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# Output feedback consensus control for fractional-order nonlinear multi-agent systems with directed topologies

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**Abstract**

This paper is devoted to the output feedback consensus control problem for a class of nonlinear fractional-order multi-agent systems (MASs) with general directed topologies. It is worth noting that the considered fractional-order MASs including the second-order MASs as special cases. By introducing a distributed filter for each agent, a control algorithm uses only relative position measurements is proposed to guarantee the global leaderless consensus can be achieved. Also the derived results are further extended to consensus tracking problem with a leader whose input is unknown and bounded. Finally, two simulation examples are provided to verify the performance of the control design.


## 1. Introduction

Consensus control of multi-agent systems (MASs) is to design appropriate controller for each agent only using local information between neighbors such that the states of all agents reach general agreement. From the viewpoint of existing number of leaders in MASs, existing consensus problem of MASs can be classified into two categories: leaderless consensus problem [1–3] and consensus tracking (or leader-follower consensus) problem with a leader [4–7]. During the last decade, consensus of MASs has received considerable attention and there

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are many results available in the literature, see the recent survey papers [8,9] and references therein.

Early work often focuses on consensus of integer-order MASs, i.e., first-order MASs [1,4,5], second-order MASs [2,3,6], and high-order MASs [7,10,11]. However, several phenomena can be explained naturally by the collective group behavior of agents with fractional-order dynamics rather than classical integer-order dynamics [12,13]. For example, the synchronization motion of multiple agents in fractional circumstances, i.e., viscoelastic materials, macromolecule fluids and porous media. Up to now, some researchers have tackled the consensus control of MASs with fractional-order dynamics. Leaderless consensus problem of nonlinear fractional-order double integrator MASs is studied in [14,15]. Both leaderless consensus problem and consensus tracking problem are considered in [16] for fractional-order MASs with input time delay. Consensus tracking control problem is also presented for fractional-order single integrator MASs with undirected topology [17] or directed topology [18,19], and for fractional-order double integrator MASs with undirected topology [20] or directed topology [14,21]. It should be emphasized that fractional-order systems are an extension to the traditional integer-order ones, which have properties of infinity memory and hereditary due to the existence of a memory term in the model [22].

Note that state feedback-based controllers are mainly based on a restrictive assumption that the state variables of each agent can be measured directly. However, in many real applications, full-state measurements are unavailable due to economical concerns or physical constraints. Especially when there exist multiple states in second-order or higher-order MASs, it is unrealistic to obtain the information of multiple states accurately. Thus, state feedback-based control in those cases should be replaced by output feedback-based control. Some results on output feedback-based consensus control problem are presented in MASs with general linear dynamics [23,24], with second-order agent dynamics [25,26] and with high-order agent dynamics [27]. So far, no author has studied the output feedback-based consensus problem for MASs with nonlinear fractional-order dynamics.

In reality, the agents might be affected by the interaction among neighboring agents, but also by its own intrinsic nonlinear dynamic. So the MASs with intrinsic nonlinear dynamics are considered recently in [2,3,5,14,18]. Since the limited view field or nonuniform sensing ranges of sensors, one agent may be able sense another agent, but not vice versa. The communication topology among the agents, in general is directed. Taking into consideration these practical cases, in this paper, we consider the consensus problem of fractional-order double integrator MASs with intrinsic nonlinear dynamics and general directed topologies using only relative output information. Due to the well-known Leibniz rule for fractional derivatives is invalid [28], how to construct a suitable Lyapunov function for analysing the stability of nonlinear fractional-order MASs is very challenging. The output feedback based consensus control of double integrator MASs in the presence of nonlinear fractional-order dynamics is even more challenging as the communication topology among the agents is not only directed but also local.

The main contributions of this paper are summarized as follows. Firstly, for the leaderless consensus problem of fractional-order MASs with intrinsic nonlinear dynamics and directed graph, a novel distributed algorithm combined with a filter is derived. By generalizing an important nonlinear fractional-order inequality, it is shown that all the agents can achieve global consensus if the interaction graph is strongly connected and the control parameters are chosen properly. Secondly, for the tracking problem with a leader whose input is bounded and unknown to any follower, a newly distributed algorithm combined with a similar filter is further developed to guarantee the tracking error and control input are uniformly ultimately bounded (UUB). Of particular interest is all the algorithms designed in this paper can be implemented only using relative out measurements between neighbors.

The rest of this paper is organized as follows. Some preliminaries are briefly presented in Section 2. Output feedback based leaderless consensus problem is first studied in Section 3. In Section 4, the results are then extended to the output feedback based tracking problem when there exists an unknown leader. Simulation examples and conclusions are outlined in Sections 5 and 6, respectively.

Let $\mathbb{R}^{m\times n}$ and $\mathbb{R}^m$ be the sets of $m\times n$ real matrices and $m$-dimensional Euclidean space, respectively. Let $\mathbf{1}_n\in\mathbb{R}^n$ ($\mathbf{0}_n\in\mathbb{R}^n$) stand for the $n\times 1$ column vector of all ones (zeros) and $I_n(O_n)$ be the $n\times n$ identity (zero) matrix. Denote by $\mathrm{diag}(d_1,\dots,d_n)\in\mathbb{R}^{n\times n}$ a diagonal matrix with diagonal entries $d_1$ to $d_n$. We use $\otimes$ to represent the Kronecker product. For a vector $x=[x_1,\dots,x_n]^T\in\mathbb{R}^n$, $\|x\|=\sqrt{x^Tx}$ denotes the Euclidean norm. Let $\overline{\lambda}(A)$ ($\underline{\lambda}(A)$) denote the maximal (minimum) eigenvalue of a positive definite matrix $A\in\mathbb{R}^{n\times n}$. Let $|b|$ and $\sigma(B)$ be the absolute value of a real number $b\in\mathbb{R}$ and the maximal singular value of a matrix $B\in\mathbb{R}^{m\times n}$, respectively.

## 2. Preliminaries

### 2.1. Graph theory

Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{A})$ denote a weighted directed graph of order $N$, where $\mathcal{V}=\{v_1,\dots,v_N\}$ and $\mathcal{E}\subseteq\mathcal{V}\times\mathcal{V}$ are, respectively, the set of nodes and the set of directed edges, and $\mathcal{A}=[a_{ij}]_{N\times N}$ is a weighted adjacency matrix with weights $a_{ij}>0$ if $(v_j,v_i)\in\mathcal{E}$ and $a_{ij}=0$ otherwise. Moreover, it is assumed that $a_{ii}=0$, $\forall i\in\mathcal{I}=\{1,\dots,N\}$. The edge $(v_j,v_i)\in\mathcal{E}$ means that the node $v_i$ can access information from $v_j$, then node $v_j$ is a neighbor of node $v_i$. The set of neighbors of node $v_i$ is denoted as $\mathcal{N}_i=\{j\in\mathcal{V}:(v_j,v_i)\in\mathcal{E}\}$. The Laplacian matrix $L=[l_{ij}]_{N\times N}$ is given by $l_{ij}=-a_{ij}$ if $i\neq j$ and $l_{ii}=\sum_{k=1}^N a_{ik}$. In the directed graph $\mathcal{G}$, a directed path from node $v_j$ to node $v_i$ is a sequence of edges $e_{i_1j}$, $e_{i_2i_1}$, …, $e_{ii_l}$ with nodes $v_{i_k}\in\mathcal{V}$, $k=1,2,\dots,l$. A directed graph is said to be strongly connected if there exists a directed path from every node to all other nodes.

**Lemma 1** [3]**.** *If the directed graph $\mathcal{G}$ is strongly connected, there exists a positive vector $\xi=[\xi_1,\dots,\xi_N]^T$ with $\mathbf{1}_N^T\xi=1$ such that $\xi^TL=0$. Moreover, for $\varsigma\in\mathbb{R}^N$ and any positive column vector $\vartheta\in\mathbb{R}^N$, it holds that*

$$\min_{\varsigma^T\vartheta=0}\varsigma^T\widehat{L}\varsigma>\frac{\lambda_2(\widehat{L})}{N}\varsigma^T\varsigma,$$

*where $\widehat{L}=\frac{1}{2}(\Xi L+L^T\Xi)$, $\Xi=\mathrm{diag}(\xi_1,\dots,\xi_N)$, and $\lambda_2(\widehat{L})$ denotes the minimum nonzero eigenvalue of $\widehat{L}$.*

### 2.2. M-*matrices*

**Definition 1** [29]**.** A nonsingular matrix $H=[h_{ij}]\in\mathbb{R}^{n\times n}$ is called an $M$-matrix if $h_{ij}\leq 0$ whenever $i\neq j$ and all elements of $H^{-1}$ are nonnegative.

