{\partial q_i} = C_L(:,i)\mathbf{e}_i^\top,\ g_i \dfrac{\partial C}{\partial g_i} = \mathbf{e}_i C_U(i,:), & i = 1, \ldots, n-1, \\ 5)\ \ b_i \dfrac{\partial C}{\partial b_i} = \begin{bmatrix} 0 & C(1:i-1, i+1:n) \\ 0 & 0 \end{bmatrix}, & i = 2, \ldots, n-1, \end{cases}$$

*where $\mathbf{e}_i$ is the $i$th column of the $n$-by-$n$ identity matrix.*

There is another important representation of $\{1;1\}$-quasiseparable matrices, which is called Givens-vector representation [34]. The numerical stability of fast matrix computations of quasiseparable matrices can be achieved through this kind of Givens-vector representation.

**Definition 2.2** Let

$$\Omega_{QS}^{GV} := (\{c_i, s_i\}_{i=2}^{n-1}, \{v_i\}_{i=1}^{n-1}, \{d_i\}_{i=1}^{n}, \{e_i\}_{i=1}^{n-1}, \{r_i, t_i\}_{i=2}^{n-1}), \tag{2.2}$$

where $(c_i, s_i)$ and $(r_i, t_i)$ are pairs of cosine-sine with $c_i^2 + s_i^2 = 1$ and $r_i^2 + t_i^2 = 1$ for every $i \in \{2, 3, \ldots, n-1\}$, and for $\{v_i\}_{i=1}^{n-1}$, $\{d_i\}_{i=1}^{n}$, $\{e_i\}_{i=1}^{n-1}$, all of them are independent real parameters. A matrix $C \in \mathbb{R}^{n\times n}$ is a $\{1; 1\}$-quasiseparable matrix if it can be represented by $\Omega_{QS}^{GV}$ as follows:

$$C = \begin{bmatrix} d_1 & e_1 r_2 & e_1 t_2 r_3 & \cdots & e_1 t_2 \cdots t_{n-2} r_{n-1} & e_1 t_2 \cdots t_{n-1} \\ c_2 v_1 & d_2 & e_2 r_3 & \cdots & e_2 t_3 \cdots t_{n-2} r_{n-1} & e_2 t_3 \cdots t_{n-1} \\ c_3 s_2 v_1 & c_3 v_2 & d_3 & \cdots & e_3 t_4 \cdots t_{n-2} r_{n-1} & e_3 t_4 \cdots t_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ c_{n-1} s_{n-2} \cdots s_2 v_1 & c_{n-1} s_{n-2} \cdots s_3 v_2 & c_{n-1} s_{n-2} \cdots s_4 v_3 & \cdots & d_{n-1} & e_{n-1} \\ s_{n-1} s_{n-2} \cdots s_2 v_1 & s_{n-1} s_{n-2} \cdots s_3 v_2 & s_{n-1} s_{n-2} \cdots s_4 v_3 & \cdots & v_{n-1} & d_n \end{bmatrix}.$$

And $\Omega_{QS}^{GV}$ is called the Givens-vector representation of $C$.

***Remark 2.1*** As discussed on [12, p. 497], it is obvious that the Givens-vector representation is a particular case of the quasiseparable representation for $\{1;1\}$-quasiseparable matrices in view of Definitions 2.1 and 2.2.

It is obvious that the pairs $\{c_i, s_i\}$ and $\{r_i, t_i\}$ of $\Omega_{QS}^{GV}$ are not independent. Thus arbitrary componentwise perturbations of $\Omega_{QS}^{GV}$ destroy the cosine-sine pairs, and it is more reasonable to restrict the perturbations to those that preserve the cosine-sine pairs. In Definition 2.3, introduced in [12], an additional parameterization by using tangents is provided in order to make explicit the correlations between $c_i, s_i$ and $r_i, t_i$.

**Definition 2.3** Let $C \in \mathbb{R}^{n\times n}$ be a $\{1;1\}$-quasiseparable matrix with the Givens-vector representation $\Omega_{QS}^{GV}$ given by (2.2), then the Givens-vector representation via tangents is given by

$$\Omega_{GV} := \left(\{w_i\}_{i=2}^{n-1},\ \{v_i\}_{i=1}^{n-1},\ \{d_i\}_{i=1}^{n},\ \{e_i\}_{i=1}^{n-1},\ \{u_i\}_{i=2}^{n-1}\right), \tag{2.3}$$

where

$$c_i = \frac{1}{\sqrt{1+w_i^2}},\quad s_i = \frac{w_i}{\sqrt{1+w_i^2}},\quad r_i = \frac{1}{\sqrt{1+u_i^2}},\quad t_i = \frac{u_i}{\sqrt{1+u_i^2}},\quad i = 2,\ldots,n-1.$$

In order to use differential calculus to deduce an explicit expression of the structured condition number of the generalized eigenvalue problem with respect to the tangent-Givens-vector representation, we will need Lemma 2.2 from [12].

**Lemma 2.2** *Let $C \in \mathbb{R}^{n\times n}$ be a {1;1}-quasiseparable matrix and let $\Omega_{GV}$ be the tangent-Givens-vector representation of C, where $\Omega_{GV}$ is given by (2.3). Then the entries of C are differentiable functions of the parameters in $\Omega_{GV}$ and their derivatives with respect to $\Omega_{GV}$ can be characterized by*

$$\begin{cases} w_i\dfrac{\partial C}{\partial w_i} = \begin{bmatrix} 0 & 0 \\ -s_i^2 C(i, 1:i-1) & 0 \\ c_i^2 C(i+1:n, 1:i-1) & 0 \end{bmatrix}, & i = 2,\ldots,n-1, \\ u_i\dfrac{\partial C}{\partial u_i} = \begin{bmatrix} 0 & -t_i^2 C(1:i-1, j) & -r_i^2 C(1:i-1, i+1:n) \\ 0 & 0 & 0 \end{bmatrix}, & i = 2,\ldots,n-1. \end{cases}$$

In the remainder of this section, we will review some previous classical results about the perturbation theory for the generalized eigenvalue problem. Then we will introduce the structured componentwise condition number for the simple and finite generalized eigenvalue of the pair $(A, B)$ with respect to the parameters that represent the pair $(A, B)$. Finally the general explicit formula of the proposed condition number is derived, which can be used to deduce the corresponding condition number expressions for the pair of $\{1;1\}$-quasiseparable matrices $A$ and $B$ in the quasiseparable representation and Givens-vector representation, respectively. First, the following lemma gives the first order expansion of a simple and finite generalized eigenvalue of the pair $(A, B)$ under perturbations on $A$ and $B$.

**Lemma 2.3** [21,31] *Let $\lambda$ be a simple and finite generalized eigenvalue of the pair $(A, B)$ with corresponding right eigenvector $x$ and left eigenvector $y$, where $A = (a_{ij})_{n\times n}$ and $B = (b_{ij})_{n\times n} \in \mathbb{C}^{n\times n}$, then for any perturbation pair $(\Delta A, \Delta B)$ with $\Delta A, \Delta B \in \mathbb{C}^{n\times n}$, when $\varepsilon := \max\{\|\Delta A\|, \|\Delta B\|\}$ is sufficient small, there exists a unique simple and finite eigenvalue $\tilde{\lambda}$ of the pair $(A + \Delta A, B + \Delta B)$ such that*

$$\widetilde{\lambda} = \lambda + \frac{y^* \Delta A x - \lambda y^* \Delta B x}{y^* B x} + \mathcal{O}(\varepsilon^2),$$

*where $\|\Delta A\|$ and $\|\Delta B\|$ are any matrix norm of $\Delta A$ and $\Delta B$, respectively.*

The relative *unstructured* componentwise condition number $\mathscr{K}(\lambda; (A, B))$ is proposed and investigated in [21], which describes the worst case sensitivity of a nonzero, finite and simple generalized eigenvalue $\lambda$ with respect to the largest relative perturbation of each of the nonzero entries of $|A|$ and $|B|$.

**Definition 2.4** With the notations of Lemma 2.3, we define the relative componentwise condition number of $\lambda$ as follows:

$$\mathscr{K}(\lambda; (A, B)) := \lim_{\eta\to 0} \sup \left\{ \frac{|\Delta\lambda|}{\eta|\lambda|} : (\lambda + \Delta\lambda) \text{ is a simple, finite, and nonzero} \right.$$
$$\left. \text{generalized eigenvalue of } (A + \Delta A, B + \Delta B), |\Delta A| \le \eta|A|,\ |\Delta B| \le \eta|B| \right\}.$$

***Remark 2.2*** Because $\lambda+\Delta\lambda$ is a generalized eigenvalue of the pair $(A+\Delta A, B+\Delta B)$, there exists a nonzero vector $x + \Delta x$ such that

$$(A + \Delta A)(x + \Delta x) = (\lambda + \Delta\lambda)(B + \Delta B)(x + \Delta x),$$

As pointed in [21, Section 2.2], Definition 2.4 is a little loose. This is because that if $\widetilde{\lambda}$ is a generalized eigenvalue of the pair $(A, B)$ with corresponding eigenvector $\tilde{x}$ such that $\widetilde{\lambda}$ is distinct from $\lambda$. Thus we can take $\Delta A = \Delta B = 0$, $x + \Delta x = \widetilde{x}$, and $\lambda + \Delta\lambda = \widetilde{\lambda}$ to obtain $\mathscr{K}(\lambda; (A, B)) = \infty$. The definition therefore needs to be augmented with the requirement that $\Delta x \to 0$ as $\eta \to 0$. For simplicity of presentation we omit this requirement from the definitions of condition numbers as done in [21].

