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On some classes of projectively flat Finsler metrics with constant flag curvature

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ABSTRACT

In this paper, we give the equivalent PDEs for projectively flat Finsler metrics with constant flag curvature defined by a Euclidean metric and two 1-forms. Furthermore, we construct some classes of new projectively flat Finsler metrics with constant flag curvature by solving these equivalent PDEs.

1. Introduction and main results

As we know, Finsler geometry is more diversified than Riemannian geometry because there are many non-Riemannian quantities on a Finsler manifold besides the Riemannian quantities ([2,1]). One of the important problems in Finsler geometry is to study and characterize the projectively flat metrics on an open subset $\Omega \subseteq \mathbb{R}^n$. A Finsler metric defined on an open subset in $\mathbb{R}^n$ is called projectively flat if its geodesics are straight lines. This is the Hilbert's 4th problem in the regular case [8]. In 1903, Hamel [7] found a system of partial differential equations

$$F_{x^k y^l} y^k = F_{x^l},$$

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which can characterize the projectively flat metrics on an open subset $\Omega \subseteq \mathbb{R}^n$. And we know that the flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry. The famous Beltrami theorem tells us that a Riemannian metric is projectively flat if and only if it has constant sectional curvature. However, this is not true in Finsler geometry.

So it is important to construct some projectively flat Finsler metrics with constant flag curvature. For example, the famous Funk metric on a unit ball $\mathbb{B}^n \subseteq \mathbb{R}^n$ in [5,6] as the following

$$F = \frac{\sqrt{|y|^2 - |x|^2|y|^2 + <x,y>^2}}{1-|x|^2} + \frac{<x,y>}{1-|x|^2}, \tag{1.1}$$

where $x \in \mathbb{R}^n$, $y \in T_x\mathbb{R}^n$, $|\cdot|$ is Euclidean norm, $<,>$ is the standard inner product of $\mathbb{R}^n$; and the Berwald's metric in [3,4] as the following

$$F = \frac{(\sqrt{|y|^2 - |x|^2|y|^2 + <x,y>^2} + <x,y>)^2}{(1-|x|^2)^2\sqrt{|y|^2 - |x|^2|y|^2 + <x,y>^2}}. \tag{1.2}$$

These two kinds of Finsler metrics can be seen as the metrics composed by the Euclidean metric $|y|$, the inner product $<x,y>$ and the Euclidean norm $|x|$ of $x \in \mathbb{R}^n$. More generally, there exist some projectively flat Finsler metrics composed by $|y|$, $<x,y>$, $|x|$, $<a,y>$ and $<a,x>$. For example, Z. Shen constructed a group of projectively flat Finsler metric with constant flag curvature $\lambda = 0$ in [10] as the following

$$\begin{aligned} F(x,y) = &\left\{1 + <a,x> + \frac{(1-|x|^2) <a,y>}{\sqrt{|y|^2 - |x|^2|y|^2 + <x,y>^2} + <x,y>}\right\} \\ &\times \frac{(\sqrt{|y|^2 - |x|^2|y|^2 + <x,y>^2} + <x,y>)^2}{(1-|x|^2)^2\sqrt{|y|^2 - |x|^2|y|^2 + <x,y>^2}}. \end{aligned} \tag{1.3}$$

By introducing new variables $u = |x|^2$, $s = \frac{<x,y>}{|y|}$, $v = <a,x>$, $t = \frac{<a,y>}{|y|}$, $|a| < 1$, W. Liu and B. Li [9] rewrote the above metric as

$$F = |y|\left\{1 + v + \frac{(1-u)t}{\sqrt{1-u+s^2}+s}\right\}\frac{(\sqrt{1-u+s^2}+s)^2}{(1-u)^2\sqrt{1-u+s^2}}.$$

This motivated them to study the following Finsler metric

$$F = |y|\phi(u,s,v,t),\ u = |x|^2,\ s = \frac{<x,y>}{|y|},\ v = <a,x>,\ t = \frac{<a,y>}{|y|}, \tag{1.4}$$

where $x \in \mathbb{R}^n$, $y \in T_x\mathbb{R}^n$, $|a| < 1$, $<a,y> = a_i y^i$ is a 1-form, ϕ is a C^∞ function. Especially, the Finsler metric F becomes a spherically symmetric when $a = 0$, because any spherically symmetric Finsler metric can be expressed by $|x|$, $|y|$ and $<x,y>$, which is proved by L. Zhou in [11]. W. Liu and B. Li [9] gave the equivalent equations of F being projectively flat.

In this paper, the equivalent PDEs for projectively flat Finsler metrics $F = |y|\phi(u,s,v,t)$ with constant flag curvature will be given. And by solving these PDEs, we can construct some classes of new projectively flat Finsler metrics with constant flag curvature. These new metrics will be provided in Sections 4 and 5.

Now we give the main results as follows.

Theorem 1.1. *Suppose $F(x,y) = |y|\phi(u,s,v,t)$ is a projectively flat Finsler metric, then it has constant flag curvature λ if and only if ϕ satisfies the following PDEs*

$$\begin{cases} 3A^2 - 2t\phi\frac{\partial A}{\partial v} - 4s\phi\frac{\partial A}{\partial u} - 8\phi\phi_u - 4\lambda\phi^4 = 0 \\ \phi_s A^2 - 8\phi\phi_u A + 4\phi^2\frac{\partial A}{\partial u} + 4\lambda\phi^4\phi_s = 0 \\ \phi_t A^2 - 4\phi\phi_v A + 2\phi^2\frac{\partial A}{\partial v} + 4\lambda\phi^4\phi_t = 0, \end{cases} \tag{1.5}$$

where $A = 2s\phi_u + \phi_s + t\phi_v$.

The conclusion in Theorem 1.1 gives the equivalent PDEs for projectively flat Finsler metrics $F = |y|\phi(u, s, v, t)$ with constant flag curvature. In order to solve these PDEs of (1.5), we introduce variables $\widetilde{u} = u - s^2$, $\widetilde{s} = s$, $\widetilde{v} = v - st$, $\widetilde{t} = t$ and a function

$$\widetilde{\phi}(\widetilde{u}, \widetilde{s}, \widetilde{v}, \widetilde{t}) = \phi(u, s, v, t).$$

So we have

Theorem 1.2. *ϕ is a solution of (1.5) if and only if $\widetilde{\phi}$ has the form*

$$\widetilde{\phi} = \frac{f}{(\widetilde{s} + g)^2 + \lambda f^2}, \tag{1.6}$$

where the C^∞ functions $f = f(\widetilde{u}, \widetilde{v}, \widetilde{t})$ and $g = g(\widetilde{u}, \widetilde{v}, \widetilde{t})$ satisfy the PDEs

$$2g_{\widetilde{t}} + (g^2 - \lambda f^2)_{\widetilde{v}} = 0, \tag{1.7}$$

$$-\widetilde{t}g_{\widetilde{v}} + 1 + (g^2 - \lambda f^2)_{\widetilde{u}} = 0, \tag{1.8}$$

$$f_{\widetilde{t}} + (fg)_{\widetilde{v}} = 0, \tag{1.9}$$

$$\widetilde{t}f_{\widetilde{v}} - 2(fg)_{\widetilde{u}} = 0. \tag{1.10}$$

Due to Theorem 1.2, one may find the projectively flat Finsler metrics $F = |y|\phi(u, s, v, t)$ with constant flag curvature only by solving the PDEs of (1.7)–(1.10), and it will be done in Sections 4 and 5. The readers can see Corollaries 4.1, 4.3, 5.1 and 5.4 for the further results.

2. Preliminaries

In this section, we shall recall some necessary notations, definitions and lemmas. Let M be a manifold of dimension n. Let $x = (x^1, \cdots, x^n)$ be a local coordinate system on M and $y = (y^1, \cdots, y^n)$ be the local fiber coordinate system defined by the local frame field $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$ on the tangent bundle TM of M. So $(x, y) = (x^1, \cdots, x^n, y^1, \cdots, y^n)$ is a local coordinate system for TM.

Definition 2.1. ([1]) A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:
(i) $G = F^2$ is smooth on $\tilde{M}$, where $\tilde{M} = TM \setminus \{0\}$;
(ii) $F(\mu y) = |\mu|F(y)$ for all $y \in TM$ and $\mu \in \mathbb{R}$;
(iii) $F(y) > 0$ for all $y \in \tilde{M}$;
(iv) The fundamental tensor g, defined locally by its components

$$g_{ij} := \frac{1}{2}G_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

is positive definite.

A manifold M endowed with a Finsler metric will be called a Finsler manifold.

In this paper, we only investigate the following form of the Finsler metric F ([9])

$$F = |y|\phi(u,s,v,t),\ u = |x|^2,\ s = \frac{<x,y>}{|y|},\ v =<a,x>,\ t = \frac{<a,y>}{|y|}, \tag{2.1}$$

where $x \in \mathbb{R}^n$, $y \in T_x\mathbb{R}^n$, $|a| < 1$, $<a,y>= a_i y^i$ is a 1-form, ϕ is a C^∞ function.

