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Stability and convergence based on the finite difference method for the nonlinear fractional cable equation on non-uniform staggered grids

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A B S T R A C T

In this article, a block-centered finite difference method for the nonlinear fractional cable equation is introduced and analyzed. The unconditional stability and the global convergence of the scheme are proved rigorously. Some a priori estimates of discrete norms with optimal order of convergence $O(\Delta t^{\alpha} + h^2 + k^2)$ both for pressure and velocity are established on non-uniform rectangular grids, where $\alpha = \min\{1+\gamma_1, 1+\gamma_2\}$, Δt, h and k are the step sizes in time, space in x- and y-direction. Moreover, the applicability and accuracy of the scheme are demonstrated by numerical experiments to support our theoretical analysis.

1. Introduction

Recently, fractional differential and integral equations have been extensively applied to numerous seemingly diverse and widespread fields of science, engineering and finance. For example, fractional calculus has been extensively and successfully applied to the problems in system biology [35], physics [2,20,22,38,39], chemistry and biochemistry [34], hydrology [3,12, 13], medicine [5,21] and finance [23,29,32].

The cable equation is one of the most fundamental equations for modeling neuronal dynamics. Langlands and his coauthors derived fractional cable equations in [8] as macroscopic models for electrodiffusion of ions in nerve cells when molecular diffusion is anomalous subdiffusion due to binding, crowding or trapping. They also introduced fractional Nernst-Planck equations and related fractional cable equations to model electrodiffusion of ions in nerve cells with anomalous subdiffusion along and across the nerve cells in [9]. Hu and Zhang [6] have studied the implicit compact difference schemes for the fractional cable equation. Two new implicit numerical methods for the fractional cable equation were considered by Liu and his coauthors [14]. A novel fully discrete Crank-Nicolson finite element method, which is obtained by finite difference in time and finite element in space, is presented to approximate the fractional Cable equation in [15]. Besides, in [31], a Galerkin finite element method combined with second-order time discrete scheme for finding the numerical solution of nonlinear time fractional Cable equation is studied and discussed. A temporal second-order fully discrete two-grid finite element scheme, in which the spatial direction is approximated by two-grid finite element method and the integer

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and fractional derivatives in time are discretized by second-order two-step backward difference method and second-order weighted and shifted Grünwald difference (WSGD) scheme, is presented to solve nonlinear fractional Cable equation in [18]. Recently, concerning the numerical treatment of the two-dimensional fractional cable equation, Yu and Jiang [33] proposed a fourth-order compact finite difference method.

Block-centered finite differences, sometimes called cell-centered finite differences, can be thought as the lowest order Raviart-Thomas mixed element method [24], with proper quadrature formulation. The application of the block-centered finite difference enables us to approximate both the pressure and velocity with second-order accuracy which is obtained the superconvergence analysis on non-uniform rectangular grids. In [1], Wheeler presented the mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences. And in 2012, a block-centered finite difference method for the Darcy-Forchheimer model was considered in [27]. Besides, in [10,16,19,26,28] block-centered finite difference methods were developed to solve linear and nonlinear equations.

As far as we know, the time fractional equations with Neumann boundary condition have recently been studied by many researchers [7,25,37]. But there is no block-centered finite difference method for the nonlinear fractional cable equation with the Neumann boundary condition on non-uniform rectangular grids. And in this paper our target is to present the block-centered finite difference method to solve the nonlinear fractional cable equation with the Neumann boundary condition on non-uniform rectangular grids. Furthermore, Unconditional stability of the method is proved. Besides, we demonstrate that the block centered finite difference scheme has α-order accuracy in time increment, where $\alpha = \min\{1+\gamma_1, 1+\gamma_2\}$ and second-order accuracy in space meshsize both for the original unknown, called pressure in porous media flow, and its derivatives, called velocity in porous media flow, in discrete L^2 norms on non-uniform rectangular grid. These error estimates are superconvergence. The key step to the superconvergence analysis, is to construct a proper relation between the velocity $\mathbf{u}$ and the difference of the pressure p. Then some numerical examples are carried to show the accuracy of the presented block-centered finite difference scheme.

The paper is organized as follows. In Sect. 2, we give the problem and some notations. In Sect. 3, we present the block-centered finite difference method. Then in Sect. 4, we present the analysis of stability and error estimates for the presented method. Some numerical experiments using the block-centered finite difference scheme are carried out in Sect. 5.

2. The problem and some notations

In this section, we first describe the problem of two-dimensional nonlinear fractional cable equation (see [6,14]) with the Neumann condition in this paper, and present some notations which will be found helpful in the following analysis.

Find $p = p(x, y, t)$ such that

$$\frac{\partial p}{\partial t} = K_0 D_t^{1-\gamma_1} \Delta p - \mu_0 D_t^{1-\gamma_2} p + f(x, y, p, t), \quad (x, y, t) \in \Omega \times J,$$

with Neumann boundary condition

$$\nabla p \cdot \mathbf{n} = 0, \quad (x, y, t) \in \partial\Omega \times J,$$

and initial condition

$$p\,|_{t=0} = p_0(x, y), \quad (x, y) \in \Omega,$$

where $\Omega = (0, 1) \times (0, 1)$, $J = (0, T]$, $\mathbf{n}$ represents the unit exterior normal vector to the boundary of Ω, $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$, $p_0(x, y)$ is given known function. $0 < \gamma_1, \ \gamma_2 < 1$, $K > 0$, $\mu > 0$ are constants, and ${}_0D_t^{1-\gamma} p$ is the Rieman-Liouville fractional partial derivative of order $1-\gamma$ defined by

$${}_0D_t^{1-\gamma} p(x, y, t) = \frac{1}{\Gamma(\gamma)} \frac{\partial}{\partial t} \int_0^t \frac{p(x, y, \tau)}{(t-\tau)^{1-\gamma}} d\tau, \quad 0 < \gamma < 1. \tag{1}$$

Introduce $\mathbf{u} = (u^x, u^y) = -\nabla p$, then the nonlinear system of above equations can be transformed into the following formulation

$$\begin{cases} \dfrac{\partial p}{\partial t} + K_0 D_t^{1-\gamma_1} \nabla \cdot \mathbf{u} + \mu_0 D_t^{1-\gamma_2} p = f(x, y, p, t), & (x, y, t) \in \Omega \times J, \\ \mathbf{u} = -\nabla p, \quad (x, y, t) \in \Omega \times J, \\ \mathbf{u} \cdot \mathbf{n} = 0, \quad (x, y, t) \in \partial\Omega \times J, \\ p\,|_{t=0} = p_0(x, y), \quad (x, y) \in \Omega. \end{cases} \tag{2}$$

In particular, the nonlinear reaction term $f(p)$ satisfies the smoothness assumptions [17]:

$A1$: $|f(p)| \leqslant C|p|$,

$A2$: The first-order derivative of $f(p)$ with respect to p is bounded, that is to say, there exists a positive constant C such that $|f'(p)| \leqslant C$.

Now, we define some notations. Let $N > 0$ be a positive integer. Set

$$\triangle t = T/N; \quad t_n = n\triangle t, \quad for \quad n \le N.$$

The two dimensional domain $\Omega = (0, 1) \times (0, 1)$ is partitioned by $\delta_x \times \delta_y$, where

$$\delta_x : 0 = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \cdots < x_{N_x-\frac{1}{2}} < x_{N_x+\frac{1}{2}} = 1,$$
$$\delta_y : 0 = y_{\frac{1}{2}} < y_{\frac{3}{2}} < \cdots < y_{N_y-\frac{1}{2}} < y_{N_y+\frac{1}{2}} = 1.$$

For $i = 1, \cdots, N_x$ and $j = 1, \cdots, N_y$, define [27,28]

$$x_i = \frac{x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}}}{2},$$
$$h_i = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}, \quad h = \max_{1\le i\le N_x} h_i,$$
$$h_{i+\frac{1}{2}} = x_{i+1} - x_i = \frac{h_i + h_{i+1}}{2},$$
$$y_j = \frac{y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}}}{2},$$
$$k_j = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}, \quad k = \max_{1\le j\le N_y} k_j,$$
$$k_{j+\frac{1}{2}} = y_{j+1} - y_j = \frac{k_j + k_{j+1}}{2}.$$

For functions $f(x, y)$ and $g(x, y)$, define the L^2 inner products and norms

$$(f, g) = \int_\Omega f(x, y)g(x, y)dxdy, \quad ||f|| = \sqrt{(f, f)}.$$

Let $g_{i,j}$, $g_{i+\frac{1}{2},j}$, $g_{i,j+\frac{1}{2}}$ denote $g(x_i, y_j)$, $g(x_{i+\frac{1}{2}}, y_j)$ and $g(x_i, y_{j+\frac{1}{2}})$. Define the discrete inner products and norms as follows:

$$(f, g)_m = \sum_{i=1}^{N_x}\sum_{j=1}^{N_y} h_i k_j f_{i,j} g_{i,j},$$
$$(f, g)_x = \sum_{i=2}^{N_x}\sum_{j=1}^{N_y} h_{i-\frac{1}{2}} k_j f_{i-\frac{1}{2},j} g_{i-\frac{1}{2},j},$$
$$(f, g)_y = \sum_{i=1}^{N_x}\sum_{j=2}^{N_y} h_i k_{j-\frac{1}{2}} f_{i,j-\frac{1}{2}} g_{i,j-\frac{1}{2}},$$
$$(\mathbf{f}, \mathbf{g})_{TM} = (f^x, g^x)_x + (f^y, g^y)_y,$$
$$||f||_i^2 = (f, f)_i, \quad i = m, x, y, TM.$$

Noting that for f, we have $\|f\|_m = \|f\|$, then we define

$$[d_x g]_{i+\frac{1}{2},j} = \frac{g_{i+1,j} - g_{i,j}}{h_{i+\frac{1}{2}}},$$
$$[d_y g]_{i,j+\frac{1}{2}} = \frac{g_{i,j+1} - g_{i,j}}{k_{j+\frac{1}{2}}},$$
$$[D_x g]_{i,j} = \frac{g_{i+\frac{1}{2},j} - g_{i-\frac{1}{2},j}}{h_i},$$
$$[D_y g]_{i,j} = \frac{g_{i,j+\frac{1}{2}} - g_{i,j-\frac{1}{2}}}{k_j},$$

$$[d_t g]^n = \frac{g^n - g^{n-1}}{\triangle t},$$
$$[D_t g]^{n+1} = \frac{3g^{n+1} - 4g^n + g^{n-1}}{2\triangle t}.$$

Next we also give some lemmas which will be used in stability analysis and error estimates. Denote by

$$\delta^n_{i,j} = \left(\frac{h^2}{8}\frac{\partial^2 p}{\partial x^2} + \frac{k^2}{8}\frac{\partial^2 p}{\partial y^2}\right)^n_{i,j} = \left[\frac{h_i^2}{8}\frac{\partial^2 p^n_{i,j}}{\partial x^2} + \frac{k_j^2}{8}\frac{\partial^2 p^n_{i,j}}{\partial y^2}\right] = O\left(h^2 + k^2\right),$$

$$\begin{aligned}\widetilde{\epsilon}^x_{i+1/2,j}(p) =& \frac{1}{2h_{i+1/2}}\int_{x_{i+1/2}}^{x_{i+1}}\left(\frac{h^2_{i+1}}{4} - (x - x_{i+1})^2\right)\frac{\partial^3 p}{\partial x^3}(x, y_j, t)dx \\ &- \frac{1}{2h_{i+1/2}}\int_{x_{i+1/2}}^{x_i}\left(\frac{h^2_i}{4} - (x - x_i)^2\right)\frac{\partial^3 p}{\partial x^3}(x, y_j, t)dx \\ &+ \frac{k_j^2}{8h_{i+1/2}}\int_{x_i}^{x_{i+1}}\frac{\partial^3 p}{\partial x\partial y^2}(x, y_j, t)dx,\end{aligned}$$

and

$$\begin{aligned}\widetilde{\epsilon}^y_{i,j+1/2}(p) =& \frac{1}{2k_{j+1/2}}\int_{y_{j+1/2}}^{y_{j+1}}\left(\frac{k^2_{j+1}}{4} - (y - y_{j+1})^2\right)\frac{\partial^3 p}{\partial y^3}(x_i, y, t)dy \\ &- \frac{1}{2k_{j+1/2}}\int_{y_{j+1/2}}^{y_j}\left(\frac{k^2_j}{4} - (y - y_j)^2\right)\frac{\partial^3 p}{\partial y^3}(x_i, y, t)dy \\ &+ \frac{h_i^2}{8k_{j+1/2}}\int_{y_j}^{y_{j+1}}\frac{\partial^3 p}{\partial y\partial x^2}(x_i, y, t)dy.\end{aligned}$$

Then we have the following Lemmas 1 and 2 which can be proven similar to the literature [28].

