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The well-posedness problem of a hyperbolic–parabolic mixed type equation on an unbounded domain

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Abstract
To study the well-posedness problem of a hyperbolic–parabolic mixed type equation, the usual boundary value condition is overdetermined. Since the equation is with strong nonlinearity, the optimal partially boundary value condition can not be expressed by Fichera function. By introducing the weak characteristic function method, a different but reasonable partial boundary value condition is found first time, basing on it, the stability of the entropy solutions is established.

Keywords Hyperbolic–parabolic mixed type equation · Unbounded domain · Optimal partially boundary value condition · The weak characteristic function method

Mathematics Subject Classification 35L65 · 35K85 · 35R35

1 Introduction

Consider the hyperbolic–parabolic mixed type equation

$$\frac{\partial u}{\partial t} = \Delta A(u) + \mathrm{div}(b(u)), \quad \text{in } Q_T = \Omega \times (0, T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is an unbounded domain with a C^2 boundary, and

$$A(u) = \int_0^u a(s)ds, \quad a(s) \geq 0. \tag{1.2}$$

Equation (1.1) arises from reaction diffusion process and many other applied fields. The Cauchy problem of Eq. (1.1) has been deeply investigated [1–13]. On

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the initial-boundary value problem, there are a lot of papers [14–20] devoting to the well-posedness of the weak solutions. In general, the initial value condition is always required

$$u(x,0)=u_0(x),\quad x\in\Omega. \tag{1.3}$$

But instead of the Dirichlet homogeneous boundary condition

$$u(x,t)=0,\quad (x,t)\in\partial\Omega\times(0,T), \tag{1.4}$$

since $a(s)\geq 0$ and the equation is degenerate, only a partial boundary value condition

$$u(x,t)=0,\quad (x,t)\in\Sigma_p\times(0,T), \tag{1.5}$$

is needed generally, where $\Sigma_p\subseteq\partial\Omega$ is a relative open subset. By using the Fechira function, we have given a conjecture to describe the geometric characteristics of Σ_p formally in [18,19], but the conjecture still remains open, only when $\Sigma_p=\partial\Omega$, the stability of the solutions had been proved. At the same time, by generalizing the definition of the trace to a weaker sense, Refs. [14–17] had succeeded to establish the well-posedness of the solutions of Eq. (1.1) in some special senses. In addition, we had studied Eq. (1.1) when $\Omega=\mathbb{R}^N_+$ is the half space [20]. However, when the domain Ω is unbounded, the problem is far to be solved.

Since Eq. (1.1) is with hyperbolic–parabolic mixed type, we should use the entropy solution to consider the uniqueness problem. We will continue to use some ideas in our previous works [7,8,18–20] to define the entropy solutions in $BV_{loc}(Q_T)$. The essential improvement lies in that the partial boundary $\Sigma_p\subseteq\partial\Omega$, where the homogeneous boundary value is imposed as (1.5), is depicted out first time. Moreover, the stability of the entropy solutions can be proved without any boundary value condition in some special cases.

2 The definition and the main results

For small $\eta>0$, let

$$S_\eta(s)=\int_0^s h_\eta(\tau)d\tau,\quad h_\eta(s)=\frac{2}{\eta}\left(1-\frac{|\,s\,|}{\eta}\right)_+. \tag{2.1}$$

The purpose of S_η is to approximate the sign function. Obviously $h_\eta(s)\in C(\mathbb{R})$, and

$$h_\eta(s)\geq 0,\quad |\,sh_\eta(s)\,|\leq 1,\ |\,S_\eta(s)\,|\leq 1;\quad \lim_{\eta\to 0}S_\eta(s)=\text{sgn}s,\ \lim_{\eta\to 0}sS'_\eta(s)=0. \tag{2.2}$$

Definition 2.1 A function u is said to be the entropy solution of Eq. (1.1) with the initial value (1.3) and with the partial boundary value (1.5), if

1. u satisfies

$$u \in BV_{loc}(Q_T) \cap L^\infty(Q_T), \quad \frac{\partial}{\partial x_i} \int_0^u \sqrt{a(s)} ds \in L^2(\Omega_R \times (0, T)), \tag{2.3}$$

   where $\Omega_R = \{x \in \Omega : |x| < R\}$, for any positive constant R.
2. For any φ, $\varphi \in C_0^2(Q_T)$, $\varphi \geq 0$, for any $k \in \mathbb{R}$, for any small $\eta > 0$, u satisfies

$$\iint_{Q_T} \left[I_\eta(u-k)\varphi_t - B_\eta^i(u,k)\varphi_{x_i} \right. \\ \left. + A_\eta(u,k)\Delta\varphi - S'_\eta(u-k) \left| \nabla \int_0^u \sqrt{a(s)} ds \right|^2 \varphi \right] dxdt \geq 0. \tag{2.4}$$