Remark. When $x = 0$, W. Liu and B. Li [9] give the conditions of $\phi = \phi(0,0,0,t)$ such that $F = |y|\phi(0,0,0,t)$ is a Finsler metric (see Lemma 2.1 in [9]).

We write

$$A_k = \frac{\partial A}{\partial y^k},\ A_{;i} = \frac{\partial A}{\partial x^i},$$

$$A_{kj} = \frac{\partial^2 A}{\partial y^k \partial y^j},\ A_{k;i} = \frac{\partial^2 A}{\partial y^k \partial x^i}, etc.,$$

to denote the differentiations of a function $A(x,y)$ with respect to y^k, x^i, y^j. We also denote the derivatives of ϕ with respect to u, s, v, t by ϕ_u, ϕ_s, ϕ_v, ϕ_t respectively.

By (2.1), it is easy to check that

$$u_{;i} = 2x^i,\ s_{;i} = \frac{y^i}{|y|},\ v_{;i} = a^i,$$

$$s_k = \frac{x^k}{|y|} - \frac{sy^k}{|y|^2},\ |y|_k = \frac{y^k}{|y|},\ t_k = \frac{a^k}{|y|} - \frac{ty^k}{|y|^2}.$$

And we need two pre-existing lemmas below for later use.

Lemma 2.2. *([9]) Let $F(x,y) = |y|\phi(u,s,v,t)$ be a Finsler metric on an open subset $\Omega \subseteq \mathbb{R}^n$, then F is projectively flat if and only if $\phi = \phi(u,s,v,t)$ satisfies the following PDEs*

$$\begin{cases} 2s\phi_{us} + \phi_{ss} + t\phi_{sv} - 2\phi_u = 0 \\ 2s\phi_{ut} + \phi_{st} + t\phi_{vt} - \phi_v = 0. \end{cases} \tag{2.2}$$

Lemma 2.3. *([4,10]) Suppose $F = F(x,y)$ is a projective flat Finsler metric on an open subset $\Omega \subseteq \mathbb{R}^n$. Then F has constant flag curvature λ if and only if*

$$P_{;k} = PP_k - \lambda F F_k, \tag{2.3}$$

where $P = \frac{F_{;m}y^m}{2F}$.

3. Proofs of Theorems 1.1 and 1.2

In this section, we will derive the equivalent PDEs for projectively flat Finsler metrics $F = |y|\phi(u,s,v,t)$ with constant flag curvature λ.

3.1. Proof of Theorem 1.1

By Lemma 2.3, F has constant flag curvature λ if and only if $P_{;k} = PP_k - \lambda FF_k$, where $P = \frac{F_{;m}y^m}{2F}$. Notice that $F(x, y) = |y|\phi(u, s, v, t)$, a straightforward calculation shows that

$$\begin{aligned}
F_{;m} &= |y|(\phi_u u_{;m} + \phi_s s_{;m} + \phi_v v_{;m}) \\
&= |y|(2\phi_u x^m + \phi_s \frac{y^m}{|y|} + a^m \phi_v), \\
F_k &= |y|_k \phi + |y|(\phi_s s_k + \phi_t t_k), \\
P &= \frac{F_{;m}y^m}{2F} = \frac{1}{2\phi}(2s\phi_u|y| + \phi_s|y| + t\phi_v|y|), \\
P_k &= -\frac{1}{2\phi^2}(\phi_s s_k + \phi_t t_k)(2s\phi_u|y| + \phi_s|y| + t\phi_v|y|) \\
&\quad + \frac{1}{2\phi}[2\phi_u|y|s_k + 2s|y|(\phi_{us}s_k + \phi_{ut}t_k) + 2s\phi_u|y|_k + (\phi_{ss}s_k + \phi_{st}t_k)|y| \\
&\quad + \phi_s|y|_k + t\phi_v|y|_k + t|y|(\phi_{vs}s_k + \phi_{vt}t_k) + \phi_v|y|t_k],
\end{aligned}$$

$$\begin{aligned}
P_{;k} &= -\frac{1}{2\phi^2}(\phi_u u_{;k} + \phi_s s_{;k} + \phi_v v_{;k})(2s\phi_u|y| + \phi_s|y| + t\phi_v|y|) \\
&\quad + \frac{1}{2\phi}[2\phi_u|y|s_{;k} + 2s|y|(\phi_{uu}u_{;k} + \phi_{us}s_{;k} + \phi_{uv}v_{;k}) + |y|(\phi_{su}u_{;k} + \phi_{ss}s_{;k} + \phi_{sv}v_{;k}) \\
&\quad + t|y|(\phi_{vu}u_{;k} + \phi_{vs}s_{;k} + \phi_{vv}v_{;k})] \\
&= -\frac{1}{2\phi^2}[(4s\phi_u^2 + 2\phi_u\phi_s + 2t\phi_u\phi_v)x^k|y| + (2s\phi_u\phi_s + \phi_s^2 + t\phi_v\phi_s)y^k \\
&\quad + (2s\phi_u\phi_v + \phi_s\phi_v + t\phi_v^2)a^k|y|] \\
&\quad + \frac{1}{2\phi}[(4s\phi_{uu} + 2\phi_{su} + 2t\phi_{vu})x^k|y| + (2\phi_u + 2s\phi_{us} + \phi_{ss} + t\phi_{vs})y^k \\
&\quad + (2s\phi_{uv} + \phi_{sv} + t\phi_{vv})a^k|y|] \\
&= [-\frac{1}{2\phi^2}(4s\phi_u^2 + 2\phi_u\phi_s + 2t\phi_u\phi_v) + \frac{1}{2\phi}(4s\phi_{uu} + 2\phi_{su} + 2t\phi_{vu})]x^k|y| \\
&\quad + [-\frac{1}{2\phi^2}(2s\phi_u\phi_s + \phi_s^2 + t\phi_v\phi_s) + \frac{1}{2\phi}(2\phi_u + 2s\phi_{us} + \phi_{ss} + t\phi_{vs})]y^k \\
&\quad + [-\frac{1}{2\phi^2}(2s\phi_u\phi_v + \phi_s\phi_v + t\phi_v^2) + \frac{1}{2\phi}(2s\phi_{uv} + \phi_{sv} + t\phi_{vv})]a^k|y|. \qquad (3.1)
\end{aligned}$$

And furthermore,

$$\begin{aligned}
\lambda FF_k &= \lambda|y|\phi[|y|_k\phi + |y|(\phi_s s_k + \phi_t t_k)] \\
&= \lambda\phi\phi_s x^k|y| + (\lambda\phi^2 - \lambda s\phi\phi_s - \lambda t\phi\phi_t)y^k + \lambda\phi\phi_t a^k|y|, \qquad (3.2)
\end{aligned}$$

$$\begin{aligned}
PP_k &= -\frac{P}{2\phi^2}[(2s\phi_u\phi_s|y| + \phi_s^2|y| + t\phi_v\phi_s|y|)s_k + (2s\phi_u\phi_t|y| + \phi_s\phi_t|y| + t\phi_v\phi_t|y|)t_k] \\
&\quad + \frac{P}{2\phi}[(2\phi_u|y| + 2s\phi_{us}|y| + \phi_{ss}|y| + t\phi_{vs}|y|)s_k + (2s\phi_{ut}|y| + \phi_{st}|y| \\
&\quad + t\phi_{vt}|y| + \phi_v|y|)t_k + (2s\phi_u + \phi_s + t\phi_v)|y|_k] \\
&= [-\frac{1}{4\phi^3}(2s\phi_u + \phi_s + t\phi_v)(2s\phi_u\phi_s + \phi_s^2 + t\phi_s\phi_v)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4\phi^2}(2s\phi_u+\phi_s+t\phi_v)(2\phi_u+2s\phi_{us}+\phi_{ss}+t\phi_{vs})]x^k|y|\\
&+[\frac{1}{4\phi^3}(2s^2\phi_u+s\phi_s+ts\phi_v)(2s\phi_u\phi_s+\phi_s^2+t\phi_s\phi_v)\\
&+\frac{1}{4\phi^3}(2st\phi_u+t\phi_s+t^2\phi_v)(2s\phi_u\phi_t+\phi_s\phi_t+t\phi_t\phi_v)\\
&-\frac{1}{4\phi^2}(2s^2\phi_u+s\phi_s+ts\phi_v)(2\phi_u+2s\phi_{us}+\phi_{ss}+t\phi_{vs})\\
&-\frac{1}{4\phi^2}(2st\phi_u+t\phi_s+t^2\phi_v)(2s\phi_{ut}+\phi_{st}+t\phi_{vt}+\phi_v)\\
&+\frac{1}{4\phi^2}(2s\phi_u+\phi_s+t\phi_v)^2]y^k\\
&+[-\frac{1}{4\phi^3}(2s\phi_u+\phi_s+t\phi_v)(2s\phi_u\phi_t+\phi_s\phi_t+t\phi_t\phi_v)+\\
&\frac{1}{4\phi^2}(2s\phi_u+\phi_s+t\phi_v)(2s\phi_{ut}+\phi_{st}+t\phi_{vt}+\phi_v)]a^k|y|. \qquad (3.3)
\end{aligned}
$$