Lemma 1. *If p is sufficiently smooth, then there holds*

$$\begin{cases} u^x_{i+1/2,j} = -[d_x(p-\delta)]_{i+1/2,j} - \widetilde{\epsilon}^x_{i+1/2,j}(p), \\ u^y_{i,j+1/2} = -[d_y(p-\delta)]_{i,j+1/2} - \widetilde{\epsilon}^y_{i,j+1/2}(p), \end{cases}$$

with the following approximate properties

$$\widetilde{\epsilon}^x_{i+1/2,j}(p) = O\left(h^2 + k^2\right), \quad \widetilde{\epsilon}^y_{i,j+1/2}(p) = O\left(h^2 + k^2\right).$$

Lemma 2. *Let $q_{i,j}$, $w^x_{i+1/2,j}$ and $w^y_{i,j+1/2}$ be any values such that $w^x_{1/2,j} = w^x_{N_x+1/2,j} = w^y_{i,1/2} = w^y_{i,N_y+1/2} = 0$, then we have*

$$(q, D_x w^x)_m = -(d_x q, w^x)_x,$$
$$(q, D_y w^y)_m = -(d_y q, w^y)_y.$$

Throughout the paper we use C, with or without subscript, to denote a positive constant, which could have different values at different appearances.

3. Block-centered finite difference method

The objective of this section is to consider the block-centered finite difference method for equation (2).

Firstly, for the convenience of theoretical analysis, we now denote

$$G^\gamma_k = (k+1)^\gamma - k^\gamma. \tag{3}$$

Then from the literature [11], it is not difficult to verify that

$$1 = G_0^\gamma > G_1^\gamma > G_2^\gamma > \cdots > G_n^\gamma > \cdots \to \Delta t^{1-\gamma} \to 0. \tag{4}$$

The following lemmas will be used in the derivation of the difference scheme.

Lemma 3 *([30]). For $0 < \gamma < 1$ and $g \in C^2[0, t_n]$, it holds that*

$$\begin{aligned}
&\frac{1}{\Gamma(\gamma)}\int_0^{t_n} \frac{g'(\tau)}{(t_n-\tau)^{1-\gamma}}d\tau - \frac{\Delta t^{\gamma-1}}{\Gamma(1+\gamma)}\left[g(t_n) - \sum_{k=1}^{n-1}(G_{n-k-1}^\gamma - G_{n-k}^\gamma)g(t_k) - G_{n-1}^\gamma g(0)\right] \\
&\quad = \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} L_{\gamma,n}(\tau)g''(\tau)d\tau,
\end{aligned} \tag{5}$$

where for $\tau \in (t_{k-1}, t_k)$,

$$L_{\gamma,n}(\tau) = \frac{1}{\Gamma(1+\gamma)}\left((t_n-\tau)^\gamma - \left[\frac{\tau - t_{k-1}}{\Delta t}(t_n - t_k)^\gamma + \frac{t_k - \tau}{\Delta t}(t_n - t_{k-1})^\gamma\right]\right).$$

Furthermore, $L_{\gamma,n}(\tau) \ge 0$, and

$$\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} L_{\gamma,n}(\tau)d\tau \le \frac{1}{\Gamma(1+\gamma)}\left[\frac{\gamma}{12} + \frac{2^{1+\gamma}}{1+\gamma} - (1+2^{\gamma-1})\right]\Delta t^{1+\gamma}. \tag{6}$$

To analyze the truncation error of the Riemann-Liouville fractional derivative, we need the following property. Suppose that the function $g(t)$ is $(m-1)$ times continuously differentiable in the interval $[0, T]$ and that $g^{(m)}(t)$ is integrable in $[0, T]$. Then for every k $(0 < k < m)$ the Riemann-Liouville fractional derivative ${}_0D_t^k g(t)$ exists, and if $0 \le m-1 \le k < m$, then for $0 < t < T$ the following equation holds [36]:

$${}_0D_t^k g(t) = \sum_{j=0}^{m-1}\frac{g^{(j)}(0)t^{j-k}}{\Gamma(1+j-k)} + \frac{1}{\Gamma(m-k)}\int_0^t \frac{g^{(m)}(\tau)}{(t-\tau)^{k-m+1}}d\tau. \tag{7}$$

Then by using Lemma 3 and equation (7), we have the following result.

Lemma 4. *Suppose $\boldsymbol{u}(x, y, t) \in C^2(J; C^3(\Omega))^2$, then it holds that*

$$\begin{aligned}
&\left[{}_0D_t^{1-\gamma}\nabla\cdot\boldsymbol{u}\right](x_i, y_j, t_n) \\
&= \frac{\Delta t^{\gamma-1}}{\Gamma(1+\gamma)}\left[\boldsymbol{Du}_{i,j}^n - \sum_{k=1}^{n-1}(G_{n-k-1}^\gamma - G_{n-k}^\gamma)\boldsymbol{Du}_{i,j}^k - G_{n-1}^\gamma \boldsymbol{Du}_{i,j}^0\right] \\
&\quad + \frac{t_n^{\gamma-1}}{\Gamma(\gamma)}\boldsymbol{Du}_{i,j}^0 + (R_1)_{i,j}^n + (R_2)_{i,j}^n, \quad 1 \le i \le N_x, 1 \le j \le N_y, 1 \le n \le N,
\end{aligned} \tag{8}$$

where $\boldsymbol{Du} = (D_x u^x, D_y u^y)$ and $(R_1)_{i,j}^n$, $(R_2)_{i,j}^n$ are defined as follows:

$$\begin{aligned}
&(R_1)_{i,j}^n \\
&= \sum_{k=1}^{n}\int_{t_{k-1}}^{t_k} L_{\gamma,n}(\tau)\left(\frac{\partial^3 u^x}{\partial x\partial t^2}(x_{i-1/2} + h_i\lambda, y_j, \tau) + \frac{\partial^3 u^y}{\partial y\partial t^2}(x_i, y_{j-1/2} + k_j\lambda, \tau)\right)d\lambda d\tau,
\end{aligned}$$

and

$$
\begin{aligned}
&(R_2)_{i,j}^n\\
=&\frac{h_i^2}{16}\int_0^1\left({}_0D_t^{1-\gamma}\frac{\partial^3 u^x}{\partial x^3}(x_i+\frac{h_i}{2}s,y_j,t_n)+D_t^{1-\gamma}\frac{\partial^3 u^x}{\partial x^4}(x_i-\frac{h_i}{2}s,y_j,t_n)\right)(1-s)^2ds\\
&+\frac{k_j^2}{16}\int_0^1\left({}_0D_t^{1-\gamma}\frac{\partial^3 u^y}{\partial y^3}(x_i,y_j+\frac{k_j}{2}s,t_n)+D_t^{1-\gamma}\frac{\partial^3 u^y}{\partial y^3}(x_i,y_j-\frac{k_j}{2}s,t_n)\right)(1-s)^2ds.
\end{aligned}
$$

If we denote

$$
C(\gamma)=\frac{1}{\Gamma(1+\gamma)}\left(\frac{\gamma}{12}+\frac{2^{1+\gamma}}{1+\gamma}-(1+2^{\gamma-1})\right),
$$

then

$$
\begin{aligned}
|(R_1)_{i,j}^n|+|(R_2)_{i,j}^n|\le&C(\gamma)\left(\|\frac{\partial^3 u^x}{\partial x\partial t^2}\|_{L^\infty(J,L^\infty(\Omega))}+\|\frac{\partial^3 u^y}{\partial y\partial t^2}\|_{L^\infty(J,L^\infty(\Omega))}\right)\Delta t^{1+\gamma}\\
&+\frac{1}{24}\|{}_0D_t^{1-\gamma}\left(\frac{\partial^3 u^x}{\partial x^3}\right)\|_{L^\infty(J,L^\infty(\Omega))}h_i^2\\
&+\frac{1}{24}\|{}_0D_t^{1-\gamma}\left(\frac{\partial^3 u^y}{\partial y^3}\right)\|_{L^\infty(J,L^\infty(\Omega))}k_j^2.
\end{aligned}
\tag{9}
$$

Proof. Let $\mathbf{v}(x,y,t)={}_0D_t^{1-\gamma}\mathbf{u}(x,y,t)$. Then we have

$$
{}_0D_t^{1-\gamma}(\nabla\cdot\mathbf{u})=\nabla\cdot\mathbf{v}.
$$

Utilizing Taylor expansion with integral remainder, we have

$$
\begin{aligned}
\nabla\cdot\mathbf{v}_{i,j}^n=&\mathbf{D}\mathbf{v}_{i,j}^n+\frac{h_i^2}{16}\int_0^1\left(\frac{\partial^3 v^x}{\partial x^3}(x_i+\frac{h_i}{2}s,y_j,t_n)+\frac{\partial^3 v^x}{\partial x^3}(x_i-\frac{h_i}{2}s,y_j,t_n)\right)(1-s)^2ds\\
&+\frac{k_j^2}{16}\int_0^1\left(\frac{\partial^3 v^y}{\partial y^3}(x_i,y_j+\frac{k_j}{2}s,t_n)+\frac{\partial^3 v^y}{\partial y^3}(x_i,y_j-\frac{k_j}{2}s,t_n)\right)(1-s)^2ds.
\end{aligned}
\tag{10}
$$

It follows from equations (5) and (7) that

$$
\begin{aligned}
\mathbf{v}_{i,j}^n=&\left[{}_0D_t^{1-\gamma}\mathbf{u}\right]_{i,j}^n\\
=&\frac{t_n^{\gamma-1}}{\Gamma(\gamma)}\mathbf{u}_{i,j}^0+\frac{1}{\Gamma(\gamma)}\int_0^{t_n}\frac{\partial\mathbf{u}(x_i,y_j,s)}{\partial\tau}(t_n-\tau)^{\gamma-1}d\tau\\
=&\frac{\Delta t^{\gamma-1}}{\Gamma(1+\gamma)}\left[\mathbf{u}_{i,j}^n-\sum_{k=1}^{n-1}(G_{n-k-1}^\gamma-G_{n-k}^\gamma)\mathbf{u}_{i,j}^k-G_{n-1}^\gamma\mathbf{u}_{i,j}^0\right]\\
&+\frac{t_n^{\gamma-1}}{\Gamma(\gamma)}\mathbf{u}_{i,j}^0+\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}L_{\gamma,n}(\tau)\frac{\partial^2\mathbf{u}}{\partial t^2}(x_i,y_j,\tau)d\tau.
\end{aligned}
\tag{11}
$$

Substituting equation (11) into (10), and using the equation

$$
\begin{aligned}
\left[\mathbf{D}\frac{\partial^2\mathbf{u}}{\partial t^2}\right](x_i,y_j,\tau)=&\int_0^1\frac{\partial^3 u^x}{\partial x\partial t^2}(x_{i-1/2}+h_i\lambda,y_j,\tau)d\lambda\\
&+\int_0^1\frac{\partial^3 u^y}{\partial y\partial t^2}(x_i,y_{j-1/2}+k_j\lambda,\tau)d\lambda
\end{aligned}
\tag{12}
$$

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we obtain the first statement.