3. For any positive constant R,

$$\lim_{t \to 0} \int_{\Omega_R} | u(x,t) - u_0(x) | \, dx = 0. \tag{2.5}$$

4. The boundary value (1.5) is satisfied in the sense of trace,

$$\gamma u \mid_{\Sigma_p} = 0. \tag{2.6}$$

Here and the after, the double indices i represents the sum from 1 to N, and

$$B_\eta^i(u,k) = \int_k^u b'^i(s) S_\eta(s-k) ds,$$
$$A_\eta(u,k) = \int_k^u a(s) S_\eta(s-k) ds,$$
$$I_\eta(u-k) = \int_0^{u-k} S_\eta(s) ds.$$

To explain the reasonableness of Definition 2.1, in one way, if Eq. (1.1) has a classical solution u, multiplying (1.1) by $\varphi S_\eta(u-k)$ and integrating over Q_T, we are able to show that u satisfies (2.4). In another way, let $\eta \to 0$ in (2.4). Then

$$\iint_{Q_T} \Big[|u-k|\varphi_t - \text{sgn}(u-k)(b^i(u) - b^i(k))\varphi_{x_i} \text{sgn}(u-k) \\ (A(u) - A(k))\Delta\varphi \Big] \quad dxdt \geq 0.$$

Thus if u is the entropy solution in Definition 2.1, then u is a entropy solution defined in [1–3] et al.

Since the domain is unbounded, we use some techniques, which had been used in considering the Cauchy problem of the equation, to prove the existence of the entropy solutions.

Theorem 2.2 *Suppose that $A(s)$ is C^3, $b^i(s)$ is C^2, $u_0(x) \in L^\infty(\Omega)$, Then Eq.* (1.1) *with the initial boundary value conditions* (1.3), (1.5) *has a entropy solution in the sense of Definition* 2.1.

The main aims of the paper are to establish the stability of the entropy solutions. For simplicity, we assume that the domain Ω can be depicted out as

$$\Omega = \{x \in \mathbb{R}^N : g(x) > 0\}, \quad \partial\Omega = \{x \in \mathbb{R}^N : g(x) = 0\}, \tag{2.7}$$

where $g(x)$ is a continuous function and is a C^2 function when x is near to the boundary $\partial\Omega$. By comparing with the usual characteristic function $\chi(x)$ of $\Omega \subset \mathbb{R}^N$, say, $\chi(x) = 1$ when $x \in \Omega$, while $\chi(x) = 0$ if $x \in \mathbb{R}^N \setminus \Omega$, we can call $g(x)$ is a weak characteristic function of Ω. Certainly, unlike the usual characteristic function, $g(x)$ is not unique.

Theorem 2.3 *Suppose that $A(s)$ is C^2, $b^i(s)$ is C^1,*

$$|b^{i\prime}(s)| \leq (1-\delta)a(s), \tag{2.8}$$

$$|A(s_1) - A(s_2)| \leq \alpha|s_1 - s_2|, \quad |b^i(s_1) - b^i(s_2)| \leq \gamma^i|s_1 - s_2|. \tag{2.9}$$

Let u, v be solutions of Eq. (1.1) *with the different initial values $u_0(x)$, $v_0(x) \in L^\infty(\Omega)$ respectively, but without any boundary value condition. If $g(x) \in C^2(\overline{\Omega})$, and*

$$\Delta g + |\nabla g| \leq 0, \tag{2.10}$$

then

$$\begin{aligned} &\int_\Omega |u(x,t) - v(x,t)|\, \nu_\delta(x) g(x) dx \\ &\quad \leq \int_\Omega |u_0(x) - v_0(x)|\, \nu_\delta(x) g(x) dx, \ \ a.e.\ t \in [0,T), \end{aligned} \tag{2.11}$$

where

$$\nu_\delta(x) = e^{-\delta\sqrt{1+|x|^2}},$$

and δ is a small positive constant, α, γ^i are constants.

Theorem 2.4 *Suppose that $A(s)$ is C^2, $b^i(s)$ is C^1, $b^{i\prime}(s) \geq 0$ such that* (2.6) *is true. Let u, v be solutions of Eq.* (1.1) *with the different initial values $u_0(x)$, $v_0(x) \in L^\infty(\Omega)$ respectively, and with the same partial homogeneous boundary value condition*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T). \tag{2.12}$$

Then

$$\int_{\Omega} |u(x,t) - v(x,t)|\, \nu_\delta(x) dx \le c \int_{\Omega} |\, u_0(x) - v_0(x)\,|\, \nu_\delta(x) dx. \tag{2.13}$$

Here

$$\Sigma_p = \{x \in \partial\Omega : \Delta g + |\nabla g| \ge 0\}. \tag{2.14}$$

Noticing that Σ_p only depends on the the function $g(x)$, we can call the method used in this paper as the weak characteristic function method. This method is easily to be generalized to study the stability of the other degenerate parabolic equations.