The explicit expressions of $\mathscr{K}(\lambda; (A, B))$ was first given in [21] as follows:

$$\mathscr{K}(\lambda; (A, B)) = \frac{|y^*||A||x| + |\lambda||y^*||B||x|}{|\lambda||y^* B x|}. \tag{2.4}$$

We will show that the above expression can be derived from a more general Theorem 2.1 direclty.

Usually, structured matrices can be represented by few parameters. In the following definition, we introduce the relative structured componentwise generalized eigenvalue condition number with respect to those parameters for the finite, nonzero and simple generalized eigenvalue $\lambda$ of the pair $(A, B)$. Here we assume that the parametrization representations $\Omega_A$ of $A$ and $\Omega_B$ of $B$ have common parameters.

**Definition 2.5** Let $A$ and $B$ be matrices whose entries are differentiable functions of these sets of parameters $\Omega_A = (\omega_1, \omega_2, \ldots, \omega_N, \omega_{N+1}, \ldots, \omega_{N+M_1})^\top$ and $\Omega_B = (\omega_1, \omega_2, \ldots, \omega_N, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{M_2})^\top$. We denote these by $A(\Omega_A)$ and $B(\Omega_B)$. Let $\lambda$ be a finite, simple and nonzero generalized eigenvalue of the pair $(A(\Omega_A),\ B(\Omega_B))$ with left eigenvector $y$ and right eigenvector $x$, separately. Then we define

$$\begin{aligned}\mathscr{K}(\lambda, (A,\ B); (\Omega_A,\ \Omega_B)) := \lim_{\eta\to 0}\sup \Bigg\{ \frac{|\Delta\lambda|}{\eta|\lambda|} &: (\lambda + \Delta\lambda) \text{ is a simple, nonzero} \\ &\text{and finite generalized eigenvalue of the pair } (A(\Omega_A + \Delta\Omega_A), \\ &B(\Omega_B + \Delta\Omega_B)),\ |\Delta\Omega_A| \le \eta|\Omega_A|,\ |\Delta\Omega_B| \le \eta|\Omega_B| \Bigg\}.\end{aligned}$$

If the matrices $A$ and $B$ are clear from the context, we usually denote the condition number $\mathscr{K}(\lambda, (A,\ B); (\Omega_A,\ \Omega_B))$ by $\mathscr{K}(\lambda; \Omega)$, where

$$\Omega := \Omega_A \cup \Omega_B = (\omega_1, \ldots, \omega_{N+M_1}, \varepsilon_1, \ldots, \varepsilon_{M_2})^\top.$$

***Remark 2.3*** The above definition is general, since when the index $N = 0$ we have $\Omega_A \cap \Omega_B = \varnothing$, which means that the parametrization representations $\Omega_A$ of $A$ and $\Omega_B$ of $B$ are independent.

In Proposition 2.1 we will give the explicit partial derivatives of a finite, simple and nonzero generalized eigenvalue $\lambda$ of the pair $(A(\Omega_A),\ B(\Omega_B))$ with respect to the parameters $\Omega_A$ and $\Omega_B$ given in Definition 2.5, which also play an important role in computing the componentwise relative eigenvalue condition number for quasiseparable matrices with respect to parameters. Theorem 2.1 is the main result of this section. The proof of Proposition 2.1 is based on Lemma 2.3 and matrix calculus, which is a generalization of the proof of [12, Proposition 2.12].

**Proposition 2.1** *With the notations of Definition 2.5, we have*

$$\frac{\partial\lambda}{\partial\omega_i} = \begin{cases} \dfrac{1}{y^*Bx}\left(y^*\dfrac{\partial A(\Omega_A)}{\partial\omega_i}x - \lambda y^*\dfrac{\partial B(\Omega_B)}{\partial\omega_i}x\right), & 1 \le i \le N, \\ \dfrac{1}{y^*Bx}\left(y^*\dfrac{\partial A(\Omega_A)}{\partial\omega_i}x\right), & N+1 \le i \le N+M_1, \end{cases}$$

$$\frac{\partial\lambda}{\partial\varepsilon_i} = -\frac{1}{y^*Bx}\left(\lambda y^*\frac{\partial B(\Omega_B)}{\partial\varepsilon_i}x\right), \quad 1 \le i \le M_2.$$

Applying Proposition 2.1 and using Definition 2.5, we can prove Theorem 2.1 directly. Since the proof of Theorem 2.1 is similar to the proof of [12, Theorem 2.13], the detailed proof of Theorem 2.1 is omitted.

**Theorem 2.1** *Under the same hypotheses of Definition 2.5, the expression of* $\mathscr{K}(\lambda;\Omega)$ *given by Definition 2.5 can be characterized by*

$$\mathscr{K}(\lambda;\Omega)=\sum_{i=1}^{N+M_1}\left|\frac{\omega_i}{\lambda}\frac{\partial\lambda}{\partial\omega_i}\right|+\sum_{i=1}^{M_2}\left|\frac{\varepsilon_i}{\lambda}\frac{\partial\lambda}{\partial\varepsilon_i}\right|. \tag{2.5}$$

In the following, we will show that the explicit expression of $\mathscr{K}(\lambda;(A,\ B))$ in (2.4) can be derived from Theorem 2.1 when we identify the parameter vector $\Omega$ defined in Definition 2.5 as $\Omega=(vec(A)^\top, vec(B)^\top)^\top$, where $vec(A)$ stacks columns of $A$ one by one. In fact,

$$a_{ij}\ \frac{\partial A}{\partial\, a_{ij}}=a_{ij}\,\mathbf{e}_i\,\mathbf{e}_j^\top,\quad b_{ij}\ \frac{\partial B}{\partial b_{ij}}=b_{ij}\,\mathbf{e}_i\,\mathbf{e}_j^\top,$$

where $\mathbf{e}_i$ and $\mathbf{e}_j$ are the $i$th and $j$th canonical vectors in $\mathbb{C}^n$, respectively. Therefore we reformulate (2.5) as

$$\begin{aligned}\mathscr{K}(\lambda;(A,\ B))&=\sum_{i=1}^{n}\sum_{j=1}^{n}\left|\frac{a_{ij}}{\lambda}\frac{\partial\lambda}{\partial a_{ij}}\right|+\sum_{i=1}^{n}\sum_{j=1}^{n}\left|-\frac{b_{ij}}{\lambda}\frac{\partial\lambda}{\partial b_{ij}}\right|\\&=\frac{\sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|\left|\overline{y_i}\right|\left|x_j\right|+|\lambda|\sum_{i=1}^{n}\sum_{j=1}^{n}|b_{ij}||\overline{y_i}||x_j|}{|\lambda||y^*Bx|}\\&=\frac{|y^*||A||x|+|\lambda||y^*||B||x|}{|\lambda||y^*Bx|},\end{aligned}$$

which is the exact expression $\mathscr{K}(\lambda;(A,\ B))$ given by (2.4).

## 3 Generalized eigenvalue condition numbers for {1;1}-quasiseparable matrices

In this section, we will focus on deriving explicit expressions of the condition number for the generalized eigenvalue problem (1.1) for {1;1}-quasiseparable matrix pairs $(A, B)$ in the quasiseparable representation and Givens-vector representation via tangents.

### 3.1 The quasiseparable representation

In this subsection, we will define the generalized eigenvalue relative componentwise condition number for a nonzero, simple and finite generalized eigenvalue $\lambda$ of the pair $(A,\ B)$, where both $A$ and $B$ are {1;1}-quasiseparable matrices, with respect to their parameters given by the quasiseparable representation below. Recalling Definition 2.1, let

$$\Omega_{QS}^{A} := \Big(\{p_i\}_{i=2}^{l_p}, \{p_i^A\}_{i=l_p+1}^{n}, \{a_i\}_{i=2}^{l_a}, \{a_i^A\}_{i=l_a+1}^{n-1}, \{q_i\}_{i=1}^{l_q}, \{q_i^A\}_{i=l_q+1}^{n-1}, \{d_i\}_{i=1}^{l_d}, \{d_i^A\}_{i=l_d+1}^{n}, \{g_i\}_{i=1}^{l_g}, \{g_i^A\}_{i=l_g+1}^{n-1}, \{b_i\}_{i=2}^{l_b}, \{b_i^A\}_{i=l_b+1}^{n-1}, \{h_i\}_{i=2}^{l_h}, \{h_i^A\}_{i=l_h+1}^{n}\Big) \tag{3.1}$$

and

$$\Omega_{QS}^{B} := \Big(\{p_i\}_{i=2}^{l_p}, \{p_i^B\}_{i=l_p+1}^{n}, \{a_i\}_{i=2}^{l_a}, \{a_i^B\}_{i=l_a+1}^{n-1}, \{q_i\}_{i=1}^{l_q}, \{q_i^B\}_{i=l_q+1}^{n-1}, \{d_i\}_{i=1}^{l_d}, \{d_i^B\}_{i=l_d+1}^{n}, \{g_i\}_{i=1}^{l_g}, \{g_i^B\}_{i=l_g+1}^{n-1}, \{b_i\}_{i=2}^{l_b}, \{b_i^B\}_{i=l_b+1}^{n-1}, \{h_i\}_{i=2}^{l_h}, \{h_i^B\}_{i=l_h+1}^{n}\Big) \tag{3.2}$$