Applying Lemma 3 and noticing that

$$\begin{aligned}&|\int_0^1 \frac{\partial^3 u^x}{\partial x\partial t^2}(x_{i-1/2}+h_i\lambda, y_j, \tau)d\lambda + \int_0^1 \frac{\partial^3 u^y}{\partial y\partial t^2}(x_i, y_{j-1/2}+k_j\lambda, \tau)d\lambda| \\ \le&\|\frac{\partial^3 u^x}{\partial x\partial t^2}\|_{L^\infty(J,L^\infty(\Omega))}+\|\frac{\partial^3 u^y}{\partial y\partial t^2}\|_{L^\infty(J,L^\infty(\Omega))},\end{aligned} \tag{13}$$

we can obtain that for $1\le i\le N_x, 1\le j\le N_y$

$$\begin{aligned}&|(R_1)^n_{i,j}| \\ \le&\frac{1}{\Gamma(1+\gamma)}\left(\frac{\gamma}{12}+\frac{2^{1+\gamma}}{1+\gamma}-(1+2^{\gamma-1})\right)\left(\|\frac{\partial^3 u^x}{\partial x\partial t^2}\|_{L^\infty(J,L^\infty(\Omega))}+\|\frac{\partial^3 u^y}{\partial y\partial t^2}\|_{L^\infty(J,L^\infty(\Omega))}\right)\Delta t^{1+\gamma}.\end{aligned} \tag{14}$$

On the other hand, it is easy to find that for $1\le i\le N_x, 1\le j\le N_y$

$$\begin{aligned}|(R_2)^n_{i,j}| \le&\frac{1}{24}\|{}_0D_t^{1-\gamma}\left(\frac{\partial^3 u^x}{\partial x^3}\right)\|_{L^\infty(J,L^\infty(\Omega))}h_i^2 \\ &+\frac{1}{24}\|{}_0D_t^{1-\gamma}\left(\frac{\partial^3 u^y}{\partial y^3}\right)\|_{L^\infty(J,L^\infty(\Omega))}k_j^2.\end{aligned} \tag{15}$$

Thus the proof is completed.

Now, denote by $\{Z^n, W^{x,n}, W^{y,n}\}_{n=1}^N$ the block-centered finite difference approximations to $\{p^n, u^{x,n}, u^{y,n}\}_{n=1}^N$, respectively. Then the scheme is as follows:

Case I: $n=0$

$$\begin{aligned}&[d_tZ]^1_{i,j}+\frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}[d_tZ]^1_{i,j}+\frac{\mu t_1^{\gamma_2-1}}{\Gamma(\gamma_2)}Z^0_{i,j}+\frac{Kt_1^{\gamma_1-1}}{\Gamma(\gamma_1)}(D_xW^x+D_yW^y)^0_{i,j} \\ &\quad+\frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\left[d_t(D_xW^x+D_yW^y)\right]^1_{i,j}=f(Z^0_{i,j}),\end{aligned} \tag{16}$$

Case II: $n\geqslant 1$

$$\begin{aligned}&[D_tZ]^{n+1}_{i,j}+\frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n}G^{\gamma_2}_{n-k}[d_tZ]^{k+1}_{i,j}+\frac{\mu t_{n+1}^{\gamma_2-1}}{\Gamma(\gamma_2)}Z^0_{i,j}+\frac{Kt_{n+1}^{\gamma_1-1}}{\Gamma(\gamma_1)}(D_xW^x+D_yW^y)^0_{i,j} \\ &+\frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n}G^{\gamma_1}_{n-k}\left[d_t(D_xW^x+D_yW^y)\right]^{k+1}_{i,j}=f(2Z^n_{i,j}-Z^{n-1}_{i,j}),\end{aligned} \tag{17}$$

and

$$W^{x,n+1}_{i+\frac12,j}=-[d_xZ]^{n+1}_{i+\frac12,j},\quad 1\le i\le N_x-1, 1\le j\le N_y, \tag{18}$$

$$W^{y,n+1}_{i,j+\frac12}=-[d_yZ]^{n+1}_{i,j+\frac12},\quad 1\le i\le N_x, 1\le j\le N_y-1. \tag{19}$$

Set the initial approximation as follows:

$$Z^0_{i,j}=p_{0_{i,j}},\quad 1\le i\le N_x, 1\le j\le N_y.$$

It is easy to see that at each time level, the difference scheme (16)-(19) is a linear system with strictly diagonally dominant coefficient matrix, thus the approximate solutions exist uniquely.

4. Stability and error analysis

In this section, the analysis of stability and the error estimates of the scheme (16)-(19) are given rigorously.

4.1. The analysis of stability for discrete scheme

In this subsection, we will give the analysis of stability for the difference scheme (16)-(19).

Theorem 5. *For the scheme (16)-(19), the numerical solutions Z^n and $\boldsymbol{W}^n$ can be bounded by the following inequality*

$$\|Z^n\|_m^2 + \left(\sum_{k=1}^{n} \Delta t \left\|Z^k\right\|_m^2\right)^{1/2} + \left(\sum_{k=1}^{n} \Delta t \left\|\boldsymbol{W}^k\right\|_{TM}^2\right)^{1/2} \le C\,\|p_0\|_m^2 + C\,\|\boldsymbol{u}_0\|_{TM}^2\,, \tag{20}$$

where C is a positive constant which is independent of $h,\ k$ and Δt.

Proof. Multiplying (17) by $Z_{i,j}^{n+1} h_i k_j$ and making summation on $i,\ j$ for $1 \le i \le N_x,\ 1 \le j \le N_y$, $n \geqslant 1$, we have

$$\begin{aligned}
&\left(D_t Z^{n+1}, Z^{n+1}\right)_m + \frac{\mu \Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)} \sum_{k=0}^{n} G_{n-k}^{\gamma_2} \left(d_t Z^{k+1}, Z^{n+1}\right)_m + \frac{\mu t_{n+1}^{\gamma_2-1}}{\Gamma(\gamma_2)} \left(Z^0, Z^{n+1}\right)_m \\
&+ \frac{K \Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)} \sum_{k=0}^{n} G_{n-k}^{\gamma_1} \left(d_t (D_x W^{x,k+1} + D_y W^{y,k+1}), Z^{n+1}\right)_m \\
&+ \frac{K t_{n+1}^{\gamma_1-1}}{\Gamma(\gamma_1)} \left(D_x W^{x,0} + D_y W^{y,0}, Z^{n+1}\right)_m \\
&= \left(f(2Z^n - Z^{n-1}), Z^{n+1}\right)_m .
\end{aligned} \tag{21}$$

Firstly, we estimate the first term in the left hand side of equation (21).

$$\begin{aligned}
\left(D_t Z^{n+1}, Z^{n+1}\right)_m = \frac{1}{4\Delta t} \Big[&\left(\|Z^{n+1}\|_m^2 + \|2Z^{n+1} - Z^n\|_m^2\right) - \left(\|Z^n\|_m^2 + \|2Z^n - Z^{n-1}\|_m^2\right) \\
&+ \|Z^{n+1} - 2Z^n + Z^{n-1}\|_m^2 \Big].
\end{aligned} \tag{22}$$

Then the second term in the left hand of equation (21) can be transformed as follows:

$$\begin{aligned}
&\frac{\mu \Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)} \sum_{k=0}^{n} G_{n-k}^{\gamma_2} \left(d_t Z^{k+1}, Z^{n+1}\right)_m \\
=& \frac{\mu \Delta t^{\gamma_2-1}}{\Gamma(1+\gamma_2)} \sum_{k=0}^{n} G_{k}^{\gamma_2} \left(Z^{n-k+1} - Z^{n-k}, Z^{n+1}\right)_m \\
=& \frac{\mu \Delta t^{\gamma_2-1}}{\Gamma(1+\gamma_2)} \left[G_0^{\gamma_2} \|Z^{n+1}\|_m^2 - \sum_{k=0}^{n-1} \left(G_k^{\gamma_2} - G_{k+1}^{\gamma_2}\right) \left(Z^{n-k}, Z^{n+1}\right)_m - G_n^{\gamma_2} \left(Z^0, Z^{n+1}\right)_m \right].
\end{aligned} \tag{23}$$

Using Lemma 2, we obtain

$$\left(D_x W^{x,n+1}, Z^{n+1}\right)_m = -\left(W^{x,n+1}, d_x Z^{n+1}\right)_x = \|W^{x,n+1}\|_x^2, \tag{24}$$

$$\left(D_y W^{y,n+1}, Z^{n+1}\right)_m = -\left(W^{y,n+1}, d_y Z^{n+1}\right)_y = \|W^{y,n+1}\|_y^2. \tag{25}$$

Then similar to the estimate of equation (23), the fourth term in the left of equation (21) can be transformed as follows:

$$\begin{aligned}
&\frac{K \Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)} \sum_{k=0}^{n} G_{n-k}^{\gamma_1} \left(d_t (D_x W^{x,k+1} + D_y W^{y,k+1}), Z^{n+1}\right)_m \\
=& \frac{K \Delta t^{\gamma_1-1}}{\Gamma(1+\gamma_1)} \sum_{k=0}^{n} G_k^{\gamma_1} \left((D_x W^{x,k+1} + D_y W^{y,k+1})^{n-k+1} - (D_x W^{x,k+1} + D_y W^{y,k+1})^{n-k}, Z^{n+1}\right)_m \\
=& \frac{K \Delta t^{\gamma_1-1}}{\Gamma(1+\gamma_1)} \left[G_0^{\gamma_1} \|\mathbf{W}^{n+1}\|_x^2 - G_n^{\gamma_1} \left(\mathbf{W}^0, \mathbf{W}^{n+1}\right)_m \right] \\
&- \frac{K \Delta t^{\gamma_1-1}}{\Gamma(1+\gamma_1)} \sum_{k=0}^{n-1} \left(G_k^{\gamma_1} - G_{k+1}^{\gamma_1}\right) \left(\mathbf{W}^{n-k}, \mathbf{W}^{n+1}\right)_m .
\end{aligned} \tag{26}$$