Since $P_{;k} = PP_k - \lambda FF_k$, it follows from (3.1)–(3.3) that the coefficients of the terms $x^k|y|$, y^k and $a^k|y|$ on both sides of $P_{;k} = PP_k - \lambda FF_k$ are equal respectively, i.e., we get the following equations

$$
\begin{aligned}
&-\frac{1}{2\phi^2}(4s\phi_u^2+2\phi_u\phi_s+2t\phi_u\phi_v)+\frac{1}{2\phi}(4s\phi_{uu}+2\phi_{su}+2t\phi_{vu})\\
=&-\frac{1}{4\phi^3}(2s\phi_u+\phi_s+t\phi_v)(2s\phi_u\phi_s+\phi_s^2+t\phi_s\phi_v)\\
&+\frac{1}{4\phi^2}(2s\phi_u+\phi_s+t\phi_v)(2\phi_u+2s\phi_{us}+\phi_{ss}+t\phi_{vs})-\lambda\phi\phi_s, \qquad (3.4)\\
&-\frac{1}{2\phi^2}(2s\phi_u\phi_s+\phi_s^2+t\phi_s\phi_v)+\frac{1}{2\phi}(2\phi_u+2s\phi_{us}+\phi_{ss}+t\phi_{vs})\\
=&\frac{1}{4\phi^3}(2s^2\phi_u+s\phi_s+ts\phi_v)(2s\phi_u\phi_s+\phi_s^2+t\phi_s\phi_v)\\
&+\frac{1}{4\phi^3}(2st\phi_u+t\phi_s+t^2\phi_v)(2s\phi_u\phi_t+\phi_s\phi_t+t\phi_t\phi_v)\\
&-\frac{1}{4\phi^2}(2s^2\phi_u+s\phi_s+ts\phi_v)(2\phi_u+2s\phi_{us}+\phi_{ss}+t\phi_{vs})\\
&-\frac{1}{4\phi^2}(2st\phi_u+t\phi_s+t^2\phi_v)(2s\phi_{ut}+\phi_{st}+t\phi_{vt}+\phi_v)\\
&+\frac{1}{4\phi^2}(2s\phi_u+\phi_s+t\phi_v)^2-\lambda\phi(\phi-s\phi_s-t\phi_t), \qquad (3.5)\\
&-\frac{1}{2\phi^2}(2s\phi_u\phi_v+\phi_s\phi_v+t\phi_v^2)+\frac{1}{2\phi}(2s\phi_{uv}+\phi_{sv}+t\phi_{vv})\\
=&-\frac{1}{4\phi^3}(2s\phi_u+\phi_s+t\phi_v)(2s\phi_u\phi_t+\phi_s\phi_t+t\phi_t\phi_v)\\
&+\frac{1}{4\phi^2}(2s\phi_u+\phi_s+t\phi_v)(2s\phi_{ut}+\phi_{st}+t\phi_{vt}+\phi_v)-\lambda\phi\phi_t. \qquad (3.6)
\end{aligned}
$$

Recall that $A = 2s\phi_u + \phi_s + t\phi_v$, and then multiply (3.5) by $4\phi^3$, together with (2.2), we get

$$\phi_s A^2 - 8\phi\phi_u A + 4\phi^2 \frac{\partial A}{\partial u} + 4\lambda\phi^4\phi_s = 0.$$

Similarly, we deduce from (3.6) that

$$\phi_t A^2 - 4\phi\phi_v A + 2\phi^2 \frac{\partial A}{\partial v} + 4\lambda\phi^4\phi_t = 0.$$

It remains to give the first equation of (1.5). By the definition of A and (2.2), it follows from (3.4) that

$$(s\phi_s + t\phi_t + \phi)A^2 - (4s\phi\phi_u + 2t\phi\phi_v - 2\phi\phi_s)A - 8\phi^2\phi_u - 4\lambda\phi^4(\phi - s\phi_s - t\phi_t) = 0,$$

which, together with two other equations of (1.5) what we have obtained above, yields the first equation of (1.5). This completes the proof.

3.2. Proof of Theorem 1.2

Notice that $\widetilde{u} = u - s^2$, $\widetilde{s} = s$, $\widetilde{v} = v - st$, $\widetilde{t} = t$, $\widetilde{\phi}(\widetilde{u}, \widetilde{s}, \widetilde{v}, \widetilde{t}) = \phi(u, s, v, t)$ and $A = 2s\phi_u + \phi_s + t\phi_v$, a direct calculation by the chain rule shows that

$$\phi_u = \widetilde{\phi}_{\widetilde{u}}, \quad \phi_v = \widetilde{\phi}_{\widetilde{v}}, \quad \phi_s = -2\widetilde{s}\widetilde{\phi}_{\widetilde{u}} + \widetilde{\phi}_{\widetilde{s}} - \widetilde{t}\widetilde{\phi}_{\widetilde{v}}, \quad \phi_t = -\widetilde{s}\widetilde{\phi}_{\widetilde{v}} + \widetilde{\phi}_{\widetilde{t}},$$
$$A = \widetilde{\phi}_{\widetilde{s}}, \quad \frac{\partial A}{\partial v} = \widetilde{\phi}_{\widetilde{s}\widetilde{v}}, \quad \frac{\partial A}{\partial u} = \widetilde{\phi}_{\widetilde{s}\widetilde{u}}.$$

Then (1.5) becomes

$$\begin{cases} 3\widetilde{\phi}_{\widetilde{s}}^2 - 2\widetilde{t}\widetilde{\phi}\widetilde{\phi}_{\widetilde{s}\widetilde{v}} - 4\widetilde{s}\widetilde{\phi}\widetilde{\phi}_{\widetilde{s}\widetilde{u}} - 8\widetilde{\phi}\widetilde{\phi}_{\widetilde{u}} - 4\lambda\widetilde{\phi}^4 = 0 \\ (-2\widetilde{s}\widetilde{\phi}_{\widetilde{u}} + \widetilde{\phi}_{\widetilde{s}} - \widetilde{t}\widetilde{\phi}_{\widetilde{v}})\widetilde{\phi}_{\widetilde{s}}^2 - 8\widetilde{\phi}\widetilde{\phi}_{\widetilde{u}}\widetilde{\phi}_{\widetilde{s}} + 4\widetilde{\phi}^2\widetilde{\phi}_{\widetilde{s}\widetilde{u}} + 4\lambda\widetilde{\phi}^4(-2\widetilde{s}\widetilde{\phi}_{\widetilde{u}} + \widetilde{\phi}_{\widetilde{s}} - \widetilde{t}\widetilde{\phi}_{\widetilde{v}}) = 0 \\ (-\widetilde{s}\widetilde{\phi}_{\widetilde{v}} + \widetilde{\phi}_{\widetilde{t}})\widetilde{\phi}_{\widetilde{s}}^2 - 4\widetilde{\phi}\widetilde{\phi}_{\widetilde{v}}\widetilde{\phi}_{\widetilde{s}} + 2\widetilde{\phi}^2\widetilde{\phi}_{\widetilde{s}\widetilde{v}} + 4\lambda\widetilde{\phi}^4(-\widetilde{s}\widetilde{\phi}_{\widetilde{v}} + \widetilde{\phi}_{\widetilde{t}}) = 0. \end{cases} \tag{3.7}$$

Similarly, (2.2) is equivalent to

$$\begin{cases} -2\widetilde{s}\widetilde{\phi}_{\widetilde{s}\widetilde{u}} + \widetilde{\phi}_{\widetilde{s}\widetilde{s}} - \widetilde{t}\widetilde{\phi}_{\widetilde{s}\widetilde{v}} = 4\widetilde{\phi}_{\widetilde{u}} \\ -\widetilde{s}\widetilde{\phi}_{\widetilde{s}\widetilde{v}} + \widetilde{\phi}_{\widetilde{s}\widetilde{t}} = 2\widetilde{\phi}_{\widetilde{v}}. \end{cases} \tag{3.8}$$

It follows from $(3.7)_1$ and $(3.8)_1$ that

$$3\widetilde{\phi}_{\widetilde{s}}^2 - 2\widetilde{\phi}\widetilde{\phi}_{\widetilde{s}\widetilde{s}} - 4\lambda\widetilde{\phi}^4 = 0. \tag{3.9}$$

Now we solve (3.9). For fixed $\widetilde{u}$, $\widetilde{v}$ and $\widetilde{t}$, let $h(\widetilde{\phi}) = \widetilde{\phi}_{\widetilde{s}}$, and then $\widetilde{\phi}_{\widetilde{s}\widetilde{s}} = h\frac{\mathrm{d}h}{\mathrm{d}\widetilde{\phi}}$. So (3.9) can be rewritten as

$$\frac{\mathrm{d}h^2}{\mathrm{d}\widetilde{\phi}} - \frac{3}{\widetilde{\phi}}h^2 = -4\lambda\widetilde{\phi}^3. \tag{3.10}$$

It is clear that (3.10) is a linear ordinary differential equation of first order, and its solution is

$$\widetilde{\phi}_{\widetilde{s}}^2 = \widetilde{\phi}^3(\eta - 4\lambda\widetilde{\phi}),$$

which leads to

$$\widetilde{\phi} = \frac{f}{(\widetilde{s}+g)^2 + \lambda f^2},$$

where η, f, g are C^∞ functions of $\widetilde{u}$, $\widetilde{v}$, $\widetilde{t}$, and $f = \frac{4}{\eta}$.