3 Proof of Theorem 2.2

Lemma 3.1 [21] *Assume that $\Omega \subset \mathbb{R}^N$ is an open set and let $f_k, f \in L^q(\Omega)$, as $k \to \infty$, $f_k \rightharpoonup f$ weakly in $L^q(\Omega)$, $1 \le q < \infty$. Then*

$$\liminf_{k\to\infty} \| f_k \|^q_{L^q(\Omega)} \ge \| f \|^q_{L^q(\Omega)}\,. \tag{3.1}$$

Consider the following regularized problem

$$\frac{\partial u}{\partial t} = \Delta A(u) + \frac{1}{n}\Delta u + \frac{\partial b^i(u)}{\partial x_i}, \quad (x,t) \in Q_{nT} = \Omega_n \times (0,T), \tag{3.2}$$

with the initial boundary conditions with initial boundary value conditions

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega_n \times (0,T), \tag{3.3}$$

$$u(x,0) = u_{0n}(x), \quad x \in \Omega_n, \tag{3.4}$$

where for large enough n, $\Omega_n = \{x \in \Omega : |x| < n\}$, and $u_{0n}(x) \in C_0^\infty(\Omega_n)$ such that $u_{0n}(x)$ locally uniform converges to $u_0(x)$.

It is well known that there are classical solutions $u_n \in C^2(\overline{Q_{nT}}) \bigcap C^3(Q_{nT})$ of this problem provided that A, b_i satisfy the assumptions in Theorem 2.2, one can refer to the eighth chapter of [22] for this fact.

We need to make some estimates for u_n of (3.2). Firstly, by the maximum principle, we have

$$|u_n| \le \|u_{0n}\|_{L^\infty} \le c. \tag{3.5}$$

Secondly, we have the following important estimate of the solutions u_n of (3.2) with the initial boundary value conditions (3.3), (3.4).

Lemma 3.2 *Let u_n be the solution of (3.2) with (3.3), (3.4). If the assumptions of Theorem 2.2 are true, then*

$$\int_\Omega |gradu_n|\nu_\delta(x)dx \le c. \tag{3.6}$$

where $|gradu|^2 = \sum_{i=1}^N |\frac{\partial u}{\partial x_i}|^2 + |\frac{\partial u}{\partial t}|^2$, c is independent of n. σ is a given positive constant, and

$$\nu_\delta(x) = \exp\left(-\sigma\sqrt{1+|x|^2}\right).$$

Proof We generalize the solution u_n to the whole space $\mathbb{R}^N$ such that when $x \in \mathbb{R}^N \setminus \Omega_n$, $u_n \equiv 0$. We denote the generalized function by $\overline{u}_n$, for simplicity, we denote it as u for the time being. Differentiate (3.2) with respect to x_s, $s = 1, 2, \cdot, N, N+1, x_{N+1} = t$, and sum up for s after multiplying the resulting relation by $u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\nu_\delta(x)$, and integrating over $\mathbb{R}^N$ yields

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} I_\eta(|\text{grad}u|)\nu_\delta(x)dx - \frac{1}{n}\int_{\mathbb{R}^N} \Delta u_{x_s}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\nu_\delta(x)dx\\
&\quad -\int_{\mathbb{R}^N}\frac{\partial}{\partial x_i}(a(u)u_{x_i}u_{x_j} + a(u)\Delta u)u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\nu_\delta(x)dx\\
&\quad -\int_{\mathbb{R}^N}\frac{\partial(b^{i'}(u)u_{x_s})}{\partial x_i}u_{x_s}\frac{S_\eta(|\text{grad}u|)}{|\text{grad}u|}\nu_\delta(x)dx\\
&= 0.
\end{aligned} \tag{3.7}$$

Here and the after, the double indices such s, p represent the sum from 1 to $N+1$, the double indices i, j represent the sum from 1 to N.

Noticing that, for any $\xi = (\xi_1, \xi_2, \cdot, \xi_N, \xi_{N+1})$,

$$\frac{\partial I_\eta}{\partial \xi_s} = S_\eta(|\xi|)\frac{\xi_s}{|\xi|},$$

$$\frac{\partial^2 I_\eta}{\partial\xi_s\partial\xi_p} = \begin{cases} \frac{S'_\eta(|\xi|)|\xi|\xi_p\xi_s - S_\eta(|\xi|)\xi_p\xi_s}{|\xi|^3}, & \text{if } s \ne p,\\ \frac{S'_\eta(|\xi|)|\xi|\xi_p\xi_s - S_\eta(|\xi|)\xi_p\xi_s}{|\xi|^3} + \frac{S_\eta(|\xi|)}{|\xi|}, & \text{if } s = p. \end{cases}$$