Then combining equation (21) with equations (22)-(26) and multiplying both sides of equation (21) by $4\Delta t$ give that

$$\begin{aligned}
&\left[\left(\|Z^{n+1}\|_m^2+\|2Z^{n+1}-Z^n\|_m^2\right)-\left(\|Z^n\|_m^2+\|2Z^n-Z^{n-1}\|_m^2\right)+\|Z^{n+1}-2Z^n+Z^{n-1}\|_m^2\right]\\
&+\frac{4\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\left[G_0^{\gamma_2}\|Z^{n+1}\|_m^2-\sum_{k=0}^{n-1}\left(G_k^{\gamma_2}-G_{k+1}^{\gamma_2}\right)\left(Z^{n-k},Z^{n+1}\right)_m-\widetilde{G}_n^{\gamma_2}\left(Z^0,Z^{n+1}\right)_m\right]\\
&+\frac{4K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\left[G_0^{\gamma_1}\|\mathbf{W}^{n+1}\|_{TM}^2-\widetilde{G}_n^{\gamma_1}\left(\mathbf{W}^0,\mathbf{W}^{n+1}\right)_m\right]\\
&-\frac{4K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n-1}\left(G_k^{\gamma_1}-G_{k+1}^{\gamma_1}\right)\left(\mathbf{W}^{n-k},\mathbf{W}^{n+1}\right)_m\\
=&4\Delta t\left(f(2Z^n-Z^{n-1}),Z^{n+1}\right)_m,
\end{aligned}\tag{27}$$

where $\widetilde{G}_n^{\gamma_l}=G_n^{\gamma_l}-\gamma_l(n+1)^{\gamma_l-1},\ l=1,2.$

In particularly, we have

$$G_k^{\gamma_l}-G_{k+1}^{\gamma_l}>0,\ \ \widetilde{G}_n^{\gamma_l}>0,\ l=1,2.\tag{28}$$

Observing the smoothness assumption $A1$, equation (28), and using Cauchy-Schwarz inequality [4], we obtain

$$\begin{aligned}
&\left[\left(\|Z^{n+1}\|_m^2+\|2Z^{n+1}-Z^n\|_m^2\right)-\left(\|Z^n\|_m^2+\|2Z^n-Z^{n-1}\|_m^2\right)\right.\\
&\quad\left.+\|Z^{n+1}-2Z^n+Z^{n-1}\|_m^2\right]+\frac{4\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\|Z^{n+1}\|_m^2+\frac{4K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\|\mathbf{W}^{n+1}\|_{TM}^2\\
\leqslant&\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\left[\sum_{k=0}^{n-1}(G_k^{\gamma_2}-G_{k+1}^{\gamma_2})\left\|Z^{n-k}\right\|_m^2+\|Z^{n+1}\|_m^2+G_n^{\gamma_2}\left\|Z^0\right\|_m^2\right]\\
&+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\left[\sum_{k=0}^{n-1}(G_k^{\gamma_1}-G_{k+1}^{\gamma_1})\left\|\mathbf{W}^{n-k}\right\|_{TM}^2+\|\mathbf{W}^{n+1}\|_{TM}^2+G_n^{\gamma_1}\left\|\mathbf{W}^0\right\|_{TM}^2\right]\\
&+C\Delta t\left(\|Z^n\|_m^2+\|Z^{n-1}\|_m^2+\|Z^{n+1}\|_m^2\right).
\end{aligned}\tag{29}$$

Consequently, we can obtain

$$\begin{aligned}
&\left(\|Z^{n+1}\|_m^2+\|2Z^{n+1}-Z^n\|_m^2\right)+\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\left(\|Z^{n+1}\|_m^2+\sum_{k=0}^{n-1}G_{k+1}^{\gamma_2}\left\|Z^{n-k}\right\|_m^2\right)\\
&+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\left(\|\mathbf{W}^{n+1}\|_{TM}^2+\sum_{k=0}^{n-1}G_{k+1}^{\gamma_1}\left\|\mathbf{W}^{n-k}\right\|_{TM}^2\right)\\
\leqslant&\left(\|Z^n\|_m^2+\|2Z^n-Z^{n-1}\|_m^2\right)+\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n-1}G_k^{\gamma_2}\left\|Z^{n-k}\right\|_m^2+\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}G_n^{\gamma_2}\left\|Z^0\right\|_m^2\\
&+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n-1}G_k^{\gamma_1}\left\|\mathbf{W}^{n-k}\right\|_{TM}^2+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}G_n^{\gamma_1}\left\|\mathbf{W}^0\right\|_{TM}^2\\
&+C\Delta t\left(\|Z^n\|_m^2+\|Z^{n-1}\|_m^2+\|Z^{n+1}\|_m^2\right).
\end{aligned}\tag{30}$$

Let

$$\begin{aligned}
Q\left(Z^n,\mathbf{W}^n\right)=&\|Z^n\|_m^2+\|2Z^n-Z^{n-1}\|_m^2+\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n-1}G_k^{\gamma_2}\left\|Z^{n-k}\right\|_m^2\\
&+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n-1}G_k^{\gamma_1}\left\|\mathbf{W}^{n-k}\right\|_{TM}^2,
\end{aligned}\tag{31}$$

then we obtain

$$Q(Z^{n+1},\mathbf{W}^{n+1})-Q(Z^n,\mathbf{W}^n)\leqslant\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}G_n^{\gamma_2}\left\|Z^0\right\|_m^2+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}G_n^{\gamma_1}\left\|\mathbf{W}^0\right\|_{TM}^2 +C\Delta t\left(\|Z^n\|_m^2+\|Z^{n-1}\|_m^2+\|Z^{n+1}\|_m^2\right). \tag{32}$$

Summing on n, $n=1,\cdots M-1 (M\leqslant N)$ and noting $\|Z^n\|_m^2\leqslant Q(Z^n)$, we give

$$Q(Z^M,\mathbf{W}^M)\leqslant Q(Z^1,\mathbf{W}^1)+\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{n=1}^{M-1}G_n^{\gamma_2}\left\|Z^0\right\|_m^2+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{n=1}^{M-1}G_n^{\gamma_1}\left\|\mathbf{W}^0\right\|_{TM}^2 +C\Delta t\sum_{n=1}^{M}Q(Z^n,\mathbf{W}^n)+C\Delta t\left\|Z^0\right\|_m^2. \tag{33}$$

We can choose Δt such that $1-C\Delta t>0$, applying Gronwall's lemma [4] gives that

$$\begin{aligned}Q(Z^M,\mathbf{W}^M)&\leqslant CQ(Z^1,\mathbf{W}^1)+C\left[\frac{\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{n=1}^{M-1}G_n^{\gamma_2}\left\|Z^0\right\|_m^2+\Delta t\left\|Z^0\right\|_m^2\right]\\&\quad+C\frac{\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{n=1}^{M-1}G_n^{\gamma_1}\left\|\mathbf{W}^0\right\|_{TM}^2\\&\leqslant CQ(Z^1,\mathbf{W}^1)+C\left[\frac{\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\left(M^{\gamma_2}-1\right)+\Delta t\right]\left\|Z^0\right\|_m^2\\&\quad+C\frac{\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\left(M^{\gamma_1}-1\right)\left\|\mathbf{W}^0\right\|_{TM}^2\\&\leqslant CQ(Z^1,\mathbf{W}^1)+C\left\|Z^0\right\|_m^2+C\left\|\mathbf{W}^0\right\|_{TM}^2.\end{aligned} \tag{34}$$

For the next procedure, we need to consider the estimation of $Q(Z^1,\mathbf{W}^1)$. Similar to the case $n\geqslant 1$, we can easily obtain that

$$Q(Z^1,\mathbf{W}^1)\leq C\left\|Z^0\right\|_m^2+C\left\|\mathbf{W}^0\right\|_{TM}^2. \tag{35}$$

Then combining the equation (34) with (35), we get

$$Q(Z^M,\mathbf{W}^M)\leqslant C\left\|Z^0\right\|_M^2++C\left\|\mathbf{W}^0\right\|_{TM}^2\leq C\|p_0\|_M^2+C\|\mathbf{u}_0\|_{TM}^2. \tag{36}$$

Note that

$$G_{N-k}^{\gamma_l}\geqslant G_{N-1}^{\gamma_l}=N^{\gamma_l}-(N-1)^{\gamma_l}>\gamma_l\Delta t^{1-\gamma_l},\ l=1,2. \tag{37}$$

This completes the proof. □

Remark. Taking notice of equation (20), we can easily obtain that the numerical solutions Z^n and $\mathbf{W}^n$ in discrete norms can be controlled by the initial condition p_0 and $\mathbf{u}_0$ under the assumption that $A1$ holds.

4.2. Optimal error results

In this subsection, error estimates using the given scheme are obtained. Firstly, we quote the following result.

Lemma 6. *If $p\in C^2(J;C^2(\Omega))$, then we have*

$$\begin{aligned}&\frac{\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n}G_{n-k}^{\gamma_2}d_t\delta_{i,j}^{k+1}\\&=\frac{h_i^2}{8}\left[\frac{\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n}G_{n-k}^{\gamma_2}d_t\left(\frac{\partial^2 p}{\partial x^2}\right)_{i,j}^{k+1}\right]+\frac{k_j^2}{8}\left[\frac{\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n}G_{n-k}^{\gamma_2}d_t\left(\frac{\partial^2 p}{\partial y^2}\right)_{i,j}^{k+1}\right]\\&=O\left(h^2+k^2\right).\end{aligned}$$

$$D_t\delta^n_{i,j} = \frac{3\delta^{n+1}_{i,j} - 4\delta^n_{i,j} + \delta^{n-1}_{i,j}}{2\Delta t} = O\left(h^2 + k^2\right),$$

and

$$d_t\delta^n_{i,j} = d_t\left(\frac{h^2}{8}\frac{\partial^2 p}{\partial x^2} + \frac{k^2}{8}\frac{\partial^2 p}{\partial y^2}\right)^n_{i,j} = \frac{h_i^2}{8}d_t\left(\frac{\partial^2 p}{\partial x^2}\right)^n_{i,j} + \frac{k_j^2}{8}d_t\left(\frac{\partial^2 p}{\partial y^2}\right)^n_{i,j} = O\left(h^2 + k^2\right).$$

The proof of this lemma can be obtained easily. From equation (2), we have

Case I: $n = 0$

$$\begin{aligned}&\left[1 + \frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\right][d_t p]^1_{i,j} + \frac{\mu t_1^{\gamma_2-1}}{\Gamma(\gamma_2)}p^0_{i,j} + \frac{K t_1^{\gamma_1-1}}{\Gamma(\gamma_1)}\mathbf{Du}^0_{i,j}\\ &\quad + \frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}[d_t\mathbf{Du}]^1_{i,j} = f(p^0_{i,j}) + \sum_{k=1}^{4}E^1_{k,i,j},\end{aligned} \tag{38}$$