Finally, substituting $\widetilde{\phi} = \frac{f}{(\widetilde{s}+g)^2+\lambda f^2}$ into $(3.7)_2$, $(3.7)_3$ and (3.8), and then multiplying all these equations by $[(\widetilde{s}+g)^2 + \lambda f^2]^4$, we immediately establish (1.7)–(1.10) by a comparison of the homogenous coefficients of $\widetilde{s}$. This completes the proof of Theorem 1.2.

4. Two classes of projectively flat Finsler metrics with constant flag curvature $\lambda \neq 0$

In this section, we will give two classes of projectively flat Finsler metrics with constant flag curvature $\lambda \neq 0$ by solving the equivalent PDEs (1.7)–(1.10).

4.1. The first class of projectively flat Finsler metrics with constant flag curvature $\lambda \neq 0$

In virtue of (1.7)–(1.10), it is easy to find that the two unknown functions f and g satisfy

$$\begin{cases} 2g_{\widetilde{t}\widetilde{u}} = -\widetilde{t}g_{\widetilde{v}\widetilde{v}} \\ 2f_{\widetilde{t}\widetilde{u}} = -\widetilde{t}f_{\widetilde{v}\widetilde{v}}. \end{cases} \tag{4.1}$$

Inspired by (4.1), in this subsection, we only consider the following special case

$$\begin{cases} 2g_{\widetilde{t}\widetilde{u}} = -\widetilde{t}g_{\widetilde{v}\widetilde{v}} = 0 \\ 2f_{\widetilde{t}\widetilde{u}} = -\widetilde{t}f_{\widetilde{v}\widetilde{v}} = 0, \end{cases} \tag{4.2}$$

namely, the functions f and g have the form

$$\begin{cases} f(\widetilde{u},\widetilde{v},\widetilde{t}) = [e(\widetilde{u}) + b(\widetilde{t})]\widetilde{v} + c(\widetilde{u}) + d(\widetilde{t}) \\ g(\widetilde{u},\widetilde{v},\widetilde{t}) = [p(\widetilde{u}) + q(\widetilde{t})]\widetilde{v} + r(\widetilde{u}) + w(\widetilde{t}). \end{cases} \tag{4.3}$$

Substituting (4.3) into (1.7)–(1.10), one can get a sequence of quadratic equations with respect to the variable $\widetilde{v}$. Comparing the homogenous coefficients of $\widetilde{v}$, we obtain the following equations

$$-\lambda[e(\widetilde{u}) + b(\widetilde{t})]^2 + [p(\widetilde{u}) + q(\widetilde{t})]^2 + q_{\widetilde{t}}(\widetilde{t}) = 0, \tag{4.4}$$

$$-\lambda[e(\widetilde{u}) + b(\widetilde{t})][c(\widetilde{u}) + d(\widetilde{t})] + [p(\widetilde{u}) + q(\widetilde{t})][r(\widetilde{u}) + w(\widetilde{t})] + w_{\widetilde{t}}(\widetilde{t}) = 0, \tag{4.5}$$

$$1 - \widetilde{t}[p(\widetilde{u}) + q(\widetilde{t})] - 2\lambda[c(\widetilde{u}) + d(\widetilde{t})]c_{\widetilde{u}}(\widetilde{u}) + 2[r(\widetilde{u}) + w(\widetilde{t})]r_{\widetilde{u}}(\widetilde{u}) = 0, \tag{4.6}$$

$$2[e(\widetilde{u}) + b(\widetilde{t})][p(\widetilde{u}) + q(\widetilde{t})] + b_{\widetilde{t}}(\widetilde{t}) = 0, \tag{4.7}$$

$$[c(\widetilde{u}) + d(\widetilde{t})][p(\widetilde{u}) + q(\widetilde{t})] + [e(\widetilde{u}) + b(\widetilde{t})][r(\widetilde{u}) + w(\widetilde{t})] + d_{\widetilde{t}}(\widetilde{t}) = 0, \tag{4.8}$$

$$-\widetilde{t}[e(\widetilde{u}) + b(\widetilde{t})] + 2[r(\widetilde{u}) + w(\widetilde{t})]c_{\widetilde{u}}(\widetilde{u}) + 2[c(\widetilde{u}) + d(\widetilde{t})]r_{\widetilde{u}}(\widetilde{u}) = 0. \tag{4.9}$$

Now we start to solve (4.4)–(4.9).

Firstly, we rewrite (4.4) and (4.7) as

$$\begin{cases} \lambda(e(\widetilde{u}) + b(\widetilde{t}))^2 - (p(\widetilde{u}) + q(\widetilde{t}))^2 = q_{\widetilde{t}}(\widetilde{t}) \\ 2(e(\widetilde{u}) + b(\widetilde{t}))(p(\widetilde{u}) + q(\widetilde{t})) = -b_{\widetilde{t}}(\widetilde{t}), \end{cases}$$

which implies that $e(\widetilde{u}) + b(\widetilde{t})$ and $p(\widetilde{u}) + q(\widetilde{t})$ do not depend on the variable $\widetilde{u}$. As a consequence,

$$p(\widetilde{u}) = p_0, \quad e(\widetilde{u}) = e_0, \tag{4.10}$$

where p_0 and e_0 are constants.

Differentiating (4.5) and (4.8) with respect to $\widetilde{u}$ and using (4.10), we derive

$$-\lambda(e(\widetilde{u}) + b(\widetilde{t}))c_{\widetilde{u}}(\widetilde{u}) + (p(\widetilde{u}) + q(\widetilde{t}))r_{\widetilde{u}}(\widetilde{u}) = 0, \tag{4.11}$$

$$(p(\widetilde{u}) + q(\widetilde{t}))c_{\widetilde{u}}(\widetilde{u}) + (e(\widetilde{u}) + b(\widetilde{t}))r_{\widetilde{u}}(\widetilde{u}) = 0. \tag{4.12}$$

Next, we are going to solve (4.4)–(4.9) in two different cases, according to whether $p(\widetilde{u}) + q(\widetilde{t}) = 0$ is valid or not.

(I) $p(\widetilde{u}) + q(\widetilde{t}) = 0$**.**

In this case, $q_{\widetilde{t}}(\widetilde{t}) = 0$. By (4.4), (4.5) and (4.8), it is easy to check that $e(\widetilde{u}) + b(\widetilde{t}) = 0$, $w_{\widetilde{t}}(\widetilde{t}) = 0$ and $d_{\widetilde{t}}(\widetilde{t}) = 0$. Together with (4.10), we have

$$q(\widetilde{t}) = -p_0, \quad b(\widetilde{t}) = -e_0, \quad w(\widetilde{t}) = w_0, \quad d(\widetilde{t}) = d_0, \tag{4.13}$$

where d_0 and w_0 are constants. As a result,

$$f = c(\widetilde{u}) + d_0, \quad g = r(\widetilde{u}) + w_0. \tag{4.14}$$

In virtue of (4.10) and (4.13), one deduces from (4.6) and (4.9) that

$$\begin{cases} 1 - 2\lambda(c(\widetilde{u}) + d_0)c_{\widetilde{u}}(\widetilde{u}) + 2(r(\widetilde{u}) + w_0)r_{\widetilde{u}}(\widetilde{u}) = 0 \\ (r(\widetilde{u}) + w_0)c_{\widetilde{u}}(\widetilde{u}) + (c(\widetilde{u}) + d_0)r_{\widetilde{u}}(\widetilde{u}) = 0, \end{cases}$$

which indicates

$$\begin{cases} (\lambda f^2 - g^2 - \widetilde{u})_{\widetilde{u}} = 0 \\ (fg)_{\widetilde{u}} = 0. \end{cases} \tag{4.15}$$