be the quasiseparable representation of $A$ and $B$, respectively. When $l_p = 1$, $l_a = 1$, $l_q = 0$, $l_d = 0$, $l_g = 0$, $l_b = 1$ and $l_h = 1$, we have that $\Omega_{QS}^{A} \cap \Omega_{QS}^{B} = \varnothing$, i.e., $\Omega_{QS}^{A}$ is independent of $\Omega_{QS}^{B}$. We denote by

$$\Omega_{QS} := \Big(\{p_i\}_{i=2}^{l_p}, \{p_i^A\}_{i=l_p+1}^{n}, \{p_i^B\}_{i=l_p+1}^{n}, \{a_i\}_{i=2}^{l_a}, \{a_i^A\}_{i=l_a+1}^{n-1}, \{a_i^B\}_{i=l_a+1}^{n-1}, \{q_i\}_{i=1}^{l_q}, \{q_i^A\}_{i=l_q+1}^{n-1}, \{q_i^B\}_{i=l_q+1}^{n-1}, \{d_i\}_{i=1}^{l_d}, \{d_i^A\}_{i=l_d+1}^{n}, \{d_i^B\}_{i=l_d+1}^{n}, \{g_i\}_{i=1}^{l_g}, \{g_i^A\}_{i=l_g+1}^{n-1}, \{g_i^B\}_{i=l_g+1}^{n-1}, \{b_i\}_{i=2}^{l_b}, \{b_i^A\}_{i=l_b+1}^{n-1}, \{b_i^B\}_{i=l_b+1}^{n-1}, \{h_i\}_{i=2}^{l_h}, \{h_i^A\}_{i=l_h+1}^{n}, \{h_i^B\}_{i=l_h+1}^{n}\Big) \tag{3.3}$$

the quasiseparable representation of the pair $(A, B)$ when $A$ and $B$ have quasiseparable representation $\Omega_{QS}^{A}$ and $\Omega_{QS}^{B}$, respectively. In the next theorem, we will apply Theorem 2.1 to derive the explicit expression for the condition number.

***Remark 3.1*** Here we assume that the quasiseparable representation of the pair $(A, B)$ has common parameters in the leading indexes. However, the proposed condition numbers and expressions with respect to arbitrary indexes of the common parameters in Theorem 3.1 can be defined and obtained with simple straightforward modifications.

**Theorem 3.1** *Let $A$ and $B$ be $\{1; 1\}$-quasiseparable matrices and let us express $A$ and $B$ as $A = A_L + A_D + A_U$ and $B = B_L + B_D + B_U$, respectively, with $A_L$ and $B_L$ strictly lower triangular, $A_D$ and $B_D$ diagonal, $A_U$ and $B_U$ strictly upper triangular. We have the following condition number expression:*

$$\begin{aligned}\mathscr{K}(\lambda; \Omega_{QS}) = \frac{1}{|\lambda||y^* B x|}\Bigg\{ & |y^*||A_D||x| + |\lambda||y^*||B_D||x| \\ & + |y^*||A_L x| + |\lambda||y^*||B_L x| \\ & + |y^* A_L||x| + |\lambda||y^* B_L||x| + |y^*||A_U x| + |\lambda||y^*||B_U x| \\ & + |y^* A_U||x| + |\lambda||y^* B_U||x| \\ & + \sum_{i=2}^{n-1}\left| y^* \begin{bmatrix} 0 & 0 \\ A(i+1:n, 1:i-1) & 0 \end{bmatrix} x \right| + \sum_{j=2}^{n-1}\left| y^* \begin{bmatrix} 0 & A(1:j-1, j+1:n) \\ 0 & 0 \end{bmatrix} x \right|\end{aligned}$$

$$+\sum_{i=2}^{n-1}|\lambda|\left|y^*\begin{bmatrix}0 & 0\\ B(i+1:n,1:i-1) & 0\end{bmatrix}x\right|+\sum_{j=2}^{n-1}|\lambda|\left|y^*\begin{bmatrix}0 & B(1:j-1,j+1:n)\\ 0 & 0\end{bmatrix}x\right|$$

$$+\alpha_d+\alpha_p+\alpha_g+\alpha_q+\alpha_h+\alpha_a+\alpha_b\Big\},$$

*where*

$$\begin{aligned}
\alpha_d &= \sum_{i=1}^{l_d}\Big(\big|\overline{y_i}A_D(i,i)x_i-\lambda\overline{y_i}B_D(i,i)x_i\big|-|\overline{y_i}A_D(i,i)x_i|-|\lambda|\,|\overline{y_i}B_D(i,i)x_i|\Big),\\
\alpha_p &= \sum_{i=2}^{l_p}\Big(|\overline{y_i}|\big|(A_L(i,:)-\lambda B_L(i,:))x\big|-|\overline{y_i}||A_L(i,:)x|-|\lambda||\overline{y_i}||B_L(i,:)x|\Big),\\
\alpha_g &= \sum_{i=1}^{l_g}\Big(|\overline{y_i}|\,|A_U(i,:)x-\lambda B_U(i,:)x|-|\overline{y_i}|\,|A_U(i,:)x|-|\lambda||\overline{y_i}|\,|B_U(i,:)x|\Big),\\
\alpha_q &= \sum_{i=1}^{l_q}\Big(|y^*(A_L(:,i)-\lambda B_L(:,i))||x_i|-|y^*A_L(:,i)||x_i|-|\lambda||y^*B_L(:,i)||x_i|\Big),\\
\alpha_h &= \sum_{i=2}^{l_h}\Big(\big|y^*\big(A_U(:,i)-\lambda B_U(:,i)\big)\big||x_i|-|y^*A_U(:,i)||x_i|-|\lambda||y^*B_U(:,i)||x_i|\Big),\\
\alpha_a &= \sum_{i=2}^{l_a}\Bigg(\left|y^*\left(\begin{bmatrix}0 & 0\\ A(i+1:n,1:i-1) & 0\end{bmatrix}-\lambda\begin{bmatrix}0 & 0\\ B(i+1:n,1:i-1) & 0\end{bmatrix}\right)x\right|\\
&\quad-\left|y^*\begin{bmatrix}0 & 0\\ A(i+1:n,1:i-1) & 0\end{bmatrix}x\right|-|\lambda|\left|y^*\begin{bmatrix}0 & 0\\ B(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\Bigg),\\
\alpha_b &= \sum_{i=2}^{l_b}\Bigg(\left|y^*\left(\begin{bmatrix}0 & A(1:i-1,i+1:n)\\ 0 & 0\end{bmatrix}-\lambda\begin{bmatrix}0 & B(1:i-1,i+1:n)\\ 0 & 0\end{bmatrix}\right)x\right|\\
&\quad-\left|y^*\begin{bmatrix}0 & A(1:i-1,i+1:n)\\ 0 & 0\end{bmatrix}x\right|-|\lambda|\left|y^*\begin{bmatrix}0 & B(1:i-1,i+1:n)\\ 0 & 0\end{bmatrix}x\right|\Bigg),
\end{aligned}\tag{3.4}$$

*and $A_D(i,i)$ is the $i$-th diagonal entry of $A_D$.*

***Proof*** From 1) of Lemma 2.1 and Proposition 2.1, for the parameters $d_i$, $d_i^A$ and $d_i^B$, we can know that

$$\begin{cases}\dfrac{d_i}{\lambda}\dfrac{\partial\lambda}{\partial d_i}=\dfrac{(1-\lambda)\overline{y_i}d_ix_i}{\lambda y^*Bx}, & i=1,\ldots,l_d,\\[2mm] \dfrac{d_i^A}{\lambda}\dfrac{\partial\lambda}{\partial d_i^A}=\dfrac{\overline{y_i}d_i^Ax_i}{\lambda y^*Bx}, & i=l_d+1,\ldots,n,\\[2mm] \dfrac{d_i^B}{\lambda}\dfrac{\partial\lambda}{\partial d_i^B}=\dfrac{-\overline{y_i}d_i^Bx_i}{y^*Bx}, & i=l_d+1,\ldots,n.\end{cases}$$

Thus from Theorem 2.1, we can derive that

$$
\begin{aligned}
K_d &= \sum_{i=1}^{l_d}\left|\frac{d_i}{\lambda}\frac{\partial\lambda}{\partial d_i}\right| + \sum_{i=l_d+1}^{n}\left|\frac{d_i^A}{\lambda}\frac{\partial\lambda}{\partial d_i^A}\right| + \sum_{i=l_d+1}^{n}\left|\frac{d_i^B}{\lambda}\frac{\partial\lambda}{\partial d_i^B}\right| \\
&= \frac{1}{|\lambda y^* Bx|}\Big(|y^*||A_D||x| + |\lambda||y^*||B_D||x|\Big) \\
&\quad + \sum_{i=1}^{l_d}\frac{1}{|\lambda y^* Bx|}\Big(\big|\overline{y_i}A_D(i,i)x_i - \lambda\overline{y_i}B_D(i,i)x_i\big| \\
&\quad - |\overline{y_i}A_D(i,i)x_i| - |\lambda|\,|\overline{y_i}B_D(i,i)x_i|\Big).
\end{aligned}
\tag{3.5}
$$