Case II: $n \geqslant 1$

$$\begin{aligned}&[D_t p]^{n+1}_{i,j} + \frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n}G^{\gamma_2}_{n-k}[d_t p]^{k+1}_{i,j} + \frac{\mu t_{n+1}^{\gamma_2-1}}{\Gamma(\gamma_2)}p^0_{i,j} + \frac{K t_{n+1}^{\gamma_1-1}}{\Gamma(\gamma_1)}\mathbf{Du}^0_{i,j}\\ &+ \frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n}G^{\gamma_1}_{n-k}[d_t\mathbf{Du}]^{k+1}_{i,j} = f(2p^n_{i,j} - p^{n-1}_{i,j}) + \sum_{k=3}^{6}E^{n+1}_{k,i,j},\end{aligned} \tag{39}$$

where

$$\begin{aligned}&E^1_{1,i,j} = [d_t p]^1_{i,j} - \left[\frac{\partial p}{\partial t}\right]^1_{i,j},\\ &E^1_{2,i,j} = f(p^1_{i,j}) - f(p^0_{i,j}),\\ &E^{n+1}_{3,i,j} = \frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n}G^{\gamma_2}_{n-k}[d_t p]^{k+1}_{i,j} + \frac{\mu t_{n+1}^{\gamma_2-1}}{\Gamma(\gamma_2)}p^0_{i,j} - \mu\left[{}_0D^{\alpha}_t p\right]^{n+1}_{i,j},\\ &E^{n+1}_{4,i,j} = \frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n}G^{\gamma_1}_{n-k}[d_t\mathbf{Du}]^{k+1}_{i,j} + \frac{K t_{n+1}^{\gamma_1-1}}{\Gamma(\gamma_1)}\mathbf{Du}^0_{i,j} - K\left[{}_0D^{1-\gamma}_t\nabla\mathbf{u}\right]^{n+1}_{i,j},\\ &E^{n+1}_{5,i,j} = [D_t p]^{n+1}_{i,j} - \left[\frac{\partial p}{\partial t}\right]^{n+1}_{i,j},\\ &E^{n+1}_{6,i,j} = f(p^{n+1}_{i,j}) - f(2p^n_{i,j} - p^{n-1}_{i,j}).\end{aligned}$$

Subtracting equation (38) from (16) and equation (39) from (17), we obtain the error equations.

Case I: $n = 0$

$$\begin{aligned}&\left[1 + \frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\right][d_t(p-Z)]^1_{i,j} + \frac{\mu t_1^{\gamma_2-1}}{\Gamma(\gamma_2)}(p-Z)^0_{i,j} + \frac{K t_1^{\gamma_1-1}}{\Gamma(\gamma_1)}\mathbf{D(u\text{-}W)}^0_{i,j}\\ &\quad + \frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}[d_t\mathbf{D(u\text{-}W)}]^1_{i,j} = f(p^0_{i,j}) - f(Z^0_{i,j}) + \sum_{k=1}^{4}E^1_{k,i,j},\end{aligned} \tag{40}$$

Case II: $n \geqslant 1$

$$\begin{aligned}&[D_t(p-Z)]^{n+1}_{i,j} + \frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n}G^{\gamma_2}_{n-k}[d_t(p-Z)]^{k+1}_{i,j} + \frac{\mu t_{n+1}^{\gamma_2-1}}{\Gamma(\gamma_2)}(p-Z)^0_{i,j} + \frac{K t_{n+1}^{\gamma_1-1}}{\Gamma(\gamma_1)}\mathbf{D(u\text{-}W)}^0_{i,j}\\ &+ \frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n}G^{\gamma_1}_{n-k}[d_t\mathbf{D(u\text{-}W)}]^{k+1}_{i,j}\\ &= f(2p^n_{i,j} - p^{n-1}_{i,j}) - f(2Z^n_{i,j} - Z^{n-1}_{i,j}) + \sum_{k=3}^{6}E^{n+1}_{k,i,j}.\end{aligned} \tag{41}$$

Then, in order to simplify the notations, we define

$$e^{p,n}_{i,j} = [p - Z - \delta]^n_{i,j},$$
$$e^{x,n}_{i+1/2,j} = \left[u^x - W^x\right]^n_{i+1/2,j},$$
$$e^{y,n}_{i,j+1/2} = \left[u^y - W^y\right]^n_{i,j+1/2},$$

thus the error equations can be rewritten as follows:

Case I: $n = 0$

$$\begin{aligned}&\left[1 + \frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\right][d_t e^p]^1_{i,j} + \frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}[d_t \mathbf{De}]^1_{i,j}\\ =&f(p^0_{i,j}) - f(Z^0_{i,j}) - d_t\delta^1_{i,j} - \frac{\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}G^{\gamma_2}_0 d_t\delta^1_{i,j} + \sum_{k=1}^{4} E^1_{k,i,j},\end{aligned} \tag{42}$$

Case II: $n \geqslant 1$

$$\begin{aligned}&[D_t e^p]^{n+1}_{i,j} + \frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n} G^{\gamma_2}_{n-k}\left[d_t e^p\right]^{k+1}_{i,j} + \frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n} G^{\gamma_1}_{n-k}[d_t\mathbf{De}]^{k+1}_{i,j}\\ =&f(2p^n_{i,j} - p^{n-1}_{i,j}) - f(2Z^n_{i,j} - Z^{n-1}_{i,j}) - D_t\delta^{n+1}_{i,j}\\ &-\frac{\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n} G^{\gamma_2}_{n-k} d_t\delta^{k+1}_{i,j} + \sum_{k=3}^{6} E^{n+1}_{k,i,j}.\end{aligned} \tag{43}$$

Theorem 7. *Suppose that $p(x, y, t) \in C^2(J; C^4(\Omega))$, then there exists a positive constant C independent of h, k and Δt such that*

$$\begin{aligned}&\left\|Z^l - p^l\right\|_m + \left(\Delta t\sum_{n=1}^{l}\|Z^n - p^n\|^2_m\right)^{1/2} + \left(\Delta t\sum_{n=1}^{l}\|\boldsymbol{W}^n - \boldsymbol{u}^n\|^2_{TM}\right)^{1/2}\\ &\le C\left(h^2 + k^2 + \Delta t^{\alpha}\right), \quad \forall 1 \le l \le N,\end{aligned} \tag{44}$$

where $\alpha = \min\{1+\gamma_1, 1+\gamma_2\}$.

Proof. Multiplying (43) by $e^{p,n+1}_{i,j} h_i k_j$ and making summation on i, j for $1 \le i \le N_x$, $1 \le j \le N_y$, we have that

$$\begin{aligned}&\left(D_t e^{p,n+1}, e^{p,n+1}\right)_m + \frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n} G^{\gamma_2}_{n-k}\left(d_t e^{p,k+1}, e^{p,n+1}\right)_m\\ &+\frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n} G^{\gamma_1}_{n-k}\left([d_t\mathbf{De}]^{k+1}, e^{p,n+1}\right)_m\\ =&\left(f(2p^n_{i,j} - p^{n-1}_{i,j}) - f(2Z^n_{i,j} - Z^{n-1}_{i,j}), e^{p,n+1}\right)_m + \sum_{k=3}^{6}\left(E^{n+1}_k, \xi^{n+1}\right)_m\\ &-\left(D_t\delta^{n+1}_{i,j} + \frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n} G^{\gamma_2}_{n-k} d_t\delta^{k+1}_{i,j}, e^{p,n+1}\right)_m.\end{aligned} \tag{45}$$

We estimate the first term in the left hand side of the equation (45).

$$\begin{aligned}\left(D_t e^{p,n+1}, e^{p,n+1}\right)_m = \frac{1}{4\Delta t}\Big[&\left(\|e^{p,n+1}\|^2_m + \|2e^{p,n+1} - e^{p,n}\|^2_m\right) - \left(\|e^{p,n}\|_m + \|2e^{p,n} - e^{p,n-1}\|^2_m\right)\\ &+\|e^{p,n+1} - 2e^{p,n} + e^{p,n-1}\|^2_m\Big].\end{aligned} \tag{46}$$

The second term in the left hand side of equation (45) can be transformed into the following:

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$$\begin{aligned}
&\frac{\mu \Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)} \sum_{k=0}^{n} G_{n-k}^{\gamma_2} \left(d_t e^{p,k+1}, e^{p,n+1} \right)_m \\
=&\frac{\mu \Delta t^{\gamma_2 - 1}}{\Gamma(1+\gamma_2)} \sum_{k=0}^{n} G_{k}^{\gamma_2} \left(e^{p,n-k+1} - e^{p,n-k}, e^{p,n+1} \right)_m \\
=&\frac{\mu \Delta t^{\gamma_2 - 1}}{\Gamma(1+\gamma_2)} \left[G_0^{\gamma_2} \left(e^{p,n+1}, e^{p,n+1} \right)_m - \sum_{k=0}^{n-1} \left(G_k^{\gamma_2} - G_{k+1}^{\gamma_2} \right) \left(e^{p,n-k}, e^{p,n+1} \right)_m \right] \\
&- \frac{\mu \Delta t^{\gamma_2 - 1}}{\Gamma(1+\gamma_2)} G_n^{\gamma_2} \left(e^{p,0}, e^{p,n+1} \right)_m .
\end{aligned} \tag{47}$$

And taking notice of Lemmas 1 and 2, we can get that

$$\left(\mathbf{D}e^{n-k}, e^{p,n+1} \right)_m = -\left(\mathbf{e}^{n-k}, \mathbf{d}e^{p,n+1} \right)_{TM} = \left(\mathbf{d}e^{p,n-k}, \mathbf{d}e^{p,n+1} \right)_{TM} + \left(\widetilde{\boldsymbol{\epsilon}}^{n-k}(p), \mathbf{d}e^{p,n+1} \right)_{TM}, \tag{48}$$

where

$$\mathbf{d}e^{p,n+1} = (d_x e^{p,n+1}, d_y e^{p,n+1}),\ \widetilde{\boldsymbol{\epsilon}}^{n-k}(p) = (\widetilde{\epsilon}^{x,n-k}(p), \widetilde{\epsilon}^{y,n-k}(p)).$$

Then the third term in the left hand side of equation (45) can be transformed into the following:

$$\begin{aligned}
&\frac{K \Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)} \sum_{k=0}^{n} G_{n-k}^{\gamma_1} \left([d_t \mathbf{D}e]^{k+1}, e^{p,n+1} \right)_m \\
=&\frac{K \Delta t^{\gamma_1 - 1}}{\Gamma(1+\gamma_1)} \left[G_0^{\gamma_1} \left(\mathbf{d}e^{p,n+1}, \mathbf{d}e^{p,n+1} \right)_{TM} - \sum_{k=0}^{n-1} \left(G_k^{\gamma_1} - G_{k+1}^{\gamma_1} \right) \left(\mathbf{d}e^{p,n-k}, \mathbf{d}e^{p,n+1} \right)_{TM} \right] \\
&- \frac{K \Delta t^{\gamma_1 - 1}}{\Gamma(1+\gamma_1)} G_n^{\gamma_1} \left(\mathbf{d}e^{p,0}, \mathbf{d}e^{p,n+1} \right)_{TM} + \frac{K \Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)} \sum_{k=0}^{n} G_{n-k}^{\gamma_1} \left(\left[d_t \widetilde{\boldsymbol{\epsilon}}(p) \right]^{k+1}, \mathbf{d}e^{p,n+1} \right)_{TM} .
\end{aligned} \tag{49}$$