By (4.14) and (4.15), we get the solutions of (1.7)–(1.10) as follows

solution 1

$$\begin{cases} f(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \sqrt{\frac{-(c_1-\widetilde{u})+\sqrt{(c_1-\widetilde{u})^2+4\lambda c_2^2}}{2\lambda}} \\ g(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \pm\sqrt{\frac{(c_1-\widetilde{u})+\sqrt{(c_1-\widetilde{u})^2+4\lambda c_2^2}}{2}} \end{cases}$$

where $c_1 \in \mathbb{R}$ if $\lambda > 0$ and $c_1 > 0$ if $\lambda < 0$, $c_2 \in \mathbb{R}$;

solution 2

$$\begin{cases} f(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \sqrt{\frac{-(c_1-\widetilde{u})-\sqrt{(c_1-\widetilde{u})^2+4\lambda c_2^2}}{2\lambda}} \\ g(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \pm\sqrt{\frac{(c_1-\widetilde{u})-\sqrt{(c_1-\widetilde{u})^2+4\lambda c_2^2}}{2}} \end{cases}$$

where $\lambda < 0$ and $c_1 > 0$, $c_2 \in \mathbb{R}$;

solution 3

$$\begin{cases} f(\widetilde{u}, \widetilde{v}, \widetilde{t}) = -\sqrt{\frac{-(c_1-\widetilde{u})+\sqrt{(c_1-\widetilde{u})^2+4\lambda c_2^2}}{2\lambda}} \\ g(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \pm\sqrt{\frac{(c_1-\widetilde{u})+\sqrt{(c_1-\widetilde{u})^2+4\lambda c_2^2}}{2}} \end{cases}$$

where $c_1 \in \mathbb{R}$ if $\lambda > 0$ and $c_1 > 0$ if $\lambda < 0$, $c_2 \in \mathbb{R}$;

solution 4

$$\begin{cases} f(\widetilde{u},\widetilde{v},\widetilde{t}) = -\sqrt{\frac{-(c_1-\widetilde{u})-\sqrt{(c_1-\widetilde{u})^2+4\lambda c_2^2}}{2\lambda}} \\ g(\widetilde{u},\widetilde{v},\widetilde{t}) = \pm\sqrt{\frac{(c_1-\widetilde{u})-\sqrt{(c_1-\widetilde{u})^2+4\lambda c_2^2}}{2}} \end{cases}$$

where $\lambda < 0$ and $c_1 > 0$, $c_2 \in \mathbb{R}$.

(II) $p(\widetilde{u}) + q(\widetilde{t}) \neq 0$.

By (4.11) and (4.12), we have

$$[\lambda(e_0 + b(\widetilde{t}))^2 + (p_0 + q(\widetilde{t}))^2]c_{\widetilde{u}}(\widetilde{u}) = 0. \tag{4.16}$$

Based on the equality (4.16), we divide our deduction into two subcases according to whether $c_{\widetilde{u}}(\widetilde{u}) = 0$ is valid or not.

(i) $c_{\widetilde{u}}(\widetilde{u}) = 0$.

By (4.11) again, $r_{\widetilde{u}}(\widetilde{u}) = 0$, and then

$$c(\widetilde{u}) = c_0, \quad r(\widetilde{u}) = r_0, \tag{4.17}$$

where c_0 and r_0 are constants.

Substituting (4.17) into (4.6) and (4.9), one gets

$$p(\widetilde{u}) + q(\widetilde{t}) = p_0 + q(\widetilde{t}) = \frac{1}{\widetilde{t}}, \quad e(\widetilde{u}) + b(\widetilde{t}) = e_0 + b(\widetilde{t}) = 0. \tag{4.18}$$

Now by (4.5) and (4.8), it indicates that

$$r(\widetilde{u}) + w(\widetilde{t}) = r_0 + w(\widetilde{t}) = \frac{c_1}{\widetilde{t}}, \quad c(\widetilde{u}) + d(\widetilde{t}) = c_0 + d(\widetilde{t}) = \frac{c_2}{\widetilde{t}}, \tag{4.19}$$

where c_1 and c_2 are constants.

Together with (4.17)–(4.19), we give a solution of the PDEs (1.7)–(1.10)

solution 5

$$\begin{cases} f = \frac{c_2}{\widetilde{t}} \\ g = \frac{\widetilde{v}+c_1}{\widetilde{t}}, \end{cases}$$

where $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}$.

(ii) $c_{\widetilde{u}}(\widetilde{u}) \neq 0$.

In virtue of (4.16), we get

$$\lambda(e_0 + b(\widetilde{t}))^2 + (p_0 + q(\widetilde{t}))^2 = 0. \tag{4.20}$$

Notice that in this subcase, (4.11) can be rewritten as

$$\frac{e_0 + b(\widetilde{t})}{p_0 + q(\widetilde{t})} = \frac{r_{\widetilde{u}}(\widetilde{u})}{\lambda c_{\widetilde{u}}(\widetilde{u})},$$

which implies that each side of this equality is a constant. Denote

$$m = \frac{e_0 + b(\widetilde{t})}{p_0 + q(\widetilde{t})} = \frac{r_{\widetilde{u}}(\widetilde{u})}{\lambda c_{\widetilde{u}}(\widetilde{u})}, \tag{4.21}$$

where the constant m will be determined later.

Substituting (4.21) into (4.20), we have

$$1+\lambda m^2=0. \tag{4.22}$$

From now on, we need $\lambda<0$, otherwise there is no solution in this subcase.

By (4.21) and (4.22), we deduce from (4.12) and (4.5), (4.8) respectively that

$$c_{\widetilde{u}}(\widetilde{u})=-mr_{\widetilde{u}}(\widetilde{u}),\quad d_{\widetilde{t}}(\widetilde{t})=mw_{\widetilde{t}}(\widetilde{t}), \tag{4.23}$$

which indicates that

$$c(\widetilde{u})=-mr_{\widetilde{u}}(\widetilde{u})+mj,\quad d(\widetilde{t})=mw(\widetilde{t})+m(k-j), \tag{4.24}$$

where k and j are constants.

Multiplying (4.9) by $\frac{1}{m}$, and then along with (4.6), we give

$$p_0+q(\widetilde{t})=\frac{1}{2\widetilde{t}}, \tag{4.25}$$

$$e_0+b(\widetilde{t})=\frac{m}{2\widetilde{t}}, \tag{4.26}$$

where in the last equality we have used (4.21).

By (4.23)–(4.26), the equations (4.5) and (4.6) can be simplified as

$$\begin{cases}\frac{k}{2\widetilde{t}}+\frac{1}{\widetilde{t}}w(\widetilde{t})+w_{\widetilde{t}}(\widetilde{t})=0\\ \frac{1}{2}+2(2r(\widetilde{u})-k)r_{\widetilde{u}}(\widetilde{u})=0,\end{cases}$$

which, together with (4.24), leads to

$$\begin{cases}r(\widetilde{u})+w(\widetilde{t})=\pm\frac{\sqrt{c_1-\widetilde{u}}}{2}+\frac{c_2}{2\widetilde{t}}\\ c(\widetilde{u})+d(\widetilde{t})=-m(\pm\frac{\sqrt{c_1-\widetilde{u}}}{2}-\frac{c_2}{2\widetilde{t}}),\end{cases} \tag{4.27}$$

where $c_1>0$, $c_2\in\mathbb{R}$.

By (4.25)–(4.27), we obtain the solution of (1.7)–(1.10) with $\lambda<0$

solution 6

$$\begin{cases}f=\pm\sqrt{\frac{-1}{\lambda}}(\frac{\sqrt{c_1-\widetilde{u}}}{2}\pm\frac{\widetilde{v}+c_2}{2\widetilde{t}})\\ g=\pm\frac{\sqrt{c_1-\widetilde{u}}}{2}+\frac{\widetilde{v}+c_2}{2\widetilde{t}}\end{cases}$$

where $c_1>0$, $c_2\in\mathbb{R}$, $\lambda<0$.

Based on the above results, we have

Corollary 4.1. *Let $B_r=\{x\in\mathbb{R}^n:|x|<r\}$, $r>0$, and let $F=|y|\widetilde{\phi}(\widetilde{u},\widetilde{s},\widetilde{v},\widetilde{t})$, where $\widetilde{\phi}$ is given by (1.6) with $\lambda\neq0$ and f, g are provided by* ***solution 1*** *or* ***solution 2****, then*

(1) F is a projectively flat Finsler metric with nonzero constant flag curvature $\lambda\neq0$ on B_ϵ if $\epsilon(>0)$ is small enough.

(2) For a fixed $r_0>0$, F is a projectively flat Finsler metric with nonzero constant flag curvature $\lambda\neq0$ on B_{r_0} if the constant c_1 in ***solution 1*** *or* ***solution 2*** *is large enough.*

Proof. For convenience, we only consider

$$\begin{cases} f(\widetilde{u},\widetilde{v},\widetilde{t}) = \sqrt{\frac{\sqrt{(1-\widetilde{u})^2+1}-(1-\widetilde{u})}{2}} \\ g(\widetilde{u},\widetilde{v},\widetilde{t}) = \sqrt{\frac{\sqrt{(1-\widetilde{u})^2+1}+(1-\widetilde{u})}{2}}. \end{cases}$$

In this case, $c_1 = 1$, $c_2 = \frac{1}{2}$, $\lambda = 1$. And it is sufficient to prove the conclusion (1). In fact, the general results of the conclusions (1) (2) can be proved in the same way by introducing the new variable $\tilde{x} = \frac{x}{\sqrt{c_1}}$.

