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A GLOBAL EXISTENCE RESULT FOR KORTEWEG SYSTEM IN THE CRITICAL L^P FRAMEWORK*

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Abstract The purpose of this work is to investigate the initial value problem for a general isothermal model of capillary fluids derived by Dunn and Serrin [12], which can be used as a phase transition model. Motivated by [9], we aim at extending the work by Danchin-Desjardins [11] to a critical framework which is not related to the energy space. For small perturbations of a stable equilibrium state in the sense of suitable L^p-type Besov norms, we establish the global existence. As a consequence, like for incompressible flows, one may exhibit a class of large highly oscillating initial velocity fields for which global existence and uniqueness holds true.

Key words Korteweg system; global existence; L^p-type Besov norms

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1 Introduction

We are interested in the following compressible capillary fluid model, which can be derived from a Cahn-Hilliard like free energy (see the pioneering works by Dunn and Serrin [12] and also [1, 4, 13]). The conservation of mass and of momentum write:

$$\begin{cases} \partial_t\rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u\otimes u) - \mu\Delta u - (\mu+\mu')\nabla \mathrm{div}\, u + \nabla P(\rho) = \kappa\rho\nabla\Delta\rho. \end{cases} \tag{1.1}$$

Here $u = u(t,x) \in \mathbb{R}^d$ $(d \geq 2)$ stands for the velocity field and $\rho = \rho(t,x) \in \mathbb{R}^+$ is the density. The pressure P is a suitably smooth functions of ρ. We denote by μ and μ' the two Lamé coefficients of the fluid, which are assumed to satisfy $\mu > 0$ and $2\mu + \mu' > 0$. The constant $\kappa > 0$ is the capillary coefficient. In this article we investigate the Cauchy problem (1.1) with the initial condition:

$$(\rho, u)|_{t=0} = (\rho_0, u_0). \tag{1.2}$$

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The formulation of the theory of capillarity with diffuse interfaces was first introduced by Korteweg a century ago [18], and derived rigorously by Dunn and Serrin [12]. Due to the the physical importance and mathematical challenges, the study on Navier-Stokes-Korteweg equstion has attracted many physicists and mathematicians. Many results concerning the existence and uniqueness of (weak, strong or smooth) solutions can be found in [3, 11, 14–16, 19, 20] and the references cited therein. Among them, we refer to [3, 14] for the global existence of weak solutions, [19] for the local existence of strong solutions, [15, 16] for the existence of classical solutions, and [11] for the existence of solutions in the critical Besov spaces.

Here, we want to investigate the well-posedness of the system (1.1)–(1.2) in critical spaces, that is, in spaces which are invariant by the scaling of Korteweg's system. This is nowadays a classical approach for achieving the largest class of data for which well-posedness may be proved. Let us explain precisely the scaling of Korteweg system. For the compressible Navier-Stokes-Korteweg equations, let us introduce the following transformation:

$$\rho_\lambda(t,x) := \rho(\lambda^2 t, \lambda x),\ \ u_\lambda := \lambda u(\lambda^2 t, \lambda x).$$

Then if (ρ, u) solves (1.1), so does $(\rho_\lambda, u_\lambda)$ provided the pressure law has been changed into $\lambda^2 P$. This motivates the following definition:

Definition 1.1 We say that a functional space is critical with respect to the scaling of the equation if the associated norm is invariant under the transformation:

$$(\rho, u) \longrightarrow (\rho_\lambda, u_\lambda)$$

(up to a constant independent of λ).

This suggests us to choose initial data (ρ_0, u_0) in spaces whose norm is invariant by

$$(\rho_0, u_0) \longrightarrow (\rho_0(\lambda\cdot), \lambda u_0(\lambda\cdot)).$$

In the homogeneous Besov spaces framework, we are thus led to take ρ_0 in $\dot{B}^{\frac{d}{p_1}}_{p_1,r_1}(\mathbb{R}^d)$ and u_0 in $\dot{B}^{\frac{d}{p_2}-1}_{p_2,r_2}(\mathbb{R}^d)$ for some $1 \le p_1, r_1, p_2, r_2 \le \infty$. However, owing to the coupling between the density and the velocity equations, it is nature to take $p_1 = p_2$. In addition, in order to obtain a L^∞ control on the density we have to take $r_1 = 1$. Finally, as regards the velocity, having $r_2 = 1$ is the only way to obtain $\nabla u \in L^1_{\rm loc}(\mathbb{R}^+; L^\infty(\mathbb{R}^d))$ by means of parabolic regularity estimates, a property which is fundamental to transport the Besov regularity of the density. Because a global in time approach does not seem to be accessible for general data, we will mainly consider the global well-posedness problem for initial data close enough to stable equilibria $(\bar\rho, 0)$, where $\bar\rho > 0$ satisfies $P'(\bar\rho) > 0$. After suitable normalization, one may assume that, without loss of generality that $\bar\rho = 1$ and that $P'(1) = 1$. So finally denoting $a := \rho - 1$, the system (1.1)–(1.2) becomes

$$\begin{cases} \partial_t a + u\cdot\nabla a + (1+a){\rm div}\ u = 0, \\ \partial_t u + u\cdot\nabla u - I(a)\mathscr{A}u + \nabla G(a) = \kappa\nabla\Delta a, \\ (a,u)|_{t=0} = (a_0, u_0) = (\rho_0 - 1, u_0) \end{cases} \tag{1.3}$$

with $\mathscr{A} := \mu\Delta + (\mu+\mu')\nabla{\rm div}$, $I(a) := 1/1{+}a$ and G, a smooth function (that may be computed from P) satisfying $G'(0) = 1$.

From now on, let us agree that if f is a tempered distribution over $\mathbb{R}^d$, then f^ℓ and $f^\hbar$ stand for the low and high frequency parts of f, respectively (the exact definition is given in (4.3)). The following statement is a consequence of results that have been proved in [11].

Theorem 1.2 Let a_0 be in $\dot{B}^{\frac{d}{2}}_{2,1}$ and satisfy $1+a_0>0$. Let $u_0\in\dot{B}^{\frac{d}{2}-1}_{2,1}$, then there exists a positive time T such that system (1.3) has a unique solution (a,u) which belongs to E_T, where E_T is the set of function which satisfy

$$a\in C([0,T];\dot{B}^{\frac{d}{2}}_{2,1})\bigcap L^1([0,T];\dot{B}^{\frac{d}{2}+2}_{2,1}),$$

$$u\in C([0,T];\dot{B}^{\frac{d}{2}-1}_{2,1})\bigcap L^1([0,T];\dot{B}^{\frac{d}{2}+1}_{2,1})$$

with $1+a$ bounded away from 0.

If in addition $a_0\in\dot{B}^{\frac{d}{2}-1}_{2,1}$, then there exists a constant c depending only on d,μ,μ' and P such that if

$$\|a_0^\ell\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}+\|a_0^\hbar\|_{\dot{B}^{\frac{d}{2}}_{2,1}}+\|u_0\|_{\dot{B}^{\frac{d}{2}-1}_{2,1}}\le c,$$

then the above solution is global and satisfies

$$a^\ell\in C_b([0,T];\dot{B}^{\frac{d}{2}-1}_{2,1})\bigcap L^1([0,T];\dot{B}^{\frac{d}{2}+1}_{2,1}),$$

$$a^\hbar\in C_b([0,T];\dot{B}^{\frac{d}{2}}_{2,1})\bigcap L^1([0,T];\dot{B}^{\frac{d}{2}+2}_{2,1}),$$

$$u\in C_b([0,T];\dot{B}^{\frac{d}{2}-1}_{2,1})\bigcap L^1([0,T];\dot{B}^{\frac{d}{2}+1}_{2,1}).$$

In the present work, we aim at extending the above statement to Besov spaces related to L^p. The motivation for this is twofold:

(1) showing that a global well-posedness result for small data with critical regularity beyond energy method;

(2) having larger spaces for which the global well-posedness for small data holds true.

In particular, as pointed out by Cannone in [5] (or more recently in [7]), owing to the fact that $d/p-1$ is negative if $p>d$, velocity fields u_0 with a larger modulus may have a small norm in $\dot{B}^{\frac{d}{p}-1}_{p,1}$ provided they have fast enough oscillation.

As in [11], the system has to be handled differently for low and high frequencies. Roughly, in the low frequency regime, the first order terms predominate so that (1.3) must be treated by means of hyperbolic energy methods. The influence of the viscous term $I(a)\mathscr{A}(u)$, however, is decisive as it supplies the parabolic decay estimates for both a and u, they are the key to global results.

In contrast, in the high frequency regime, two types of modes coexist: the parabolic one (for the velocity) and the damped one (for the density), and a L^p approach may be used. More explanation will be given in Section 3. For the time being, let us introduce some notation (the reader is referred to the next section for the definition of spaces $\widetilde{C}_b(\dot{B}^s_{p,r})$ and $\widetilde{L}^1(\dot{B}^s_{p,r})$ and to (4.3) for the definition of $a^\ell,\ a^\hbar$ and $u^\hbar$.

Notation For $s\in\mathbb{R}$ and $1\le p,r\le\infty$, we denote by $E^s_{p,r}$ the set of functions (a,u) which satisfy

$$a^\ell\in\widetilde{C}_b(\dot{B}^s_{2,r})\bigcap\widetilde{L}^1(\dot{B}^{s+2}_{2,r}),\ a^\hbar\in\widetilde{C}_b(\dot{B}^{s+1}_{2,r}\bigcap\dot{B}^{\frac{d}{p}}_{p,1})\bigcap\widetilde{L}^1(\dot{B}^{s+3}_{2,r})\bigcap\dot{B}^{\frac{d}{p}+2}_{p,1}),$$

$$u\in\widetilde{C}_b(\dot{B}^s_{2,r})\bigcap\widetilde{L}^1(\dot{B}^{s+2}_{2,r}),\quad\text{and}\quad u^\hbar\in\widetilde{C}_b(\dot{B}^{\frac{d}{p}-1}_{p,1})\bigcap L^1(\dot{B}^{\frac{d}{p}+1}_{p,1}).$$

If $T>0$ then we denote by $E^s_{p,r}(T)$ the restriction to $[0,T]$ of functions of $E^s_{p,r}$. If $r=\infty$ then we replaced the strong continuity in $\dot{B}^s_{2,r}$ or in $\dot{B}^{s+1}_{2,r}$, above, by the weak continuity.

We shall repeatedly use the idea that in the case $p\geq 2$, owing to Bernstein's inequality (see Lemma below), the space $E^s_{p,r}$ is a subset of the set of functions (a,u) which satisfy

$$a\in\widetilde{C}_b(\dot{B}^{\frac{d}{p}}_{p,1})\bigcap L^2(\dot{B}^{\frac{d}{p}}_{p,1}),a^\ell\in\widetilde{C}_b(\dot{B}^s_{2,r})\bigcap\widetilde{L}^1(\dot{B}^{s+2}_{2,r}),\ a^\hbar\in\widetilde{C}_b(\dot{B}^{s+1}_{2,r})\bigcap\widetilde{L}^1(\dot{B}^{s+3}_{2,r}),$$

$$u\in\widetilde{C}_b(\dot{B}^s_{2,r})\bigcap\widetilde{L}^1(\dot{B}^{s+2}_{2,r}),\quad\text{and}\quad u\in\widetilde{C}_b(\dot{B}^{\frac{d}{p}-1}_{p,1})\bigcap L^1(\dot{B}^{\frac{d}{p}+1}_{p,1}).$$

Let us now state our main result.

Theorem 1.3 Assume that $a_0\in\dot{B}^{\frac{d}{p_1}}_{p_1,1}$ and $u_0\in\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}$ for some $p_1\in[2,2d)$. Furthermore we make the following addition (lower order) assumption:

$$a_0^\ell\in\dot{B}^s_{2,r},\ a_0^\hbar\in\dot{B}^{s+1}_{2,r}\text{ and }u_0\in\dot{B}^s_{2,r}$$

for some $r\in[1,\infty]$ and $s\in\mathbb{R}$ such that

$$-\min(1,d/p_2)<s<d/2-1\ \text{ if } r>1,\tag{1.4}$$

$$-\min(1,d/p_2)<s\leq d/2-1\ \text{ if } r\geq 1,\tag{1.5}$$

where $p_2\in[p_1,2d)$. Then there exist two constants c and M depending only on d,p_2,s and on the physical parameters of the system such that if

$$\|a_0^\ell\|_{\dot{B}^s_{2,r}}+\|a_0^\hbar\|_{\dot{B}^{s+1}_{2,r}}+\|a_0^\hbar\|_{\dot{B}^{\frac{d}{p_2}}_{p_2,1}}+\|u_0\|_{\dot{B}^s_{2,r}}+\|u_0^\hbar\|_{\dot{B}^{\frac{d}{p_2}-1}_{p_2,1}}\leq c,$$

then system (1.3) has a unique global-in-time solution (a,u) in $E^s_{p_1,r}$. In addition, the norm of the solution in $E^s_{p_i,r}$ $(i=1,2)$ by M times the norm of the data in the corresponding space.