Next we estimate the terms in the right hand side of equation (45). First, using triangle inequality and the smoothness assumption $A2$, we obtain that

$$\begin{aligned}
&\left| \left(f(2p_{i,j}^n - p_{i,j}^{n-1}) - f(2Z_{i,j}^n - Z_{i,j}^{n-1}), e^{p,n+1} \right)_m \right| \\
\leqslant& C \left(\|e^{p,n}\|_m^2 + \|e^{p,n-1}\|_m^2 + \|e^{p,n+1}\|_m^2 \right) + O(h^2 + k^2).
\end{aligned} \tag{50}$$

Furthermore, observing Lemmas 3 and 4, we get that

$$\left| \left(E_3^{n+1}, e^{p,n+1} \right)_m \right| \leqslant C \|e^{p,n+1}\|_m^2 + O\left(\Delta t^{2+2\gamma_2} \right). \tag{51}$$

$$\left| \left(E_4^{n+1}, e^{p,n+1} \right)_m \right| \leqslant C \|e^{p,n+1}\|_m^2 + O\left(\Delta t^{2+2\gamma_1} + h^4 + k^4 \right). \tag{52}$$

We use Taylor's expansion and Cauchy-Schwarz inequality to get the estimates of the following terms in the right hand side of equation (45).

$$\begin{aligned}
&\left| \left(E_5^{n+1}, \xi^{n+1} \right)_m \right| \leqslant C \|\xi^{n+1}\|_m^2 + O\left(\Delta t^4 \right), \\
&\left| \left(E_6^{n+1}, \xi^{n+1} \right)_m \right| \leqslant C \|\xi^{n+1}\|_m^2 + O\left(\Delta t^4 \right).
\end{aligned} \tag{53}$$

Recalling Lemma 6 and using Cauchy-Schwarz inequality, the last term in the right hand side of equation (45) can be estimated as follows:

$$\left| \left(D_t \delta_{i,j}^{n+1} + \frac{\mu \Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)} \sum_{k=0}^{n} G_{n-k}^{\gamma_2} d_t \delta_{i,j}^{k+1}, e^{p,n+1} \right)_m \right| \leqslant C \|\xi^{n+1}\|_m^2 + O\left(h^4 + k^4 \right). \tag{54}$$

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Combining equation (45) with equations (46)-(53) and noting $G_k^{\alpha}-G_{k+1}^{\alpha}>0$, we obtain

$$\begin{aligned}
&\frac{1}{4\Delta t}\left[\left(\|e^{p,n+1}\|_m^2+\|2e^{p,n+1}-e^{p,n}\|_m^2\right)-\left(\|e^{p,n}\|_m+\|2e^{p,n}-e^{p,n-1}\|_m^2\right)+\|e^{p,n+1}-2e^{p,n}+e^{p,n-1}\|_m^2\right]\\
&\quad+\frac{\mu\Delta t^{\gamma_2-1}}{\Gamma(1+\gamma_2)}G_0^{\gamma_2}\|e^{p,n+1}\|_m^2+\frac{K\Delta t^{\gamma_1-1}}{\Gamma(1+\gamma_1)}G_0^{\gamma_1}\|\mathbf{d}e^{p,n+1}\|_{TM}^2\\
&\le\frac{\mu\Delta t^{\gamma_2-1}}{\Gamma(1+\gamma_2)}\left[\sum_{k=0}^{n-1}\left(G_k^{\gamma_2}-G_{k+1}^{\gamma_2}\right)\left(e^{p,n-k},e^{p,n+1}\right)_m+G_n^{\gamma_2}\left(e^{p,0},e^{p,n+1}\right)_m\right]\\
&\quad+\frac{K\Delta t^{\gamma_1-1}}{\Gamma(1+\gamma_1)}\left[\sum_{k=0}^{n-1}\left(G_k^{\gamma_1}-G_{k+1}^{\gamma_1}\right)\left(\mathbf{d}e^{p,n-k},\mathbf{d}e^{p,n+1}\right)_{TM}+G_n^{\gamma_1}\left(\mathbf{d}e^{p,0},\mathbf{d}e^{p,n+1}\right)_{TM}\right]\\
&\quad-\frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n}G_{n-k}^{\gamma_1}\left(\left[d_t\widetilde{\boldsymbol{\epsilon}}(p)\right]^{k+1},\mathbf{d}e^{p,n+1}\right)_{TM}\\
&\quad+C\left(\|e^{p,n}\|_m^2+\|e^{p,n-1}\|_m^2+\|e^{p,n+1}\|_m^2\right)+O\left(h^4+k^4+\Delta t^{2\alpha}\right),
\end{aligned}\tag{55}$$

where $\alpha=\min\{1+\gamma_1,1+\gamma_2\}$. Multiplying both sides of equation (55) by $4\Delta t$ and using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left(\|e^{p,n+1}\|_m^2+\|2e^{p,n+1}-e^{p,n}\|_m^2\right)+\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\left(\|e^{p,n+1}\|_m^2+\sum_{k=0}^{n-1}G_{k+1}^{\gamma_2}\left\|e^{p,n-k}\right\|_m^2\right)\\
&\quad+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\left(\|\mathbf{d}e^{p,n+1}\|_{TM}^2+\sum_{k=0}^{n-1}G_{k+1}^{\gamma_1}\left\|\mathbf{d}e^{p,n-k}\right\|_{TM}^2\right)\\
&\leqslant\left(\|e^{p,n}\|_m^2+\|2e^{p,n}-e^{p,n-1}\|_m^2\right)+\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{k=0}^{n-1}G_k^{\gamma_2}\left\|e^{p,n-k}\right\|_m^2+\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}G_n^{\gamma_2}\left\|e^{p,0}\right\|_m^2\\
&\quad+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n-1}G_k^{\gamma_1}\left\|\mathbf{d}e^{p,n-k}\right\|_{TM}^2+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}G_n^{\gamma_1}\left\|\mathbf{d}e^{p,0}\right\|_{TM}^2\\
&\quad+\frac{K\gamma_1\Delta t}{\Gamma(1+\gamma_1)}\|\mathbf{d}e^{p,n+1}\|_{TM}^2+C\left(\|e^{p,n}\|_m^2+\|e^{p,n-1}\|_m^2+\|e^{p,n+1}\|_m^2\right)\\
&\quad+C\Delta t\left(h^4+k^4+\Delta t^{2\alpha}\right),
\end{aligned}\tag{56}$$

where the following equation is used.

$$\left|\frac{4K\Delta t^{\gamma_1+1}}{\Gamma(1+\gamma_1)}\sum_{k=0}^{n}G_{n-k}^{\gamma_1}\left(\left[d_t\widetilde{\boldsymbol{\epsilon}}(p)\right]^{k+1},\mathbf{d}e^{p,n+1}\right)_{TM}\right|\le\frac{K\Delta t}{\Gamma(1+\gamma_1)}\|\mathbf{d}e^{p,n+1}\|_{TM}^2+C\Delta t(h^4+k^4).$$

Summing on n, $n=1,\cdots,M-1$ $(M\leqslant N)$ gives that

$$\begin{aligned}
Q\left(e^{p,M},\mathbf{d}e^{p,M}\right)\leqslant&\,Q\left(e^{p,1},\mathbf{d}e^{p,1}\right)+\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{n=1}^{M-1}G_n^{\gamma_2}\left\|e^{p,0}\right\|_m^2+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{n=1}^{M-1}G_n^{\gamma_1}\left\|\mathbf{d}e^{p,0}\right\|_{TM}^2\\
&+C\Delta t\sum_{n=1}^{M}Q\left(e^{p,n},\mathbf{d}e^{p,n}\right)+\frac{K\gamma_1}{\Gamma(1+\gamma_1)}\sum_{n=1}^{M-1}\Delta t\|\mathbf{d}e^{p,n+1}\|_{TM}^2\\
&+C\Delta t\left\|e^{p,0}\right\|_m^2+O(h^4+k^4+\Delta t^{2\alpha}).
\end{aligned}\tag{57}$$

For the next estimation, we have to give error analysis for $Q\left(e^{p,1},\mathbf{d}e^{p,1}\right)$. Multiplying (42) by $2e_{i,j}^{p,1}h_ik_j\Delta t$, making summation on i, j for $1\le i\le N_x$, $1\le j\le N_y$ and using Cauchy-Schwarz inequality, we have that

$$\begin{aligned}&\left(1+\frac{\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\right)\|e^{p,1}\|_m^2+\frac{K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\|\mathbf{d}e^{p,1}\|_{TM}^2\\ \leqslant&\left(1+\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\right)\left\|e^{p,0}\right\|_m^2+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\|\mathbf{d}e^{p,0}\|_{TM}^2+\frac{K\Delta t}{4\Gamma(1+\gamma_1)}\|\mathbf{d}e^{p,1}\|_{TM}^2\\ &+C\left\|e^{p,0}\right\|_m^2+C\Delta t^2\left(\|E_1\|_m^2+\|E_2\|_m^2\right)+\frac{1}{2}\|e^{p,1}\|_m^2\\ &+O\left(h^4+k^4+\Delta t^{2\alpha}\right),\end{aligned}\tag{58}$$

i.e.

$$Q\left(e^{p,1},\mathbf{d}e^{p,1}\right)\leq\frac{K\Delta t}{\Gamma(1+\gamma_1)}\|\mathbf{d}e^{p,1}\|_{TM}^2+O\left(h^4+k^4+\Delta t^{2\alpha}\right).\tag{59}$$

By equations (57) and (59), we have that

$$\begin{aligned}Q\left(e^{p,M},\mathbf{d}e^{p,M}\right)\leqslant&\frac{2\mu\Delta t^{\gamma_2}}{\Gamma(1+\gamma_2)}\sum_{n=1}^{M-1}G_n^{\gamma_2}\left\|e^{p,0}\right\|_m^2+\frac{2K\Delta t^{\gamma_1}}{\Gamma(1+\gamma_1)}\sum_{n=1}^{M-1}G_n^{\gamma_1}\left\|\mathbf{d}e^{p,0}\right\|_{TM}^2\\ &+C\Delta t\sum_{n=1}^{M}Q\left(e^{p,n},\mathbf{d}e^{p,n}\right)+\frac{K\gamma_1}{\Gamma(1+\gamma_1)}\sum_{n=0}^{M-1}\Delta t\|\mathbf{d}e^{p,n+1}\|_{TM}^2\\ &+C\Delta t\left\|e^{p,0}\right\|_m^2+O(h^4+k^4+\Delta t^{2\alpha}).\end{aligned}$$

Taking notice of the equation (37), choosing Δt sufficiently small and applying Gronwall's inequality, we obtain

$$\|e^{p,M}\|_m^2+\sum_{n=1}^{M}\Delta t\|e^{p,n}\|_m^2+\sum_{n=1}^{M}\Delta t\|\mathbf{d}e^{p,n}\|_{TM}^2\leq O\left(h^4+k^4+\Delta t^{2\alpha}\right).\tag{60}$$

Noting that $\mathbf{d}e^{p,n}=-(\mathbf{e}^n+\widetilde{\boldsymbol{\varepsilon}}^n(p))$ and the definition of δ, we can obtain equation (44). □

5. Numerical tests

In this section, some numerical experiments using the block-centered finite difference method have been carried out.