Set $\epsilon < 1$. A direct computation shows that on B_ϵ,

$$\begin{aligned} &\frac{1}{4} \le f(\widetilde{u}) \le 1, \quad |f_{\widetilde{u}}(\widetilde{u})| \le 1, \quad |f_{\widetilde{u}\widetilde{u}}(\widetilde{u})| \le 1, \\ &\frac{1}{2} \le g(\widetilde{u}) \le 2, \quad |g_{\widetilde{u}}(\widetilde{u})| \le 1, \quad |g_{\widetilde{u}\widetilde{u}}(\widetilde{u})| \le 1, \end{aligned} \tag{4.28}$$

and there exists a positive constant C independent of ϵ such that

$$\begin{aligned} &\frac{1}{40} \le \widetilde{\phi} \le 4, \quad |(\widetilde{\phi}^2)_{\widetilde{u}}| \le C\widetilde{\phi}^2, \quad |(\widetilde{\phi}^2)_{\widetilde{u}\widetilde{u}}| \le C\widetilde{\phi}^2, \\ &|(\widetilde{\phi}^2)_{\widetilde{s}}| \le C\widetilde{\phi}^2, \quad |(\widetilde{\phi}^2)_{\widetilde{s}\widetilde{s}}| \le C\widetilde{\phi}^2, \quad |(\widetilde{\phi}^2)_{\widetilde{u}\widetilde{s}}| \le C\widetilde{\phi}^2. \end{aligned} \tag{4.29}$$

Observe that $\widetilde{u} = |x|^2 - \frac{<x,y>^2}{|y|^2}$, $\widetilde{s} = \frac{<x,y>}{|y|}$, it is easy to check that

$$|\widetilde{s}_i| \le \frac{2\epsilon}{|y|}, \quad |\widetilde{s}_{ij}| \le \frac{4\epsilon}{|y|^2}, \quad |\widetilde{u}_i| \le \frac{4\epsilon}{|y|}, |\widetilde{u}_{ij}| \le \frac{8\epsilon}{|y|^2}. \tag{4.30}$$

Notice that

$$\begin{aligned} (F^2)_{ij} &= (|y|^2\widetilde{\phi}^2)_{ij} = \widetilde{\phi}^2(|y|^2)_{ij} + (\widetilde{\phi}^2)_i(|y|^2)_j + (\widetilde{\phi}^2)_j(|y|^2)_i + |y|^2(\widetilde{\phi}^2)_{ij} \\ &= \widetilde{\phi}^2[(|y|^2)_{ij} + \widetilde{\phi}^{-2}(2y^i(\widetilde{\phi}^2)_{\widetilde{u}} \cdot \widetilde{u}_j + 2y^j(\widetilde{\phi}^2)_{\widetilde{u}} \cdot \widetilde{u}_i + 2y^i(\widetilde{\phi}^2)_{\widetilde{s}} \cdot \widetilde{s}_j + 2y^j(\widetilde{\phi}^2)_{\widetilde{s}} \cdot \widetilde{s}_i \\ &+ |y|^2((\widetilde{\phi}^2)_{\widetilde{u}\widetilde{u}} \cdot \widetilde{u}_i\widetilde{u}_j + (\widetilde{\phi}^2)_{\widetilde{u}\widetilde{s}} \cdot \widetilde{u}_i\widetilde{s}_j + (\widetilde{\phi}^2)_{\widetilde{u}\widetilde{s}} \cdot \widetilde{u}_j\widetilde{s}_i + (\widetilde{\phi}^2)_{\widetilde{s}\widetilde{s}} \cdot \widetilde{s}_i\widetilde{s}_j \\ &+ (\widetilde{\phi}^2)_{\widetilde{u}} \cdot \widetilde{u}_{ij} + (\widetilde{\phi}^2)_{\widetilde{s}} \cdot \widetilde{s}_{ij}))]. \end{aligned}$$

Now denote $M_{ij} = (\widetilde{\phi}^2)_i(|y|^2)_j + (\widetilde{\phi}^2)_j(|y|^2)_i + |y|^2(\widetilde{\phi}^2)_{ij}$ and the symmetric matrix $M = (M_{ij})$. By (4.29) and (4.30), we get $|M_{ij}| \le C\epsilon$ for any i and j, where the constant C is independent of ϵ. As a result, $((F^2)_{ij}) = \widetilde{\phi}^2(2I + M)$ is positive definite when ϵ is small enough, here the symbol I denotes the unit matrix. On the other hand, one can immediately verify that $F(y) > 0$ for any $y \neq 0$ and $F(ky) = |k|F(y)$ for any $k \in \mathbb{R}$. Hence, F is indeed a Finsler metric. Due to Theorems 1.1 and 1.2, it is also a projectively flat Finsler metric with nonzero constant flag curvature $\lambda = 1$. This completes the proof.

Remark 4.2. We have to remove **solutions 3-6**, because all these solutions do not guarantee that the metric $F \ge 0$.

4.2. The second class of projectively flat Finsler metrics with constant flag curvature $\lambda \neq 0$

In this subsection, we will solve (1.7)–(1.10) under the following assumption

$$\begin{cases} (g^2 - \lambda f^2)_{\widetilde{v}} = (g^2 - \lambda f^2)_{\widetilde{u}} \\ (fg)_{\widetilde{v}} = (fg)_{\widetilde{u}}, \end{cases} \tag{4.31}$$

and then give another class of projectively flat Finsler metrics with constant flag curvature $\lambda \neq 0$.

By (4.31), it is obvious that f and g have the form

$$\begin{cases} f = \widetilde{\phi}_1(\widetilde{u} + \widetilde{v}, \widetilde{t}) \\ g = \widetilde{\phi}_2(\widetilde{u} + \widetilde{v}, \widetilde{t}). \end{cases}$$

Using (4.31) again and by (1.7)–(1.10), we get

$$\begin{cases} 2g_{\widetilde{t}} + \widetilde{t} g_{\widetilde{v}} - 1 = 0 \\ 2f_{\widetilde{t}} + \widetilde{t} f_{\widetilde{v}} = 0, \end{cases} \tag{4.32}$$

together with (1.9) and (1.10), which yields

$$\begin{cases} (-\widetilde{t} f + 2fg)_{\widetilde{t}} = 0 \\ (-\widetilde{t} f + 2fg)_{\widetilde{v}} = 0. \end{cases}$$

Thus

$$-\widetilde{t} f + 2fg = c_2, \tag{4.33}$$

where $c_2 \in \mathbb{R}$.

Similarly, by (4.32), we deduce from (1.7) and (1.8) that

$$\begin{cases} (\widetilde{u} + \widetilde{v} - \widetilde{t} g + g^2 - \lambda f^2)_{\widetilde{t}} = 0 \\ (\widetilde{u} + \widetilde{v} - \widetilde{t} g + g^2 - \lambda f^2)_{\widetilde{v}} = 0. \end{cases}$$

Consequently,

$$\widetilde{u} + \widetilde{v} - \widetilde{t} g + g^2 - \lambda f^2 = c_1, \tag{4.34}$$

where $c_1 \in \mathbb{R}$.

By (4.33) and (4.34), we obtain the solutions of (1.7)–(1.10) as follows

solution 7

$$\begin{cases} f(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \sqrt{\frac{-l + \sqrt{l^2 + \lambda c_2^2}}{2\lambda}} \\ g(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \frac{\widetilde{t}}{2} \pm \sqrt{\frac{l + \sqrt{l^2 + \lambda c_2^2}}{2}}, \end{cases}$$

where $l := l(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \frac{\widetilde{t}^2}{4} - (\widetilde{u} + \widetilde{v}) + c_1$, $c_1 \in \mathbb{R}$ if $\lambda > 0$ and $c_1 > 0$ if $\lambda < 0$, $c_2 \in \mathbb{R}$;

solution 8

$$\begin{cases} f(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \sqrt{\frac{-l - \sqrt{l^2 + \lambda c_2^2}}{2\lambda}} \\ g(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \frac{\widetilde{t}}{2} \pm \sqrt{\frac{l - \sqrt{l^2 + \lambda c_2^2}}{2}}, \end{cases}$$

where $l := l(\widetilde{u}, \widetilde{v}, \widetilde{t}) = \frac{\widetilde{t}^2}{4} - (\widetilde{u} + \widetilde{v}) + c_1$, $c_1 > 0$, $\lambda < 0$ and $c_2 \in \mathbb{R}$;