Let us introduce some notations for the use throughout this article. C stands for a "harmless" constant and we will sometimes use the notation $A\lesssim B$ equivalently to $A\leq CB$. The notation $A\approx B$ means that $A\lesssim B$ and $B\lesssim A$. If $f:\mathbb{R}^d\to\mathbb{R}^m$ is a differential function then Df denotes the Jacobean matrix of f, and ∇f is the transposed matrix of Df.

We conclude this section by stating the arrangement of the rest of this article. In Section 2, we recall some basic facts about Littlewood-Paley decomposition and Besov spaces. Section 3 is devoted to the proof a priori estimates, first for the linearized system and next for the paralinearized system. The proof of global well-posedness is carried over to the last section.

2 Littlewood-Paley Theory and Functional Spaces

Let us introduce the Littlewood-Paley decomposition. Choose a radial function $\varphi\in S(\mathbb{R}^d)$ supported in $\mathcal{C}=\{\xi\in\mathbb{R}^d,\frac{3}{4}\leq|\xi|\leq\frac{8}{3}\}$ such that

$$\sum_{q\in\mathbb{Z}}\varphi(2^{-q}\xi)=1,\quad\text{if}\ \ \xi\neq 0.$$

The frequency localization operator Δ_q and S_q are defined by

$$\dot{\Delta}_q f=\varphi(2^{-q}D)f,\qquad S_q f=\sum_{j\leq q-1}\dot{\Delta}_j f\ \ \text{for all}\quad q\in\mathbb{Z}.$$

With our choice of φ, one can easily verify that

$$\dot{\Delta}_j\Delta_q = 0 \ \text{ if } \ |j-k| \geq 2$$

and

$$\dot{\Delta}_j(S_{k-1}f\dot{\Delta}_k f) = 0 \ \text{ if } \ |j-k| \geq 5.$$

We denote the space $\mathcal{Z}'(\mathbb{R}^d)$ by the dual space of $\mathcal{Z}(\mathbb{R}^d) = \{f \in S(\mathbb{R}^d); D^\alpha \hat{f}(0) = 0; \forall\alpha \in \mathbb{N}^N$ multi-index$\}$, it also can be identified by the quotient space of $S'(\mathbb{R}^d)/\mathcal{P}$ with polynomials space $\mathcal{P}$. The formal equality

$$f = \sum_{q\in\mathbb{Z}} \Delta_q f$$

holds true for $f \in \mathcal{Z}'(\mathbb{R}^d)$ and is called the homogeneous Littlewood-Paley decomposition.

The operators $\dot{\Delta}$ help us recall the definition of the Besov space (see also [21]).

Definition 2.1 Let $s \in \mathbb{R}, 1 \leq p, r \leq +\infty$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is defined by

$$\dot{B}^s_{p,r} = \{f \in \mathcal{Z}'(\mathbb{R}^N) : \|f\|_{\dot{B}^s_{p,r}} < +\infty\}$$

where

$$\|f\|_{\dot{B}^s_{p,r}} := \|2^{qs}\|\dot{\Delta}_q f\|_{L^p}\|_{l^r}.$$

Let us now state some basic properties for $\dot{B}^s_{p,r}$ spaces.

Proposition 2.2 The following some basic properties for $\dot{B}^s_{p,r}$ hold

• for any $p \in [1,\infty]$ we have the following chain of continuous embedding:

$$\dot{B}^0_{p,1} \hookrightarrow L^p \hookrightarrow \dot{B}^0_{p,\infty};$$

• if $p < \infty$ then $\dot{B}^{\frac{d}{p}}_{p,1}$ is an algebra continuously embedded in the set of continuous function decaying to 0 at infinity;

• the following real interpolation property is satisfied for all $1 \leq p, r_1, r_2, r \leq \infty,\ s_1 \neq s_2$ and $\theta \in (0,1)$:

$$[\dot{B}^{s_1}_{p,r_1}, \dot{B}^{s_2}_{p,r_2}]_{(\theta,r)} = \dot{B}^{\theta s_2+(1-\theta)s_1}_{p,r};$$

• for any smooth homogeneous of degree m function F on $\mathbb{R}^d\backslash\{0\}$ the operator $F(D)$ maps $\dot{B}^s_{p,r}$ in $\dot{B}^{s-m}_{p,r}$. In particular, as the Leray projector $\mathbb{P}$ over divergence free vector-fields and $\mathbb{P}^\perp := Id - \mathbb{P}$ satisfies the above assumptions with $m = 0$ (for, in Fourier variables, we have $\mathbb{P}^\perp u(\xi) = -\frac{\xi}{|\xi|^2}\xi \cdot \hat{u}(\xi)$), they map $\dot{B}^s_{p,r}$ in itself. Note also that the above property implies that the gradient operator maps $\dot{B}^s_{p,r}$ in $\dot{B}^{s-1}_{p,r}$.

The following lemma (referred to in what follows as Bernstein's inequalities) describes the way derivatives act on spectrally localized functions.

Lemma 2.3 Let $0 < r < R$. There exists a constant C such that, for any nonnegative integer k, any couple (p,q) in $[1,\infty]^2$ with $1 \leq p \leq q$ and any function u of L^p, we have for all $\lambda > 0$,

$$\begin{aligned}
&\text{Supp } \hat{u} \subset B(0,\lambda R) \Rightarrow \|D^k u\|_{L^q} \leq C^{k+1}\lambda^{k+d(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p};\\
&\text{Supp } \hat{u} \subset \{\xi \in \mathbb{R}^d : r\lambda \leq |\xi| \leq R\lambda\} \Rightarrow C^{-k-1}\lambda^k\|u\|_{L^p} \leq \|D^k u\|_{L^p}\\
&\qquad \leq C^{k+1}\lambda^k\|u\|_{L^p}.
\end{aligned}$$

The first Bernstein's inequality entails the following embedding result.

Proposition 2.4 For all $s\in\mathbb{R}, 1\le p_1\le p_2\le\infty$ and $1\le r_1\le r_2\le\ \infty$, the space $\dot{B}^{s}_{p_1,r_1}$ is continuously embedded in the space $\dot{B}^{s-d(\frac{1}{p}-\frac{1}{p})}_{p_2,r_2}$.

In this article, we shall work mainly with functions or distributions depending on both the time variable t and the space variable x. More often, these functions will be seen as defined on some time interval I and valued in some Banach space X. We shall denote by $C(I;X)$ (resp. $C_b(I;X)$) the set of continuous (resp. continuous bounded) functions on I with values in X. For $p\in[1,\infty]$, the notation $L^p(I;X)$ stands for the set of measurable functions on I with values in X such that $t\mapsto\|f(t)\|_X$ belongs to $L^p(I)$. We denote by $L^p_{\rm loc}(I;X)$ the set of those functions defined on I and valued in X which, restricted to any compact subset J of I, are in $L^p(J;X)$. In the case where $I=[0,T]$, the space $L^p([0,T];X)$ (resp. $C([0,T];X)$) will also be denoted by $L^p_T(X)$ (resp. $C_T(X)$). Finally, if $I=\mathbb{R}^+$ we shall alternately use the notation $L^p(X)$.

We next introduce the Besov-Chemin-Lerner space which is initiated in [8].

Definition 2.5 For $T>0, s\in\mathbb{R}$ and $1\le p,q,r\le\infty$.The space $\widetilde{L}^q_T(\dot{B}^s_{p,r})$ is defined as the set of all the the distributions f satisfying

$$\|f\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})}<\infty,$$

where

$$\|f\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})}:=\|2^{2ks}\|\dot{\Delta}_k f(t)\|_{L^q(0,T;L^p)}\|_{\ell^r}.$$

The letter T is omitted for functions defined over $\mathbb{R}^+$. We shall also adopt the notation

$$\widetilde{C}_T(\dot{B}^s_{p,r}):=\widetilde{L}^\infty_T(\dot{B}^s_{p,r})\cap C([0,T];\dot{B}^s_{p,r})$$

and

$$\widetilde{C}_b(\dot{B}^s_{p,r}):=\widetilde{L}^\infty(\dot{B}^s_{p,r})\cap C_b(\mathbb{R}^+;\dot{B}^s_{p,r}).$$

The spaces $\widetilde{L}^q_T(\dot{B}^s_{p,r})$ may be compared with the spaces $L^q_T(\dot{B}^s_{p,r})$ through the Minkowski inequality: we have

$$\|f\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})}\le\|f\|_{L^q_T(\dot{B}^s_{p,r})}\quad\text{if}\quad r\ge q$$

and

$$\|f\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})}\ge\|f\|_{L^q_T(\dot{B}^s_{p,r})}\quad\text{if}\quad r\le q.$$

The general principle is that all the properties of continuity for the product and composition which are true in Besov spaces (see blow) remain true in the above spaces. The time exponent simply behaves according to Hölder's inequality.

In the sequel, we will constantly use the Bony's decomposition from [2] that

$$uv=T_uv+T_vu+R(u,v)\tag{2.1}$$

with

$$T_uv=\sum_q S_{q-1}u\dot{\Delta}_qv$$

and

$$R(u,r):=\sum_q\sum_{|q'-q|\le1}\dot{\Delta}_qu\dot{\Delta}_{q'}v.$$

The above operator T is called the "paraproduct" whereas R is called the "remainder". We shall sometimes use the notation

$$T'_u v := T_u v + R(u,v).$$

The following properties of continuity for the paraproduct and remainder operators (sometimes adapted to $\widetilde{L}^q_T(\dot{B}^s_{p,r})$ spaces) will be of constant use in the article.

Proposition 2.6 For any $(s,p,r)\in\mathbb{R}\times[1,\infty]^2$ and $t<0$, there exists a constant C such that

$$\|T_u v\|_{\dot{B}^s_{p,r}} \le C\|u\|_{L^\infty}\|v\|_{\dot{B}^s_{p,r}}$$

and

$$\|T_u v\|_{\dot{B}^{s+t}_{p,r}} \le C\|u\|_{\dot{B}^t_{\infty,\infty}}\|v\|_{\dot{B}^s_{p,r}}.$$

For any (s_1,p_1,r_1) and (s_2,p_2,r_2) in $\mathbb{R}\times[1,\infty]^2$ there exists a constant C such that

(1) if $s_1+s_2>0, 1/p:=1/p_1+1/p_2\le 1$ and $1/r:=1/r_1+1/r_2\le 1$ then

$$\|R(u,v)\|_{\dot{B}^{s_1+s_2}_{p,r}} \le \|u\|_{\dot{B}^{s_1}_{p_1,r_1}}\|v\|_{\dot{B}^{s_2}_{p_2,r_2}};$$

(2) if $s_1+s_2=0, 1/p:=1/p_1+1/p_2\le 1$ and $1/r_1+1/r_2\ge 1$ then

$$\|R(u,v)\|_{\dot{B}^0_{p,\infty}} \le \|u\|_{\dot{B}^{s_1}_{p_1,r_1}}\|v\|_{\dot{B}^{s_2}_{p_2,r_2}}.$$

Combining the above proposition with (2.1) yields the following "tame estimate".

Corollary 2.7 Let f and g be in $L^\infty\cap\dot{B}^s_{p,r}$ for some $s>0$ and $(p,r)\in[1,\infty]^2$. Then there exists a constant C depending only on d,p and s and such that

$$\|fg\|_{\dot{B}^s_{p,r}} \le \|f\|_{L^\infty}\|g\|_{\dot{B}^s_{p,r}} + \|g\|_{L^\infty}\|f\|_{\dot{B}^s_{p,r}}.$$

The following result pertaining to the composition of functions in Besov spaces will be needed for handling the pressure and the viscosity terms.

Proposition 2.8 Let I be a bounded interval of $\mathbb{R}$ and $F: I\mapsto\mathbb{R}$ be a smooth function vanishing at 0. Then for all compact subset J of I, all $(p,r)\in[1,\infty]^2$ and all positive s with $s<d/p$ if $r>1$ and $s\le d/p$ if $r=1$, there exists a constant C such that for all $a\in\dot{B}^s_{p,r}$ with values in J, we have $F(a)\in\dot{B}^s_{p,r}$ and

$$\|F(a)\|_{\dot{B}^s_{p,r}} \le C\|a\|_{\dot{B}^s_{p,r}}.$$

We recall classical estimates for the flow of a smooth vector-field with bounded spatial derivatives. The following proposition may be easily deduced from Proposition 8 in [9] and Lemma 2.3.