**Lemma 2** [30]. *If $H = [h_{ij}] \in \mathbb{R}^{n\times n}$ is an M-matrix, then there exists a positive diagonal matrix $\Theta$ such that $\Theta H + H^T\Theta > 0$. One such $\Theta$ is give by $\mathrm{diag}(\theta_1, \ldots, \theta_N)$, where $[\theta_1, \ldots, \theta_N]^T = (H^T)^{-1}\mathbf{1}_N$.*

### 2.3. Caputo fractional derivative and Mittag–Leffler function

**Definition 2** [22]. The Riemann–Liouville fractional integral of function $f(t) \in C^n([0, \infty), \mathbb{R})$ is defined as

$$I^q f(t) = I^q[f(\cdot)](t) = \int_0^t \frac{(t-\tau)^{q-1}}{\Gamma(q)} f(\tau)d\tau,$$

where $n-1 < q \le n$, $n \in \{1, 2, \ldots\}$ and $\Gamma(\cdot)$ is the well-known Gamma function, defined as $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$.

**Definition 3** [22]. The Caputo fractional derivative of function $f(t) \in C^n([0, \infty), \mathbb{R})$ is defined as

$$D^q f(t) = I^{n-q} f^{(n)}(t) = \int_0^t \frac{(t-\tau)^{n-q-1}}{\Gamma(n-q)} f^{(n)}(\tau)d\tau,$$

where $n-1 < q \le n$, $n \in \{1, 2, \ldots\}$.

**Definition 4** [22]. The Mittag–Leffler function with two positive parameter $a$ and $b$ is defined as

$$E_{a,b}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(ka+b)},$$

where $z$ is a complex number. Let $E_{a,1}(z) = E_a(z)$ as $b = 1$, further, $E_{1,1}(z) = e^z$.

**Lemma 3** [22]. *If $a<2$, $b$ is an arbitrary real number, $\mu$ satisfies $a\pi/2<\mu<\min\{\pi, a\pi\}$and $C$ is a positive real constant, then the estimate of Mittag–Leffler function is*

$$|E_{a,b}(z)| \le \frac{C}{1+|z|}, \ \mu \le |\arg(z)| \le \pi, |z| \ge 0, \tag{1}$$

*where $\arg(z)$ denotes the argument of complex number $z$.*

**Lemma 4** [22]. *Let $0<q\le 1$, $p \in \mathbb{R}$, and $h(t)$ be a given continuous function. The solution of the initial value problem $D^q x(t) = px(t) + h(t)$ can be expressed by*

$$x(t) = x(t_0)E_q(p(t-t_0)^q) + \int_{t_0}^t (t-\tau)^{q-1}E_{q,q}(p(t-\tau)^q)h(\tau)d\tau. \tag{2}$$

*In particular, if $q = 1$, then $x(t) = x(t_0)e^{p(t-t_0)} + \int_{t_0}^t e^{p(t-\tau)}h(\tau)d\tau$.*

Some properties of the Mittag–Leffler function are given below, which can be seen in [31].

**Property 1.** *If $0<a\le 1$, $b>0$ and $t>0$, then $0 < E_a(-bt^a) < 1$ and $E_{a,a}(-bt^a) > 0$. In addition, $t^aE_{a,a+1}(-bt^a)$ is the monotone increasing function satisfying $0 < t^aE_{a,a+1}(-bt^a) < \frac{1}{b}$.*

### 2.4. Some lemmas

**Lemma 5** [32]. *Let $x(t) \in \mathbb{R}^n$ be a real continuous and differentiable vector function. Then, for any time instant $t \geq t_0$, the following relationship holds:*

$$\frac{1}{2}D^q\left[x^T(t)Px(t)\right] \leq x^T(t)P[D^q x(t)], \quad 0 < q \leq 1, \tag{3}$$

*where $P \in \mathbb{R}^{n\times n}$ is a positive definite symmetric matrix.*

**Lemma 6** (Schur Complement [33]). *The following linear matrix inequality (LMI)*

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0,$$

*where $Q(x) = Q^T(x), R(x) = R^T(x)$, is equivalent to either of the following conditions:*

1) $Q(x) > 0, R(x) - S^T(x)Q^{-1}(x)S(x) > 0;$
2) $R(x) > 0, Q(x) - S(x)R^{-1}(x)S^T(x) > 0.$

## 3. Output feedback based leaderless consensus control

We first consider the leaderless consensus problem for a group of $N$ agents distributed on a communication graph $\mathcal{G}$. The dynamics of each agent is described by

$$\begin{cases} D^q x_i = w_i, \\ D^q w_i = f(t, x_i, w_i) + u_i, \\ y_i = x_i, \quad i \in \mathcal{I} = \{1, 2, \dots, N\}, \end{cases} \tag{4}$$

where $0 < q \leq 1$ $x_i \in \mathbb{R}^n$, $w_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^n$ and $y_i \in \mathbb{R}^n$ denote, respectively, the position, velocity, control input and output of agent $i$, and $f(t, x_i, w_i) \in \mathbb{R}^n$ is the intrinsic nonlinear dynamics of agent $i$.

**Assumption 1.** Suppose that there exist two constants $\rho_1$ and $\rho_2$ such that, $\forall x, v, y, z \in \mathbb{R}^n$ and $t \geq 0$

$$\|f(t, x, v) - f(t, y, z)\| \leq \rho_1\|x - y\| + \rho_2\|v - z\|. \tag{5}$$

**Remark 1.** Note that the Lipschitz assumption condition in Assumption 1 is mild, see [2,3,5]. It is easy to verify that a class of Chaotic systems satisfy this assumption, such as Chua's oscillator, Chua's circuit, Chen system and Lorenz system, to name a few. Due to the agents often have the same dynamics in a MAS, each agent has exactly the same intrinsic nonlinear dynamics is reasonable in this paper.

In this section, the control objective is to design $u_i$ using only local output information such that the leaderless consensus is achieved asymptotically, that is $\lim_{t\to\infty}\|x_i(t) - x_j(t)\| = \lim_{t\to\infty}\|D^q x_i(t) - D^q x_j(t)\| = 0$ holds for any initial values, $\forall i, j \in \mathcal{I}$.

It is clear that Lyapunov's direct method provides a powerful tool to analyze the stability of nonlinear system. For a continuous function $V(t) : [0, \infty) \to \mathbb{R}$, if $\dot{V}(t) \leq -k_1 V(t) + k_2$, where $k_1$ and $k_2$ are constants satisfying $k_1 > 0$ and $k_2 \geq 0$, by using the Comparison

Lemma (Lemma 3.4 in [34]), the following inequality is derived:

$$V(t) \le \left(V(0) - \frac{k_2}{k_1}\right)e^{-k_1 t} + \frac{k_2}{k_1} \le V(0)e^{-k_1 t} + \frac{k_2}{k_1}. \tag{6}$$

If $D^q V(t) \le -k_1 V(t) + k_2$, however, the above inequality (6) is no longer applicable. To analyze the nonlinear fractional-order system stability, we need to extend the first-order inequality (6) to the case of fractional-order.

**Lemma 7.** *If $0 < q \le 1$ and the q-order derivative of a continuous function $V(t) : [0, \infty) \to \mathbb{R}$ satisfying*

$$D^q V(t) \le -k_1 V(t) + k_2 \tag{7}$$

*with $k_1 > 0$ and $k_2 \ge 0$ being two constants, then*

$$V(t) \le V(0)E_q(-k_1 t^q) + \frac{k_2}{k_1}, \quad \forall t \ge 0. \tag{8}$$

**Proof.** From Eq. (7), we have that there exists a function $M(t) \ge 0$ such that

$$D^q V(t) = -k_1 V(t) + k_2 - M(t). \tag{9}$$

According to Lemma 4, the solution of Eq. (9) is written as

$$V(t) = V(0)E_q(-k_1 t^q) + \int_0^t (t-\tau)^{q-1} E_{q,q}(-k_1 (t-\tau)^q)[k_2 - M(\tau)]d\tau. \tag{10}$$

In virtue of Property 1, one has that $E_{q,q}(-k_1 t^q) > 0$ and $0 < t^q E_{q,q+1}(-k_1 t^q) < \frac{1}{k_1}$. It thus follows from Eq. (10) that

$$\begin{aligned} V(t) &\le V(0)E_q(-k_1 t^q) + k_2 \int_0^t (t-\tau)^{q-1} E_{q,q}(-k_1 (t-\tau)^q) d\tau \\ &= V(0)E_q(-k_1 t^q) + k_2 t^q E_{q,q+1}(-k_1 t^q) \\ &\le V(0)E_q(-k_1 t^q) + \frac{k_2}{k_1}, \end{aligned} \tag{11}$$

where the first equality holds due to the fact in [22,31] that $\int_0^t E_{a,b}(ps^a)s^{b-1}ds = t^b E_{a,b+1}(pt^a)$ with $a > 0$, $b > 0$ and $p \in \mathbb{R}$ being some constants. □

**Remark 2.** It is worth mentioning that when $q = 1$, Lemma 7 reduces to the first-order inequality (6) in [34]. By using Lemma 3, we have that $E_q(-k_1 t^q) \to 0$ as $t \to \infty$, so $V(t) \to \frac{k_2}{k_1}$ as $t \to \infty$. When $k_2 = 0$, Lemma 7 reduces to the fractional-order inequality in [14], in this case $V(t) \to 0$ as $t \to \infty$. Thus, the new proposed Lemma 7 can be used to analyze the convergence problem of fractional-order system, which generalizes the results in [14,34].