For the partial derivative of $\lambda$ with respect to parameters $p_i$, $p_i^A$ and $p_i^B$, from 2) of Lemma 2.1 and Proposition 2.1, it can be derived that

$$
\begin{cases}
\dfrac{p_i}{\lambda}\dfrac{\partial\lambda}{\partial p_i} = \dfrac{1}{\lambda y^* Bx}\big(\overline{y_i}A_L(i,:)x - \lambda\overline{y_i}B_L(i,:)x\big), & i = 2,\dots,l_p, \\
\dfrac{p_i^A}{\lambda}\dfrac{\partial\lambda}{\partial p_i^A} = \dfrac{1}{\lambda y^* Bx}\overline{y_i}A_L(i,:)x, & i = l_p+1,\dots,n, \\
\dfrac{p_i^B}{\lambda}\dfrac{\partial\lambda}{\partial p_i^B} = \dfrac{-1}{y^* Bx}\overline{y_i}B_L(i,:)x, & i = l_p+1,\dots,n.
\end{cases}
$$

Therefore, from Theorem 2.1 we can obtain that

$$
\begin{aligned}
K_p &= \sum_{i=2}^{l_p}\left|\frac{p_i}{\lambda}\frac{\partial\lambda}{\partial p_i}\right| + \sum_{i=l_p+1}^{n-1}\left|\frac{p_i^A}{\lambda}\frac{\partial\lambda}{\partial p_i^A}\right| + \sum_{i=l_p+1}^{n-1}\left|\frac{p_i^B}{\lambda}\frac{\partial\lambda}{\partial p_i^B}\right| \\
&= \frac{1}{|\lambda y^* Bx|}\big(|y^*||A_L x| + |\lambda||y^*||B_L x|\big) \\
&\quad + \frac{1}{|\lambda y^* Bx|}\sum_{i=2}^{l_p}\Big(|\overline{y_i}|\big|A_L(i,:)x - \lambda B_L(i,:)x\big| \\
&\quad - \big|\overline{y_i}\big|\,|A_L(i,:)x| - |\lambda|\big|\overline{y_i}\big|\,|B_L(i,:)x|\Big).
\end{aligned}
\tag{3.6}
$$

With 3) in Lemma 2.1 and Theorem 2.1, it is also easy to verify that

$$
\begin{aligned}
K_a &= \sum_{i=2}^{l_a}\left|\frac{a_i}{\lambda}\frac{\partial\lambda}{\partial a_i}\right| + \sum_{i=l_a+1}^{n-1}\left|\frac{a_i^A}{\lambda}\frac{\partial\lambda}{\partial a_i^A}\right| + \sum_{i=l_a+1}^{n-1}\left|\frac{a_i^B}{\lambda}\frac{\partial\lambda}{\partial a_i^B}\right| \\
&= \frac{1}{|\lambda y^* Bx|}\sum_{i=2}^{n-1}\left(\left|y^*\begin{bmatrix} 0 & 0 \\ A(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right.
\end{aligned}
$$

$$+ |\lambda| \left| y^* \begin{bmatrix} 0 & 0 \\ B(i+1:n, 1:i-1) & 0 \end{bmatrix} x \right| \Bigg)$$

$$+ \frac{1}{|\lambda y^* B x|} \sum_{i=2}^{l_a} \Bigg( \Bigg| y^* \begin{bmatrix} 0 & 0 \\ A(i+1:n, 1:i-1) & 0 \end{bmatrix} x$$

$$- \lambda y^* \begin{bmatrix} 0 & 0 \\ B(i+1:n, 1:i-1) & 0 \end{bmatrix} x \Bigg|$$

$$- \left| y^* \begin{bmatrix} 0 & 0 \\ A(i+1:n, 1:i-1) & 0 \end{bmatrix} x \right|$$

$$- |\lambda| \left| y^* \begin{bmatrix} 0 & 0 \\ B(i+1:n, 1:i-1) & 0 \end{bmatrix} x \right| \Bigg). \tag{3.7}$$

Again from 4) in Lemma 2.1 and Theorem 2.1, we know that

$$\begin{aligned} K_q &= \sum_{i=1}^{l_q} \left| \frac{q_i}{\lambda} \frac{\partial \lambda}{\partial q_i} \right| + \sum_{i=l_q+1}^{n-1} \left| \frac{q_i^A}{\lambda} \frac{\partial \lambda}{\partial q_i^A} \right| + \sum_{i=l_q+1}^{n-1} \left| \frac{q_i^B}{\lambda} \frac{\partial \lambda}{\partial q_i^B} \right| \\ &= \frac{1}{|\lambda y^* B x|} \left( |y^* A_L||x| + |\lambda||y^* B_L||x| \right) \\ &\quad + \frac{1}{|\lambda y^* B x|} \sum_{i=1}^{l_q} \Bigg( \left| y^* A_L(:, i) - \lambda y^* B_L(:, i) \right| |x_i| \\ &\quad - |y^* A_L(:, i)||x_i| - |\lambda||y^* B_L(:, i)||x_i| \Bigg). \end{aligned} \tag{3.8}$$

Similarly, repeating the above procedures with 5), 6) and 7) in Lemma 2.1, Proposition 2.1 and Theorem 2.1, we obtain that

$$\begin{aligned} K_g &= \sum_{i=1}^{l_g} \left| \frac{g_i}{\lambda} \frac{\partial \lambda}{\partial g_i} \right| + \sum_{i=l_g+1}^{n-1} \left| \frac{g_i^A}{\lambda} \frac{\partial \lambda}{\partial g_i^A} \right| + \sum_{i=l_g+1}^{n-1} \left| \frac{g_i^B}{\lambda} \frac{\partial \lambda}{\partial g_i^B} \right| \\ &= \frac{1}{|\lambda y^* B x|} \Bigg( |y^*||A_U x| + |\lambda||y^*||B_U x| \Bigg) \\ &\quad + \frac{1}{|\lambda y^* B x|} \sum_{i=1}^{l_g} \Bigg( |\overline{y_i}| \left| A_U(i, :)x - \lambda B_U(i, :)x \right| \\ &\quad - |\overline{y_i}|\, |A_U(i, :)x| - |\lambda|\, |B_U(i, :)x| \Bigg), \\ K_b &= \sum_{i=2}^{l_b} \left| \frac{b_i}{\lambda} \frac{\partial \lambda}{\partial b_i} \right| + \sum_{i=l_b+1}^{n-1} \left| \frac{b_i^A}{\lambda} \frac{\partial \lambda}{\partial b_i^A} \right| + \sum_{i=l_b+1}^{n-1} \left| \frac{b_i^B}{\lambda} \frac{\partial \lambda}{\partial b_i^B} \right| \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{|\lambda y^* B x|} \sum_{i=2}^{n-1} \left( \left| y^* \begin{bmatrix} 0 & A(1:i-1, i+1:n) \\ 0 & 0 \end{bmatrix} x \right| \right. \\
&\left. + |\lambda| \left| y^* \begin{bmatrix} 0 & B(1:i-1, i+1:n) \\ 0 & 0 \end{bmatrix} x \right| \right) \\
&+ \frac{1}{|\lambda y^* B x|} \sum_{i=2}^{l_b} \left( \left| y^* \begin{bmatrix} 0 & A(1:i-1, i+1:n) \\ 0 & 0 \end{bmatrix} x \right. \right. \\
&\left. - \lambda y^* \begin{bmatrix} 0 & B(1:i-1, i+1:n) \\ 0 & 0 \end{bmatrix} x \right| \\
&- \left| y^* \begin{bmatrix} 0 & A(1:i-1, i+1:n) \\ 0 & 0 \end{bmatrix} x \right| \\
&\left. - |\lambda| \left| y^* \begin{bmatrix} 0 & B(1:i-1, i+1:n) \\ 0 & 0 \end{bmatrix} x \right| \right), \\
K_h &= \sum_{i=2}^{l_h} \left| \frac{h_i}{\lambda} \frac{\partial \lambda}{\partial h_i} \right| + \sum_{i=l_h+1}^{n} \left| \frac{h_i^A}{\lambda} \frac{\partial \lambda}{\partial h_i^A} \right| + \sum_{i=l_h+1}^{n} \left| \frac{h_i^B}{\lambda} \frac{\partial \lambda}{\partial h_i^B} \right| \\
&= \frac{1}{|\lambda y^* B x|} \sum_{i=2}^{n} \left( |y^* A_U||x| + |\lambda||y^* B_U||x| \right) \\
&+ \frac{1}{|\lambda y^* B x|} \sum_{i=2}^{l_h} \left( \left| y^* A_U(:, i) - \lambda y^* B_U(:, i) \right| |x_i| \right. \\
&\left. - \left| y^* A_U(:, i) \right| |x_i| - |\lambda| \left| y^* B_U(:, i) \right| |x_i| \right).
\end{aligned}
$$

Then, by observing that

$$
\mathscr{K}(\lambda; \Omega_{QS}) = K_d + K_p + K_a + K_g + K_q + K_b + K_h, \tag{3.9}
$$

we complete the proof of this theorem. □

The explicit expression of $\mathscr{K}(\lambda; \Omega_{QS})$ given by Theorem 3.1 does not depend on the quasiseparable representation (3.3) of the pair $(A, B)$, but it only relies on the entries of the matrices, the simple and finite generalized eigenvalue $\lambda$, and the left and right eigenvectors. This property of $\mathscr{K}(\lambda; \Omega_{QS})$ is stated in the following proposition.