Proposition 2.9 Let v be a smooth globally Lipschitz time dependent vector-field. Let $V(t)=\int_0^t\|\nabla v(t')\|_{L^\infty}\mathrm{d}t'$. Let

$$\Psi_q(t,x) = x + \int_0^t S_q v(t',\Psi(t',x))\mathrm{d}t'.$$

Then for all $t\in\mathbb{R}$, the flow Ψ_q is a smooth diffeomorphism over $\mathbb{R}^d$. Moreover there exists a constant C and one has if $t\ge 0$,

$$\|g\circ\Psi_q\|_{L^p} \le \mathrm{e}^{CV}\|g\|_{L^p} \text{ for all function } g \text{ in } L^p,$$
$$\|D\Psi_q^{\pm}\|_{L^\infty} \le \mathrm{e}^{CV},$$

$$\|D\Psi_q^{\pm}-Id\|_{L^\infty}\le \mathrm{e}^{CV}-1,$$
$$\|D^k\Psi_q^{\pm}\|_{L^\infty}\le C2^{(k-1)q}(\mathrm{e}^{CV}-1)\ \text{ for }\ k=2,3,4.$$

3 The Paralinearized System

Let us fix some suitably smooth vector-field v. The key to the proof of Theorem 1.3 is a new estimate in Besov spaces for the following paralinearization of system (1.3)

$$\begin{cases}\partial_t a+\mathrm{div}(T_v a)+\mathrm{div}\ u=F,\\ \partial_t u+T_v\cdot\nabla u-\mathscr{A}u+\nabla a-\kappa\nabla\Delta a=G,\end{cases}\tag{3.1}$$

with $\mathrm{div}(T_v a):=\partial_i(T_{v_i}a)$, and $T_v\cdot\nabla u:=T_{v_i}\partial_i u$. Here the summation convention over repeated indices is used, so does the following article.

We shall make an extensive use of the following notation. For $(p,r)\in[1,\infty]^2$ and $s\in\mathbb{R}$:

$$\|f\|^{\ell}_{\dot B^s_{p,r}}:=\Big(\sum_{q<N}(2^{qs}\|\dot\Delta_q u\|_{L^p})^r\Big)^{\frac1r}\ \text{ and }\ \|f\|^{\hbar}_{\dot B^s_{p,r}}:=\Big(\sum_{q\ge N}(2^{qs}\|\dot\Delta_q u\|_{L^p})^r\Big)^{\frac1r}.\tag{3.2}$$

For notational simplicity, the dependence on N is omitted. In the following statements, the value of N will depend only on the viscosity coefficients μ,μ' and the capillary coefficient κ.

Let us first recall a priori estimates for (3.1) in Besov spaces modeled on L^2:

Proposition 3.1 Let $s\in\mathbb{R}$ and $N\in\mathbb{N}$. Let (a,u) be a solution of (3.1). There exists a constant C depending only on μ,μ',κ,N,d and s, such that the following estimate holds:

$$\begin{aligned}&\|(a,u)\|^{\ell}_{\widetilde L^\infty_t(\dot B^s_{2,r})\cap\widetilde L^1_t(\dot B^{s+2}_{2,r})}+\|a\|^{\hbar}_{\widetilde L^\infty_t(\dot B^{s+1}_{2,r})\cap\widetilde L^1_t(\dot B^{s+3}_{2,r})}+\|u\|^{\hbar}_{\widetilde L^\infty_t(\dot B^{s}_{2,r})\cap\widetilde L^1_t(\dot B^{s+2}_{2,r})}\\ \le{}& C\mathrm{e}^{CV(t)}(\|(a_0,u_0)\|^{\ell}_{\dot B^s_{2,r}}+\|a_0\|^{\hbar}_{\dot B^{s+1}_{2,r}}+\|u_0\|^{\hbar}_{\dot B^s_{2,r}}+\|(F,G)\|^{\ell}_{\widetilde L^1_t(\dot B^s_{2,r})}\\ &+\|F\|^{\hbar}_{\widetilde L^1_t(\dot B^{s+1}_{2,r})}+\|G\|^{\hbar}_{\widetilde L^1_t(\dot B^{s}_{2,r})}),\end{aligned}$$

with $V(t):=\int_0^t\|\nabla v(\tau)\|_{L^\infty}\mathrm{d}\tau$.

Proof The case $r=1$ has been proved in [11]. The general case $r\in[1,\infty]$ may be obtained by means of real interpolation. □

The rest of this section is devoted to extending the previous proposition to the L^p framework. We first consider the linearized homogeneous system (that is $v\equiv0,F\equiv0$, and $G\equiv0$). The general case will be treated in the second part of this section.

3.1 Linearized Homogeneous System

We consider the following linear system

$$\begin{cases}\partial_t a+\mathrm{div}\ u=0,\\ \partial_t u-\mathscr{A}u+\nabla a-\kappa\nabla\Delta a=0.\end{cases}\tag{3.3}$$

Let $\nu:=2\mu+\mu'$. Introducing the Leray projector $\mathbb{P}:=Id+\nabla(-\Delta)^{-1}\mathrm{div}$ on divergence-free vector-fields, and $\mathbb{P}^\perp:=Id-\mathbb{P}$, the above system translates into

$$\begin{cases}\partial_t a+\mathrm{div}\mathbb{P}^\perp u=0,\\ \partial_t\mathbb{P}^\perp u-\nu\mathbb{P}^\perp u+\nabla a-\kappa\nabla\Delta a=0,\\ \partial_t\mathbb{P}u-\mu\Delta\mathbb{P}u=0.\end{cases}\tag{3.4}$$

Note that the equation for $\mathbb{P}u$ reduces to an ordinary equation, independent from the others, Moreover, if we denote by Λ^s the pseudo differential operator defined by $\Lambda^s u = \mathscr{F}^{-1}(|\xi|^s \hat{u}(\xi))$, it is equivalent to study $\mathbb{P}^{\perp}u$ or $v := \Lambda^{-1}\text{div } u$ and $\mathbb{P}u$ or $w := \Lambda^{-1}\text{curl } u$ (with $(\text{curl } z)_i^j = \partial_j z^i - \partial_i z^j$). So we are led to consider:

$$\begin{cases} \partial_t a + \Lambda v = 0, \\ \partial_t u - \nu\Delta v - \Lambda a - \kappa\Lambda^3 a = 0, \\ \partial_t w - \mu w = 0. \end{cases} \tag{3.5}$$

Indeed, as the definition of v and w, and relation $u = -\Lambda^{-1}\nabla v - \Lambda^{-1}\text{div } w$ involve pseudo-differential operators of degree zero, the estimates in Besov spaces for the original function u will be same as for (v, w).

This section is devoted to the proof of the following result.

Lemma 3.2 Let (a, u) satisfy system (3.3) with the initial data (a_0, u_0). Let $v := \Lambda^{-1}\text{div } u$ and $w := \Lambda^{-1}\text{curl } u$. There exist two constants c and C such that

(1) for all $j \in \mathbb{Z}$ and $p \in [1, \infty]$, we have

$$\mathrm{e}^{ct2^{2j}} \|\dot{\Delta}_j w(t)\|_{L^p} \le C \|\dot{\Delta}_j w_0\|_{L^p}; \tag{3.6}$$

(2) there is $M > 0$ such that for $j \ge M$ then for all $p \in [1, \infty]$,

$$\begin{cases} \|\dot{\Delta}_j v(t)\|_{L^p} \le C\mathrm{e}^{-ct2^{2j}}(\|\dot{\Delta}_j a_0\|_{L^p} + \|\dot{\Delta}_j v_0\|_{L^p}), \\ \|\dot{\Delta}_j a(t)\|_{L^p} \le C\mathrm{e}^{-ct2^{2j}}(\|\dot{\Delta}_j a_0\|_{L^p} + (2^j)^{-1}\|\dot{\Delta}_j v_0\|_{L^p}). \end{cases} \tag{3.7}$$

For all $m \ge 1$ there exist two constants c and C depending only on m such that if $j \le m$ then

$$\|\dot{\Delta}_j a(t), \dot{\Delta}_j v(t)\|_{L^2} \le C\mathrm{e}^{-ct2^{2j}} \|\dot{\Delta}_j a_0, \dot{\Delta}_j v_0\|_{L^p}. \tag{3.8}$$

Proof The estimate for w was proved by Chenmin in [6], so let us focus on the first two equations of (3.5). Taking the Fourier transform with respect to the space variable yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} \hat{a} \\ \hat{v} \end{pmatrix} = A(\xi)\begin{pmatrix} \hat{a} \\ \hat{v} \end{pmatrix} \quad \text{with} \quad A(\xi) := \begin{pmatrix} 0 & -|\xi| \\ |\xi| + \kappa|\xi|^3 & -\nu|\xi|^2 \end{pmatrix}. \tag{3.9}$$

The characteristic polynomial of $A(\xi)$ is $\lambda^2 + \nu|\xi|^2\lambda + |\xi|^2 + \kappa|\xi|^4$ and has two distinct roots

$$\lambda_{\pm}(\xi) := -\frac{\nu|\xi|^2}{2}(1 \pm R(\xi)) \quad \text{with} \quad R(\xi) := \sqrt{1 - \frac{4\kappa}{\nu^2} - \frac{4}{\nu^2|\xi|^2}}.$$

The matrix $A(\xi)$ is diagonalizable and after computing the associated eigenvalues, we find that

$$\begin{aligned} \hat{a}(t,\xi) = {} & \mathrm{e}^{t\lambda_-(\xi)}\Big(\frac{1}{2}(1 + \frac{1}{R(\xi)})\hat{a}_0(\xi) - \frac{1}{\nu|\xi|R(\xi)}\hat{v}_0(\xi)\Big) \\ & + \mathrm{e}^{t\lambda_+(\xi)}\Big(\frac{1}{2}(1 - \frac{1}{R(\xi)})\hat{a}_0(\xi) + \frac{1}{\nu|\xi|R(\xi)}\hat{v}_0(\xi)\Big), \\ \hat{v}(t,\xi) = {} & \mathrm{e}^{t\lambda_-(\xi)}\Big(\frac{1}{\nu|\xi|R(\xi)}\hat{a}_0(\xi) + \frac{1}{2}(1 - \frac{1}{R(\xi)})\hat{v}_0(\xi)\Big) \\ & + \mathrm{e}^{t\lambda_+(\xi)}\Big(-\frac{1}{\nu|\xi|R(\xi)}\hat{a}_0(\xi) + \frac{1}{2}(1 + \frac{1}{R(\xi)})\hat{v}_0(\xi)\Big). \end{aligned}$$

Remark that $\lambda_+(\xi) \sim -\frac{\nu|\xi|^2}{2}(1+\sqrt{1-\frac{4\kappa}{\nu^2}}\,)$ and that $\lambda_-(\xi) \sim -\frac{\nu|\xi|^2}{2}(1-\sqrt{1-\frac{4\kappa}{\nu^2}}\,)$ when $|\xi|$ goes to infinity. This is reminiscent of parabolic regularization for the high frequency of v and the damping for the high frequencies of a.

Let us now tackle the proof of (3.7) and (3.8). The proof of (3.8) relies on the energy method in the Fourier space and the proof of (3.7) relies on the use of explicit expression for $\dot{\Delta}_j a$ and $\dot{\Delta}_j v$. □

The low frequency regime For all $m \geq 1$, let us observe that for all $j \leq m$ we have

$$\begin{cases} \partial_t \widehat{\dot{\Delta}_j a}(t,\xi) + |\xi|\widehat{\dot{\Delta}_j v}(t,\xi) = 0, \\ \partial_t \widehat{\dot{\Delta}_j v}(t,\xi) + \nu|\xi|^2\widehat{\dot{\Delta}_j v}(t,\xi) - |\xi|\widehat{\dot{\Delta}_j a}(t,\xi) - \kappa|\xi|^3\widehat{\dot{\Delta}_j a}(t,\xi) = 0. \end{cases} \tag{3.10}$$

If we apply the energy method in the Fourier space to (3.10), then there is a time-frequency Lyapunov functional $\mathcal{E}(\widehat{\dot{\Delta}_j a}(t,\xi), \widehat{\dot{\Delta}_j v}(t,\xi))$ with

$$\mathcal{E}(\widehat{\dot{\Delta}_j a}(t,\xi), \widehat{\dot{\Delta}_j v}(t,\xi)) \sim |\widehat{\dot{\Delta}_j a}(t,\xi)|^2 + |\widehat{\dot{\Delta}_j v}(t,\xi)|^2 \tag{3.11}$$

satisfying that there is $c > 0$ depend only on m, ν and κ such that the Lyapunov inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}(\widehat{\dot{\Delta}_j a}(t,\xi), \widehat{\dot{\Delta}_j v}(t,\xi))^2 + c|\xi|^2\mathcal{E}(\widehat{\dot{\Delta}_j a}(t,\xi), \widehat{\dot{\Delta}_j v}(t,\xi)) \leq 0. \tag{3.12}$$

Combing (3.11), (3.12) and Plancherel theorem, we infer that inequality (3.8) holds for $j \leq m$.