Integrating by part, we have

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^N} I_\eta(|\text{grad}u|)\nu_\delta(x)dx - \frac{1}{n}\int_{\mathbb{R}^N}\frac{\partial^2 I_\eta}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}\nu_\delta(x)dx\\
&\quad -\int_{\mathbb{R}^N} a(u)\frac{\partial^2 I_\eta}{\partial\xi_s\partial\xi_p}u_{x_sx_i}u_{x_px_i}\nu_\delta(x)dx\\
&\quad -\int_{\mathbb{R}^N}\frac{\partial a'(u)}{\partial x_i}u_{x_i}(|\text{grad}u|S_\eta(|\text{grad}u|) - I_\eta(|\text{grad}u|))\nu_\delta(x)dx
\end{aligned}$$

$$\begin{aligned}&-\int_{\mathbb{R}^N}\frac{\partial b^{i'}(u)}{\partial x_i}(|\mathrm{grad}u|S_\eta(|\mathrm{grad}u|)-I_\eta(|\mathrm{grad}u|))\nu_\delta(x)dx\\&-\int_{\mathbb{R}^N}a'(u)\Delta u(|\mathrm{grad}u|S_\eta(|\mathrm{grad}u|)-I_\eta(|\mathrm{grad}u|))\nu_\delta(x)dx\\&=0.\end{aligned}\tag{3.8}$$

Letting $\eta\to 0$, and noticing that

$$\lim_{\eta\to 0}\left[|\mathrm{grad}u|S_\eta(|\mathrm{grad}u|)-I_\eta(|\mathrm{grad}u|)\right]=0,$$

similar as the proofs of [17,18,20]. We can show that

$$\frac{d}{dt}\int_{\mathbb{R}^N}|\mathrm{grad}u|\nu_\delta(x)dx\le c_1+c_2\int_{\mathbb{R}^N}|\mathrm{grad}u|\nu_\delta(x)dx,$$

by the well-known Gronwall Lemma, we have

$$\int_{\mathbb{R}^N}|\mathrm{grad}u|\nu_\delta(x)dx\le c.$$

Accordingly, denoting back u as $\overline{u}_n$, we have

$$\int_{\mathbb{R}^N}|\mathrm{grad}\overline{u}_n|\nu_\delta(x)dx\le c.\tag{3.9}$$

By (3.2), (3.9), it is not difficult to show that

$$\int_0^T\int_{\mathbb{R}^N}\left[a(\overline{u}_n)+\frac{1}{n}\right]|\nabla u_n|^2dxdt\le c.\tag{3.10}$$

Then, by (3.9)–(3.10), we have

$$\int_\Omega|\mathrm{grad}\bar{u_n}|\nu_\delta(x)dxdt\le c.\tag{3.11}$$

$$\int_0^T\int_\Omega\left[a(\bar{u_n})+\frac{1}{n}\right]|\nabla\bar{u_n}|^2\nu_\delta(x)dxdt\le c.\tag{3.12}$$

The proof is complete. □

Thus there exists a subsequence of $\{u_n\}$ (we still denote it as u_n) and a function $u\in BV_{loc}(Q_T)\cap L^\infty(Q_T)$ such that $u_n\to u$ a.e. on Q_T.

Proof of Theorem 2.2 First of all, by (3.12)

$$\frac{\partial}{\partial x_i}\int_0^{u_n}\sqrt{a(s)}ds\rightharpoonup\frac{\partial}{\partial x_i}\int_0^{u}\sqrt{a(s)}ds\ \text{weakly in}\ L^2(\Omega_R\times(0,T)),\quad\forall R>0,$$

$$\frac{\partial}{\partial x_i}\int_0^u \sqrt{a(s)}ds \in L^2(\Omega_R \times (0,T)), \quad \forall\, R>0, i=1,2,\ldots,N.$$

Let $\varphi \geq 0$, $\varphi \in C^2(Q_T)$. Multiplying (2.2) by $\varphi S_\eta(u_n-k)$, and integrating over Q_T, we obtain

$$\begin{aligned}&\iint_{Q_T}\frac{\partial u_n}{\partial t}\varphi S_\eta(u_n-k)dxdt = \iint_{Q_T}\Delta A(u_n)\varphi S_\eta(u_n-k)dxdt\\ &+\frac{1}{n}\iint_{Q_T}\Delta u_n\varphi S_\eta(u_n-k)dxdt + \iint_{Q_T}\frac{\partial b^i(u_n)}{\partial x_i}\varphi S_\eta(u_n-k)dxdt.\end{aligned} \tag{3.13}$$

Let's calculate every term in (3.13).