solution 9

$$\begin{cases} f(\widetilde{u},\widetilde{v},\widetilde{t}) = -\sqrt{\frac{-l+\sqrt{l^2+\lambda c_2^2}}{2\lambda}} \\ g(\widetilde{u},\widetilde{v},\widetilde{t}) = \frac{\widetilde{t}}{2} \pm \sqrt{\frac{l+\sqrt{l^2+\lambda c_2^2}}{2}}, \end{cases}$$

where $l := l(\widetilde{u},\widetilde{v},\widetilde{t}) = \frac{\widetilde{t}^2}{4} - (\widetilde{u}+\widetilde{v}) + c_1$, $c_1 \in \mathbb{R}$ if $\lambda > 0$ and $c_1 > 0$ if $\lambda < 0$, $c_2 \in \mathbb{R}$;

solution 10

$$\begin{cases} f(\widetilde{u},\widetilde{v},\widetilde{t}) = -\sqrt{\frac{-l-\sqrt{l^2+\lambda c_2^2}}{2\lambda}} \\ g(\widetilde{u},\widetilde{v},\widetilde{t}) = \frac{\widetilde{t}}{2} \pm \sqrt{\frac{l-\sqrt{l^2+\lambda c_2^2}}{2}}, \end{cases}$$

where $l := l(\widetilde{u},\widetilde{v},\widetilde{t}) = \frac{\widetilde{t}^2}{4} - (\widetilde{u}+\widetilde{v}) + c_1$, $c_1 > 0$, $\lambda < 0$ and $c_2 \in \mathbb{R}$.

Now, by Theorems 1.1 and 1.2, we conclude that

Corollary 4.3. *Let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, $r > 0$, and $F = |y|\widetilde{\phi}(\widetilde{u},\widetilde{s},\widetilde{v},\widetilde{t})$, where $\widetilde{\phi}$ is given by (1.6) with $\lambda \neq 0$ and f, g are presented by* ***solution 7*** *or* ***solution 8****, then*

(1) F is a projectively flat Finsler metric with nonzero constant flag curvature $\lambda \neq 0$ on B_ϵ if $\epsilon(> 0)$ and $|a|$ are small enough.

(2) For a fixed $r_0 > 0$, F is a projectively flat Finsler metric with nonzero constant flag curvature $\lambda \neq 0$ on B_{r_0} if the constant c_1 in ***solution 7*** *or* ***solution 8*** *is large enough.*

The proof is completely similar to that of Corollary 4.1, we omit it.

Remark 4.4. We discard **solutions 9, 10** since they do not assure that the metric $F \geq 0$.

In a word, by Corollaries 4.1 and 4.3, we have constructed two classes of new projectively flat Finsler metrics with nonzero constant flag curvature.

5. Two classes of projectively flat Finsler metrics with constant flag curvature $\lambda = 0$

When $\lambda = 0$, (1.6)–(1.10) in Theorem 1.2 are reduced to

$$\widetilde{\phi} = \frac{f}{(\widetilde{s}+g)^2}, \tag{5.1}$$

where the functions f and g satisfy

$$\begin{cases} 2g_{\widetilde{t}} + (g^2)_{\widetilde{v}} = 0 \\ -\widetilde{t}g_{\widetilde{v}} + 1 + (g^2)_{\widetilde{u}} = 0 \\ f_{\widetilde{t}} + (fg)_{\widetilde{v}} = 0 \\ \widetilde{t}f_{\widetilde{v}} - 2(fg)_{\widetilde{u}} = 0. \end{cases} \tag{5.2}$$

In this section, using an argument similar to that in Section 4, we will give two classes of projectively flat Finsler metrics with constant flag curvature $\lambda = 0$ by solving the PDEs (5.2).

5.1. The first class of projectively flat Finsler metric with constant flag curvature $\lambda = 0$

In this subsection, we are going to present a class of projectively flat Finsler metric with constant flag curvature $\lambda = 0$ by the method applied in Subsection 4.1.

By (5.2), we still have

$$\begin{cases} 2g_{\widetilde{t}\widetilde{u}} = -\widetilde{t}g_{\widetilde{v}\widetilde{v}} \\ 2f_{\widetilde{t}\widetilde{u}} = -\widetilde{t}f_{\widetilde{v}\widetilde{v}}. \end{cases}$$

And now we only discuss (5.2) under the assumption as follows

$$\begin{cases} 2g_{\widetilde{t}\widetilde{u}} = -\widetilde{t}g_{\widetilde{v}\widetilde{v}} = 0 \\ 2f_{\widetilde{t}\widetilde{u}} = -\widetilde{t}f_{\widetilde{v}\widetilde{v}} = 0, \end{cases} \tag{5.3}$$

which implies that f and g have the form

$$\begin{cases} f(\widetilde{u},\widetilde{v},\widetilde{t}) = [e(\widetilde{u}) + b(\widetilde{t})]\widetilde{v} + c(\widetilde{u}) + d(\widetilde{t}) \\ g(\widetilde{u},\widetilde{v},\widetilde{t}) = [p(\widetilde{u}) + q(\widetilde{t})]\widetilde{v} + r(\widetilde{u}) + w(\widetilde{t}). \end{cases} \tag{5.4}$$

Substituting (5.3) to (5.2), we obtain the following equations by a comparison of the homogeneous coefficients of $\widetilde{v}$

$$[p(\widetilde{u}) + q(\widetilde{t})]^2 + q_{\widetilde{t}}(\widetilde{t}) = 0, \tag{5.5}$$

$$[p(\widetilde{u}) + q(\widetilde{t})][r(\widetilde{u}) + w(\widetilde{t})] + w_{\widetilde{t}}(\widetilde{t}) = 0, \tag{5.6}$$

$$1 - \widetilde{t}[p(\widetilde{u}) + q(\widetilde{t})] + 2[r(\widetilde{u}) + w(\widetilde{t})]r_{\widetilde{u}}(\widetilde{u}) = 0, \tag{5.7}$$

$$2[e(\widetilde{u}) + b(\widetilde{t})][p(\widetilde{u}) + q(\widetilde{t})] + b_{\widetilde{t}}(\widetilde{t}) = 0, \tag{5.8}$$

$$[c(\widetilde{u}) + d(\widetilde{t})][p(\widetilde{u}) + q(\widetilde{t})] + [e(\widetilde{u}) + b(\widetilde{t})][r(\widetilde{u}) + w(\widetilde{t})] + d_{\widetilde{t}}(\widetilde{t}) = 0, \tag{5.9}$$

$$-\widetilde{t}[e(\widetilde{u}) + b(\widetilde{t})] + 2[r(\widetilde{u}) + w(\widetilde{t})]c_{\widetilde{u}}(\widetilde{u}) + 2[c(\widetilde{u}) + d(\widetilde{t})]r_{\widetilde{u}}(\widetilde{u}) = 0. \tag{5.10}$$

In order to solve (5.5)–(5.10), just as what we have done in Subsection 4.1, we still start with (5.5) and (5.8), which can be rewritten as

$$(p(\widetilde{u}) + q(\widetilde{t}))^2 = -q_{\widetilde{t}}(\widetilde{t}), \tag{5.11}$$

$$2(e(\widetilde{u}) + b(\widetilde{t}))(p(\widetilde{u}) + q(\widetilde{t})) = -b_{\widetilde{t}}(\widetilde{t}). \tag{5.12}$$

It follows from (5.11) that

$$p(\widetilde{u}) = p_0, \tag{5.13}$$

where p_0 is a constant. However, one can not conclude from (5.12) that $e(\widetilde{u})$ must be a constant since $p(\widetilde{u}) + q(\widetilde{t})$ maybe be zero. This is different from (4.11) and (4.12) in Subsection 4.1.

Differentiating (5.6) with respect to $\widetilde{u}$ and by (5.13), we derive

$$(p(\widetilde{u}) + q(\widetilde{t}))r_{\widetilde{u}}(\widetilde{u}) = 0. \tag{5.14}$$

Now we divide our discussion into two cases.

(I) $p(\widetilde{u}) + q(\widetilde{t}) = 0$**.**

In this case, by (5.5) (5.6) and (5.8), it is obvious that $q_{\widetilde{t}}(\widetilde{t}) = 0$, $w_{\widetilde{t}}(\widetilde{t}) = 0$, $b_{\widetilde{t}}(\widetilde{t}) = 0$, and then

$$q(\widetilde{t}) = -p_0, \quad b(\widetilde{t}) = b_0, \quad w(\widetilde{t}) = w_0, \tag{5.15}$$

where $b_0, w_0 \in \mathbb{R}$.