The high frequency regime To finish, we have to prove inequality (3.7) for $j \geq M$ if M is large enough. In the case $p \neq 2$, we do not expect the above method to give an L^p estimate. We shall, rather, adapt the approach used in [6] for the heat equation. We divide into two case:

Case 1 $1 - \frac{4\kappa}{\nu^2} \neq 0$. We let $\phi_0 \in C_c^\infty(\mathbb{R}^d)$ with $\mathrm{supp}\phi_0 \subset \mathcal{C}' = \{\xi \in \mathbb{R}^d, \frac{1}{2} \leq |\xi| \leq 3\}$, and $\phi_0 = 1$ on $\mathcal{C}$. Observing that

$$\frac{1}{2}(1 - \frac{1}{R(\xi)}) = -(\frac{2}{\nu^2|\xi|^2} + \frac{2\kappa}{\nu^2})\frac{1}{R(\xi)(R(\xi)+1)},$$

we have

$$\begin{aligned} \widehat{\dot{\Delta}_j v}(t,\xi) = {} & \mathrm{e}^{t\lambda_-(\xi)}(\frac{\phi_0(2^{-j}\xi)}{\nu|\xi|R(\xi)}\widehat{\dot{\Delta}_j a_0}(\xi) - 2\phi_0(2^{-j}\xi)(\frac{1}{\nu^2|\xi|^2} + \frac{\kappa}{\nu^2})\frac{1}{R(\xi)(R(\xi)+1)}\widehat{\dot{\Delta}_j v_0}(\xi) \\ & + \mathrm{e}^{t\lambda_+(\xi)}(-\frac{\phi_0(2^{-j}\xi)}{\nu|\xi|R(\xi)}\widehat{\dot{\Delta}_j a_0}(\xi) + \frac{\phi_0(2^{-j}\xi)}{2}(1 + \frac{1}{R(\xi)})\hat{v}_0(\xi)), \end{aligned}$$

which we rewrite into

$$\begin{aligned} \dot{\Delta}_j v(t) = {} & h_1(t) * (|\nu D|^{-1}\dot{\Delta}_j a_0) + h_2(t) * (|\nu D|^{-2} + \frac{\kappa}{\nu^2})\dot{\Delta}_j v_0) \\ & + h_3(t) * (|\nu D|^{-1}\dot{\Delta}_j a_0) + h_4(t) * \dot{\Delta}_j v_0 \end{aligned} \tag{3.13}$$

with

$$\begin{cases} h_1(t,x) := -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi}\frac{\phi_0(2^{-j}\xi)}{R(\xi)}\mathrm{e}^{t\lambda_-(\xi)}\mathrm{d}\xi, \\ h_2(t,x) := -\frac{2}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi}\phi_0(2^{-j}\xi)(\frac{1}{R(\xi)} - \frac{1}{R(\xi)+1})\mathrm{e}^{t\lambda_-(\xi)}\mathrm{d}\xi, \\ h_3(t,x) := -\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi}\frac{\phi_0(2^{-j}\xi)}{R(\xi)}\mathrm{e}^{t\lambda_+(\xi)}\mathrm{d}\xi, \\ h_4(t,x) := -\frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi}\phi_0(2^{-j}\xi)(1 + \frac{1}{R(\xi)})\mathrm{e}^{t\lambda_+(\xi)}\mathrm{d}\xi. \end{cases}$$

We claim that there exist two positive constants c and C such that for all $t \geq 0$,

$$\|h_1(t), h_2(t), h_3(t), h_4(t)\|_{L^p} \leq C\mathrm{e}^{-c2^{2j}}.$$

To prove this, we shall study the following functions

$$\begin{cases} h^1(t,x) := \displaystyle\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi} \frac{\phi_0(2^{-j}\xi)}{R(\xi)} \mathrm{e}^{t\lambda_-(\xi)}\mathrm{d}\xi, \\ h^2(t,x) := \displaystyle\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi} \frac{\phi_0(2^{-j}\xi)}{R(\xi)+1} \mathrm{e}^{t\lambda_-(\xi)}\mathrm{d}\xi, \\ h^3(t,x) := \displaystyle\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi} \frac{\phi_0(2^{-j}\xi)}{R(\xi)} \mathrm{e}^{t\lambda_+(\xi)}\mathrm{d}\xi, \\ h^4(t,x) := \displaystyle\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi} \phi_0(2^{-j}\xi) \mathrm{e}^{t\lambda_+(\xi)}\mathrm{d}\xi. \end{cases}$$

We just consider the function h^1, and the proof works the same for h^2, h^3 and h^4. Using the change of variable $\xi = 2^j\eta$ we obtain that $h^1(t,x) = 2^{jd}h_t(2^jx)$ with

$$h_t(x) := \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\eta}\phi_0(\eta)A(2^j\eta)\mathrm{e}^{-\frac{\nu t2^{2j}}{2}|\eta|^2B(2^j\eta)}\mathrm{d}\eta$$

with

$$A(\xi) := \frac{1}{\sqrt{1-\frac{4\kappa}{\nu^2}-\frac{4}{\nu^2|\xi|^2}}} \quad \text{and} \quad B(\xi) := 1-\sqrt{1-\frac{4\kappa}{\nu^2}-\frac{4}{\nu^2|\xi|^2}}.$$

Note that $\|h^2(t)\|_{L^1} = \|h_t\|_{L^1}$. In order to estimate the L^1-norms, we shall exhibit a suitable bound for $(1+|x|^2)^d h(x)$. By definition of h_t, we have

$$(1+|x|^2)^d h_t(x) = \int_{\mathbb{R}^d} [(Id-\Delta_\xi)^d \mathrm{e}^{\mathrm{i}x\cdot\xi}]\phi_0(\xi)A(2^j\xi)\mathrm{e}^{-\frac{\nu t2^{2j}}{2}|\xi|^2B(2^j\xi)}\mathrm{d}\xi,$$

so that, performing integrations by parts:

$$(1+|x|^2)^d h_t(x) = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}x\cdot\xi}(Id-\Delta_\xi)^d[\phi_0(\xi)A(2^j\xi)\mathrm{e}^{-\frac{\nu t2^{2j}}{2}|\xi|^2B(2^j\xi)}]\mathrm{d}\xi. \tag{3.14}$$

From the Leibniz formula, we see that there exist integer numbers $C_{\alpha,\beta}$ so that

$$\begin{aligned} &(Id-\Delta_\xi)^d[\phi_0(\xi)A(2^j\xi)\mathrm{e}^{-\frac{\nu t2^{2j}}{2}|\xi|^2B(2^j\xi)}] \\ =& \sum_{|\alpha|+|\beta|\leq 2d} C_{\alpha,\beta}\partial^\alpha(\mathrm{e}^{-\frac{\nu t2^{2j}}{2}|\xi|^2B(2^j\xi)})\partial^\beta(\phi_0(\xi)A(2^j\xi)). \end{aligned}$$

Now, the Leibniz formula gives:

$$\partial^\beta(\phi_0(\xi)A(2^j\xi)) = \sum_{|\gamma|\leq|\beta|} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} \partial^{\beta-\gamma}\phi_0(\xi)(2^j)^{|\gamma|}(\partial^\gamma A)(2^j\xi),$$

and the Faá-Bruno formula yields that the following quantity:

$$\mathrm{e}^{\frac{\nu t2^{2j}}{2}|\xi|^2B(2^j\xi)}\partial^\alpha(\mathrm{e}^{-\frac{\nu t2^{2j}}{2}|\xi|^2B(2^j\xi)})$$

is a sum of terms of the form:

$$\sum_{\alpha_1+\cdots+\alpha_m=\alpha, |\alpha_j|\geq 1} (-\frac{\nu t2^{2j}}{2})^m \prod_{j=1}^m \partial^{\alpha_j}(|\xi|^2B(2^j\xi)).$$

Once more using the Leibniz formula we obtain:

$$\partial^{\alpha_j}(|\xi|^2 B(2^j\xi)) = \sum_{\gamma_j\le\alpha_j}\begin{pmatrix}\alpha_j\\ \gamma_j\end{pmatrix}\partial^{\alpha_j-\gamma_j}(|\xi|^2)(2^j)^{|\gamma_j|}(\partial^{\gamma_j}B)(2^j\xi).$$

Recall that in (3.14) integration may be restricted to $\mathcal{C}'$ thanks to the cut-off function ϕ_0. So it suffices to bound the integrand on $\mathcal{C}'$. Thanks to the exponential term, we can absorb the the powers of $\nu t 2^{2^j}$, as the real part of $B(2^j\xi)$ is positive if j large enough. For the powers of 2^j, it's easy to get rid of these term is to take advantage of the following properties of functions A and B:

Proposition 3.3 There exist two constants $C>0$ and $M>0$ such that, for all $\xi\in\mathcal{C}'$ and $j>M$, and all nonzero multi-index γ, we have:

$$|A(2^j\xi)|+|B(2^j\xi)|+(2^j)^{|\gamma|}(|(\partial^\gamma A)(2^j\xi)|+|(\partial^\gamma B)(2^j\xi)|)\le C.$$

The above proposition may be easily proved by induction. It is only a matter of computing the derivatives of A and B and using the fact that, for $j>M$, the terms of the form $\frac{1}{\sqrt{1-\frac{4\kappa}{\nu^2}-\frac{4}{\nu^2|2^j\xi|^2}}}$ can be bounded by a constant if M large enough.

Gathering all this information and using (3.14), we conclude that for all $x\in\mathbb{R}^d$:

$$(1+|x|^2)^d|h_t(x)|\le C(1+2^{2j}t)^{2d}\int_{\mathcal{C}'}\mathrm{e}^{-\frac{\nu t2^{2j}}{2}|\xi|^2(1-\sqrt{1-\frac{4\kappa}{\nu^2}-\frac{4}{\nu^2|2^j\xi|^2}})}\mathrm{d}\xi.$$

So finally, there exist two constants c and C such that for all $x\in\mathbb{R}^d$,

$$(1+|x|^2)^d|h_t(x)|\le C\mathrm{e}^{-ct2^{2j}},$$

and we can conclude that h^1 satisfies the desired inequality and the claim is proved.

Now, convolution inequalities enable us to bound the L^p norm of all the terms in the decomposition (3.13). We eventually obtain two positive constants c and C such that for all $j>M$,

$$\|\dot\Delta_j v(t)\|_{L^p}\le C\mathrm{e}^{-ct2^{2j}}(\|\dot\Delta_j a_0\|_{L^p}+\|\dot\Delta_j v_0\|_{L^p}).$$

Taking the same method to estimate a we have

$$\|\dot\Delta_j a(t)\|_{L^p}\le C\mathrm{e}^{-ct2^{2j}}(\|\dot\Delta_j a_0\|_{L^p}+(2^j)^{-1}\|\dot\Delta_j v_0\|_{L^p}).$$

Case 2 $1-\frac{4\kappa}{\nu^2}=0$. If $1-\frac{4\kappa}{\nu^2}=0$, we rewrite $\hat a(t,\xi)$ and $\hat v(t,\xi)$ into

$$\begin{aligned}\hat a(t,\xi)&=\cos(|\xi|t)\mathrm{e}^{-\frac{\nu|\xi|^2t}{2}}\hat a_0(\xi)-\frac12\frac{\sin(|\xi|t)}{|\xi|t}\nu|\xi|^2t\mathrm{e}^{-\frac{\nu|\xi|^2t}{2}}\hat a_0(\xi)\\&\quad+\frac{1}{2\nu}\frac{\sin(|\xi|t)}{|\xi|t}\nu|\xi|^2t\mathrm{e}^{-\frac{\nu|\xi|^2t}{2}}\frac{1}{|\xi|}\hat v_0(\xi),\\ \hat v(t,\xi)&=-\frac{1}{2\nu}\frac{\sin(|\xi|t)}{|\xi|t}\nu|\xi|^2t\mathrm{e}^{-\frac{\nu|\xi|^2t}{2}}\frac{1}{|\xi|}\hat a_0(\xi)+\cos(|\xi|t)\mathrm{e}^{-\frac{\nu|\xi|^2t}{2}}\hat v_0(\xi)\\&\quad+\frac12\frac{\sin(|\xi|t)}{|\xi|t}\nu|\xi|^2t\mathrm{e}^{-\frac{\nu|\xi|^2t}{2}}\hat v_0(\xi).\end{aligned}$$

Because there exist constants c,C and M such that

$$|D^\alpha(\frac{\sin(|\xi|t)}{|\xi|t}\nu|\xi|^2t\mathrm{e}^{-\frac{\nu|\xi|^2t}{2}}+\cos(|\xi|t)\mathrm{e}^{-\frac{\nu|\xi|^2t}{2}})|<C\mathrm{e}^{-c|\xi|^2t},$$

if $|\xi|>M$. Then discuss just as above, we have (3.7).

3.2 The Variable Coefficients Case

This section is devoted to the proof of the following result which is the key to obtain Theorem 1.3.