**Proposition 3.1** *Let $A$ and $B \in \mathbb{R}^{n\times n}$ be {1;1}-quasiseparable matrices and $\lambda \neq 0$ be a simple and finite generalized eigenvalue of the pair $(A, B)$ with parameters $\Omega_A$ and $\Omega_B$, for any sets $\Omega_{QS} = \Omega_A \cup \Omega_B$ and $\Omega_{QS}^{'} = \Omega_A' \cup \Omega_B'$ of quasiseparable parameters of $(A, B)$, under the assumption that the numbers of common parameters of each type in $\Omega_{QS}$ and $\Omega_{QS}^{'}$ are the same, the following property holds*

$$
\mathscr{K}(\lambda; \Omega_{QS}) = \mathscr{K}(\lambda; \Omega_{QS}^{'}).
$$

For the classical eigenvalue problem (1.2), $A = A(\Omega_{QS})$ is a $\{1;1\}$-quasiseparable matrix, and the corresponding relative componentwise eigenvalue condition number $\mathrm{cond}(\lambda; \Omega_{QS})$ is defined and expressed in Theorem 4.4 of [12] as follows

$$\begin{aligned}\mathrm{cond}(\lambda; \Omega_{QS}) = \frac{1}{|\lambda||y^*x|}\Bigg\{&|y^*||A_D||x| + |y^*||A_L x| + |y^* A_L||x| \\ &+ |y^*||A_U x| + |y^* A_U||x| \\ &+ \sum_{i=2}^{n-1}\left|y^*\begin{bmatrix} 0 & 0 \\ A(i+1:n, 1:i-1) & 0\end{bmatrix} x\right| \\ &+ \sum_{j=2}^{n-1}\left|y^*\begin{bmatrix} 0 & A(1:j-1, j+1:n) \\ 0 & 0\end{bmatrix} x\right|\Bigg\}.\end{aligned} \tag{3.10}$$

Dopico and Pomés [12] propose fast computation routines to compute the componentwise eigenvalue condition number $\mathrm{cond}(\lambda; \Omega_{QS})$ given by (3.10) in $\mathcal{O}(n)$ flops in view of the recursive structure of the $\{1;1\}$-quasiseparable matrix; see [12, Proposition 4.12] for more details. Thus, using the idea of Algorithm 1 of [12] for fast computations for $\mathrm{cond}(\lambda; \Omega_{QS})$, similar techniques to compute $\mathscr{K}(\lambda; \Omega_{QS})$ in $\mathcal{O}(n)$ flops for a specific finite and simple generalized eigenvalue $\lambda$ of the $\{1;1\}$-quasiseparable matrices pair $(A, B)$ in the quasiseparable representation may also be developed. For brevity, we will not give the detailed descriptions for computing $\mathscr{K}(\lambda; \Omega_{QS})$.

### 3.2 The Givens-vector representation via tangents

This subsection is devoted to the explicit expression of the generalized eigenvalue condition number for the $\{1;1\}$-quasiseparable matrices pair $(A, B)$ when $A$ and $B$ are represented by the Givens-vector parameters [34]. First, in Definition 2.2 the Givens-vector representation [34] for a $\{1;1\}$-quasiseparable matrix is described. Recalling Definition 2.3, suppose that

$$\begin{aligned}\Omega_{GV}^A := \Big(&\{w_i\}_{i=2}^{l_w}, \{w_i^A\}_{i=l_w+1}^{n-1}, \{v_i\}_{i=1}^{l_v}, \{v_i^A\}_{i=l_v+1}^{n-1}, \{d_i\}_{i=1}^{l_d}, \{d_i^A\}_{i=l_d+1}^{n}, \{e_i\}_{i=1}^{l_e}, \\ &\{e_i^A\}_{i=l_e+1}^{n-1}, \{u_i\}_{i=2}^{l_u}, \{u_i^A\}_{i=l_u+1}^{n-1}\Big),\end{aligned} \tag{3.11}$$

and

$$\begin{aligned}\Omega_{GV}^B := \Big(&\{w_i\}_{i=2}^{l_w}, \{w_i^B\}_{i=l_w+1}^{n-1}, \{v_i\}_{i=1}^{l_v}, \{v_i^B\}_{i=l_v+1}^{n-1}, \{d_i\}_{i=1}^{l_d}, \{d_i^B\}_{i=l_d+1}^{n}, \{e_i\}_{i=1}^{l_e}, \\ &\{e_i^B\}_{i=l_e+1}^{n-1}, \{u_i\}_{i=2}^{l_u}, \{u_i^B\}_{i=l_u+1}^{n-1}\Big),\end{aligned} \tag{3.12}$$

are the Givens-vector representation via tangents of $A$ and $B$, respectively. When $l_w = 1$, $l_v = 0$, $l_d = 0$, $l_e = 0$, and $l_u = 1$, we have that $\Omega_{GV}^A \cap \Omega_{GV}^B = \varnothing$, which implies that $\Omega_{GV}^A$ is independent of $\Omega_{GV}^B$. Moreover, the set of parameters

$$\Omega_{GV} := \Big(\{w_i\}_{i=2}^{l_w}, \{w_i^A\}_{i=l_w+1}^{n-1}, \{w_i^B\}_{i=l_w+1}^{n-1}, \{v_i\}_{i=2}^{l_v}, \{v_i^A\}_{i=l_v+1}^{n-1}, \{v_i^B\}_{i=l_v+1}^{n-1}, \{d_i\}_{i=1}^{l_d}, \{d_i^A\}_{i=l_d+1}^{n}, \{d_i^B\}_{i=l_d+1}^{n}, \{e_i\}_{i=1}^{l_e}, \{e_i^A\}_{i=l_e+1}^{n-1}, \{e_i^B\}_{i=l_e+1}^{n-1}, \{u_i\}_{i=2}^{l_u}, \{u_i^A\}_{i=l_u+1}^{n-1}, \{u_i^B\}_{i=l_u+1}^{n-1}\Big) \tag{3.13}$$

is the Givens-vector representation via tangents of the pair $(A, B)$ when $A$ and $B$ have quasiseparable representations $\Omega_{GV}^A$ and $\Omega_{GV}^B$, respectively. From Definition 2.3, in the following of the paper, we denote by

$$\begin{cases} c_i = c_i^A = c_i^B = \dfrac{1}{\sqrt{1+w_i^2}}, \quad s_i = s_i^A = s_i^B = \dfrac{w_i}{\sqrt{1+w_i^2}}, & i = 2, \ldots, l_w, \\ c_i^A = \dfrac{1}{\sqrt{1+(w_i^A)^2}}, \quad s_i^A = \dfrac{w_i^A}{\sqrt{1+(w_i^A)^2}}, & i = l_w+1, \ldots, n-1, \\ c_i^B = \dfrac{1}{\sqrt{1+(w_i^B)^2}}, \quad s_i^B = \frac{w_i^B}{\sqrt{1+(w_i^B)^2}}, & i = l_w+1, \ldots, n-1, \\ r_i = r_i^A = r_i^B = \dfrac{1}{\sqrt{1+u_i^2}}, \quad t_i = t_i^A = t_i^B = \dfrac{u_i}{\sqrt{1+u_i^2}}, & i = 2, \ldots, l_u, \\ r_i^A = \dfrac{1}{\sqrt{1+(u_i^A)^2}}, \quad t_i^A = \dfrac{u_i^A}{\sqrt{1+(u_i^A)^2}}, & i = l_u+1, \ldots, n-1, \\ r_i^B = \dfrac{1}{\sqrt{1+(u_i^B)^2}}, \quad t_i^B = \dfrac{u_i^B}{\sqrt{1+(u_i^B)^2}}, & i = l_u+1, \ldots, n-1. \end{cases} \tag{3.14}$$

In the next theorem, we will give an explicit expression for the componentwise generalized eigenvalue condition number with respect to the Givens-vector representation via tangents. By applying Theorem 2.1 and Lemma 2.2, we derive the explicit expression for the condition number in the next theorem. Since its proof is similar to Theorem 3.1, we omit its proof here.

**Theorem 3.2** *With the notations in Theorem 3.1, we have*

$$\begin{aligned} \mathscr{K}(\lambda; \Omega_{GV}) = \frac{1}{|\lambda||y^* B x|} \Bigg\{ & |y^*||A_D||x| + |\lambda||y^*||B_D||x| \\ & + |y^* A_L||x| + |\lambda||y^* B_L||x| \\ & + |y^*||A_U x| + |\lambda||y^*||B_U x| + \sum_{i=2}^{n-1} \left( \left| y^* \begin{bmatrix} 0 & 0 \\ -(s_i^A)^2 A(i, 1:i-1) & 0 \\ (c_i^A)^2 A(i+1:n, 1:i-1) & 0 \end{bmatrix} x \right| \right. \\ & \left. + |\lambda| \left| y^* \begin{bmatrix} 0 & 0 \\ -(s_i^B)^2 B(i, 1:i-1) & 0 \\ (c_i^B)^2 B(i+1:n, 1:i-1) & 0 \end{bmatrix} x \right| \right) + \beta_d + \beta_e + \beta_v + \beta_w \end{aligned}$$

$$+\sum_{i=2}^{n-1}\left(\left|y^*\begin{bmatrix}0 & -(t_i^A)^2A(1:i-1,i) & (r_i^A)^2A(1:i-1,i+1:n)\\ 0 & 0 & 0\end{bmatrix}x\right|\right.$$
$$\left.+|\lambda|\left|y^*\begin{bmatrix}0 & -(t_i^B)^2B(1:i-1,i) & (r_i^B)^2B(1:i-1,i+1:n)\\ 0 & 0 & 0\end{bmatrix}x\right|\right)+\beta_u,\Bigg\},$$