To derive the main results of this section, we need the following assumption.

**Assumption 2.** The directed graph $\mathcal{G}$ is strongly connected.

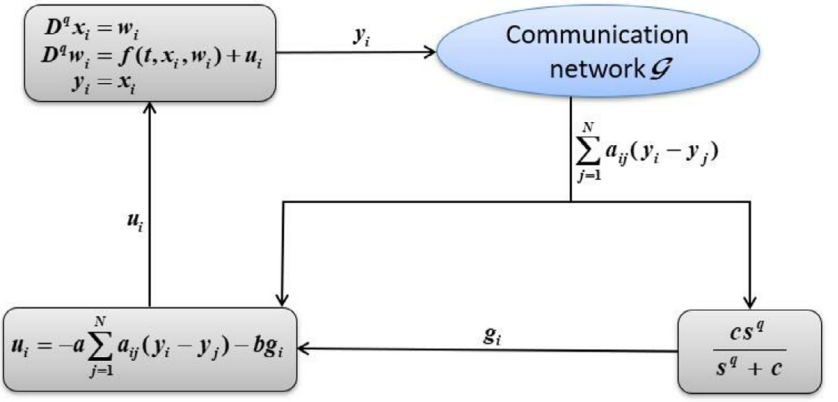


Fig. 1. Framework of closed-loop system.

Consider the leaderless consensus of fractional-order MASs (4) under the following control law:

$$\begin{cases} u_i = -a\sum_{j=1}^{N} a_{ij}(y_i - y_j) - bg_i, \\ g_i = c\sum_{j=1}^{N} a_{ij}(y_i - y_j) - z_i, \\ D^q z_i = cg_i, \end{cases} \tag{12}$$

where $g_i \in \mathbb{R}^n$ and $z_i \in \mathbb{R}^n$ are the filter output and an auxiliary filter vector, respectively, $a$, $b$, $c$ are positive design parameters to be determined later.

**Remark 3.** The framework of closed-loop system is shown in Fig. 1. It is readily seen from Fig. 1 that only relative positive coupling measurements between neighbors are utilized in the algorithm (12). This is in contrast to the existing consensus algorithms in [14,15,20,21] for fractional-order MASs with double-integrator dynamics, where not only the relative positive coupling measurements but also the relative velocity coupling measurements are utilized in the algorithm design. Due to the limitation of communication capability, the relative velocity measurements are general very difficult to obtain in practice for second-order MASs, especially for fractional-order MASs with double-integrator dynamics.

For simplicity in this paper, we assume that $n = 1$. All the results can be extended to the case as $n > 1$ by using the Kronecker product.

**Theorem 1.** *Under Assumptions 1 and 2, the leaderless consensus of fractional-order MASs (4) using protocol (12) is achieved if*

$$\gamma_1 = \frac{a^2}{N}\lambda_2(\widehat{L}) - \left(a\rho_1 + \frac{a\rho_2 + b\rho_1}{2}\right)\overline{\xi} > 0, \tag{13}$$

$$\gamma_2 = \frac{b^2}{N}\lambda_2(\widehat{L}) - \left(a + b\rho_2 + \frac{a\rho_2 + b\rho_1}{2}\right)\overline{\xi} > 0, \tag{14}$$

$$c > b\sigma(L) + \frac{\gamma_3^2}{\gamma_1} + \frac{\gamma_4^2}{\gamma_2}, \tag{15}$$

*where* $\gamma_3 = -\frac{ab\bar{\xi}}{2} - \frac{a}{2}\sigma(L^2) - \frac{\sqrt{2}\rho_1}{2}\sigma(L)$, $\gamma_4 = -\frac{b^2\bar{\xi}}{2} - \frac{b}{2}\sigma(L^2) - \frac{\sqrt{2}\rho_2}{2}\sigma(L)$, $\widehat{L}$ *defined as in Lemma 1 and* $\bar{\xi} = \max_{i\in\mathcal{I}}\{\xi_i\}$. *Moreover,* $u_i$ *in (12) is bounded and continuous everywhere.*

**Proof.** Introduce some error variables $\hat{x}_i = x_i - \bar{x}$ and $\hat{w}_i = w_i - \bar{w}$, where $\bar{x} = \sum_{k=1}^N \xi_k x_k$, $\bar{w} = \sum_{k=1}^N \xi_k w_k$, and $\xi_i$ is clearly defined in Lemma 1. For notational brevity, let $\hat{x}$, $\hat{w}$, $u$, $f$ and $g$ be the column stack vectors of $\hat{x}_i$, $\hat{w}_i$, $u_i$, $f(t, x_i, w_i)$ and $g_i$, respectively. Since $\xi^T L = 0$, then we can get from Eqs. (4) and (12) that

$$\begin{cases} D^q\hat{x} = \hat{w}, \\ D^q\hat{w} = -aL\hat{x} + (I_N - \mathbf{1}_N\xi^T)(f - bg), \\ D^q g = cL\hat{w} - cg. \end{cases} \tag{16}$$

Choose $\phi = g - L\hat{w}$, it thus follows from Eq. (16) that

$$\begin{cases} D^q\hat{x} = \hat{w}, \\ D^q\hat{w} = -aL\hat{x} - bL\hat{w} + (I_N - \mathbf{1}_N\xi^T)(f - b\phi), \\ D^q\phi = -c\phi - LD^q\hat{w}. \end{cases} \tag{17}$$

Note that $L\mathbf{1}_N = \mathbf{0}_N$, we can further represent Eq. (17) as

$$\begin{cases} D^q e = Ae + G(t), \\ D^q\phi = -c\phi + aL^2\hat{x} + bL^2\hat{w} - L(f - b\phi), \end{cases} \tag{18}$$

where $e = [\hat{x}^T, \hat{w}^T]^T$, $A = \begin{bmatrix} O_N & I_N \\ -aL & -bL \end{bmatrix}$ and $G(t) = \begin{bmatrix} \mathbf{0}_N \\ (I_N - \mathbf{1}_N\xi^T)(f - b\phi) \end{bmatrix}$.

The Lyapunov function is constructed as follows:

$$V(t) = \frac{1}{2}\phi^T\phi + \frac{1}{2}e^T Pe, \tag{19}$$

where $P = \begin{bmatrix} 2ab\widehat{L} & a\Xi \\ a\Xi & b\Xi \end{bmatrix}$, $\widehat{L}$ and $\Xi$ are clearly defined in Lemma 1. According to Lemma 1, one gets $\hat{x}^T\widehat{L}\hat{x} > \frac{\lambda_2(\widehat{L})}{N}\hat{x}^T\hat{x} \geq \frac{\lambda_2(\widehat{L})}{N\bar{\xi}}\hat{x}^T\Xi\hat{x}$, so

$$V(t) \geq \frac{1}{2}\phi^T\phi + \frac{1}{2}e^T(\widehat{P}\otimes\Xi)e,$$

with $\widehat{P} = \begin{bmatrix} \frac{2ab}{N\bar{\xi}}\lambda_2(\widehat{L}) & a \\ a & b \end{bmatrix}$. By using Lemma 6, we can derive that $\widehat{P} > 0$ if $\lambda_2(\widehat{L}) > \frac{a}{2b^2}N\bar{\xi}$. It follows from Eq. (14) that $\lambda_2(\widehat{L}) > \frac{a}{b^2}N\bar{\xi} > \frac{a}{2b^2}N\bar{\xi}$, thus $P \geq \widehat{P}\otimes\Xi > 0$. Clearly, $V(t) \geq 0$ and $V(t) = 0$ if and only if $\hat{x} = \hat{w} = \phi = \mathbf{0}_N$.