Proposition 3.4 Let $(s,s')\in\mathbb{R}^2$, $r\in[1,\infty]$ and (a,u) be a solution of (3.1). Assume that $p\in[2,\infty]$. Let $V(t):=\int_0^t\|\nabla v(\tau)\|_{L^\infty}\mathrm{d}\tau$. There exists a constant C depending only on $(\mu,\mu'),d$ and (s,s'), and on integer N depending only on ν such that under Notation (3.2) the following estimate holds for all $t\geq 0$:

$$\begin{aligned}&\|(a,u)\|^{\ell}_{\widetilde{L}^\infty_t(\dot{B}^{s'}_{2,r})\cap\widetilde{L}^1_t(\dot{B}^{s'+2}_{2,r})}+\|a\|^{\hbar}_{\widetilde{L}^\infty_t(\dot{B}^{s'+1}_{2,r})\cap\widetilde{L}^1_t(\dot{B}^{s'+3}_{2,r})}+\|u\|^{\hbar}_{\widetilde{L}^\infty_t(\dot{B}^{s'}_{2,r})\cap\widetilde{L}^1_t(\dot{B}^{s'+2}_{2,r})}\\&\quad+\|(a)\|^{\hbar}_{\widetilde{L}^\infty_t(\dot{B}^{s+1}_{p,1})\cap L^1_t(\dot{B}^{s+3}_{p,1})}+\|(u)\|^{\hbar}_{\widetilde{L}^\infty_t(\dot{B}^{s}_{p,1})\cap L^1_t(\dot{B}^{s+2}_{p,1})}\\&\leq C\mathrm{e}^{CV(t)}(\|(a_0,u_0)\|^{\ell}_{\dot{B}^{s'}_{2,r}}+\|(F,G)\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{s'}_{2,r})}+\|a_0\|^{\hbar}_{\dot{B}^{s'+1}_{2,r}}+\|u_0\|^{\hbar}_{\dot{B}^{s'}_{2,r}}+\|F\|^{\hbar}_{\widetilde{L}^1_t(\dot{B}^{s'+1}_{2,r})}\\&\quad+\|G\|^{\hbar}_{\widetilde{L}^1_t(\dot{B}^{s'}_{2,r})}+\|a_0\|^{\hbar}_{\dot{B}^{s+1}_{p,1}}+\|u_0\|^{\hbar}_{\dot{B}^{s}_{p,1}}+\|F\|^{\hbar}_{L^1_t(\dot{B}^{s+1}_{p,1})}+\|G\|^{\hbar}_{L^1_t(\dot{B}^{s}_{p,1})}).\end{aligned}$$

Proof Owning to the hyperbolic nature of the first equation of (3.1), one cannot expect to have any estimate for (a,u) by a direct application of Lemma 3.2 treating the convection terms as source terms. To overcome this difficulty, we shall adapt the method introduced by Hmidi in [17] for transport-diffusion equations with divergence free vector-fields, and extended in [10] for general vector-fields.

First, we apply operator $\dot{\Delta}_q$ to (3.1) in order to obtain a system of equations for $(a_q,u_q):=(\dot{\Delta}_q a,\dot{\Delta}_q u)$. We have

$$\begin{cases}\partial_t a_q+v_q\cdot\nabla a_q+\mathrm{div}\ u_q=f_q,\\ \partial_t u_q+v_q\cdot\nabla u_q-\mathscr{A}u_q+\nabla a_q-\kappa\nabla\Delta a_q=g_q.\end{cases}$$

with $v_q:=S_{q-1}v$ and

$$\begin{aligned}f_q&:=\dot{\Delta}_qF+\mathrm{div}(S_{q-1}v\dot{\Delta}_qa-\dot{\Delta}_qT_va)-\dot{\Delta}_qa\mathrm{div}\ v_q,\\ g_q&:=\dot{\Delta}_qG+S_{q-1}v\cdot\nabla\dot{\Delta}_qu-\dot{\Delta}_q(T_v\cdot\nabla u).\end{aligned}$$

We have the following estimates in [9] for f_q and g_q:

$$||\nabla f_q||_{L^p}\leq||\nabla\dot{\Delta}_qF||_{L^p}+C||\nabla v||_{L^\infty}\sum_{q'\sim q}||\nabla\dot{\Delta}_{q'}a||_{L^p},\tag{3.15}$$

and

$$||g_q||_{L^p}\leq||\dot{\Delta}_qG||_{L^p}+C||\nabla v||_{L^\infty}\sum_{q'\sim q}||\dot{\Delta}_{q'}u||_{L^p}.\tag{3.16}$$

Where $q\sim q'$ means that the summation is restricted over $q'\in\{q-4,\cdots,q+4\}$. This is a consequence of the spectral localization properties of the Littlewood-Paley decomposition.

In order to handle he convection terms, we perform the Lagrangian change of variable $(\tau,x)=(t,\psi_q(t,y))$ where ψ_q stands for the flow of v_q. Let $\phi_q:=\psi_q^-$, $\widetilde{f}:=f_q\circ\psi_q$ and $\widetilde{g}:=g_q\circ\psi_q$. Obviously $(\widetilde{a}_q,\widetilde{u}_q):=(a_q\circ\psi_q,u_q\circ\psi_q)$ satisfies

$$\begin{cases}\partial_t\widetilde{a}_q+\mathrm{div}\ \widetilde{u}_q=\widetilde{f}_q+R^1_q,\\ \partial_t\widetilde{u}_q-\mathscr{A}\widetilde{u}_q+\nabla\widetilde{a}_q-\kappa\nabla\Delta\widetilde{a}_q=\widetilde{g}_q+R^2_q+\kappa R^3_q+R^4_q,\end{cases}\tag{3.17}$$

where the remainder terms R_q^1, R_q^2, R_q^3 and R_q^4 are defined by:

$$\begin{aligned}
R_q^1(t,x) &:= Tr[\nabla\widetilde{u}_q(t,x)\cdot(Id-\nabla\phi_q(t,\psi_q(t,x)))),\\
R_q^2(t,x) &:= \nabla\widetilde{a}_q(t,x)\cdot(Id-\nabla\phi_q(t,\psi_q(t,x))),\\
R_q^{3,i}(t,x) &:= \{Tr[\nabla D_k\phi_q(t,\psi_q(t,x))\cdot\nabla D\widetilde{a}_q(t,x)\cdot D\phi_q(t,\psi_q(t,x))\\
&\quad+(\nabla\phi_q(t,\psi_q(t,x))-Id)\cdot\nabla DD_k\widetilde{a}_q(t,x)\cdot D\phi_q(t,\psi_q(t,x))\\
&\quad+(\nabla\phi_q(t,\psi_q(t,x))-Id)\cdot\nabla D\widetilde{a}_q(t,x)\cdot DD_k\phi_q(t,\psi_q(t,x))\\
&\quad+\nabla DD_k\widetilde{a}_q(t,x)\cdot(D\phi_q(t,\psi_q(t,x))-Id)+\nabla D\widetilde{a}_q(t,x)\\
&\quad\cdot DD_k\phi_q(t,\psi_q(t,x))]+\nabla D_k\widetilde{a}_q(t,x)\cdot\Delta\phi_q(t,\psi_q(t,x))\\
&\quad+\nabla\widetilde{a}_q(t,x)\cdot\Delta D_k\phi_q(t,\psi_q(t,x))\}\cdot\nabla\phi_q(t,\psi_q(t,x))\\
&\quad+\Delta D_k\widetilde{a}_q(t,x)(D_k\phi_q^i(t,\psi_q(t,x))-\delta_{ik}Id)
\end{aligned}$$

and $R_q^4 := \mu R_q^5 + (\mu+\mu')R_q^6$ with

$$\begin{aligned}
R_q^{5,i}(t,x) &:= Tr[(\nabla\phi_q(t,\psi_q(t,x))-Id)\cdot\nabla D\widetilde{u}_q^i(t,x)\cdot D\phi_q(t,\psi_q(t,x))\\
&\quad+\nabla D\widetilde{u}_q^i(t,x)\cdot(D\phi_q(t,\psi_q(t,x))-Id)]+\nabla\widetilde{u}_q^i(t,x)\cdot\Delta\phi_q(t,\psi_q(t,x)),\\
R_q^{6,i}(t,x) &:= Tr[D\widetilde{u}_q(t,x)\cdot\partial_i D\phi_q(t,\psi_q(t,x))]\\
&\quad+\sum_{j,l,k,j\neq k,l\neq i}\partial_{lj}^2\widetilde{u}_q^k(t,x)\cdot\partial_i\phi_q^l(t,\psi_q(t,x))\cdot\partial_k\phi_q^j(t,\psi_q(t,x))\\
&\quad+\sum_{k=1}^d\partial_{ik}^2\widetilde{u}_q^k(t,x)\cdot[(\partial_i\phi_q^i(t,\psi_q(t,x))-Id)\cdot\partial_k\phi_q^k(t,\psi_q(t,x))\\
&\quad+(\partial_k\phi_q^k(t,\psi_q(t,x))-Id)].
\end{aligned}$$

Now we expect Lemma 3.2 to provide us with the desired estimate for $(\widetilde{a}_q,\widetilde{u}_q)$. As the change of variable destroys the spectral localization of (a_q,u_q), this is not so straightforward, though. To handle frequencies of $\widetilde{u}_q$ which are very small compare to 2^q , we are going to take advantage of Lemma A.1 in [10], which ensures that there exists a constant C such that for all $N_0\in\mathbb{N}, q\in\mathbb{N}$ and $t\in\mathbb{R}^+$, we have

$$||S_{q-N_0}\widetilde{u}_q(t)||_{L^p}\leq C(2^{-N_0}\mathrm{e}^{CV(t)}+(\mathrm{e}^{CV(t)}-1))||u_q(t)||_{L^p}.$$

Bounding the low of frequencies of $\nabla\widetilde{a}_q$ stems from a similar argument; indeed, using the Bernstein's inequality, one may write

$$||S_{q-N_0}D\widetilde{a}_q(t)||_{L^p}\leq C2^{q-N_0}||S_{q-N_0}\widetilde{a}_q(t)||_{L^p}.$$

Hence, using the aforementioned lemma and the Bernstein's inequality, one may conclude that

$$||S_{q-N_0}D\widetilde{a}_q(t)||_{L^p}\leq C(2^{-2N_0}\mathrm{e}^{CV(t)}+2^{-N_0}(\mathrm{e}^{CV(t)}-1))||Da_q(t)||_{L^p}. \tag{3.18}$$

In order to bound the high frequency part of $(D\widetilde{a}_q,\widetilde{u}_q)$, we shall use a decomposition into dyadic blocks and bound each block by means of Lemma 3.2. More precisely, if we denote by $(\mathrm{e}^{tL})_{t\geq0}$ the semi-group with generator L then we may write

$$\begin{pmatrix}\dot\Delta_j\widetilde{a}_q(t)\\ \dot\Delta_j\widetilde{u}_q(t)\end{pmatrix}=\mathrm{e}^{tL}\begin{pmatrix}\dot\Delta_j\dot\Delta_q a_0(t)\\ \dot\Delta_j\dot\Delta_q a_0(t)\end{pmatrix}+\int_0^t\mathrm{e}^{(t-\tau)L}\begin{pmatrix}\dot\Delta_j(\widetilde{f}_q(t)+R_q^1)\\ \dot\Delta_j(\widetilde{g}_q(t)+R_q^2+R_q^3+R_q^4)\end{pmatrix}\mathrm{d}\tau.$$

Hence combined Lemma 3.2 with Bernstein's inequality ensures that there exists $N_1 \in \mathbb{Z}$ depending only on ν, κ and two constants c and C depending only on (μ, μ') and κ, such that for all $j \geq N_1$, we have

$$\begin{aligned}||\dot{\Delta}_j D\widetilde{a}_q(t)||_{L^p} \lesssim\ & \mathrm{e}^{-c2^{2j}t}(||\dot{\Delta}_j\dot{\Delta}_q Da_0||_{L^p} + ||\dot{\Delta}_j\dot{\Delta}_q u_0||_{L^p}) \\ & + \int_0^t \mathrm{e}^{-c2^{2j}(t-\tau)}(||\dot{\Delta}_j D\widetilde{f}_q(t)||_{L^p} + ||\dot{\Delta}_j DR_q^1(t)||_{L^p} \\ & + ||\dot{\Delta}_j \widetilde{g}_q(t)||_{L^p} + ||\dot{\Delta}_j R_q^2(t)||_{L^p} + ||\dot{\Delta}_j R_q^3(t)||_{L^p} \\ & + ||\dot{\Delta}_j R_q^4(t)||_{L^p} + ||\dot{\Delta}_j R_q^5(t)||_{L^p})\mathrm{d}\tau, \end{aligned} \tag{3.19}$$

$$\begin{aligned}||\dot{\Delta}_j \widetilde{u}_q(t)||_{L^p} \lesssim\ & \mathrm{e}^{-c2^{2j}t}(||\dot{\Delta}_j\dot{\Delta}_q Da_0||_{L^p} + ||\dot{\Delta}_j\dot{\Delta}_q u_0||_{L^p}) \\ & + \int_0^t \mathrm{e}^{-c2^{2j}(t-\tau)}(||\dot{\Delta}_j \widetilde{D} f_q(t)||_{L^p} + ||\dot{\Delta}_j DR_q^1(t)||_{L^p} \\ & + ||\dot{\Delta}_j \widetilde{g}_q(t)||_{L^p} + ||\dot{\Delta}_j R_q^2(t)||_{L^p} + ||\dot{\Delta}_j R_q^3(t)||_{L^p} \\ & + ||\dot{\Delta}_j R_q^4(t)||_{L^p} + ||\dot{\Delta}_j R_q^5(t)||_{L^p})\mathrm{d}\tau. \end{aligned} \tag{3.20}$$