$$\iint_{Q_T}\frac{\partial u_n}{\partial t}\varphi S_\eta(u_n-k)dxdt = -\iint_{Q_T} I_\eta(u_n-k)\varphi_t dxdt. \tag{3.14}$$

$$\begin{aligned}&\frac{1}{n}\iint_{Q_T}\Delta u_n\varphi S_\eta(u_n-k)dxdt\\ &=-\frac{1}{n}\iint_{Q_{nT}}\nabla u_n(S_\eta(u_n-k)\nabla\varphi+\varphi S_\eta'(u_n-k)\nabla u_n)dxdt\\ &=-\frac{1}{n}\iint_{Q_{nT}}\nabla u_n S_\eta(u_n-k)\nabla\varphi dxdt - \frac{1}{n}\iint_{Q_T} |\nabla u_n|^2 S_\eta'(u_n-k)\varphi dxdt,\end{aligned} \tag{3.15}$$

$$\begin{aligned}&\iint_{Q_T}\Delta A(u_n)\varphi S_\eta(u_n-k)dxdt\\ &=-\iint_{Q_T}\nabla A(u_n)(S_\eta(u_n-k)\nabla\varphi+\varphi S_\eta'(u_n-k)\nabla u_n)dxdt\\ &=-\iint_{Q_T}\nabla A(u_n)S_\eta(u_n-k)\nabla\varphi dxdt - \iint_{Q_T} a(u_n)\,|\nabla u_n|^2 S_\eta'(u_n-k)\varphi dxdt\\ &=\iint_{Q_T} A_\eta(u_n,k)\triangle\varphi dxdt - \iint_{Q_T} a(u_n)\,|\nabla u_n|^2 S_\eta'(u_n-k)\varphi dxdt,\end{aligned} \tag{3.16}$$

$$\begin{aligned}&\iint_{Q_T}\frac{\partial b^i(u_n)}{\partial x_i}\varphi S_\eta(u_n-k)dxdt\\ &=-\iint_{Q_{nT}}[b_i(u_n)-b^i(k)]\left[\frac{\partial\varphi}{\partial x_i}S_\eta(u_n-k)+\varphi S_\eta'(u_n-k)\frac{\partial u_n}{\partial x_i}\right]dxdt\\ &=-\iint_{Q_T} B_\eta^i(u_n,k)\varphi_{x_i}dxdt.\end{aligned} \tag{3.17}$$

From (3.13)–(3.17), we have

$$\iint_{Q_{nT}} I_\eta(u_n-k)\varphi_t dxdt + \iint_{Q_T} A_\eta(u_n,k)\triangle\varphi dxdt - \iint_{Q_T} B^i_\eta(u_n,k)\varphi_{x_i} dxdt$$
$$-\frac{1}{n}\iint_{Q_T} \nabla u_n\cdot\nabla\varphi S_\eta(u_n-k)dxdt - \frac{1}{n}\iint_{Q_T} |\nabla u_n|^2 S'_\eta(u_n-k)\varphi dxdt$$
$$-\iint_{Q_T} a(u_n)|\nabla u_n|^2 S'_\eta(u_n-k)\varphi dxdt$$
$$=0. \tag{3.18}$$

By that

$$\lim_{n\to\infty}\frac{1}{n}\iint_{Q_T}\nabla u_n\cdot\nabla\varphi S_\eta(u_n-k)dxdt=0, \tag{3.19}$$
$$-\frac{1}{n}\iint_{Q_T}|\nabla u_n|^2 S'_\eta(u_n-k)\varphi dxdt\le 0, \tag{3.20}$$

and using Lemma 3.1,

$$\liminf_{n\to\infty}\iint_{Q_T} S'_\eta(u_n-k)a(u_n)\frac{\partial u_n}{\partial x_i}\frac{\partial u_n}{\partial x_i}\varphi dxdt$$
$$\ge\iint_{Q_T} S'_\eta(u-k)\left|\nabla\int_0^u\sqrt{a(s)}ds\right|^2\varphi dxdt. \tag{3.21}$$

Let $n\to\infty$ in (3.18). By (3.19)–(3.21), we get (2.4).

At last, we can prove that the initial value (1.3) is satisfies in the sense of Definition 2.1 in a similar way as [2,8], we omit the details here. Thus, we have accomplished the proof of Theorem 2.2. □

4 Proof of Theorem 2.3

Let Γ_u be the set of all jump points of $u\in BV(Q_T)$, v the normal of Γ_u at $X=(x,t)$, $u^+(X)$ and $u^-(X)$ the approximate limits of u at $X\in\Gamma_u$ with respect to $(v, Y-X)>0$ and $(v, Y-X)<0$ respectively. For continuous function $p(u,x,t)$ and $u\in BV(Q_T)$, define

$$\widehat{p}(u,x,t)=\int_0^1 p(\tau u^+ +(1-\tau)u^-,x,t)d\tau,$$

which is called the composite mean value of p. Moreover, if $f(s)\in C^1(\mathbb{R})$, $u\in BV(Q_T)$, then $f(u)\in BV(Q_T)$ and

$$\frac{\partial f(u)}{\partial x_i}=\widehat{f'}(u)\frac{\partial u}{\partial x_i},\quad i=1,2,\dots,N,N+1, \tag{4.1}$$

where $x_{N+1}=t$ as usual.