Let $f_* = f(t, \bar{x}, \bar{w})$ and $Q = \frac{1}{2}(PA + A^T P)$. In virtue of Lemma 5, taking the $q$-order derivative of $V$ yields

$$
\begin{aligned}
D^q V(t) &\le \phi^T D^q \phi + e^T P D^q e \\
&= \phi^T D^q \phi + e^T \frac{PA + A^T P}{2} e + \left(a\hat{x}^T + b\hat{w}^T\right)\Xi(I_N - \mathbf{1}_N \xi^T)(f - b\phi) \\
&= \phi^T D^q \phi + e^T Q e + \left(a\hat{x}^T + b\hat{w}^T\right)\Xi(f - \mathbf{1}_N \otimes f_* - b\phi),
\end{aligned} \tag{20}
$$

where the last equality holds due to

$$
\begin{aligned}
\left(a\hat{x}^T + b\hat{w}^T\right)\Xi \mathbf{1}_N \xi^T (f - b\phi) &= \left(ax^T + bw^T\right)\left(I_N - \xi \mathbf{1}_N^T\right)\Xi \mathbf{1}_N \xi^T (f - b\phi) \\
&= \left(ax^T + bw^T\right)\left(\xi\xi^T - \xi\xi^T\right)(f - b\phi) = 0
\end{aligned}
$$

and

$$
\begin{aligned}
\left(a\hat{x}^T + b\hat{w}^T\right)\Xi(\mathbf{1}_N \otimes f_*) &= \left(ax^T + bw^T\right)\left(I_N - \xi \mathbf{1}_N^T\right)\Xi\left[\mathbf{1}_N \otimes f_*\right] \\
&= \left(ax^T + bw^T\right)(\xi - \xi) \otimes f_* = 0.
\end{aligned}
$$

In view of $\widehat{L} = \frac{1}{2}(\Xi L + L^T \Xi)$, one has

$$
Q = \frac{1}{2}(PA + A^T P) = \begin{bmatrix} -a^2 \widehat{L} & O_N \\ O_N & a\Xi - b^2 \widehat{L} \end{bmatrix}. \tag{21}
$$

Since $L\mathbf{1}_N = \mathbf{0}_N$, one can obtain from Eq. (18) that

$$
\begin{aligned}
\phi^T D^q \phi &= \phi^T \left[-c\phi + aL^2 \hat{x} + bL^2 \hat{w} - L(f - b\phi)\right] \\
&= -c\phi^T \phi + a\phi^T L^2 \hat{x} + b\phi^T L^2 \hat{w} - \phi^T L(f - \mathbf{1}_N \otimes f_* - b\phi) \\
&\le -c\|\phi\|^2 + a\sigma(L^2)\|\phi\|\|\hat{x}\| + b\sigma(L^2)\|\phi\|\|\hat{w}\| \\
&\quad + \sigma(L)\|\phi\|\|f - \mathbf{1}_N \otimes f_*\| + b\sigma(L)\|\phi\|^2.
\end{aligned} \tag{22}
$$

It easy to get from Assumption 1 that

$$
\begin{aligned}
\|f - \mathbf{1}_N \otimes f_*\| &= \sqrt{\sum_{i=1}^{N} |f(t, x_i, w_i) - f(t, \bar{x}, \bar{w})|^2} \\
&\le \sqrt{\sum_{i=1}^{N} (\rho_1 |\hat{x}_i| + \rho_2 |\hat{w}_i|)^2} \\
&\le \sqrt{\sum_{i=1}^{N} 2\left(\rho_1^2 |\hat{x}_i|^2 + \rho_2^2 |\hat{w}_i|^2\right)} \\
&\le \sqrt{\sum_{i=1}^{N} 2\rho_1^2 |\hat{x}_i|^2} + \sqrt{\sum_{i=1}^{N} 2\rho_2^2 |\hat{w}_i|^2} \\
&= \sqrt{2}\rho_1 \|\hat{x}\| + \sqrt{2}\rho_2 \|\hat{w}\|
\end{aligned} \tag{23}
$$

and

$$\begin{aligned}&\left(a\hat{x}^T + b\hat{w}^T\right)\Xi(f - \mathbf{1}_N \otimes f_*)\\&= \sum_{i=1}^{N}\left(a\hat{x}_i + b\hat{w}_i\right)\xi_i[f(t, x_i, w_i) - f(t, \bar{x}, \bar{w})]\\&\le \sum_{i=1}^{N}\xi_i\left[a\rho_1|\hat{x}_i|^2 + b\rho_2|\hat{w}_i|^2 + (a\rho_2 + b\rho_1)|\hat{x}_i||\hat{w}_i|\right]\\&\le \sum_{i=1}^{N}\overline{\xi}\left(a\rho_1 + \frac{a\rho_2 + b\rho_1}{2}\right)|\hat{x}_i|^2 + \sum_{i=1}^{N}\overline{\xi}\left(b\rho_2 + \frac{a\rho_2 + b\rho_1}{2}\right)|\hat{w}_i|^2\\&= e^T Be, \qquad (24)\end{aligned}$$

where

$$B = \begin{bmatrix} a\rho_1 + \dfrac{a\rho_2 + b\rho_1}{2} & 0 \\ 0 & b\rho_2 + \dfrac{a\rho_2 + b\rho_1}{2} \end{bmatrix} \otimes \overline{\xi}.$$

Combining Eqs. (20)–(24), one can conclude that

$$\begin{aligned}D^q V(t) \le& -\frac{a^2\lambda_2(\widehat{L})}{N}\|\hat{x}\|^2 + \left(a\overline{\xi} - \frac{b^2\lambda_2(\widehat{L})}{N}\right)\|\hat{w}\|^2 + e^T Be + (b\sigma(L) - c)\|\phi\|^2\\&+ \left(ab\overline{\xi} + a\sigma(L^2) + \sqrt{2}\rho_1\sigma(L)\right)\|\hat{x}\|\|\phi\|\\&+ \left(b^2\overline{\xi} + b\sigma(L^2) + \sqrt{2}\rho_2\sigma(L)\right)\|\hat{w}\|\|\phi\|\\ \le& -[\|\hat{x}\|, \|\hat{w}\|, \|\phi\|]C[\|\hat{x}\|, \|\hat{w}\|, \|\phi\|]^T, \qquad (25)\end{aligned}$$

with

$$C = \begin{bmatrix} \gamma_1 & 0 & \gamma_3 \\ 0 & \gamma_2 & \gamma_4 \\ \gamma_3 & \gamma_4 & c - b\sigma(L) \end{bmatrix}$$

where $\gamma_i$ $(i = 1, 2, 3, 4)$ defined as in Theorem 1. It follows from Lemma 6 that $C > 0$ if

$$\begin{cases} \gamma_1, \gamma_2 > 0, \\ c - b\sigma(L) - \begin{bmatrix}\gamma_3 & \gamma_4\end{bmatrix}\begin{bmatrix} \dfrac{1}{\gamma_1} & 0 \\ 0 & \dfrac{1}{\gamma_2} \end{bmatrix}\begin{bmatrix}\gamma_3 \\ \gamma_4\end{bmatrix} > 0. \end{cases} \qquad (26)$$

Clearly $C > 0$ if inequalities (13)–(15) hold. It follows from (19) and (25) that

$D^q V(t) \le -\underline{\lambda}(C)(\|e\|^2 + \|\phi\|^2) \le -\theta V(t),$

where $\theta = \frac{2\underline{\lambda}(C)}{\max\{\overline{\lambda}(P), 1\}} > 0$. By using Lemma 7, we have $V(t) \le V(0)E_q(-\theta t^q)$, $t \ge 0$. It thus follows from Eq. (25) that

$$\|e\| \le \sqrt{\frac{2V(t)}{\underline{\lambda}(P)}} \le \sqrt{\frac{2V(0)}{\underline{\lambda}(P)}E_q(-\theta t^q)} \to 0 \qquad (27)$$

as $t\to\infty$. Therefore, the fractional-order consensus is achieved in fractional-order MASs (4) with the protocol (12).

From Property 1, one gets $V(t)\leq V(0)E_q(-\theta t^q)\leq V(0)$ for $t\geq 0$. Due to $u=-aL\hat{x}-bg$ and $g=\phi+L\hat{w}$, it holds that

$$\begin{aligned} u &\leq a\sigma(L)\|\hat{x}\|+b\big(\|\phi\|+\sigma(L)\|\hat{w}\|\big)\\ &\leq (a+b)\sigma(L)\|e\|+b\|\phi\|\\ &\leq (a+b)\sigma(L)\sqrt{\frac{2V(0)}{\underline{\lambda}(P)}}+b\sqrt{2V(0)} \end{aligned} \tag{28}$$

for $t\geq 0$ and $u\to \mathbf{0}_N$ as $t\to\infty$. Thus $u$ is bounded and continuous everywhere. This completes the proof. □

**Remark 4.** The conditions (13) and (14) are equivalent to

$$\frac{\lambda_2(\widehat{L})}{N\overline{\xi}}>\max\left\{\frac{\rho_1}{a}+\frac{\rho_2}{2a}+\frac{b\rho_1}{2a^2},\frac{\rho_2}{b}+\frac{\rho_1}{2b}+\frac{a\rho_2}{2b^2}+\frac{a}{b^2}\right\}. \tag{29}$$

It should be noted that $\frac{\rho_1}{a}+\frac{\rho_2}{2a}+\frac{b\rho_1}{2a^2}\to 0^+$ and $\frac{\rho_2}{b}+\frac{\rho_1}{2b}+\frac{a\rho_2}{2b^2}+\frac{a}{b^2}\to 0^+$ as $a,b\to+\infty$. Therefore, one can always choose two sufficiently large parameters $a$ and $b$ such that Eqs. (13) and (14) hold. Also, one can always choose a large parameter $c$ such that Eq. (15) holds. In particular, if $b=a$, one can choose a large parameter $a$ satisfying $a>\frac{N\overline{\xi}}{\lambda_2(\widehat{L})}\max\left\{\frac{3\rho_1+\rho_2}{2},\frac{\rho_1+3\rho_2}{2}+1\right\}$ such that Eqs. (13) and (14) hold.