By (5.13) and (5.15), equations (5.7) (5.9) and (5.10) are simplified as

$$1 + 2(r(\widetilde{u}) + w_0)r_{\widetilde{u}}(\widetilde{u}) = 0, \tag{5.16}$$

$$(e(\widetilde{u}) + b_0)(r(\widetilde{u}) + w_0) = -d_{\widetilde{t}}(\widetilde{t}), \tag{5.17}$$

$$-\widetilde{t}(e(\widetilde{u}) + b_0) + 2(r(\widetilde{u}) + w_0)c_{\widetilde{u}}(\widetilde{u}) + 2(c(\widetilde{u}) + d(\widetilde{t}))r_{\widetilde{u}}(\widetilde{u}) = 0. \tag{5.18}$$

We deduce from (5.16) that

$$r(\widetilde{u}) + w(\widetilde{t}) = r(\widetilde{u}) + w_0 = \pm\sqrt{c_1 - \widetilde{u}}, \tag{5.19}$$

where $c_1 > 0$.

Next, (5.17) indicates that $d_{\widetilde{t}}(\widetilde{t})$ is always a constant $-c_2$, that is,

$$d(\widetilde{t}) = -c_2 t + m, \tag{5.20}$$

where $c_2, m \in \mathbb{R}$.

Substituting (5.19) and (5.20) into (5.17), we get

$$e(\widetilde{u}) + b(\widetilde{t}) = \pm\frac{c_2}{\sqrt{c_1 - \widetilde{u}}}, \tag{5.21}$$

where c_2 is a constant.

Multiplying (5.18) by $r(\widetilde{u}) + w_0$, and by (5.16), (5.17), (5.19), we have

$$2(c_1 - \widetilde{u})c_{\widetilde{u}}(\widetilde{u}) - c(\widetilde{u}) = m,$$

which leads to

$$c(\widetilde{u}) = -m + \frac{c_3}{\sqrt{c_1 - \widetilde{u}}}, \tag{5.22}$$

where c_3 is a constant.

Together with (5.19)–(5.22), we obtain the following solutions of (5.2)

solution 1#

$$\begin{cases} f = \frac{c_2\widetilde{v}}{\sqrt{c_1 - \widetilde{u}}} + \frac{c_3}{\sqrt{c_1 - \widetilde{u}}} - c_2\widetilde{t} \\ g = \sqrt{c_1 - \widetilde{u}}, \end{cases}$$

where $c_1 > 0, c_2, c_3 \in \mathbb{R}$;

solution 2#

$$\begin{cases} f = -\frac{c_2\widetilde{v}}{\sqrt{c_1 - \widetilde{u}}} + \frac{c_3}{\sqrt{c_1 - \widetilde{u}}} - c_2\widetilde{t} \\ g = -\sqrt{c_1 - \widetilde{u}}, \end{cases}$$

where $c_1 > 0, c_2, c_3 \in \mathbb{R}$.

(II) $p(\widetilde{u}) + q(\widetilde{t}) \neq 0$.

By (5.14), $r_{\widetilde{u}}(\widetilde{u}) = 0$, and then

$$r(\widetilde{u}) = r_0. \tag{5.23}$$

Consequently, we deduce from (5.7) that

$$p(\widetilde{u}) + q(\widetilde{t}) = \frac{1}{\widetilde{t}}. \tag{5.24}$$

Substituting (5.23) (5.24) into (5.8) and (5.9), we find that

$$e(\widetilde{u}) = e_0, \quad c(\widetilde{u}) = c_0, \tag{5.25}$$

where $c_0, e_0 \in \mathbb{R}$. And further, we deduce from (5.10) that

$$e(\widetilde{u}) + b(\widetilde{t}) = 0. \tag{5.26}$$

Applying all the results above to (5.6) and (5.9), by a direct computation, we derive

$$\begin{cases} r(\widetilde{u}) + w(\widetilde{t}) = \frac{c_1}{\widetilde{t}} \\ c(\widetilde{u}) + d(\widetilde{t}) = \frac{c_2}{\widetilde{t}}. \end{cases} \tag{5.27}$$

Together with (5.23)–(5.27), we get the following solution of (5.2)

solution 3#

$$\begin{cases} f = \frac{c_2}{\widetilde{t}} \\ g = \frac{\widetilde{v}+c_1}{\widetilde{t}}, \end{cases}$$

where $c_1, c_2 \in \mathbb{R}$.

Now, by Theorems 1.1 and 1.2, we have

Corollary 5.1. *Let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, $r > 0$, and let $F = |y|\widetilde{\phi}(\widetilde{u}, \widetilde{s}, \widetilde{v}, \widetilde{t})$, where $\widetilde{\phi}$ is given by (5.1) and f, g are provided by* ***solution 1#*** *with $c_3 > 0$ or* ***solution 2#*** *with $c_3 > 0$, then*

(1) F is a projectively flat Finsler metric with zero flag curvature on B_ϵ if $\epsilon(> 0)$ and $|a|$ are small enough.

(2) For a fixed $r_0 > 0$, F is a projectively flat Finsler metric with zero flag curvature on B_{r_0} if the constant c_1 in ***solution 1#*** *or* ***solution 2#*** *is large enough and $|a|$ is small enough.*

The proof is analogous to that of Corollary 4.1, so we omit it too.

Remark 5.2. Solution 3# have to be removed because it does not guarantee that the metric $F \geq 0$.

Remark 5.3. The metric F determined by **solution 1#** with the constants $c_2 = 0$ and $c_3 = 1$ has been given by L. Zhou in [12]. Z. Shen [10] has found the metric F derived by **solution 2#** with the constants $c_1 = 1$, $c_2 = -1$, $c_3 = 1$.

5.2. The second class of projectively flat Finsler metric with constant flag curvature $\lambda = 0$

In this subsection, we give another class of projectively flat Finsler metric with constant flag curvature $\lambda = 0$ under the assumption

$$\begin{cases}(g^2)_{\widetilde{v}} = (g^2)_{\widetilde{u}} \\ (fg)_{\widetilde{v}} = (fg)_{\widetilde{u}},\end{cases} \tag{5.28}$$

which shows that the functions f and g have the form

$$\begin{cases}f = \widetilde{\phi}_1(\widetilde{u}+\widetilde{v},\widetilde{t}) \\ g = \widetilde{\phi}_2(\widetilde{u}+\widetilde{v},\widetilde{t}).\end{cases}$$

Combining $(5.2)_1$ and $(5.2)_2$ with (5.28), we have

$$\begin{cases}(g^2 - g\widetilde{t} + \widetilde{u} + \widetilde{v})_{\widetilde{t}} = 0 \\ (g^2 - g\widetilde{t} + \widetilde{u} + \widetilde{v})_{\widetilde{v}} = 0.\end{cases}$$

Then

$$g^2 - g\widetilde{t} = -(\widetilde{u}+\widetilde{v}) + c_1,$$

where $c_1 \in \mathbb{R}$.

Hence

$$g = \frac{\widetilde{t} \pm \sqrt{\widetilde{t}^2 - 4(\widetilde{u}+\widetilde{v}) + 4c_1}}{2}. \tag{5.29}$$

Similarly, along with $(5.2)_3$ and $(5.2)_4$, by (5.28), we get

$$\begin{cases}(\widetilde{t}f - 2fg)_{\widetilde{t}} = 0 \\ (\widetilde{t}f - 2fg)_{\widetilde{v}} = 0,\end{cases}$$

which implies that

$$\widetilde{t}f - 2fg = c_2,$$

where $c_2 \in \mathbb{R}$. Namely, $f = \frac{c_2}{\widetilde{t} - 2g}$.

By (5.29), it yields

$$f = \frac{c_2}{\sqrt{\widetilde{t}^2 - 4(\widetilde{u}+\widetilde{v}) + 4c_1}}, \tag{5.30}$$

where $c_1 \in \mathbb{R}$, $c_2 \in \mathbb{R}$. So we obtain one of the solutions of (5.2) again

solution 4#

$$\begin{cases}f = \frac{c_2}{\sqrt{\widetilde{t}^2 - 4(\widetilde{u}+\widetilde{v}) + 4c_1}} \\ g = \frac{\widetilde{t} \pm \sqrt{\widetilde{t}^2 - 4(\widetilde{u}+\widetilde{v}) + 4c_1}}{2}\end{cases}$$

where $c_1 > 0$, $c_2 \in \mathbb{R}$.

Now, due to Theorems 1.1 and 1.2 again, we conclude

Corollary 5.4. *Let $B_r = \{x \in \mathbb{R}^n : |x| < r\}$, $r > 0$, and let $F = |y|\widetilde{\phi}(\widetilde{u}, \widetilde{s}, \widetilde{v}, \widetilde{t})$, where $\widetilde{\phi}$ is given by (5.1) and f, g are provided by* ***solution 4#*** *with $c_2 > 0$, then*

(1) F is a projectively flat Finsler metric with zero flag curvature on B_ϵ if $\epsilon(> 0)$ and $|a|$ are small enough.

(2) For a fixed $r_0 > 0$, F is a projectively flat Finsler metric with zero flag curvature on B_{r_0} if the constant c_1 in ***solution 4#*** *is large enough.*

We still omit the proof since it is also similar to that of Corollary 4.1.

In summary, by Corollaries 5.1 and 5.4, we have also obtained two classes of new projectively flat Finsler metrics with zero flag curvature.

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