*where*

$$\beta_d=\alpha_d,\ \beta_e=\alpha_g,\ \beta_v=\alpha_q, \tag{3.15}$$

*and*

$$\beta_w=\sum_{i=2}^{l_w}\left(\left|y^*\begin{bmatrix}0 & 0\\ -(s_i)^2A(i,1:i-1) & 0\\ (c_i)^2A(i+1:n,1:i-1) & 0\end{bmatrix}x\right.\right.$$
$$\left.-\lambda y^*\begin{bmatrix}0 & 0\\ -(s_i)^2B(i,1:i-1) & 0\\ (c_i)^2B(i+1:n,1:i-1) & 0\end{bmatrix}x\right|$$
$$-\left|y^*\begin{bmatrix}0 & 0\\ -(s_i)^2A(i,1:i-1) & 0\\ (c_i)^2A(i+1:n,1:i-1) & 0\end{bmatrix}x\right|$$
$$\left.-|\lambda|\left|y^*\begin{bmatrix}0 & 0\\ -(s_i)^2B(i,1:i-1) & 0\\ (c_i)^2B(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right),$$
$$\beta_u=\sum_{i=2}^{l_u}\left(\left|y^*\begin{bmatrix}0 & -t_i^2A(1:i-1,i) & r_i^2A(1:i-1,i+1:n)\\ 0 & 0 & 0\end{bmatrix}x\right.\right.$$
$$\left.-\lambda y^*\begin{bmatrix}0 & -t_i^2B(1:i-1,i) & r_i^2B(1:i-1,i+1:n)\\ 0 & 0 & 0\end{bmatrix}x\right|$$
$$-\left|y^*\begin{bmatrix}0 & -t_i^2A(1:i-1,i) & r_i^2A(1:i-1,i+1:n)\\ 0 & 0 & 0\end{bmatrix}x\right|$$
$$\left.-|\lambda|\left|y^*\begin{bmatrix}0 & -t_i^2B(1:i-1,i) & r_i^2B(1:i-1,i+1:n)\\ 0 & 0 & 0\end{bmatrix}x\right|\right), \tag{3.16}$$

*and $\alpha_d$, $\alpha_g$, $\alpha_q$ are defined in* (3.4)*, $c_i$, $s_i$, $c_i^A$, $s_i^B$, $r_i$, $t_i$ $r_i^A$, $t_i^A$, $r_i^B$ and $t_i^B$ are defined in* (3.14).

It is easy to see that $\mathscr{K}(\lambda;\,\Omega_{GV})$ given in Theorem 3.2 depends on entries of the pair $(A,\ B)$, the generalized eigenvalue $\lambda$, the left and right eigenvectors $x$ and $y$, and the parameters $\{c_i^A, s_i^A\}$, $\{c_i^B, s_i^B\}$, $\{r_i^A, t_i^A\}$ and $\{r_i^B, t_i^B\}$.

In the previous subsection, we have discussed that $\mathscr{K}(\lambda;\,\Omega_{QS})$ can be computed in $\mathcal{O}(n)$ flops. Using the techniques developed in [12, Proposition 5.6], we can compute $\mathscr{K}(\lambda;\,\Omega_{GV})$ in $\mathcal{O}(n)$ flops by utilizing the recursive structures of $A$ and $B$. For brevity, we will not give the detailed procedures to compute $\mathscr{K}(\lambda;\,\Omega_{GV})$ here.

## 4 Relationships between various condition numbers for the generalized structured eigenvalue problem with {1;1}-quasiseparable matrices

In this section we will investigate relationships between various condition numbers for (1.1) with {1;1}-quasiseparable matrices from their explicit expressions.

In the following proposition we will prove that the structured relative componentwise condition number $\mathscr{K}(\lambda; \Omega_{QS})$ given in Theorem 3.1 is smaller than the unstructured relative componentwise condition number $\mathscr{K}(\lambda; (A, B))$ given by (2.4) from their explicit expressions up to a constant $n$.

**Proposition 4.1** *Let $A$ and $B \in \mathbb{R}^{n\times n}$ be {1;1}-quasiseparable matrices and consider a set of quasiseparable parameters $\Omega_{QS}$ given by* (3.3)*. Let $\lambda \neq 0$ be a simple and finite generalized eigenvalue of the pair $(A,\ B)$. Then, the following relation holds,*

$$\mathscr{K}(\lambda; \Omega_{QS}) \le n\, \mathscr{K}(\lambda; (A, B)).$$

***Proof*** Recalling $\alpha_d$, $\alpha_p$, $\alpha_g$, $\alpha_q$, $\alpha_h$, $\alpha_a$, $\alpha_b$ are defined in (3.4), noting that

$$\alpha_d \le 0, \quad \alpha_p \le 0, \quad \alpha_g \le 0, \quad \alpha_q \le 0, \quad \alpha_h \le 0, \quad \alpha_a \le 0, \quad \alpha_b \le 0 \tag{4.1}$$

and standard inequalities of absolute values, for $\mathscr{K}(\lambda; \Omega_{QS})$ given in Theorem 3.1, we prove that

$$\begin{aligned}
\mathscr{K}(\lambda; \Omega_{QS}) \le \frac{1}{|\lambda||y^*Bx|} \Bigg\{ & |y^*||A_D||x| + |y^*||A_L x| \\
& + |y^* A_L||x| + |y^*||A_U x| + |y^* A_U||x| \\
& + \sum_{i=2}^{n-1} |y^*||A_L||x| + \sum_{j=2}^{n-1} |y^*||A_U||x| + \lambda |y^*||B_D||x| \\
& + |\lambda||y^*||B_L x| + |\lambda||y^* B_L||x| \\
& + |\lambda||y^*||B_U x| + |\lambda||y^* B_U||x| + \sum_{i=2}^{n-1} |\lambda||y^*||B_L||x| + \sum_{j=2}^{n-1} |\lambda||y^*||B_U||x| \Bigg\} \\
\le \ & \frac{1}{|\lambda||y^*Bx|} \Big\{ |y^*||A_D||x| + n|y^*||A_L||x| + n\, |y^*||A_U x| + |\lambda||y^*||B_D||x| \\
& + n\, |\lambda||y^*||B_L||x| + n\, |\lambda||y^*||B_U||x| \Big\} = n\ \mathscr{K}(\lambda; (A, B)).
\end{aligned}$$

□

Because the Givens-vector representation is a particular case of the quasiseparable representation, and perturbations on $\Omega_{GV}$ are only restricted to preserve the cosine-sine relations in the parameters $\{c_i^A, s_i^A\}$, $\{r_i^A, t_i^A\}$, $\{r_i^B, t_i^B\}$, and $\{c_i^B, s_i^B\}$ of $\Omega_{QS}^{GV}$, it is natural to expect $\mathscr{K}(\lambda; \Omega_{GV})$ given in Theorem 3.2 not to be larger than

$\mathscr{K}(\lambda; \Omega_{QS})$ given in Theorem 3.1. In [12, Section 5] and [11, Section 6], structured condition numbers with respect to Givens-vector representation via tangents for eigenvalue computations and linear system solving for $\{1;1\}$-quasiseparable matrices are proved to be smaller than the corresponding structured condition numbers with respect to the general quasiseparable representation. In Theorem 4.1, the corresponding result is proved for generalized eigenvalue problems, that is, $\mathscr{K}(\lambda; \Omega_{GV})$ cannot be larger than $\mathscr{K}(\lambda; \Omega_{QS})$. Moreover in Theorem 4.1, we claim that the relative condition number of a simple and finite generalized eigenvalue of quasiseparable matrices pair $(A, B)$ with respect to a general quasiseparable representation can only be smaller than the corresponding condition number with respect to the Givens-vector representation via tangents up to a factor of $3(n-2)$. Therefore, both representations can be considered equivalent from the point of view of the accuracy of the generalized eigenvalue computations that they allow. Before Theorem 4.1, we need Lemma 4.1, which is proved inside in the proof of Theorem 6.3 in [12].

Let a $\{1; 1\}$-quasiseparable matrix $C = C(\Omega_{GV}^{C})$ be defined by the vector $\Omega_{GV}^{C}$, where $\Omega_{GV}^{C}$ is the Givens-vector representation via tangents of $C$; see Definition 2.3. From Definition 2.3, the Givens-vector representation $\Omega_{QS,C}^{GV}$ of $C$ can be constructed from $\Omega_{GV}^{C}$. Moreover, $\Omega_{QS,C}^{GV}$ is also a quasiseparable representation of $C$. Thus if there is a quasiseparable perturbation $\delta\Omega_{QS,C}^{GV}$ of the parameters in $\Omega_{QS,C}^{GV}$ such that $|\delta\Omega_{QS,C}^{GV}| \leq \eta|\Omega_{QS,C}^{GV}|$, we may obtain a resulting quasiseparable matrices $\widetilde{C} := C(\Omega_{QS,C}^{GV} + \delta\Omega_{QS,C}^{GV})$. Generally, $\delta\Omega_{QS,C}^{GV}$ does not preserve the pair cosine-sine of $\Omega_{QS}^{GV,C}$. From Definition 2.3, $\widetilde{C}$ can be represented by a vector

$$\Omega_{GV}^{C}{}' = \left(\{w_i'\}_{i=2}^{n-1}, \{v_i'\}_{i=2}^{n-1}, \{d_i'\}_{i=1}^{n}, \{e_i'\}_{i=1}^{n-1}, \{u_i'\}_{i=2}^{n-1}\right)$$

of tangent-Givens-vector parameters and let $\delta'\Omega_{GV}^{C} = \Omega_{GV}^{C}{}' - \Omega_{GV}^{C}$. In the following lemma we have the perturbation magnitude relationship between $\delta'\Omega_{GV}^{C}$ and $\delta\Omega_{QS,C}^{GV}$.