As given in Remark 1, although many systems satisfy the Lipschitz-type nonlinear condition in Assumption 1. In the following part of this section, a more relaxed assumption given as below is considered.

**Assumption 3.** Suppose that there exist two constants $\rho_1$ and $\rho_2$ such that, $\forall x,v,y,z\in\mathbb{R}^n$ and $t\geq 0$

$$\|f(t,x,v)-f(t,y,z)\|\leq\rho_1\|x-y\|+\rho_2\|v-z\|+\eta(t), \tag{30}$$

where $\eta(t)\geq 0$ is continuous and bounded.

Compared with Assumption 1, the condition (30) is more relaxed with an additional bound $\eta(t)$. For example, the systems can be those in Remark 1 with perturbations resulting from uncertainties, modeling errors, process noise and disturbances that exist in many realistic problems. Under Assumption 3, the following theorem shows that both positive and velocity consensus errors $\hat{x}$ and $\hat{w}$ are globally UUB.

**Theorem 2.** *Under Assumptions 2 and 3, both position and velocity consensus errors of fractional-order MASs (4) using protocol (12) are globally UUB if*

$$\delta_1=\frac{a^2}{N}\lambda_2(\widehat{L})-(2a\rho_1+a\rho_2+b\rho_1)\overline{\xi}>0, \tag{31}$$

$$\delta_2=\frac{b^2}{N}\lambda_2(\widehat{L})-(a+2b\rho_2+a\rho_2+b\rho_1)\overline{\xi}>0, \tag{32}$$

$$c>b\sigma(L)+\frac{\delta_3^2}{\delta_1}+\frac{\delta_4^2}{\delta_2}, \tag{33}$$

*where* $\delta_3 = -\frac{ab\overline{\xi}}{2} - \frac{a}{2}\sigma(L^2) - \rho_1\sigma(L)$, $\delta_4 = -\frac{b^2\overline{\xi}}{2} - \frac{b}{2}\sigma(L^2) - \rho_2\sigma(L)$, $\widehat{L}$ *defined as in Lemma 1 and* $\overline{\xi} = \max_{i\in\mathcal{I}}\{\xi_i\}$*. Moreover,* $u_i$ *in (12) is bounded and continuous everywhere.*

**Proof.** Consider the same Lyapunov function candidate $V(t)$ as in Eq. (19). Under Assumption 3, we can derive that

$$\begin{aligned}\|f - \mathbf{1}_N \otimes f_*\| &= \sqrt{\sum_{i=1}^{N} |f(t, x_i, w_i) - f(t, \bar{x}, \bar{w})|^2} \\ &\leq \sqrt{\sum_{i=1}^{N} [\rho_1|\hat{x}_i| + \rho_2|\hat{w}_i| + \eta]^2} \\ &\leq \sqrt{\sum_{i=1}^{N} \left[4\left(\rho_1^2|\hat{x}_i|^2 + \rho_2^2|\hat{w}_i|^2\right) + 2\eta^2\right]} \\ &\leq 2\rho_1\|\hat{x}\| + 2\rho_2\|\hat{w}\| + \sqrt{2N}\eta, \end{aligned} \tag{34}$$

where we have used the fact that $(x + y + z)^2 \leq 2[(x + y)^2 + z^2] \leq 4(x^2 + y^2) + 2z^2$ for $x, y, z \in \mathbb{R}$. In light of Eq. (34), we can further derive that

$$\begin{aligned}\left(a\hat{x}^T + b\hat{w}^T\right)\Xi(f - \mathbf{1}_N \otimes f_*) &\leq \|a\hat{x} + b\hat{w}\|\sigma(\Xi)\|f - \mathbf{1}_N \otimes f_*\| \\ &\leq \overline{\xi}\left(a\|\hat{x}\| + b\|\hat{w}\|\right)\left(2\rho_1\|\hat{x}\| + 2\rho_2\|\hat{w}\| + \sqrt{2N}\eta\right) \\ &\leq (2a\rho_1 + a\rho_2 + b\rho_1)\overline{\xi}\|\hat{x}\|^2 + (2b\rho_2 + a\rho_2 + b\rho_1)\overline{\xi}\|\hat{w}\|^2 \\ &\quad + \sqrt{2N}\overline{\xi}\left(a\|\hat{x}\| + b\|\hat{w}\|\right)\eta. \end{aligned} \tag{35}$$

Combining Eqs. (20)–(22), (34) and (35) yields

$$\begin{aligned} D^q V(t) \leq &- \left(\frac{a^2}{N}\lambda_2(\widehat{L}) - (2a\rho_1 + a\rho_2 + b\rho_1)\overline{\xi}\right)\|\hat{x}\|^2 \\ &- \left(\frac{b^2}{N}\lambda_2(\widehat{L}) - (a + 2b\rho_2 + a\rho_2 + b\rho_1)\overline{\xi}\right)\|\hat{w}\|^2 \\ &+ (b\sigma(L) - c)\|\phi\|^2 + \left(ab\overline{\xi} + a\sigma(L^2) + 2\rho_1\sigma(L)\right)\|\hat{x}\|\|\phi\| \\ &+ \left(b^2\overline{\xi} + b\sigma(L^2) + 2\rho_2\sigma(L)\right)\|\hat{w}\|\|\phi\| \\ &+ \sqrt{2N}\overline{\xi}\left(a\|\hat{x}\| + b\|\hat{w}\|\right)\eta + \sqrt{2N}\sigma(L)\|\phi\|\eta \\ = &- [\|\hat{x}\|, \|\hat{w}\|, \|\phi\|]\Delta[\|\hat{x}\|, \|\hat{w}\|, \|\phi\|]^T \\ &+ \sqrt{2N}\overline{\xi}\left(a\|\hat{x}\| + b\|\hat{w}\|\right)\eta + \sqrt{2N}\sigma(L)\|\phi\|\eta \end{aligned} \tag{36}$$

with $\Delta = \begin{bmatrix} \delta_1 & 0 & \delta_3 \\ 0 & \delta_2 & \delta_4 \\ \delta_3 & \delta_4 & c - b\sigma(L) \end{bmatrix}$, where $\delta_i$ $(i = 1, 2, 3, 4)$ defined as in Theorem 2. It follows from Lemma 6 that $\Delta > 0$ if conditions (31)–(33) hold. By using the Young's inequality, i.e., $2xy \leq ax^2 + \frac{1}{a}y^2$ for any $a > 0$, one has

$$\sqrt{2N}a\overline{\xi}\|\hat{x}\|\eta \leq \frac{\underline{\lambda}(\Delta)}{2}\|\hat{x}\|^2 + \frac{a^2\overline{\xi}^2}{\underline{\lambda}(\Delta)}N\eta^2, \tag{37}$$

$$\sqrt{2N}b\bar{\xi}\|\hat{w}\|\eta \leq \frac{\underline{\lambda}(\Delta)}{2}\|\hat{w}\|^2 + \frac{b^2\bar{\xi}^2}{\underline{\lambda}(\Delta)}N\eta^2, \tag{38}$$

$$\sqrt{2N}\sigma(L)\|\phi\|\eta \leq \frac{\underline{\lambda}(\Delta)}{2}\|\phi\|^2 + \frac{\sigma^2(L)}{\underline{\lambda}(\Delta)}N\eta^2. \tag{39}$$