Lemma 4.1 *Let u be a solution of* (1.1). *Then*

$$a(s)=0,\ \ s\in I(u^{+}(x,t),u^{-}(x,t))\ \ \ a.e.\ on\ \Gamma_u, \tag{4.2}$$

which $I(\alpha,\beta)$ denote the closed interval with endpoints α and β, and (4.2) *is in the sense of Hausdorff measure $H_N(\Gamma_u)$.*

The lemma and its proof can be found in [7].

Proof of Theorem 2.3 Let u, v be two entropy solutions of (1.1) with initial values

$$u(x,0)=u_0(x),\quad v(x,0)=v_0(x), \tag{4.3}$$

but without any boundary value condition.

For any $\eta>0$, $k,l\in\mathbb{R}$, for any $0\le\varphi\in C_0^2(Q_T)$, we have

$$\iint_{Q_T}\Bigg[I_\eta(u-k)\varphi_t-B_\eta^i(u,k)\varphi_{x_i} \\ +A_\eta(u,k)\Delta\varphi-S'_\eta(u-k)\left|\nabla\int_0^u\sqrt{a(s)}ds\right|^2\varphi\Bigg]dxdt\ge 0, \tag{4.4}$$

$$\iint_{Q_T}\Bigg[I_\eta(v-l)\varphi_\tau-B_\eta^i(v,l)\varphi_{y_i} \\ +A_\eta(v,l)\Delta\varphi-S'_\eta(v-l)\left|\nabla\int_0^v\sqrt{a(s)}ds\right|^2\varphi\Bigg]dyd\tau\ge 0. \tag{4.5}$$

Let $\psi(x,t,y,\tau)=\phi(x,t)j_h(x-y,t-\tau)$. Here $\phi(x,t)\ge 0,\ \phi(x,t)\in C_0^\infty(Q_T)$, and

$$j_h(x-y,t-\tau)=\omega_h(t-\tau)\prod_{i=1}^{N}\omega_h(x_i-y_i), \tag{4.6}$$

$$\omega_h(s)=\frac{1}{h}\omega\left(\frac{s}{h}\right),\ \omega(s)\in \overset{\infty}{\underset{0}{C}}(\mathbb{R}),\ \omega(s)\ge 0,\ \omega(s)=0 \\ \text{if } |s|>1,\ \int_{-\infty}^{\infty}\omega(s)ds=1. \tag{4.7}$$

We choose $k=v(y,\tau),\ \ l=u(x,t),\ \ \varphi=\psi(x,t,y,\tau)$ in (4.4) (4.5), integrate over Q_T, to obtain

$$\iint_{Q_T}\iint_{Q_T}\Bigg\{\Big[I_\eta(u-v)(\psi_t+\psi_\tau)-(B_\eta^i(u,v)\psi_{x_i}+B_\eta^i(v,u)\psi_{y_i}) \\ +A_\eta(u,v)\Delta_x\psi+A_\eta(v,u)\Delta_y\psi\Big]$$

$$- \; S_\eta'(u-v)\left(\left|\nabla\int_0^u\sqrt{a(s)}ds\right|^2+\left|\nabla\int_0^v\sqrt{a(s)}ds\right|^2\right)\psi\Bigg\}\,dxdtdyd\tau\geq 0. \quad (4.8)$$

By a complicated calculation (similar as [20]), letting $\eta\to 0$, $h\to 0$ in (4.8), using Lemma 4.1, we can deduce that

$$\begin{aligned}\iint_{Q_T}&\{|u(x,t)-v(x,t)|\phi_t-\operatorname{sgn}(u-v)[b_i(u)\\&-b_i(v)]\phi_{x_i}+|A(u)-A(v)|\Delta\phi\Big\}\,dxdt\geq 0.\end{aligned} \quad (4.9)$$

For $0<\tau<s<T$, we choose

$$\eta(t)=\int_{\tau-t}^{s-t}\alpha_\epsilon(\sigma)d\sigma,\quad \epsilon<\min\{\tau,T-s\},$$

where $\alpha_\epsilon(t)$ is the kernel of mollifier with $\alpha_\epsilon(t)=0$ for $t\notin(-\epsilon,\epsilon)$. Let us chose the test function

$$\phi=\eta(t)\xi(x),$$

in (4.9), in which $\xi(x)\in C_0^\infty(\Omega)$. Then

$$\begin{aligned}&\int_\Omega|u(x,s)-v(x,s)|\xi(x)dx-\int_\Omega|u(x,\tau)-v(x,\tau)|\xi(x)dx\\&\leq-\int_\tau^s\int_\Omega\operatorname{sgn}(u-v)[b_i(u)-b_i(v)]\xi_{x_i}dxdt+\int_\tau^s\int_\Omega|A(u)-A(v)|\Delta\xi dxdt,\end{aligned} \quad (4.10)$$

By a process of limit, we can choose

$$\xi=\nu_\delta(x)g(x), \quad (4.11)$$

where

$$\nu_\delta=e^{-\delta\sqrt{1+|x|^2}}.$$

as before.