**Lemma 4.1** [12, Lemma 6.2] *With the notations before, we have*

$$\left|\delta\Omega_{QS,C}^{GV}\right| \leq \eta\left|\Omega_{QS,C}^{GV}\right| \Rightarrow |\delta'\Omega_{GV}^{C}| \leq (3(n-2)\eta + \mathcal{O}(\eta^2))\left|\Omega_{GV}^{C}\right|.$$

**Theorem 4.1** *With the notations in Theorems 3.1 and 3.2, assume that $l_g = l_e$, $l_q = l_v$, $l_a = l_p = l_w$ and $l_h = l_b = l_u$, then we have*

$$\mathscr{K}(\lambda; \Omega_{GV}) \leq \mathscr{K}(\lambda; \Omega_{QS}) \leq (3n-2)\mathscr{K}(\lambda; \Omega_{GV}).$$

***Proof*** Recall $\alpha_d$, $\alpha_p$, $\alpha_g$, $\alpha_q$, $\alpha_h$, $\alpha_a$, $\alpha_b$ are defined in (3.4), $\beta_w$ and $\beta_u$ are defined in (3.15). First, it can be verified that

$$\sum_{i=2}^{l_w}\left(\left|y^*\begin{bmatrix}0 & 0\\ -(s_i)^2A(i,1:i-1) & 0\\ (c_i)^2A(i+1:n,1:i-1) & 0\end{bmatrix}x\right.\right.$$
$$\left.\left.-\lambda y^*\begin{bmatrix}0 & 0\\ -(s_i)^2B(i,1:i-1) & 0\\ (c_i)^2B(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right)$$
$$\leq\sum_{i=2}^{l_w}\left(\left|y^*\begin{bmatrix}0 & 0\\ A(i,1:i-1) & 0\\ 0 & 0\end{bmatrix}x-\lambda y^*\begin{bmatrix}0 & 0\\ B(i,1:i-1) & 0\\ 0 & 0\end{bmatrix}x\right|\right.$$
$$\left.+\left|y^*\begin{bmatrix}0 & 0\\ A(i+1:n,1:i-1) & 0\end{bmatrix}x-\lambda y^*\begin{bmatrix}0 & 0\\ B(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right)$$
$$=\alpha_p+\sum_{i=2}^{l_w}\left(\left|y^*\begin{bmatrix}0 & 0\\ A(i+1:n,1:i-1) & 0\end{bmatrix}x\right.\right.$$
$$\left.\left.-\lambda y^*\begin{bmatrix}0 & 0\\ B(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right),$$

$$\sum_{i=l_w+1}^{n-1}\left(\left|y^*\begin{bmatrix}0 & 0\\ -(s_i^A)^2A(i,1:i-1) & 0\\ (c_i^A)^2A(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right.$$
$$\leq\sum_{i=l_w+1}^{n-1}\left(\left|y^*\begin{bmatrix}0 & 0\\ A(i,1:i-1) & 0\\ 0 & 0\end{bmatrix}x\right|\right.$$
$$\left.+\left|y^*\begin{bmatrix}0 & 0\\ A(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right)$$
$$=\sum_{i=l_w+1}^{n-1}\left(|y_iA_L(i,:)x|+\left|y^*\begin{bmatrix}0 & 0\\ A(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right)$$
$$\leq\left|y^*\right|\left|A_Lx\right|+\sum_{i=l_w+1}^{n-1}\left(\left|y^*\begin{bmatrix}0 & 0\\ A(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right),$$

$$\sum_{i=l_w+1}^{n-1}\left(|\lambda|\left|y^*\begin{bmatrix}0 & 0\\ -(s_i^B)^2B(i,1:i-1) & 0\\ (c_i^B)^2B(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right)\leq|\lambda||y^*||B_Lx|$$
$$+\sum_{i=l_w+1}^{n-1}\left(|\lambda|\left|y^*\begin{bmatrix}0 & 0\\ B(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right).$$

From the above three inequalities, we prove that

$$\sum_{i=2}^{n-1}\left(\left|y^*\begin{bmatrix}0 & 0\\ -(s_i^A)^2A(i,1:i-1) & 0\\ (c_i^A)^2A(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right.$$
$$\left.+|\lambda|\left|y^*\begin{bmatrix}0 & 0\\ -(s_i^B)^2B(i,1:i-1) & 0\\ (c_i^B)^2B(i+1:n,1:i-1) & 0\end{bmatrix}x\right|\right)+\beta_w$$
$$\leq |y^*||A_Lx|+|\lambda||y^*||B_Lx|+\sum_{i=2}^{n-1}\left|y^*\begin{bmatrix}0 & 0\\ A(i+1:n,1:i-1) & 0\end{bmatrix}x\right|$$
$$+\sum_{i=2}^{n-1}|\lambda|\left|y^*\begin{bmatrix}0 & 0\\ B(i{+}1:n,1:i{-}1) & 0\end{bmatrix}x\right|+\alpha_a. \tag{4.2}$$

Using similar techniques, we can also derive that

$$\sum_{i=2}^{n-1}\left(\left|y^*\begin{bmatrix}0 & -(t_i^A)^2A(1:i-1,i) & (r_i^A)^2A(1:i-1,i+1:n)\\ 0 & 0 & 0\end{bmatrix}x\right|\right.$$
$$\left.+|\lambda|\left|y^*\begin{bmatrix}0 & -(t_i^B)^2B(1:i-1,i) & (r_i^B)^2B(1:i-1,i+1:n)\\ 0 & 0 & 0\end{bmatrix}x\right|\right)+\beta_u$$
$$\leq |y^*||A_Ux|+|\lambda||y^*||B_Ux|+\sum_{j=2}^{n-1}\left|y^*\begin{bmatrix}0 & A(1:j{-}1,j{+}1:n)\\ 0 & 0\end{bmatrix}x\right|$$
$$+\sum_{j=2}^{n-1}|\lambda|\left|y^*\begin{bmatrix}0 & B(1:j-1,j+1:n)\\ 0 & 0\end{bmatrix}x\right|+\alpha_b. \tag{4.3}$$

Comparing the expressions of $\mathscr{K}(\lambda;\Omega_{QS})$ and $\mathscr{K}(\lambda;\Omega_{GV})$ given in Theorem 3.1 and Theorem 3.2 respectively, from (3.15), using (4.2) and (4.3), we prove that

$$\mathscr{K}(\lambda;\Omega_{GV})\leq\mathscr{K}(\lambda;\Omega_{QS}).$$

On the other hand, from Definition 2.5 and Lemma 4.1, we have

$$\mathscr{K}(\lambda;\Omega_{QS})\leq\lim_{\eta\to 0}\sup\left\{\frac{|\delta\lambda|}{\eta|\lambda|}:(\lambda+\delta\lambda)\text{ is a generalized eigenvalue of}\right.$$
$$\left.(A,B)(\Omega_{GV}+\delta\Omega_{GV}),\ |\delta\Omega_{GV}|\leq(3(n{-}2)\eta+\mathcal{O}(\eta^2))|\Omega_{GV}|\right\}$$

By considering the change of variable $\eta'=3(n{-}2)\eta+\mathcal{O}(\eta^2)$, we obtain

$$\mathscr{K}(\lambda;\,\Omega_{QS}) \le \lim_{\eta'\to 0}\,\sup\left\{\frac{3(n-2)|\delta\,\lambda|}{\eta'|\lambda|} : (\lambda+\delta\,\lambda)\text{ is a generalized eigenvalue of } (A,\,B)(\Omega_{GV}+\delta\,\Omega_{GV}),\,|\delta\,\Omega_{GV}|\le\eta'|\Omega_{GV}|\right\} = 3(n-2)\mathscr{K}(\lambda;\,\Omega_{GV}),$$

which finishes the proof. □

## 5 Numerical experiments

In this section, we demonstrate our test results of some numerical examples to illustrate structured condition numbers presented in the previous sections. All the computations were carried out by using MATLAB 8.1 with machine precision about $2.2\times 10^{-16}$.

First, we do numerical experiments for the case that the tangent-Givens-vector representation $\Omega_{GV}^{A}$ of $A$ is independent of the one of $B$. The vectors

$$\begin{aligned} &w^A\in\mathbb{R}^{n-2},\quad v^A\in\mathbb{R}^{n-1},\quad d^A\in\mathbb{R}^{n},\quad e^A\in\mathbb{R}^{n-1},\quad u^A\in\mathbb{R}^{n-2},\\ &w^B\in\mathbb{R}^{n-2},\quad v^B\in\mathbb{R}^{n-1},\quad d^B\in\mathbb{R}^{n},\quad e^B\in\mathbb{R}^{n-1},\quad u^B\in\mathbb{R}^{n-2}, \end{aligned} \tag{5.1}$$

which are the tangent-Givens-vector representations in Definition 2.3 for {1;1}-quasi-separable matrices $A$ and $B$, are generated from the standard Gaussian distribution by using MATLAB command's `randn`. Using Definitions 2.2 and 2.3, we can construct the corresponding pairs $(A,\,B)$ according to the above generated parameters of the tangent-Givens-vector representation. The generalized eigenvalue $\lambda$ and its associated right and left eigenvectors $x$ and $y$ are computed through calling MATLAB's routine `eig`. Then we compute condition numbers $\mathscr{K}(\lambda;\,(A,\,B))$, $\mathscr{K}(\lambda;\,\Omega_{QS})$ and $\mathscr{K}(\lambda;\,\Omega_{GV})$ from their explicit formulas via matrix-vector multiplications. We do numerical experiments for parameters generated by (5.1) to compare the values $\mathscr{K}(\lambda;\,(A,\,B))$, $\mathscr{K}(\lambda;\,\Omega_{QS})$ and $\mathscr{K}(\lambda;\,\Omega_{GV})$. The results show that the differences are marginal, i.e., $\mathscr{K}(\lambda;\,(A,\,B))\approx\mathscr{K}(\lambda;\,\Omega_{QS})$ and $\mathscr{K}(\lambda;\,(A,\,B))\approx\mathscr{K}(\lambda;\,\Omega_{GV})$ and all of them are very often moderate.