In virtue of Eqs. (37) and (38), we can further represent Eq. (36) as

$$D^qV(t) \leq -\frac{\underline{\lambda}(\Delta)}{2}(\|e\|^2 + \|\phi\|^2) + \frac{(a^2+b^2)\bar{\xi}^2 + \sigma^2(L)}{\underline{\lambda}(\Delta)}N\eta^2,$$

this together with Eq. (19) ensure that

$$D^qV(t) \leq -k_1V(t) + k_2, \tag{40}$$

where $k_1 = \frac{\underline{\lambda}(\Delta)}{\max\{\bar{\lambda}(P),1\}} > 0$, $k_2 = \frac{(a^2+b^2)\bar{\xi}^2+\sigma^2(L)}{\underline{\lambda}(\Delta)}N\bar{\eta}^2$ and $\bar{\eta} = \max_{t\geq 0}\eta(t)$. It thus follows from Lemma 7 that $V(t) \leq V(0)E_q(-k_1t^q) + \frac{k_2}{k_1}$, which means that

$$\begin{aligned}\|e\| &\leq \sqrt{\frac{2V(t)}{\underline{\lambda}(P)}} = \sqrt{\frac{2}{\underline{\lambda}(P)}\left(V(0)E_q(-k_1t^q) + \frac{k_2}{k_1}\right)} \\ &\leq \sqrt{\frac{2}{\underline{\lambda}(P)}\left(V(0) + \frac{k_2}{k_1}\right)},\ t\geq 0,\end{aligned} \tag{41}$$

and $\|e\| \leq \sqrt{\frac{2k_2}{k_1\underline{\lambda}(P)}}$ as $t\to\infty$. Therefore, for all $i\in\mathcal{I}$, the consensus errors $\hat{x}_i$ and $\hat{w}_i$ are cooperative UUB. Similar to the proof procedure of Theorem 1, we can conclude that $u_i$ is continuous and bounded. This completes the proof. □

**Remark 5.** In some practical applications, the state consensus error can hardly converge to zero exactly under the influences of uncertainties, modeling errors, process noise and disturbances. If both of the position and velocity consensus errors converge to a small residual set around the origin asymptotically, then fractional-order MASs (4) is said to achieve the consensus with small residual error, which is acceptable in most practical circumstances. It follows from Eq. (41) that the consensus error $e$ is globally UUB and can be made smaller by letting $k_2$ smaller, i.e., making $a$, $b$, $N$ and $\bar{\eta}$ smaller.

**Remark 6.** Different with the existing consensus algorithms in [14,15,20,21] for fractional-order MASs with double-integrator dynamics, the term associated with the relative velocity measurements $\sum_{j\in\mathcal{N}_i} a_{ij}(w_i - w_j)$ is replaced with a filter output $g_i$ in this paper. For any given initial value $z_i(0)$, $g_i$ can be calculated by Eq. (12) only using the relative position measurements, which resulting in a local output feedback based consensus algorithm. It is shown in Theorem 2 that the proposed distributed algorithm with filter can guarantee the global stability of the closed-loop system even if there exist bounded noise and disturbances. In fact, we can get from Eq. (41) that both positive and velocity consensus errors are not only UUB but also bounded at any time.

## 4. Output feedback based tracking control

In this section, we focus on the output consensus tracking problem for a fractional-order MAS consisting of $N$ followers and one leader. Let the leader indexed by 0 and all followers indexed by $1, \ldots, N$. The dynamics of each agent is governed by

$$\begin{cases} D^q x_i = w_i, \\ D^q w_i = f(t, x_i, w_i) + u_i, \\ y_i = x_i, \quad i \in \mathcal{I} \cup \{0\}, \end{cases} \tag{42}$$

where we consider a more general case that the leader's input $u_0$ is possibly time-varying and nonzero. Moreover, the leader's input information without being effected by those of the followers or unknown to any follower.

The control objective of this section is to design a control algorithm for Eq. (42) only using relative out measurements between neighbors such that both position and velocity tracking errors are globally UUB. Before addressing the consensus tracking problem, the following two assumptions on the leader's input and network topology among the $N+1$ agents are needed.

**Assumption 4.** There exists a constant $\rho_0$ such that $\|u_0\| \le \rho_0$, namely $u_0$ is bounded.

**Assumption 5.** Suppose that there exists a directed path from the leader to all other followers.

The output feedback based consensus tracking control algorithm for Eq. (42) is designed as

$$\begin{cases} u_i = -a\left[\sum_{j=1}^{N} a_{ij}(y_i - y_j) + d_i(y_i - y_0)\right] - bg_i, \\ g_i = c\left[\sum_{j=1}^{N} a_{ij}(y_i - y_j) + d_i(y_i - y_0)\right] - z_i, \\ D^q z_i = cg_i, \end{cases} \tag{43}$$

in which $d_i > 0$ whenever the follower $i$ is a neighbor of the leader, otherwise $d_i = 0$, and the rest of the variables are the same as in Eq. (12).

Let $\tilde{x}_i = x_i - x_0$ and $\tilde{w}_i = w_i - w_0$ represent the position and velocity tracking errors. After some manipulation, we can derive that

$$\begin{cases} D^q \tilde{x}_i = \tilde{w}_i, \\ D^q \tilde{w}_i = f(t, x_i, w_i) - f_0 - u_0 - a\sum_{j=1}^{N} h_{ij}\tilde{x}_j - bg_i, \\ D^q g_i = c\left(\sum_{j=1}^{N} h_{ij}\tilde{w}_j - g_i\right), \end{cases} \tag{44}$$

where $f_0 = f(t, x_0, w_0)$, $H = [h_{ij}]_{N\times N} = L + D$ and $D = \mathrm{diag}(d_1, \ldots, d_N)$. As shown in [29], $H$ is a nonsingular $M$-matrix if and only if Assumption 5 holds.

For brevity, let $\tilde{x}$, $\tilde{w}$, $f$ and $g$ be the column stack vectors of $\tilde{x}_i$, $\tilde{w}_i$, $f(t, x_i, w_i)$ and $g_i$, respectively. Choose $\varphi = g - H\tilde{w}$, it follows from Eq. (44) that

$$\begin{cases} D^q \tilde{x} = \tilde{w}, \\ D^q \tilde{w} = -aH\tilde{x} - bH\tilde{w} + f - \mathbf{1}_N \otimes (f_0 + u_0) - b\varphi, \\ D^q \varphi = -c\varphi + aH^2\tilde{x} + bH^2\tilde{w} - H\big[f - \mathbf{1}_N \otimes (f_0 + u_0) - b\varphi\big]. \end{cases} \tag{45}$$

**Theorem 3.** *Under Assumptions 3–5, both position and velocity consensus tracking errors of fractional-order MASs (42) using Eq. (43) are globally UUB if*

$$\zeta_1 = a^2\underline{\lambda}(\widehat{H}) - (2a\rho_1 + a\rho_2 + b\rho_1)\overline{\theta} > 0, \tag{46}$$

$$\zeta_2 = b^2\underline{\lambda}(\widehat{H}) - (a + 2b\rho_2 + a\rho_2 + b\rho_1)\overline{\theta} > 0, \tag{47}$$

$$c > b\sigma(H) + \frac{\zeta_3^2}{\zeta_1} + \frac{\zeta_4^2}{\zeta_2}, \tag{48}$$

*where* $\widehat{H} = \frac{1}{2}(\Theta H + H^T\Theta)$, $\Theta$ *defined as in Lemma 2,* $\overline{\theta} = \max_{i\in\mathcal{I}}\{\theta_i\}$, $\zeta_3 = -\frac{ab\overline{\theta}}{2} - \frac{a}{2}\sigma(H^2) - \rho_1\sigma(H)$ *and* $\zeta_4 = -\frac{b^2\overline{\theta}}{2} - \frac{b}{2}\sigma(H^2) - \rho_2\sigma(H)$. *Moreover,* $u_i$ *in Eq. (43) is bounded and continuous everywhere.*

**Proof.** Assumption 5 holds implies that $H$ is an $M$-matrix, this together with Lemma 2 ensure that there exists a positive definite diagonal matrix $\Theta = \text{diag}(\theta_1, \cdots, \theta_N)$ such that $\widehat{H} = \frac{1}{2}(\Theta H + H^T\Theta) > 0$, so $\underline{\lambda}(\widehat{H}) > 0$.

Choose a Lyapunov function

$$V(t) = \frac{1}{2}\tilde{e}^T\Omega\tilde{e} + \frac{1}{2}\varphi^T\varphi, \tag{49}$$

where $\tilde{e} = [\tilde{x}^T, \tilde{w}^T]^T$ and $\Omega = \begin{bmatrix} 2ab\widehat{H} & a\Theta \\ a\Theta & b\Theta \end{bmatrix}$. Under Assumptions 3 and 4, it thus follows from Eq. (34) that

$$\begin{aligned} \|f - \mathbf{1}_N \otimes (f_0 + u_0)\| &\leq \sqrt{\sum_{i=1}^{N} |f(t, x_i, w_i) - f_0 - u_0|^2} \\ &\leq \sqrt{\sum_{i=1}^{N} [|f(t, x_i, w_i) - f_0| + |u_0|]^2} \\ &\leq \sqrt{\sum_{i=1}^{N} [\rho_1|\tilde{x}_i| + \rho_2|\tilde{w}_i| + \eta + \rho_0]^2} \\ &\leq 2\rho_1\|\hat{x}\| + 2\rho_2\|\hat{w}\| + \sqrt{2N}(\eta + \rho_0). \end{aligned} \tag{50}$$

The result is then established by following the similar steps as those in Theorem 2. We thus omit it here for brevity. This completes the proof. □

**Remark 7.** Similar to the analysis procedure of Remark 4, one can always choose three large enough parameters $a$, $b$, $c$ such that conditions (46)–(48) hold. Although some global information ($\underline{\lambda}(\widehat{H})$, $\overline{\theta}$, $\sigma(H)$ and $\sigma(H^2)$) and the Lipschitz constants ($\rho_1$ and $\rho_2$) are used to determine the control parameters. In fact, the conditions for the parameters in Theorem 3 might be conservative. When implementing the control algorithm in practice, the control parameters can be tuned according to the performance of the whole system, which might actually be chosen much smaller than what are given in Theorem 3. So the accurate global graph information might not be needed in practice.