In the first place, we have the following direct calculations

$$\begin{aligned}\nu_{\delta x_i}&=-\delta\nu_\delta\frac{x_i}{\sqrt{1+|x|^2}},\\\nu_{\delta x_ix_i}&=\delta^2\nu_\delta\frac{x_i^2}{1+|x|^2}-\delta\nu_\delta\frac{1+\sum_{j=1,j\neq i}^N|x_j|^2}{(1+|x|^2)^{\frac{3}{2}}}.\end{aligned} \quad (4.12)$$

$$\xi_{x_i} = \nu_{\delta x_i} g(x) + \nu_\delta g_{x_i}(x) = -\delta\nu_\delta \frac{x_i g(x)}{\sqrt{1+|x|^2}} + \nu_\delta g_{x_i}(x).$$

$$\begin{aligned}\Delta\xi &= \Delta\nu_\delta g(x) + 2\nu_{\delta x_i} g_{x_i} + \nu_\delta \Delta g \\ &= \left[\delta^2\nu_\delta \frac{|x|^2}{1+|x|^2} - \delta\nu_\delta \frac{N+(N-1)|x|^2}{(1+|x|^2)^{\frac{3}{2}}}\right] g(x) - 2\delta\nu_\delta \frac{x_i g_{x_i}}{\sqrt{1+|x|^2}} + \nu_\delta \Delta g.\end{aligned} \tag{4.13}$$

Since we assume that $0 \le |b^{i'}(s)| \le (1-\delta)a(s)$,

$$\begin{aligned}
&-\mathrm{sgn}(u-v)[b_i(u)-b_i(v)]\xi_{x_i} + |A(u)-A(v)|\Delta\xi \\
&= -\mathrm{sgn}(u-v)(b_i(u)-b_i(v))\left[-\delta\nu_\delta \frac{x_i g(x)}{\sqrt{1+|x|^2}} + \nu_\delta g_{x_i}(x)\right] \\
&\quad + |A(u)-A(v)|\left[\delta^2\nu_\delta \frac{|x|^2}{1+|x|^2} - \delta\nu_\delta \frac{N+(N-1)|x|^2}{(1+|x|^2)^{\frac{3}{2}}}\right] g(x) \\
&\quad + |A(u)-A(v)|\left[-2\delta\nu_\delta \frac{x_i g_{x_i}}{\sqrt{1+|x|^2}} + \nu_\delta \Delta g\right] \\
&\le \nu_\delta \left[|A(u)-A(v)|\Delta g - \delta|A(u)-A(v)| \frac{2x_i}{\sqrt{1+|x|^2}} g_{x_i} + |b_i(u)-b_i(v)| g_{x_i}\right] \\
&\quad + c\delta(\alpha+\gamma^i)\nu_\delta g(x)|u-v|. \\
&= \nu_\delta |A(u)-A(v)| \left[\Delta g + \left(\delta \frac{2|x_i|}{\sqrt{1+|x|^2}} + \frac{|b^{i'}(\zeta)|}{a(\zeta)}\right)|g_{x_i}|\right] \\
&\quad + c\delta(\alpha+\gamma^i)\nu_\delta g(x)|u-v| \\
&\le \nu_\delta |A(u)-A(v)|[\Delta g + (\delta + 1 - \delta)|g_{x_i}|] + c\delta(\alpha+\gamma^i)\nu_\delta g(x)|u-v| \\
&\le \nu_\delta |A(u)-A(v)|(\Delta g + |\nabla g|) + c\delta(\alpha+\gamma^i)\nu_\delta g(x)|u-v|,
\end{aligned} \tag{4.14}$$

where the Cauchy mean value theorem is used,

$$\left|\frac{b^i(u)-b^i(v)}{A(u)-A(v)}\right| = \frac{|b^{i'}(\zeta)|}{a(\zeta)} \le 1-\delta.$$

Since $\Delta g + |\nabla g| \le 0$, by (4.10)–(4.14), we have

$$\begin{aligned}&\int_\Omega |u(x,s)-v(x,s)|\nu_\delta g(x)dx - \int_\Omega |u(x,\tau)-v(x,\tau)|\nu_\delta g(x)dx \\ &\quad \le c\int_\tau^s \int_\Omega \nu_\delta g(x)|u-v|dxdt,\end{aligned} \tag{4.15}$$