As in [12, Section 7], in order to make the unstructured condition number $\mathscr{K}(\lambda;\,(A,\,B))$ to be much larger than the structured ones, we rescale the parameters in (5.1) as follows. We generate standard Gaussian random parameters of (5.1) by using MATLAB function randn. First, set $\ell=\lfloor 0.3\times(n-1)\rfloor$, where $\lfloor z\rfloor$ is the largest integer which is smaller than or equal to $z$. Suppose $\alpha_i=1+(i-1)\cdot 4/(\ell-1)$ and $\beta_i=\alpha_{\ell-i+1}$, where $i=1,\ldots,\ell$. We randomly select $\ell$ indexes of $e^A$ and $v^A$, separately. For the selected component $\ell$ indexes of $e^A$ in the ascending order, we multiply the corresponding component of $e^A$ by the weight $10^{\alpha_i+3}$. On the other hand, in a similar way, we multiply the select component of $v^A$ by the weight $10^{\beta_i+3}$. The vectors of $d^A$ and $u^A$ are rescaled by factors $10^{-3}$ and $10^{3}$, respectively. For parameters for $B$, similarly, let $\gamma_i=1+4\cdot(i-1)/(n-1)$ and $\delta_i=\gamma_{n-i}$, where $i=1,\ldots,n-1$. For $i$-th components of $e^B$ and $v^B$, we rescale them by factors $10^{\gamma_i-3}$ and $10^{\delta_i+3}$, respectively. Also, for each components of $d^B$ and $u^B$, they are rescaled by factors $10^{3}$ and $10^{-3}$, respectively. For each generated pairs $(A,\,B)$, we

**Table 1** The maximum ratios between $\mathscr{K}(\lambda;\ (A,\ B))$ and $\mathscr{K}(\lambda;\ \Omega_{QS})$ with the corresponding generalized eigenvalues $\lambda_{\rm opt}$ for $n=200$ and $n=300$ when parameters of $A$ are independent of ones of $B$

| $n$ | $\max\limits_{\lambda}\left\{\frac{\mathscr{K}(\lambda;(A,B))}{\mathscr{K}(\lambda;\Omega_{QS})}\right\}$ | $\lambda_{\rm opt}$ | $\mathscr{K}(\lambda_{\rm opt};\ (A,\ B))$ | $\mathscr{K}(\lambda_{\rm opt};\ \Omega_{QS})$ | $\mathscr{K}(\lambda_{\rm opt};\ \Omega_{GV})$ |
|---|---|---|---|---|---|
| 200 | $1.1586\cdot 10^7$ | $-181.2307$ | $8.4965\cdot 10^7$ | 7.3336 | 4.1710 |
| 300 | $3.2096\cdot 10^7$ | 0.1283 | $2.0818\cdot 10^8$ | 6.4862 | 3.9548 |

compute $\mathscr{K}(\lambda;\ (A,\ B))$, $\mathscr{K}(\lambda;\ \Omega_{QS})$ and $\mathscr{K}(\lambda;\ \Omega_{GV})$ when $\lambda$ is a simple, nonzero and finite eigenvalue of $(A,\ B)$. In Table 1, we report the maximum ratios between $\mathscr{K}(\lambda;\ (A,\ B))$ and $\mathscr{K}(\lambda;\ \Omega_{QS})$, where $\lambda_{\rm opt}$ is the optimal generalized eigenvalue of $\max\limits_{\lambda}\left\{\frac{\mathscr{K}(\lambda;(A,B))}{\mathscr{K}(\lambda;\Omega_{QS})}\right\}$. Also the structured and unstructured generalized eigenvalue condition numbers of $\lambda_{\rm opt}$ are displayed. From Table 1, it can be seen that the structured condition number $\mathscr{K}(\lambda;\ \Omega_{QS})$ can be much smaller than the unstructured one $\mathscr{K}(\lambda;\ (A,\ B))$. Thus the forward error determined by unstructured condition numbers are pessimistic and may severely overestimate the exact relative error of the computed generalized eigenvalue.

In the following, we will do numerical experiments for the case that $A$ and $B$ have common parameter representations. Set $\ell=\lfloor 0.05\times n\rfloor$. We choose $l_w=l_v=l_d=l_e=l_u=\ell$. Recall that the Givens-vector representation via tangents $\Omega_{GV}$ of the pair $(A,\ B)$ is given by (3.13). Using MATLAB notations, we generate the parameters as follows:

$$
\begin{aligned}
v&=(v_i)_{i=2}^{\ell}=10^8*\texttt{randn}(\ell-1,1),\ e=(e_i)_{i=1}^{\ell}=\texttt{randn}(\ell,1).*z_1,\\
v^A&=(v_i^A)_{i=\ell+1}^{n-1}=\texttt{randn}(n-\ell-1,1).*z_2,\\
e^A&=(e_i^A)_{i=\ell+1}^{n-1}=\texttt{randn}(n-\ell-1,1).*z_3,\\
v^B&=(v_i^B)_{i=\ell+1}^{n-1}=\texttt{randn}(n-\ell-1,1).*z_2,\\
e^B&=(e_i^B)_{i=\ell+1}^{n-1}=\texttt{randn}(n-\ell-1,1).*z_3,\\
w&=(w_i)_{i=2}^{\ell}=\texttt{randn}(\ell-1,1),\\
w^A&=(w_i^A)_{i=\ell+1}^{n-1}=\texttt{randn}(n-\ell-1,1).*z_2,\\
d&=(d_i)_{i=1}^{\ell}=\texttt{randn}(\ell,1),\ d^A=(d_i^A)_{i=\ell+1}^{n}=\texttt{randn}(n-\ell,1),\\
u&=(u_i)_{i=2}^{\ell}=\texttt{randn}(\ell-1,1),\ u^A=(u_i^A)_{i=\ell+1}^{n-1}=\texttt{randn}(n-\ell,1),\\
d^A&=(d_i^A)_{i=\ell+1}^{n}=\texttt{randn}(n-\ell,1),\ u^B=(u_i^B)_{i=\ell+1}^{n-1}=\texttt{randn}(n-\ell,1),\\
w^B&=(w_i^B)_{i=\ell+1}^{n-1}=\texttt{randn}(n-\ell-1,1),
\end{aligned}
$$

where $z_1=10.\hat{}(1:7/(\ell-1):8)$, $z_2=10.\hat{}(1:7/(n-\ell-2):8)$, $z_3=10.\hat{}(1:-7/(n-\ell-2):8)$. Again from Table 2, there exists a particular situation such that $\mathscr{K}(\lambda;\ (A,\ B))$ can be much larger than $\mathscr{K}(\lambda;\ \Omega_{QS})$ and $\mathscr{K}(\lambda;\ \Omega_{GV})$, which means that it is necessary to measure the conditioning of (1.1) through structured componentwise perturbation analysis when $A$ and $B$ are $\{1;1\}$-quasiseparable matrices.

**Table 2** The maximum ratios between $\mathscr{K}(\lambda;\ (A,\ B))$ and $\mathscr{K}(\lambda;\ \Omega_{QS})$ with the corresponding generalized eigenvalues $\lambda_{\text{opt}}$ for $n = 100$ and $n = 200$ when $A$ and $B$ have common parameters

| $n$ | $\max_{\lambda}\left\{\frac{\mathscr{K}(\lambda;(A,B))}{\mathscr{K}(\lambda;\Omega_{QS})}\right\}$ | $\lambda_{\text{opt}}$ | $\mathscr{K}(\lambda_{\text{opt}};\ (A,\ B))$ | $\mathscr{K}(\lambda_{\text{opt}};\ \Omega_{QS})$ | $\mathscr{K}(\lambda_{\text{opt}};\ \Omega_{GV})$ |
|---|---|---|---|---|---|
| 100 | $1.2920 \cdot 10^5$ | 0.6549 − 0.4274i | $2.8877 \cdot 10^6$ | 22.3508 | 12.7448 |
| 200 | $1.0799 \cdot 10^7$ | 0.0014 + 0.0003i | $1.2065 \cdot 10^8$ | 22.1646 | 11.1724 |

## 6 Concluding remarks

In this paper, we have introduced structured componentwise generalized eigenvalue condition numbers when matrices of the pair $(A,\ B)$ have parameterized representations, where we assume there are common parameters describing the pair $(A,\ B)$, simultaneously. Especially, $A$ and $B$ are $\{1;\ 1\}$-quasiseparable matrices, the explicit formulas of structured componentwise generalized eigenvalue condition numbers with respect to general quasiseparable representation and the Givens-vector representation via tangents for $\{1;1\}$-quasiseparable matrices are derived. Their properties and relationships with each other and the unstructured componentwise generalized eigenvalue condition numbers were investigated. Moreover, we can compute our proposed condition numbers efficiently by considering the recursive structure of quasiseparable matrices. Numerical experiments showed that when parameters of the Givens-vector representation via tangents are unbalanced, the structured condition numbers can be much smaller than the corresponding unstructured condition numbers.

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