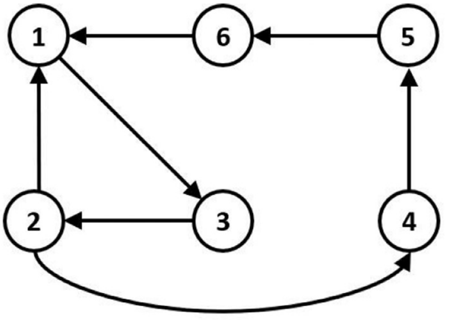


Fig. 2. Directed communication topology in Example 1.

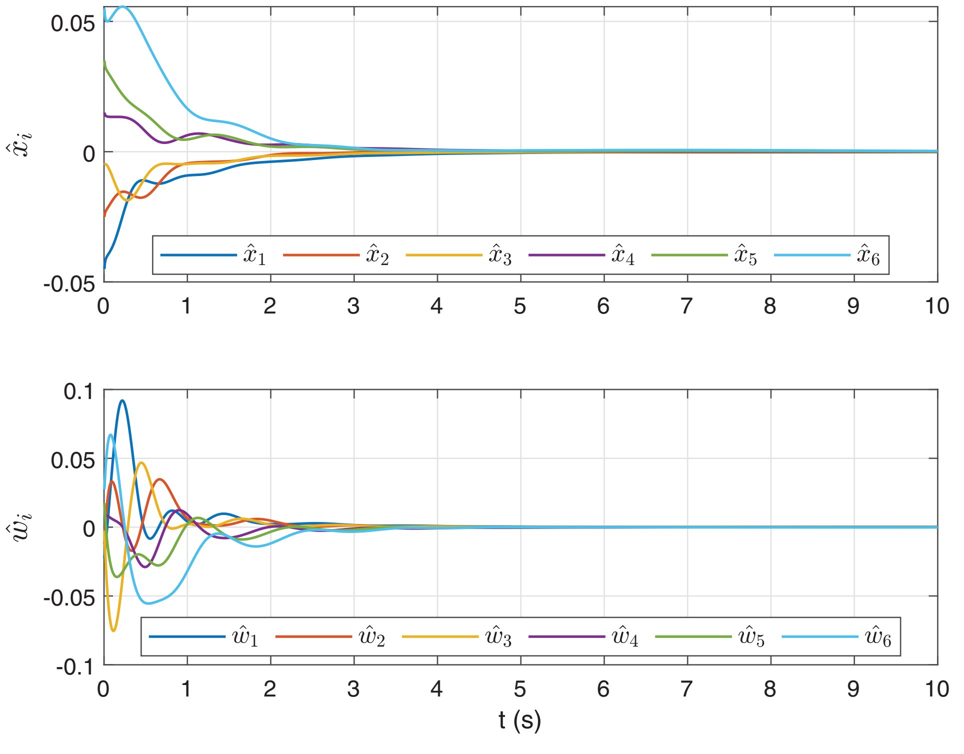


Fig. 3. Trajectories of consensus errors $\hat{x}_i$ and $\hat{w}_i$.

**Remark 8.** Note that the consensus tracking error $\tilde{e}$ in Theorem 3 depends on $\eta$ and $\rho_0$, that is decreasing $\eta$ and/or $\rho_0$ leads to a smaller error $\tilde{e}$. And specifically when $\eta = \rho_0 = 0$, $\tilde{e}$ will converge to zero asymptotically.

**Remark 9.** For the case where $q = 1$, the fractional-order MASs (42) reduces to the second-order MASs. And hence the existing consensus problem of second-order MASs in [2,6] can be regarded as a special case of this paper. Therefore, the results presented in this paper are extensions of the results studied in [2,6] to fractional-order MASs. What's more, the proposed

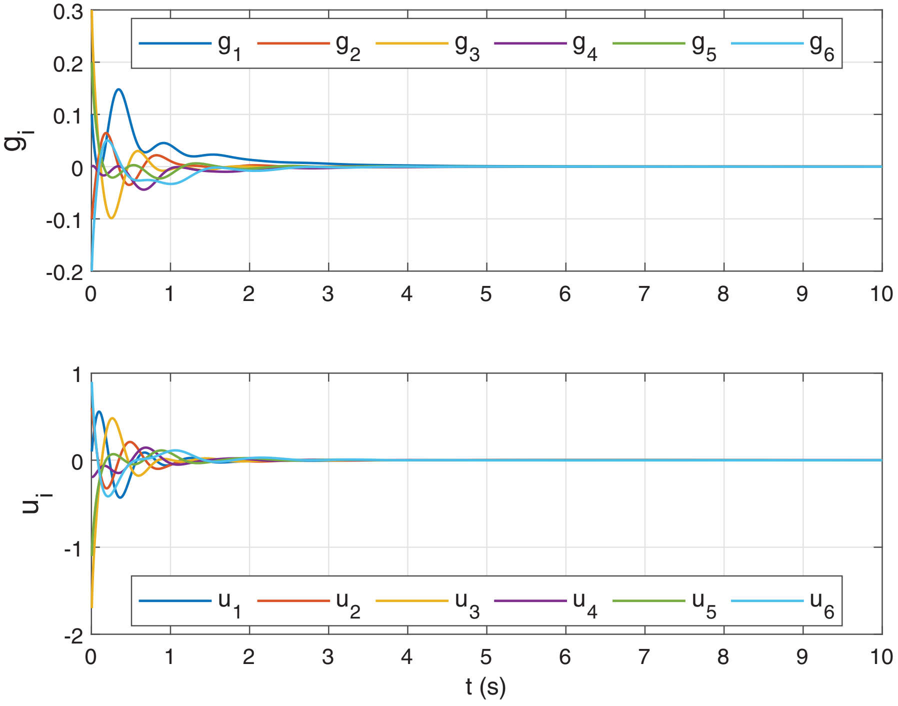


Fig. 4. Trajectories of filter output $g_i$ and control input $u_i$ in Example 1.

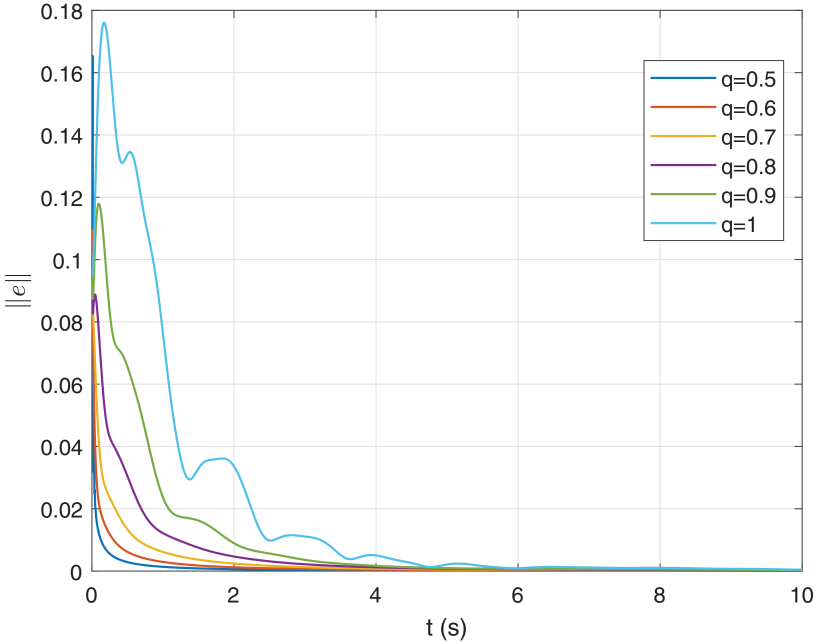


Fig. 5. Trajectories of consensus error $\|e\|$ with different orders.

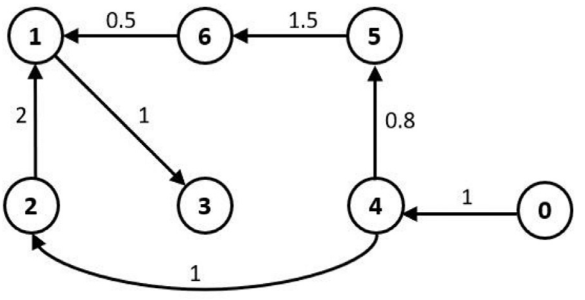


Fig. 6. Directed communication topology in Example 2.