We test Examples 1 and 2 to verify the convergence rates where $\Omega=(0,1)\times(0,1)$ and $K=\mu=1$. In Example 1, the initial spatial partition is a 5×5 grid. Then the grid is refined by dividing each edge into two equal parts and the nonuniform meshes are used which are generated from the corresponding uniform mesh by adding a random in both x and y directions (cf. Fig. 1). The numerical results are listed in Tables 1 and 3, where Table 1 is given to show the space convergence rate and Table 3 is presented in order to show the time convergence rate. And in Example 2, the domain is uniformly divided by the rectangles decomposition. The numerical results are listed in Tables 2 and 4, where Table 2 is given to show the space convergence rate and Table 4 is presented in order to show the time convergence rate. In these tables, we choose the optimal step size $M=\left[N^{\frac{\alpha}{2}}\right]$, where M is the spatial partition and N is the temporal partition. Moreover, as to report the features of the block-centered finite difference method vividly, we give the scalar and flux solution figures of Example 2. Here, just as observed that $u^x(x,y,t)=u^y(y,x,t)$, thus only the exact and numerical solutions of p and u_x are presented. They are shown in Figs. 2 and 3 at $T=1$ with the mesh size $M=20,\ T=1$ for Example 2.

For simplicity, we define

$$\begin{cases}\|Z-p\|_{L^\infty(L^2)}=\max\limits_{1\leq n\leq M}\left\{\|(Z-p)^n\|_m\right\},\\ \|Z-p\|_{L^2(L^2)}=\left(\Delta t\sum\limits_{n=1}^{l}\left\|Z^n-p^n\right\|_m^2\right)^{1/2},\\ \|\mathbf{W}-\mathbf{u}\|_{L^2(L^2)}=\left(\Delta t\sum\limits_{n=1}^{l}\left\|\mathbf{W}^n-\mathbf{u}^n\right\|_{TM}^2\right)^{1/2}.\end{cases}$$

Example 1. The initial condition and the right side of the equation are computed according to the analytic solution given as below.

$$\begin{cases}p(x,y,t)=-t^3cos(2\pi x)cos(2\pi y),\\ f(x,y,t,p)=g(x,y,t),\end{cases}$$

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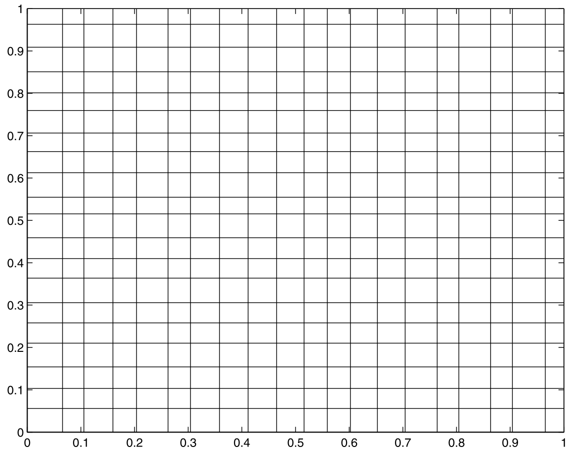

Fig. 1. The non-uniform mesh generated from the 20×20 uniform mesh.

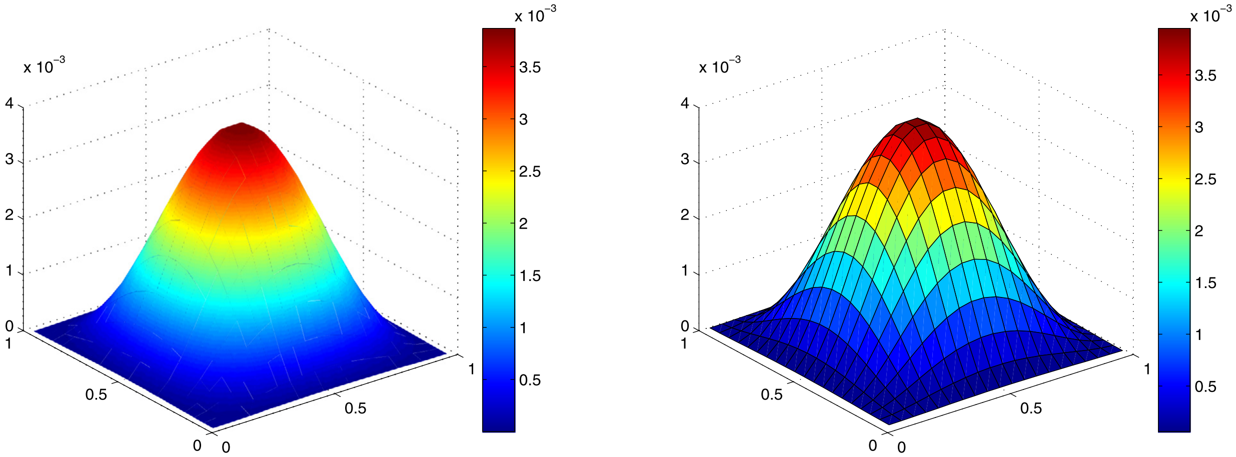

Fig. 2. The scalar solution figures. (a): the exact solution p, (b): the numerical solution Z. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

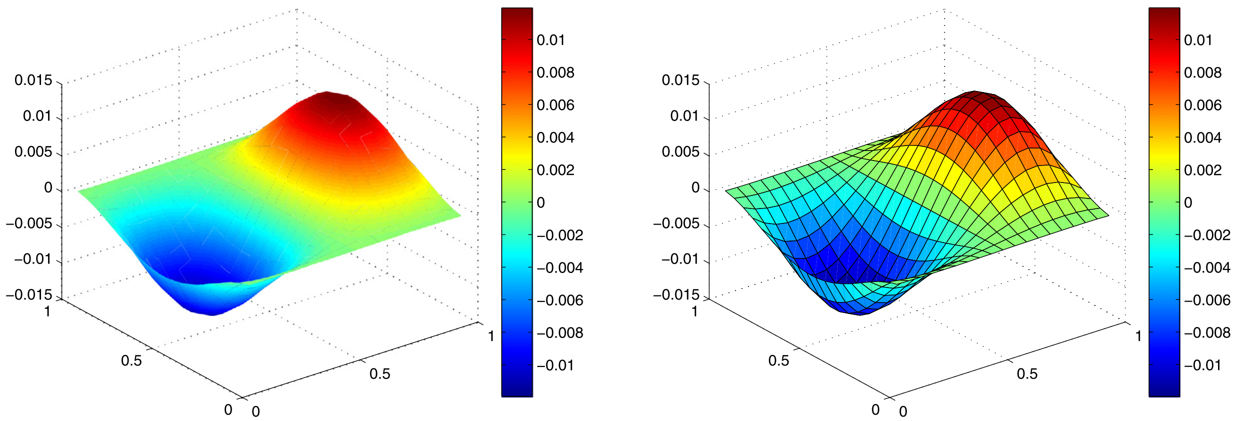

Fig. 3. The flux solution figures. (a): the exact solution u^x, (b): the numerical solution W^x.

where

$$g(x, y, t) = (-3t^2 - \frac{6t^{2+\gamma_2}}{\Gamma(3+\gamma_2)} - 8\pi^2 \frac{6t^{2+\gamma_1}}{\Gamma(3+\gamma_1)}) \cos(2\pi x) \cos(2\pi y).$$

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Table 1
Errors and convergence rates in space of Example 1 with $\gamma_1 = \frac{1}{2},\ \gamma_2 = \frac{1}{4}$.

$M \times M$	$\frac{\|Z-p\|_{L^\infty(L^2)}}{\|p\|_{L^\infty(L^2)}}$	Rate	$\frac{\|Z-p\|_{L^2(L^2)}}{\|p\|_{L^2(L^2)}}$	Rate	$\frac{\|\mathbf{W}-\mathbf{u}\|_{L^2(L^2)}}{\|\mathbf{u}\|_{L^2(L^2)}}$	Rate
5×5	1.45E-1	–	1.49E-1	–	1.25E-1	–
10×10	3.86E-2	1.91	3.95E-2	1.92	3.13E-2	2.00
20×20	9.18E-3	2.07	9.36E-3	2.08	7.27E-3	2.11
40×40	2.28E-3	2.01	2.31E-3	2.02	1.76E-3	2.05
80×80	5.56E-4	2.03	5.62E-4	2.04	4.23E-4	2.06

Table 2
Errors and convergence rates in space of Example 2 with $\gamma_1 = \frac{1}{2},\ \gamma_2 = \frac{1}{4}$.

$M \times M$	$\frac{\|Z-p\|_{L^\infty(L^2)}}{\|p\|_{L^\infty(L^2)}}$	Rate	$\frac{\|Z-p\|_{L^2(L^2)}}{\|p\|_{L^2(L^2)}}$	Rate	$\frac{\|\mathbf{W}-\mathbf{u}\|_{L^2(L^2)}}{\|\mathbf{u}\|_{L^2(L^2)}}$	Rate
5×5	5.52E-1	–	5.37E-1	–	6.12E-2	–
10×10	1.34E-1	2.05	1.29E-1	2.06	1.34E-2	2.19
20×20	3.32E-2	2.01	3.18E-2	2.02	3.04E-3	2.14
40×40	8.27E-3	2.00	7.92E-3	2.01	6.94E-4	2.13
80×80	2.07E-3	2.00	1.98E-3	2.00	1.61E-4	2.11

Table 3
Errors and convergence rates in time of Example 1 with $\alpha = \gamma_1 = \gamma_2$.