In the final step of the proof, it will be possible to obtain an estimate for a_q and u_q just by using the fact

$$\begin{aligned}\nabla a_q &= (S_{q-N_0}\nabla\widetilde{a}_q \circ \phi_q + \sum_{j \geq q-N_0} \dot{\Delta}_j \nabla\widetilde{a}_q \circ \phi_q) \cdot \nabla\phi_q \\ u_q &= S_{q-N_0}\widetilde{u}_q \circ \phi_q + \sum_{j\geq q-N_0} \dot{\Delta}_j \widetilde{u}_q \circ \phi_q. \end{aligned} \tag{3.21}$$

So now we are left with bounding all the terms in the right-hand side of (3.19) and (3.20). According to [9], we have the following estimates

$$\begin{aligned}||\dot{\Delta}_j D\widetilde{f}_q(t)||_{L^p} &\lesssim 2^{q-j}\mathrm{e}^{CV(t)}(||\dot{\Delta}_j DF_q(t)||_{L^p} + C||Dv(t)||_{L^\infty} \sum_{q'\sim q} ||\dot{\Delta}_{q'} Da(t)||), \\ ||\dot{\Delta}_j \widetilde{g}_q(t)||_{L^p} &\lesssim 2^{q-j}\mathrm{e}^{CV(t)}(||\dot{\Delta}_j G_q(t)||_{L^p} + C||Dv(t)||_{L^\infty} \sum_{q'\sim q} ||\dot{\Delta}_{q'} u(t)||), \\ ||\dot{\Delta}_j DR_q^1(t)||_{L^p} &\lesssim 2^{q-j}(\mathrm{e}^{CV(t)}-1)\mathrm{e}^{CV(t)}2^{2q}||u_q||_{L^p}, \\ ||\dot{\Delta}_j R_q^2(t)||_{L^p} &\lesssim 2^{q-j}(\mathrm{e}^{CV(t)}-1)\mathrm{e}^{CV(t)}||Da_q||_{L^p}, \\ ||\dot{\Delta}_j R_q^4(t)||_{L^p} &\lesssim 2^{q-j}(\mathrm{e}^{CV(t)}-1)\mathrm{e}^{CV(t)}2^{2q}||u_q||_{L^p}. \end{aligned} \tag{3.22}$$

Let us now bound the term R_q^3, we use Bernstein's inequality and write

$$||\dot{\Delta}_j R_q^3||_{L^p} \lesssim 2^{-j}||\dot{\Delta}_j DR_q^3||_{L^p}.$$

Therefore, using the chain rule and Hölder's inequality, we obtain

$$\begin{aligned}||\dot{\Delta}_j R_q^3||_{L^p} \lesssim\ & 2^{-j}\{||D^4\widetilde{a}_q||_{L^p}||D\phi_q\circ\psi_q - Id||_{L^\infty}(1+||D\phi_q\circ\psi_q||_{L^\infty})^2 \\ & + ||D^3\widetilde{a}_q||_{L^p}||D^2\phi_q\circ\psi_q||_{L^\infty}(||D\phi_q\circ\psi_q||_{L^\infty}+1)^2 \\ & \times(1+||\psi_q||_{L^\infty}) + ||D^2\widetilde{a}_q||_{L^p}((||D^3\phi_q\circ\psi_q||_{L^\infty} \\ & + ||D^2\phi_q\circ\psi_q||^2_{L^\infty})(||D\phi_q\circ\psi_q||_{L^\infty}+1)^2(1+||\psi_q||_{L^\infty})) \\ & + ||D\widetilde{a}_q||_{L^p}(||D^3\phi_q\circ\psi_q||_{L^\infty}||D\phi_q\circ\psi_q||_{L^\infty} \\ & + ||D^2\phi_q\circ\psi_q||^2_{L^\infty})||\psi_q||_{L^\infty}\}. \end{aligned}$$

So, taking advantage of Proposition 2.9, we conclude that

$$||\dot{\Delta}_j R_q^3||_{L^p} \lesssim 2^{q-j}(\mathrm{e}^{CV}-1)2^{2q}||Da_q||_{L^p}. \tag{3.23}$$

Now, combining (3.19) and (3.20) with convolution inequalities with respect to time yields

$$\begin{aligned} &||\dot{\Delta}_j D\widetilde{a}_q, \dot{\Delta}_j \widetilde{u}_q||_{L_t^\infty(L^p)} + 2^{2^j}||\dot{\Delta}_j D\widetilde{a}_q, \dot{\Delta}_j \widetilde{u}_q||_{L_t^1(L^p)} \\ \lesssim\; & ||\dot{\Delta}_j\dot{\Delta}_q Da_0||_{L^p} + ||\dot{\Delta}_j\dot{\Delta}_q u_0||_{L^p} + ||\dot{\Delta}_j D\widetilde{f}_q||_{L_t^1(L^p)} + ||\dot{\Delta}_j DR_q^1||_{L_t^1(L^p)} \\ &+ ||\dot{\Delta}_j\widetilde{g}_q||_{L_t^1(L^p)} + ||\dot{\Delta}_j R_q^2||_{L_t^1(L^p)} + ||\dot{\Delta}_j R_q^3||_{L_t^1(L^p)} + ||\dot{\Delta}_j R_q^4||_{L_t^1(L^p)}. \end{aligned}$$

Hence, inserting inequalities (3.22) and (3.23) in the above inequality, we end up with

$$\begin{aligned} &||\dot{\Delta}_j D\widetilde{a}_q, \dot{\Delta}_j \widetilde{u}_q||_{L_t^\infty(L^p)} + 2^{2^j}||\dot{\Delta}_j D\widetilde{a}_q, \dot{\Delta}_j \widetilde{u}_q||_{L_t^1(L^p)} \\ \lesssim& ||\dot{\Delta}_j\dot{\Delta}_q Da_0, \dot{\Delta}_j\dot{\Delta}_q u_0||_{L^p} + 2^{q-j}\mathrm{e}^{CV(t)}||\dot{\Delta}_j DF, \dot{\Delta}_j G||_{L_t^1(L^p)} \\ &+ 2^{q-j}\sum_{q'\sim q}\int_0^t V'\mathrm{e}^{CV}||(Da_{q'}, u_{q'})||_{L^p}\mathrm{d}\tau + 2^{q-j}(\mathrm{e}^{CV(t)}-1)\mathrm{e}^{CV(t)}2^{2q}(||Da_q, u_q||_{L_t^1(L^p)}). \end{aligned}$$

Let

$$\begin{aligned} U_q(t) &:= ||(Da_q, u_q)||_{L_t^\infty(L^p)} + 2^{2q}||(Da_q, u_q)||_{L_t^1(L^p)}, \\ U_q^0(t) &:= ||(Da_q(0), u_q(0))||_{L^p} + ||(D\dot{\Delta}_q F, \dot{\Delta}_q G)||_{L_t^1(L^p)}. \end{aligned}$$

From the previous inequality and (3.21), we see that for all $q \geq N_0 + N_1$ and $t \geq 0$, we have

$$U_q(t) \leq C\mathrm{e}^{CV(t)}(2^{3N_0}(U_q^0(t) + \sum_{q'\sim q}\int_0^t V'U_{q'}\mathrm{d}\tau) + C(2^{3N_0}(\mathrm{e}^{CV(t)}-1) + 2^{-N_0})U_q(t)).$$

Let us fix some T such that $CV(T) \leq \log 2$ (so that in particular $\mathrm{e}^{CV(t)} - 1 \leq 2CV(t)$ for $t \in [0, T]$). Next, choose N_0 to be a unique integer such that $2C2^{-N_0} \in (1/4, 1/2)$. Finally, let us assume, in addition, that $16C^2V(T)2^{3N_0} \leq 1$. Then the last term of the above inequality may be absorbed by the left-hand side. Hence, if T has been defined by

$$T := \sup\{t \leq 0 : CV(t) \leq \log 2 \quad \text{and} \quad 16C^2V(T)2^{3N_0} \leq 1\} \tag{3.24}$$

then for all $t \in [0, T]$ and $q \geq N_0 + N_1$, we have

$$U_q(t) \leq C(U_q^0 + \sum_{q'\sim q}\int_0^t V'U_{q'}\mathrm{d}\tau).$$

So multiplying both sides by 2^{qs}, we have

$$2^{qs}U_q(t) \leq C(2^{qs}U_q^0 + \sum_{q'\sim q}\int_0^t V'2^{q's}U_{q'}\mathrm{d}\tau). \tag{3.25}$$

In order to estimate the Besov norm of the high frequency part of the solution, the natural next step would be to perform a summation over $q \leq N_0 + N_1$ in (3.25) the use the Gronwall lemma. However, owing to the summation over q', the right-hand side involves a finite number of terms $U_{q'}$ with $q' < N_0 + N_1$. In other words, there is a slight overlap between the low and high frequency parts of the solution. Here comes into play the L^2 type assumption on the data. Indeed, taking $N = N_0 + N_1$ in (3.2), Proposition 3.1 ensures that for all $q < N_0 + N_1$ and $s' \in \mathbb{R}$, we have for all $t \in [0, T]$,

$$2^{qs'}||(\dot{\Delta}_q a, \dot{\Delta}_q u)||_{L_t^\infty(L^2)} + 2^{q(s'+2)}||(\dot{\Delta}_q a, \dot{\Delta}_q u)||_{L_t^1(L^2)}$$

$$\leq C(\|a_0\|^{\ell}_{\dot{B}^{s'}_{2,r}} + \|a_0\|^{\hbar}_{\dot{B}^{s'+1}_{2,r}} + \|u_0\|_{\dot{B}^{s'}_{2,r}} + \|F\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{s'}_{2,r})} + \|F\|^{\hbar}_{\widetilde{L}^1_t(\dot{B}^{s'+1}_{2,r})} + \|G\|_{\widetilde{L}^1_t(\dot{B}^{s'}_{2,r})}).$$

Owing to the Bernstein's inequality (recall that $p \geq 2$), the above inequality ensures that for $N_0 + N_1 - 4 \leq q < N_0 + N_1$, we have

$$2^{qs}U_q(t) \lesssim \|a_0\|^{\ell}_{\dot{B}^{s'}_{2,r}} + \|a_0\|^{\hbar}_{\dot{B}^{s'+1}_{2,r}} + \|u_0\|_{\dot{B}^{s'}_{2,r}} + \|F\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{s'}_{2,r})} + \|F\|^{\hbar}_{\widetilde{L}^1_t(\dot{B}^{s'+1}_{2,r})} + \|G\|_{\widetilde{L}^1_t(\dot{B}^{s'}_{2,r})}.$$

So, summing up over $q \geq N_0 + N_1$ in (3.25), plugging in the above inequality and bearing in mind the definition of T in (3.24), we have for all $t \in [0, T]$,

$$\begin{aligned}\sum_{q\geq N_0+N_1} 2^{qs}U_q(t) \lesssim{}& \|a_0\|^{\ell}_{\dot{B}^{s'}_{2,r}} + \|a_0\|^{\hbar}_{\dot{B}^{s'+1}_{2,r}} + \|u_0\|_{\dot{B}^{s'}_{2,r}} + \|F\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{s'}_{2,r})}\\ &+ \|F\|^{\hbar}_{\widetilde{L}^1_t(\dot{B}^{s'+1}_{2,r})} + \|G\|_{\widetilde{L}^1_t(\dot{B}^{s'}_{2,r})} + \sum_{q\geq N_0+N_1} 2^{qs}U^0_q(t)\\ &+ \int_0^t \|\nabla v\|_{L^\infty} \sum_{q\geq N_0+N_1} 2^{qs}U_q(t)\mathrm{d}\tau.\end{aligned}$$

Then applying the Gronwall lemma, we conclude that for all $t \in [0, T]$,

$$\begin{aligned}&\|a\|^{\hbar}_{\widetilde{L}^\infty_t(\dot{B}^{s+1}_{p,1})\cap\widetilde{L}^1_t(\dot{B}^{s+3}_{p,1})} + \|u\|^{\hbar}_{\widetilde{L}^\infty_t(\dot{B}^{s}_{p,1})\cap\widetilde{L}^1_t(\dot{B}^{s+2}_{p,1})}\\ \lesssim{}& \mathrm{e}^{CV(t)}(\|a_0\|^{\ell}_{\dot{B}^{s'}_{2,r}} + \|a_0\|^{\hbar}_{\dot{B}^{s'+1}_{2,r}} + \|u_0\|_{\dot{B}^{s'}_{2,r}} + \|F\|^{\ell}_{\widetilde{L}^1_t(\dot{B}^{s'}_{2,r})} + \|F\|^{\hbar}_{\widetilde{L}^1_t(\dot{B}^{s'+1}_{2,r})}\\ &+ \|G\|_{\widetilde{L}^1_t(\dot{B}^{s'}_{2,r})} + \|a_0\|^{\hbar}_{\dot{B}^{s+1}_{p,1}} + \|u_0\|^{\hbar}_{\dot{B}^{s}_{p,1}} + \|F\|^{\hbar}_{L^1_t(\dot{B}^{s+1}_{p,1})} + \|G\|_{L^1_t(\dot{B}^{s}_{p,1})}).\end{aligned}$$

This yields the desired L^p type estimate for $t \in [0, T]$. Then a standard bootstrap arguments leads to the result for all positive time. As for the L^2 type estimate, it has already been proved in Proposition 3.1. □

4 Proof of Global Well-posedness

This section is devoted to the proof of Theorem 1.3. In the first paragraph, we establish new a priori estimates for the solution to (1.3). Global solutions are constructed in the second paragraph.