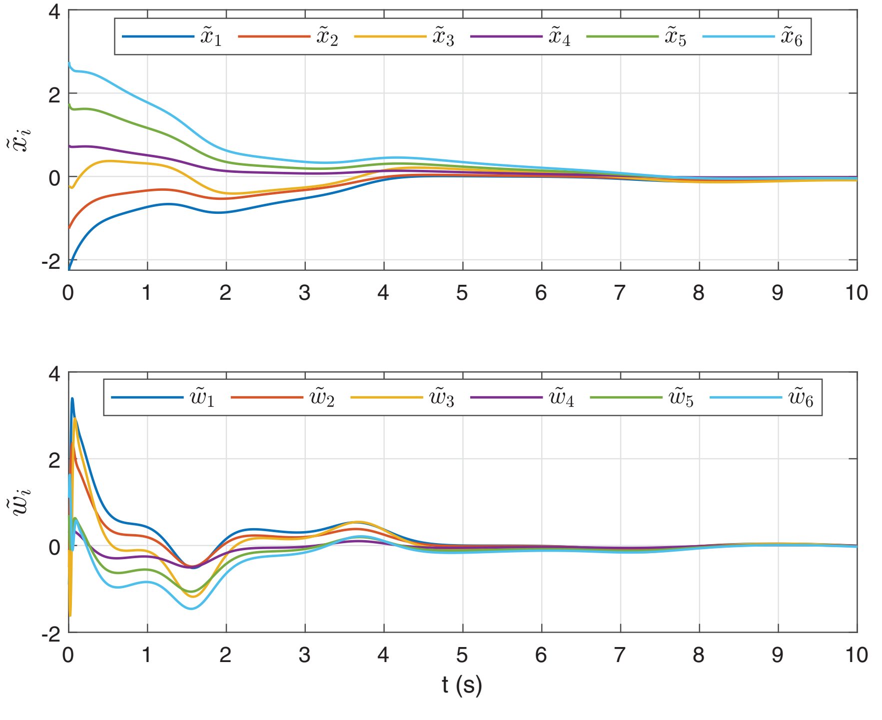


Fig. 7. Trajectories of tracking errors $\tilde{x}_i$ and $\tilde{w}_i$.

consensus algorithm of this paper exhibits several salient advantages, such as (1) structurally simple; (2) robustness; and (3) less demanding in its design.

## 5. Simulation examples

To illustrate the effectiveness of our proposed algorithms, we provide the following two simulation examples.

**Example 1** (Leaderless consensus problem)**.** Consider the leaderless consensus problem for a group of six agents modeled by fractional-order MASs (4). The topology structure $\mathcal{G}$ among the six agents is shown in Fig. 2, where each nonzero edge weight is assumed to be 1. The

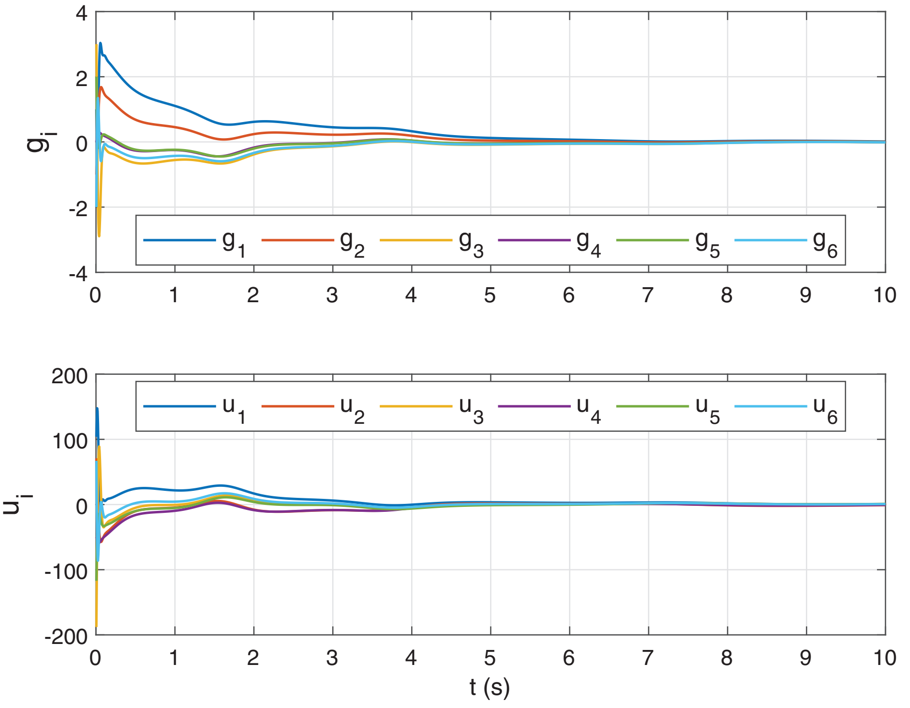


Fig. 8. Trajectories of filter output $g_i$ and control input $u_i$ in Example 2.

nonlinear function is given by $f(t, x_i, w_i) = -0.1i \sin x_i - 0.5w_i$. Choose $q = 0.95$, $a = b = 5$ and $c = 10$. The initial states are chosen as $x_i(0) = 0.02i$ and $w_i(0) = 0.01i$, $i = 1, 2, \ldots, 6$. The evolutions of the position and velocity consensus errors $\hat{x}_i$, $\hat{w}_i$, the filter output $g_i$ and the control input $u_i$ are provided in Figs. 3 and 4, respectively, which implying that the leaderless consensus problem is indeed solved and all control inputs are bounded and smooth.

Next, we consider the consensus error convergence speed of fractional-order MASs (4) with different orders by using Eq. (12). The trajectories of consensus error $\|e\|$ for fractional-order MASs (4) using Eq. (12) with different orders and all other parameters given above are plotted in Fig. 5. It can be seen from Fig. 5 that the smaller $q$ is the faster the decaying speed of $\|e\|$ will be. So the consensus control of fractional-order MAS has better convergence performance than the one of second-order MAS.

**Example 2** (Consensus tracking problem)**.** To illustrate the obtained theoretical results of Theorem 3, numerical simulation on a group of six followers and one leader is conducted. The directed communication topology among the seven agents is shown in Fig. 6, where the weights are indicated on edges. Let $f_0 = x_0 - 0.18w_0 + 0.3\cos t$, $u_0 = -x_0^3$ and $f_i = x_i - 0.1iw_i + 0.04i \sin t$, $i = 1, 2, \ldots, 6$. Then the dynamics of the leader is described by the following fractional-order chaotic Duffing system [20]:

$$\begin{cases} D^q x_0 = w_0, \\ D^q w_0 = x_0 - x_0^3 - 0.18w_0 + 0.3\cos t, \\ y_0 = x_0, \end{cases} \tag{51}$$

and the dynamics of each follower is described as

$$\begin{cases} D^q x_i = w_i, \\ D^q w_i = x_i - 0.1 i w_i + 0.04 i \sin t + u_i, \\ y_i = x_i, \ \ i = 1, \dots, 6. \end{cases} \tag{52}$$

As given in [20], the Duffing system (51) exhibits chaotic behaviors when $q = 0.97$ and the initial conditions are chosen as $[x_0(0), w_0(0)]^T = [0.21, 0.13]^T$. The initial states of the other followers are set as $x_i(0) = i$ and $w_i(0) = 0.4i$, $i = 1, 2, \dots, 6$. The control parameters are given as $a = 25$, $b = 46$ and $c = 65$. The trajectories of tracking errors $\tilde{x}_i$, $\tilde{w}_i$, the filter output $g_i$ and the control input $u_i$ are provided in Figs. 7 and 8, respectively, which meaning that both position and velocity consensus tracking errors are UUB, and all control inputs are bounded and smooth. Therefore, the correctness of Theorem 3 is verified by these simulation results.

## 6. Conclusions

This paper focus on designing output feedback based consensus protocols for nonlinear fractional-order MASs with double-integrator dynamics under general directed topologies. The leaderless consensus problem and consensus tracking problem are addressed in Sections 3 and 4, respectively, by proposing a distributed algorithm only using relative positive measurements between neighbors. By using the fractional Lyapunov direct method, some sufficient conditions are derived to guarantee the global leaderless consensus can be achieved for any strongly connected directed topology. The results can also be extended to tracking problem with an unknown leader for any communication digraph that there exists a directed path from the leader to each follower. Future work will be focused on designing fixed-time consensus algorithm in fractional-order nonlinear MASs with general directed (switching) topology via output feedback control. It is interesting to solve the problem of this paper via event-triggered control.

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