α	Δt	$\frac{\|Z-p\|_{L^\infty(L^2)}}{\|p\|_{L^\infty(L^2)}}$	Rate	$\frac{\|Z-p\|_{L^2(L^2)}}{\|p\|_{L^2(L^2)}}$	Rate	$\frac{\|\mathbf{W}-\mathbf{u}\|_{L^2(L^2)}}{\|\mathbf{u}\|_{L^2(L^2)}}$	Rate
0.25	1/10	3.10E-1	–	3.22E-1	–	2.70E-1	–
	1/20	9.93E-2	1.64	1.05E-1	1.62	9.70E-2	1.48
	1/40	4.56E-2	1.12	4.82E-2	1.13	4.40E-2	1.14
	1/80	1.97E-2	1.21	2.09E-2	1.21	1.91E-2	1.21
0.50	1/10	1.27E-1	–	1.32E-1	–	1.16E-1	–
	1/20	5.10E-2	1.32	5.34E-2	1.31	4.55E-2	1.35
	1/40	1.64E-2	1.64	1.73E-2	1.63	1.53E-2	1.57
	1/80	5.75E-3	1.51	6.12E-3	1.50	5.45E-3	1.49
0.75	1/10	7.62E-2	–	7.80E-2	–	5.98E-2	–
	1/20	2.10E-2	1.86	2.13E-2	1.87	1.68E-2	1.83
	1/40	5.97E-3	1.81	6.22E-3	1.78	5.06E-3	1.73
	1/80	1.77E-3	1.75	1.86E-3	1.74	1.53E-3	1.73

Example 2. The initial condition and the right side of the equation are computed according to the analytic solution given as below.

$$\begin{cases} p(x,y,t) = t^3x^2(x-1)^2y^2(y-1)^2, \\ f(x,y,t,p) = -(p^2+p^3) + g(x,y,t), \end{cases}$$

where

$$\begin{aligned} g(x,y,t) &= 3t^2x^2(x-1)^2y^2(y-1)^2 + x^2(x-1)^2y^2(y-1)^2\frac{6t^{2+\gamma_2}}{\Gamma(3+\gamma_2)} \\ &+ (t^3x^2(x-1)^2y^2(y-1)^2)^2 + (t^3x^2(x-1)^2y^2(y-1)^2)^3 - \Big(2x^2y^2(x-1)^2 + 2x^2y^2(y-1)^2 \\ &+2x^2(x-1)^2(y-1)^2 + 2(x-1)^2y^2(y-1)^2) + 8xy^2(x-1)(y-1)^2 + 8x^2y(y-1)(x-1)^2\Big)\frac{6t^{2+\gamma_1}}{\Gamma(3+\gamma_1)}. \end{aligned}$$

From Tables 1-4 and Figs. 2-3, we can see that the block-centered finite difference approximations for the pressure and velocity have the $(h^2 + k^2 + \Delta t^\alpha)$ accuracy in discrete norms, where $\alpha = \min\{1+\gamma_1, 1+\gamma_2\}$. These results are in consistent with the error estimates in Theorem 7.

6. Conclusions

In this article, we develop a block-centered finite difference method for the fractional cable equation. Unconditional stability of the method is proved. Moreover, we demonstrate that the block centered finite difference scheme has α-order accuracy in time increment, where $\alpha = \min\{1+\gamma_1, 1+\gamma_2\}$ and second-order accuracy in x- and y-direction. We verify

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Table 4
Errors and convergence rates in time of Example 2 with $\alpha = \gamma_1 = \gamma_2$.

α	Δt	$\frac{\|Z-p\|_{L^\infty(L^2)}}{\|p\|_{L^\infty(L^2)}}$	Rate	$\frac{\|Z-p\|_{L^2(L^2)}}{\|p\|_{L^2(L^2)}}$	Rate	$\frac{\|\mathbf{W}-\mathbf{u}\|_{L^2(L^2)}}{\|\mathbf{u}\|_{L^2(L^2)}}$	Rate
0.25	1/10	1.20E-0	–	1.20E-0	–	1.62E-1	–
	1/20	3.83E-1	1.65	3.80E-1	1.66	6.33E-2	1.35
	1/40	1.84E-1	1.05	1.82E-1	1.06	2.80E-2	1.18
	1/80	8.15E-2	1.18	8.04E-2	1.18	1.20E-2	1.22
0.50	1/10	4.34E-1	–	4.27E-1	–	6.55E-2	–
	1/20	1.87E-1	1.21	1.82E-1	1.23	2.59E-2	1.34
	1/40	5.90E-2	1.67	5.72E-2	1.67	9.18E-3	1.50
	1/80	2.07E-2	1.51	2.00E-2	1.52	3.32E-3	1.47
0.75	1/10	2.52E-1	–	2.44E-1	–	2.82E-2	–
	1/20	6.24E-2	2.02	5.99E-2	2.03	8.24E-3	1.78
	1/40	1.94E-2	1.69	1.84E-2	1.70	2.63E-3	1.65
	1/80	5.70E-3	1.77	5.40E-3	1.77	8.10E-4	1.70

the applicability and accuracy of the scheme by numerical experiments. In our future work, we will consider two-grid block-centered finite difference method for the nonlinear fractional cable equation.

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References

[1] T. Arbogast, M.F. Wheeler, I. Yotov, Mixed finite elements for elliptic problems with tensor coefficients as cell-centered finite differences, SIAM J. Numer. Anal. 34 (2) (1997) 828–852.
[2] E. Barkai, R. Metzler, J. Klafter, From continuous time random walks to the fractional Fokker-Planck equation, Phys. Rev. E 61 (1) (2000) 132.
[3] D.A. Benson, S.W. Wheatcraft, M.M. Meerschaert, Application of a fractional advection-dispersion equation, Water Resour. Res. 36 (6) (2000) 1403–1412.
[4] S.C. Brenner, L.R. Scott, The Mathematical Theory of Finite Element Methods, Springer-Verlag, New York, 1994.
[5] B. Henry, T. Langlands, S. Wearne, Fractional cable models for spiny neuronal dendrites, Phys. Rev. Lett. 100 (12) (2008) 128103.
[6] X. Hu, L. Zhang, Implicit compact difference schemes for the fractional cable equation, Appl. Math. Model. 36 (9) (2012) 4027–4043.
[7] T. Langlands, B.I. Henry, The accuracy and stability of an implicit solution method for the fractional diffusion equation, J. Comput. Phys. 205 (2) (2005) 719–736.
[8] T. Langlands, B. Henry, S. Wearne, Fractional cable equation models for anomalous electrodiffusion in nerve cells: infinite domain solutions, J. Math. Biol. 59 (6) (2009) 761–808.
[9] T. Langlands, B. Henry, S.L. Wearne, Fractional cable equation models for anomalous electrodiffusion in nerve cells: finite domain solutions, SIAM J. Appl. Math. 71 (4) (2011) 1168–1203.
[10] X. Li, H. Rui, A two-grid block-centered finite difference method for nonlinear non-Fickian flow model, Appl. Math. Comput. 281 (2016) 300–313.
[11] Y. Lin, C. Xu, Finite difference/spectral approximations for the time-fractional diffusion equation, J. Comput. Phys. 225 (2) (2007) 1533–1552.
[12] F. Liu, V.V. Anh, I. Turner, P. Zhuang, Time fractional advection-dispersion equation, J. Appl. Math. Comput. 13 (1–2) (2003) 233–245.
[13] F. Liu, V. Anh, I. Turner, Numerical solution of the space fractional Fokker–Planck equation, J. Comput. Appl. Math. 166 (1) (2004) 209–219.
[14] F. Liu, Q. Yang, I. Turner, Two new implicit numerical methods for the fractional cable equation, J. Comput. Nonlinear Dyn. 6 (1) (2011) 011009.
[15] J. Liu, H. Li, Y. Liu, A new fully discrete finite difference/element approximation for fractional cable equation, J. Appl. Math. Comput. (2015) 1–17.
[16] W. Liu, J. Cui, J. Xin, A block-centered finite difference method for an unsteady asymptotic coupled model in fractured media aquifer system, J. Comput. Appl. Math. 337 (2018) 319–340.
[17] Y. Liu, Y. Du, H. Li, S. He, W. Gao, Finite difference/finite element method for a nonlinear time-fractional fourth-order reaction–diffusion problem, Comput. Math. Appl. 70 (4) (2015) 573–591.
[18] Y. Liu, Y.-W. Du, H. Li, J.-F. Wang, A two-grid finite element approximation for a nonlinear time-fractional cable equation, Nonlinear Dyn. (2015) 1–14.
[19] Z. Liu, X. Li, A parallel CGS block-centered finite difference method for a nonlinear time-fractional parabolic equation, Comput. Methods Appl. Mech. Eng. 308 (2016) 330–348.
[20] Z. Liu, X. Li, Efficient modified techniques of invariant energy quadratization approach for gradient flows, Appl. Math. Lett. 98 (2019) 206–214.
[21] R. Magin, X. Feng, D. Baleanu, Solving the fractional order Bloch equation, Concepts Magn. Reson., Part A, Bridg. Educ. Res. 34 (1) (2009) 16.
[22] R. Metzler, J. Klafter, Boundary value problems for fractional diffusion equations, Phys. A, Stat. Mech. Appl. 278 (1) (2000) 107–125.
[23] M. Raberto, E. Scalas, F. Mainardi, Waiting-times and returns in high-frequency financial data: an empirical study, Phys. A, Stat. Mech. Appl. 314 (1) (2002) 749–755.
[24] P.-A. Raviart, J.-M. Thomas, A mixed finite element method for 2-nd order elliptic problems, in: Mathematical Aspects of Finite Element Methods, Springer, 1977, pp. 292–315.
[25] J. Ren, Z.-Z. Sun, X. Zhao, Compact difference scheme for the fractional sub-diffusion equation with Neumann boundary conditions, J. Comput. Phys. 232 (1) (2013) 456–467.
[26] H. Rui, W. Liu, A two-grid block-centered finite difference method for Darcy-Forchheimer flow in porous media, SIAM J. Numer. Anal. 53 (4) (2015) 1941–1962.
[27] H. Rui, H. Pan, A block-centered finite difference method for the Darcy–Forchheimer model, SIAM J. Numer. Anal. 50 (5) (2012) 2612–2631.
[28] H. Rui, H. Pan, Block-centered finite difference methods for parabolic equation with time-dependent coefficient, Jpn. J. Ind. Appl. Math. 30 (3) (2013) 681–699.
[29] E. Scalas, R. Gorenflo, F. Mainardi, Fractional calculus and continuous-time finance, Phys. A, Stat. Mech. Appl. 284 (1) (2000) 376–384.

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[30] Z.-z. Sun, X. Wu, A fully discrete difference scheme for a diffusion-wave system, Appl. Numer. Math. 56 (2) (2006) 193–209.
[31] Y. Wang, Y. Liu, H. Li, J. Wang, Finite element method combined with second-order time discrete scheme for nonlinear fractional cable equation, Eur. Phys. J. Plus 131 (3) (2016) 1–16.
[32] W. Wyss, The fractional Black-Scholes equation, Fract. Calc. Appl. Anal. 3 (1) (2000) 51–62.
[33] B. Yu, X. Jiang, Numerical identification of the fractional derivatives in the two-dimensional fractional cable equation, J. Sci. Comput. (2015) 1–21.
[34] S. Yuste, L. Acedo, K. Lindenberg, Reaction front in an $a+b \to c$ reaction-subdiffusion process, Phys. Rev. E 69 (3) (2004) 036126.
[35] S.B. Yuste, K. Lindenberg, Subdiffusion-limited $a+a$ reactions, Phys. Rev. Lett. 87 (11) (2001) 118301.
[36] Y.-n. Zhang, Z.-z. Sun, H.-w. Wu, Error estimates of Crank-Nicolson-type difference schemes for the subdiffusion equation, SIAM J. Numer. Anal. 49 (6) (2011) 2302–2322.
[37] X. Zhao, Z.-z. Sun, A box-type scheme for fractional sub-diffusion equation with Neumann boundary conditions, J. Comput. Phys. 230 (15) (2011) 6061–6074.
[38] Z. Zhou, Z. Tan, Finite element approximation of optimal control problem governed by space fractional equation, J. Sci. Comput. 78 (3) (2019) 1840–1861.
[39] Z. Zhou, G. Wei, Finite element approximation of optimal control problems governed by time fractional diffusion equation, Comput. Math. Appl. 71 (1) (2016) 301–318.

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