4.1 A Priori Estimates

Consider a solution (a, u) to (1.3). In order to take advantage of Proposition 3.4, we rewrite the system satisfied by (a, u) as follows:

$$\begin{cases}\partial_t a + \mathrm{div}(T_u a) + \mathrm{div}\ u = F,\\ \partial_t u + T_u \cdot \nabla u - \mathscr{A}u + \nabla a - \kappa\nabla\Delta a = G\end{cases} \tag{4.1}$$

with

$$F := -\mathrm{div}T'_a u \ \text{ and } \ G := \nabla(aK(a)) - J(a)\mathscr{A}u - T'_{\nabla u}\cdot u,$$

where $J(a) := a/(1 + a)$ and K is a smooth function vanishing at 0 (recall that $P'(1) = 1$).

Throughout, we make the following assumption on a:

$$\forall (t, x) \in [0, T] \times \mathbb{R}^d, \quad |a(t, x)| \leq \frac{1}{2} \tag{$\mathcal{P}$}$$

and we set for $p=p_1,p_2$,

$$\begin{aligned}X_p(t):=&||a||^{\ell}_{\widetilde{L}^{\infty}_t(\dot B^s_{2,r})\cap\widetilde{L}^1_t(\dot B^{s+2}_{2,r})}+||a||^{\hbar}_{\widetilde{L}^{\infty}_t(\dot B^{s+1}_{2,r}\cap\dot B^{\frac{d}{p}}_{p,1})}+||a||^{\hbar}_{\widetilde{L}^1_t(\dot B^{s+3}_{2,r}\cap\dot B^{\frac{d}{p}+2}_{p,1})}\\&+||u||_{\widetilde{L}^{\infty}_t(\dot B^s_{2,r})\cap\widetilde{L}^1_t(\dot B^{s+2}_{2,r})}+||u||^{\hbar}_{\widetilde{L}^{\infty}_t(\dot B^{\frac{d}{p}-1}_{p,1})\cap L^1_t(\dot B^{\frac{d}{p}+1}_{p,1})},\\X_{p,0}:=&||a_0||^{\ell}_{\dot B^s_{2,r}}+||a_0||^{\hbar}_{\dot B^{s+1}_{2,r}\cap\dot B^{\frac{d}{p}}_{p,1}}+||u_0||_{\dot B^s_{2,r}}+||u_0||^{\hbar}_{\dot B^{\frac{d}{p}-1}_{p,1}}.\end{aligned}$$

Let $U(t):=\int_0^t||\nabla u||_{L^\infty}\mathrm{d}\tau$. For $i=1,2$, according to Proposition 3.4, we have for some constant C depending only on s,d and p_i,

$$\begin{aligned}X_{p_i}(t)\le C\mathrm{e}^{CU(t)}(&X_{p_i,0}(t)+||F||^{\ell}_{\widetilde{L}^1_t(\dot B^s_{2,r})}+||F||^{\hbar}_{\widetilde{L}^1_t(\dot B^{s+1}_{2,r})}\\&+||G||_{\widetilde{L}^1_t(\dot B^s_{2,r})}+||F||^{\hbar}_{L^1_t(\dot B^{\frac{d}{p_i}}_{p_i,1})}+||G||_{L^1_t(\dot B^{\frac{d}{p_i}-1}_{p_i,1})}).\end{aligned}\tag{4.2}$$

If we denoting

$$f^{\ell}:=\sum_{q\le N}\dot\Delta_q f\ \text{ and }\ f^{\hbar}:=\sum_{q\ge N}\dot\Delta_q f\ \text{ for }\ f\in\ S'(\mathbb{R}^d).\tag{4.3}$$

Taking the same argument in [9] to estimate F and G, we have the following result.

Proposition 4.1 Let (a,u) be a solution of (1.3) which belongs to $E^s_{p_1,r}(T)$ with s,p_1 and r satisfying the condition of Theorem 1.3. There exist two constants c and C depending only on d,p_2,s and r such that if $X_{p_2,0}(T)\le c$ then

$$X_{p_i}(t)\le CX_{p_i,0}\ \text{ for all } t\in[0,T] \text{ and }\ i=1,2.$$

4.2 The Proof of the Global Existence Theorem

For all $p>0$, Danchin-Desjardins [11] has shown that uniqueness holds true in the set of functions (a,u) such that

$$||a||_{\widetilde{L}^{\infty}_T(\dot B^{\frac{d}{p}}_{p,1})}+||u||_{\widetilde{L}^{\infty}_T(\dot B^{\frac{d}{p}-1}_{p,1})}\le c$$

for some small enough constant c depending only on p and d. So let us proceed to the proof of global existence under assumptions of Theorem 1.3. To simplify the presentation, we assume throughout that $r<\infty$. The case $r=\infty$ follows from the similar arguments. It is only a matter of replacing the strong topology in $\dot B^s_{2,r}$ or $\dot B^{s+1}_{2,r}$ by weak topology. We proceed in three steps:

Step 1 smooth solutions

We smooth out the data so as to obtain a sequence $(a_{0,n},u_{0,n})_{n\in\mathbb{N}}$ such that

$$a_{0,n}\in\dot B^s_{2,r}\cap\dot B^{\frac{d}{2}}_{2,r}\ \text{ and }\ u_{0,n}\in\dot B^s_{2,r}\cap\dot B^{\frac{d}{2}-1}_{2,r}$$

and that

$$\begin{aligned}&a^{\ell}_{0,n}\longrightarrow a^{\ell}_0 \text{ in } \dot B^s_{2,r},\quad a^{\hbar}_{0,n}\longrightarrow a^{\hbar}_0 \text{ in } \dot B^{s+1}_{2,r}\cap\dot B^{\frac{d}{p_1}}_{p_1,1},\\&u^{\ell}_{0,n}\longrightarrow u^{\ell}_0 \text{ in } \dot B^s_{2,r},\quad u^{\hbar}_{0,n}\longrightarrow u^{\hbar}_0 \text{ in } \dot B^{\frac{d}{p_1}-1}_{p_1,1}.\end{aligned}\tag{4.4}$$

Note that, owing to Bernstein's inequality, one may merely take

$$a_{0,n}:=S_na_0\ \text{ and }\ u_{0,n}:=S_nu_0.$$

Now, from the Theorem 1.2, for all $n\in\mathbb{N}$, we have a maximal solution (a_n,u_n) over the time interval $[0,T_n^*)$ such that for all $T<T_n^*$ we have

$$a_n\in\widetilde{C}_T(\dot{B}^{\frac{d}{2}}_{2,1})\cap L^1_T(\dot{B}^{\frac{d}{2}+2}_{2,1})\ \text{and}\ u_n\in\widetilde{C}_T(\dot{B}^{\frac{d}{2}-1}_{2,1})\cap L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1}).$$

We claim that $(a_n,)$ is in $E^s_{p_1,r}(T)$ (and thus also in $E^s_{p_2,r}(T)$) for all $T<T*$. Note that, because $p_1\geq 2$, Proposition 2.4 ensures that

$$a_n^\hbar\in\widetilde{C}_T(\dot{B}^{\frac{d}{p_1}}_{p_1,1}\cap\dot{B}^s_{2,r})\cap L^1_T(\dot{B}^{\frac{d}{p_1}+2}_{p_1,1}\cap\dot{B}^{s+2}_{2,r})\ \text{and}\ u_n^\hbar\in\widetilde{C}_T(\dot{B}^{\frac{d}{p_1}-1}_{p_1,1})\cap L^1_T(\dot{B}^{\frac{d}{p_1}+1}_{p_1,1}).$$

So, in order to complete the justification of our claim, it is only a matter of showing that the low frequencies of a_n and of u_n are in $\widetilde{C}_T(\dot{B}^s_{2,r})$ (hence also in $\widetilde{L}^1_T(\dot{B}^{s+2}_{2,r})$) for all $T<T_n^*$.

For the time being, let us assume for simplicity that, in addition to condition (1.4) or (1.5), we have

$$s>\frac{d}{2}-2. \tag{4.5}$$

In order to establish the property for a_n, one may use the fact that

$$\partial_t a_n=-\text{div } u_n-\text{div}(a_nu_n).$$

Fix some $T\in(0,T_n^*)$. As u_n is in $\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})$ and as a_n is in $\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})$ the product laws in Besov spaces ensure that $\partial_t a_n$ is in $\widetilde{L}^\infty_T(B^{\frac{d}{2}-2}_{2,1})$. Hence $\partial_t a_n^\ell$ is in $\widetilde{L}^\infty_T(B^{\frac{d}{2}}_{2,1})$, owing to (4.5). As $a^\ell_{n,0}$ belongs to $\dot{B}^s_{2,r}$ we thus have $a_n^\ell\in\widetilde{C}_T(\dot{B}^s_{2,r})$.

Let us now check that $u_n\in\widetilde{C}_T(\dot{B}^s_{2,r})$. Owing to (4.5), it suffices to establish that ∂_tu_n belongs to $L^1_T(\dot{B}^{\frac{d}{2}-2}_{2,1})$. Now, we have

$$\partial_tu_n=(1-J(a_n))\mathscr{A}u_n-\nabla((1+K(a_n))a_n)-u_n\cdot\nabla u_n-\kappa\nabla\Delta a_n.$$

In the case $d\geq 3$, one may use the fact that a_n (hence also $J(a_n)$ and $K(a_n)$) belongs to $\widetilde{L}^\infty_T(\dot{B}^{\frac{d}{2}}_{2,1})$ and that $\mathscr{A}u_n$ belongs to $L^2_T(\dot{B}^{\frac{d}{2}-2}_{2,1})$ to deduce that the first term of the right-hand side is in $L^2_T(\dot{B}^{\frac{d}{2}-2}_{2,1})$. Next, as a_n is also in $L^\infty_T(\dot{B}^{\frac{d}{2}-1}_{2,1})$, the second term is in $L^2_T(\dot{B}^{\frac{d}{2}-2}_{2,1})$. A similar argument works for the third term. Finally, as a_n is belong to $L^1_T(\dot{B}^{\frac{d}{2}+1}_{2,1})$, the last term is belong $L^1_T(\dot{B}^{\frac{d}{2}-2}_{2,1})$.

In the case $d=2$, owing to the fact that the product only maps $\dot{B}^{-1}_{2,1}\times\dot{B}^1_{2,1}$ in the larger Besov space $\dot{B}^{-1}_{2,\infty}$, we obtain, rather, that ∂_tu_n belongs to $L^1_T(\dot{B}^{-1}_{2,\infty})$. In any case, as (4.5) is satisfied and $u_{0,n}$ is in $\dot{B}^s_{2,r}$, one may conclude $u_n^\ell\in\widetilde{C}_T(\dot{B}^s_{2,r})$. This completes the proof that $(a_n,u_n)\in E^s_{p,r}(T)$ for all $T<T^*$ under assumption (4.5).

The case of smaller values of s may be treated by bootstrapping as, from the previous discussion, we already know that a_n and u_n are in $\widetilde{C}_T(\dot{B}^{\frac{d}{2}-2}_{2,r})$. Then one may argue as above to show that ∂_ta_n and ∂_tu_n are in $L^1_T(\dot{B}^{max(s,\frac{d}{2}-2)}_{2,r})$, and so on.

Step 2 global existence and uniform bounds

According to Proposition 4.1 and (4.4), there exists a constant c such that(with obvious notation) if $X^n_{p_2}\leq c$ then

$$X^n_p(t)\leq CX^n_{p,0}\ \text{ for all }\ n\in\mathbb{N},\ \ t\in[0,T_n^*)\ \text{ and }\ p=p_1,p_2. \tag{4.6}$$

Note that if we suppose that $X_{p_2,0}\leq\frac{c}{2}$, then property (4.4) guarantees that the above smallness condition for the smoothed out date is satisfied for all large enough n.

Taking the standard method in [2, Chap. 10], we have the following continuation criterion.

Lemma 4.2 Under the hypotheses of Theorem 1.2, assume that the system (1.3) has a solution (a,u) on $[0,T)\times\mathbb{R}^d$ which belongs to $E_{T'}$ for all $T'<T$ and satisfies

$$\int_0^T ||\nabla u||_{L^\infty}\mathrm{d}t<\infty \quad\text{and}\quad \begin{cases} ||a||_{L^\infty_T(\dot B^{\frac{d}{2}}_{2,1})}\le\eta, & \text{if } d\ge 3,\\ ||a||_{\widetilde L^\infty_T(\dot B^{1}_{2,1})}\le\eta, & \text{if } d=2,\end{cases}$$

where η is a constant depending only on μ and μ'. Then there exists some $T^*>T$ such that (a,u) may be continued on $[0,T^*]\times\mathbb{R}^d$ to a solution of system (1.3) which belongs to E_{T^*}.

By contradiction, we assume that T_n^* is finite. Then applying Proposition 4.1 with Lebesgue exponents 2 and p_2 ensures that

$$X_2^n(t)\le CX^n_{2,0}\quad\text{for all}\quad t\in[0,T_n^*).$$

In order words, a_n belongs to $\widetilde L^\infty_{T_n^*}(\dot B^{\frac{d}{2}}_{2,1})$ and u_n belongs to $\widetilde L^\infty_{T_n^*}(\dot B^{\frac{d}{2}-1}_{2,1})$. Then, from Lemma 4.2, we conclude that (a_n,u_n) may be continued beyond T_n^* into a solution $(\widetilde a_n,\widetilde u_n)$ of (1.3) which coincides with (a_n,u_n) on $[0,T_n^*)$ and such that, for some $T>T_n^*$,

$$\widetilde a_n\in\widetilde C_T(\dot B^{\frac{d}{2}}_{2,1})\quad\text{and}\quad \widetilde u_n\in\widetilde C_T(\dot B^{\frac{d}{2}-1}_{2,1})\cap L^1_T(\dot B^{\frac{d}{2}+1}_{2,1}).$$

Then the previous step ensures that $(\widetilde a_n,\widetilde u_n)\in E^s_{p_1,r}(T)$. This stands in contradiction with the definition of T_n^*. Hence $T_n^*=\infty$ and (4.6) holds true globally.

Step 3 passing to the limit

Let us first focus on the convergence of $(a_n)_{n\in\mathbb{N}}$. We claim that, up to extraction, $(a_n)_{n\in\mathbb{N}}$ converges in the distributional sense to some function a such that

$$a^\ell\in\widetilde L^\infty(\dot B^s_{2,r})\cap\widetilde L^1(\dot B^{s+2}_{2,r}), a^\hbar\in\widetilde L^\infty(\dot B^{s+1}_{2,r})\cap\widetilde L^1(\dot B^{s+3}_{2,r})\cap\in\widetilde L^\infty(\dot B^{\frac{d}{p_1}}_{p_1,1})\cap\widetilde L^1(\dot B^{\frac{d}{p_1}+2}_{p_1,1}). \tag{4.7}$$

The proof relies on an Aubin-lions type argument. Indeed, let us admit for a while that

$$(\partial_t a_n)_{n\in\mathbb{N}}\quad\text{is bounded in } \widetilde L^2(\dot B^s_{2,r}). \tag{4.8}$$

Then, as $(a_{0,n})_{n\in\mathbb{N}}$ is bounded in $\dot B^s_{2,r}$, we deduce that $(a_n)_{n\in\mathbb{N}}$ seen as a sequence of $\dot B^s_{2,r}$ valued functions, is equicontinuous on $\mathbb{R}^+$. In addition, according to the previous step, $(a_n)_{n\in\mathbb{N}}$ is bounded in $C(\mathbb{R}^+;\dot B^s_{2,r}\cap\dot B^{s+1}_{2,r})$. Let $(\mathcal{X}_k)_{k\in\mathbb{N}}$ be a sequence of $C_0^\infty(\mathbb{R}^d)$ cut-off functions supported in the ball $B(0,k+1)$ of $\mathbb{R}^d$ and equal to 1 in a neighborhood of $B(0,p)$. Let us observe now that the application $u\longmapsto\mathcal{X}_p u$ is compact from $\dot B^s_{2,r}\cap\dot B^{s+1}_{2,r}$ into $\dot B^s_{2,r}$. Therefore, Ascoli's theorem ensures that there exists some function a^k such that, up to extraction, $(\mathcal{X}_k a_n)_{n\in\mathbb{N}}$ converges to a^k in $C([0,k];\dot B^s_{2,r})$. Using the Cantor diagonal extraction process, we can then find a subsequence $(a_n)_{n\in\mathbb{N}}$ (still denoted by $(a_n)_{n\in\mathbb{N}}$) such that for all $k\in\mathbb{N}$, $\mathcal{X}_k a_n$ converges to a^k in $C([0,k];\dot B^s_{2,r})$. As $\mathcal{X}_k\mathcal{X}_{k+1}=\mathcal{X}_k$, we have, in addition, $a^k=\mathcal{X}_k a^{k+1}$. From that, we can easily deduce that there exists some function a such that for $(\mathcal{X}a_n)_{n\in\mathbb{N}}$ converges to $\mathcal{X}a$ in $C(\mathbb{R}^+;\dot B^s_{2,r})$ for all $\mathcal{X}\in C_0^\infty(\mathbb{R}^+)$. Then, by using the so-called Fatou property [2,Theorem 2.25] for the Besov spaces, we can conclude that (4.7) is satisfied.

For the sake of completeness, let us now establish (4.8). We write

$$\partial_t a_n=-\mathrm{div}\ u_n-\mathrm{div}(a_n u_n).$$

From the previous step and interpolation, we know that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\widetilde L^2(\dot B^{s+1}_{2,r})$. Hence the first term of the right-hand side is bounded in $\widetilde L^2(\dot B^s_{2,r})$. Using embedding, (1.4) and

(1.5), we see that $(a_n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty(\dot{B}^{\frac{d}{p_1}}_{p_1,1}\cap\dot{B}^{s+1}_{2,r})$ and that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^2(\dot{B}^{\frac{d}{p_1}}_{p_1,1}\cap\dot{B}^{s+1}_{2,r})$. Hence Proposition 2.6 ensures that $\mathrm{div}(a_nu_n)$ is bounded in $\widetilde{L}^2(\dot{B}^s_{2,r})$.

We now want to prove that $(u_n)_{n\in\mathbb{N}}$ converges up to extraction and, in the distributional sense, to some function u such that

$$u\in\widetilde{L}^\infty(\dot{B}^s_{2,r})\cap\widetilde{L}^1(\dot{B}^{s+2}_{2,r})\quad\text{and}\quad u^\hbar\in\widetilde{L}^\infty(\dot{B}^{\frac{d}{p_1}-1}_{p_1,1})\cap\widetilde{L}^1(\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}).\tag{4.9}$$

For expository purpose, in what follows we agree that the notation $\widetilde{L}^{1^+}(\dot{B}^{s^-}_{2,r})$ stands for the space $\widetilde{L}^{\frac{2}{2-\varepsilon}}(\dot{B}^{s-\varepsilon}_{2,r})$ for suitably small ε. Admit for a while that

$$(\partial_tu_n)_{n\in\mathbb{N}}\ \text{ is bounded in } \widetilde{L}^{1^+}_{\mathrm{loc}}(\dot{B}^{s^-}_{2,r}).\tag{4.10}$$

Then $(u_n)_{n\in\mathbb{N}}$ is equicontinuous on $\mathbb{R}^+$ with values in $\dot{B}^{s^-}_{2,r}$. By adapting the arguments that have been used for handling $(a_n)_{n\in\mathbb{N}}$, one can conclude that $(u_n)_{n\in\mathbb{N}}$ converges up to extraction to some distribution u satisfying (4.9).

Let us now establish (4.10). For the momentum equation, we may write

$$\partial u_n=\mathscr{A}u_n-J(a_n)\mathscr{A}u_n-\nabla(a_n(1+K(a_n)))-u_n\cdot\nabla u_n-\kappa\nabla\Delta a_n.\tag{4.11}$$

Let us first point out that as $(u_n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty(\dot{B}^s_{2,r})\cap\widetilde{L}^1(\dot{B}^{s+2}_{2,r})$ interpolation ensures that $(\mathscr{A}u_n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^{1^+}(\dot{B}^{s^-}_{2,r})$. As $(\mathscr{A}u_n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^{1^+}(\dot{B}^{s^-}_{2,r}\cap\dot{B}^{(\frac{d}{p_1}-1)^-}_{p_1,1})$ and as $(J(a_n))_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty(\dot{B}^{s+1}_{2,r}\cap\dot{B}^{\frac{d}{p_1}}_{p_1,1})$, we conclude that the second terms of the right-hand side of (4.11) is bounded in $\widetilde{L}^{1^+}(\dot{B}^{s^-}_{2,r})$. In addition, as $(a_n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty(\dot{B}^{1+s^-}_{2,r})$, we see that the third of (4.11) belongs to $\widetilde{L}^\infty(\dot{B}^{s^-}_{2,r})$.

Next, using the fact that $(u_n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty(\dot{B}^{\frac{d}{p_1}-1}_{p_1,1})$ and that $(\nabla u_n)_{n\in\mathbb{N}}$ is bounded in $L^{1^+}(\dot{B}^{1+s^-}_{2,r})$, owing to (1.4) and (1.5), we see that $u_n\cdot\nabla u_n$ is bounded in $\widetilde{L}^{1^+}(\dot{B}^{s^-}_{2,r})$. Finally, as $(a_n)_{n\in\mathbb{N}}$ is bounded in $\widetilde{L}^\infty(\dot{B}^{s+1}_{2,r})\cap\widetilde{L}^1(\dot{B}^{s+3}_{2,r})$, we conclude that the last term of (4.11) is bounded in $\widetilde{L}^{1^+}(\dot{B}^{s^-}_{2,r})$.

Note that by interpolating between the local results of convergence that have been proved so far, and the uniform bounds establish in the previous step, one can obtain stronger results of convergence so that one may pass to the limit in the system satisfied by (a_n,u_n). As a conclusion, we discover that (a,u) is, indeed, a solution to (1.3) with data (a_0,u_0). In order to complete the proof of the existence part of Theorem 1.3, it is only a matter of checking the continuity properties with respect to time, namely that

$$a^\ell\in C(\mathbb{R}^+;\dot{B}^s_{2,r}),\ a^\hbar\in C(\mathbb{R}^+;\dot{B}^{s+1}_{2,r}\cap\dot{B}^{\frac{d}{p_1}}_{p_1,1})$$

and

$$u\in C(\mathbb{R}^+;\dot{B}^s_{2,r}),\ u^\hbar\in C(\mathbb{R}^+;\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}).$$

In fact, owing to $p_1\ge2$ and to Bernstein's inequality, it is equivalent to establish that

$$a\in C(\mathbb{R}^+;\dot{B}^s_{2,r}\cap\dot{B}^{s+1}_{2,r}\cap\dot{B}^{\frac{d}{p_1}}_{p_1,1})\text{ and }u\in C(\mathbb{R}^+;\dot{B}^s_{2,r}\cap\dot{B}^{\frac{d}{p_1}-1}_{p_1,1}).$$

As regards a, it suffices to notice that, according to (4.7), (4.9) and to the product laws in the Besov spaces, we have

$$\partial_ta+u\cdot\nabla a=-(1+a)\mathrm{div}\ u\in\widetilde{L}^1_{\mathrm{loc}}(\dot{B}^s_{2,r}\cap\dot{B}^{s+1}_{2,r})\cap L^1(\dot{B}^{\frac{d}{p_1}}_{p_1,1}).$$

As $a_0 \in \dot{B}^s_{2,r} \cap \dot{B}^{s+1}_{2,r} \cap \dot{B}^{\frac{d}{p_1}}_{p_1,1}$, classical results for the transport equation (see for example [2, Chap. 3]) ensures that, indeed, a is in $C(\mathbb{R}^+; \dot{B}^s_{2,r} \cap \dot{B}^{s+1}_{2,r} \cap \dot{B}^{\frac{d}{p_1}}_{p_1,1})$.

To obtain the continuity result for u, one may use the fact

$$\partial u_n - \mathscr{A} u_n = -J(a_n)\mathscr{A} u_n - \nabla(a_n(1+K(a_n))) - u_n \cdot \nabla u_n - \kappa \nabla \Delta a_n.$$

The right-hand of the above equation belongs to $\widetilde{L}^1(\dot{B}^s_{2,r}) \cap L^1(\dot{B}^{\frac{d}{p_1}-1}_{p_1,1})$, then from the standard properties for the heat equation (see for example [6]), we thus have $u \in C(\mathbb{R}^+; \dot{B}^s_{2,r} \cap \dot{B}^{\frac{d}{p_1}-1}_{p_1,1})$. Therefore, we complete the proof of the main result of Theorem 1.3 in this article.

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