By the Gronwall inequality, we have

$$\int_{\Omega} |u(x,s)-v(x,s)|\nu_\delta(x)g(x)dx \le c\int_{\Omega} |u(x,\tau)-v(x,\tau)|\nu_\delta(x)g(x)dx. \quad (4.16)$$

Let $\tau \to 0$. We have the conclusion. □

Proof of Theorem 2.4 Let u, v be two entropy solutions of (1.1) with the different initial values $u(x,0)$, $v(x,0)$, and with the same partial homogeneous boundary values

$$u(x,t)=v(x,t)=0, \quad (x,t)\in \Sigma_p\times(0,T), \quad (4.17)$$

where

$$\Sigma_p=\{x\in\partial\Omega : \Delta g+|\nabla g|\ge 0\}.$$

Similar as the proof of Theorem 2.3, we have (4.10).

For small enough λ, we set

$$\varphi_\lambda(x)=\begin{cases}\frac{g(x)}{\lambda}, & \text{if } \ 0\le g(x)\le\lambda,\\ 1, & \text{if } \ g(x)>\lambda,\end{cases} \quad (4.18)$$

let

$$\xi=\varphi_\lambda(x)\nu_\delta(x), \quad (4.19)$$

and denote

$$\Omega_\lambda=\{x\in\Omega : g(x)\le\lambda\}.$$

Then, when $x\in\Omega\backslash\Omega_\lambda$,

$$\varphi_\lambda(x)_{x_i}=0, \quad \Delta\varphi_\lambda(x)=0. \quad (4.20)$$

When $x\in\Omega_\lambda$,

$$\varphi_\lambda(x)_{x_i}=\frac{g_{x_i}}{\lambda}, \quad \Delta\varphi_\lambda(x)=\frac{\Delta g}{\lambda}. \quad (4.21)$$

Since we assume that $0\le|b^{i'}(s)|\le(1-\delta)a(s)$, we have

$$\begin{aligned}&-\text{sgn}(u-v)[b_i(u)-b_i(v)]\xi_{x_i}+|A(u)-A(v)|\Delta\xi\\ &\quad=-\text{sgn}(u-v)(b_i(u)-b_i(v))\left[-\delta\nu_\delta\frac{x_i\varphi_\lambda(x)}{\sqrt{1+|x|^2}}+\nu_\delta\varphi_{\lambda x_i}(x)\right]\end{aligned}$$

$$
\begin{aligned}
&+|A(u)-A(v)|\left[\delta^2\nu_\delta\frac{|x|^2}{1+|x|^2}-\delta\nu_\delta\frac{N+(N-1)|x|^2}{(1+|x|^2)^{\frac{3}{2}}}\right]\varphi_\lambda(x)\\
&+|A(u)-A(v)|\left[-2\delta\nu_\delta\frac{x_i\varphi_{\lambda x_i}}{\sqrt{1+|x|^2}}+\nu_\delta\Delta\varphi_\lambda(x)\right]\\
\le\ &\nu_\delta\Bigg[|A(u)-A(v)|\Delta\varphi_\lambda(x)-\delta|A(u)-A(v)|\frac{2x_i}{\sqrt{1+|x|^2}}\varphi_{\lambda x_i}\\
&+\operatorname{sgn}(u-v)(b_i(u)-b_i(v))\varphi_{\lambda x_i}(x)\Bigg]\\
&+c\delta(\alpha+\gamma^i)\nu_\delta\varphi_\lambda(x)|u-v|\\
\le\ &\nu_\delta|A(u)-A(v)|\left[\Delta\varphi_\lambda(x)+(\delta\frac{2|x_i|}{\sqrt{1+|x|^2}}+\frac{|b^{i\prime}(\zeta)|}{a(\zeta)})|\varphi_{\lambda x_i}|\right]\\
&+c\delta(\alpha+\gamma^i)\nu_\delta\varphi_\lambda(x)|u-v|\\
\le\ &\nu_\delta|A(u)-A(v)|[\Delta\varphi_\lambda(x)+(\delta+1-\delta)|\varphi_{\lambda x_i}|+c\delta(\alpha+\gamma^i)\nu_\delta\varphi_\lambda(x)|u-v|\\
\le\ &\nu_\delta|A(u)-A(v)|[\Delta\varphi_\lambda(x)+|\nabla\varphi_\lambda(x)|]+c\delta(\alpha+\gamma^i)\nu_\delta g(x)|u-v|, \qquad (4.22)
\end{aligned}
$$

where

$$
\left|\frac{b^i(u)-b^i(v)}{A(u)-A(v)}\right|=\frac{|b^{i\prime}(\zeta)|}{a(\zeta)},
$$

as before.

Noticing (4.19)–(4.21), we have

$$
\Delta\varphi_\lambda(x)+|\nabla\varphi_\lambda(x)|=\frac{1}{\lambda}(\Delta g+|\nabla g|)
$$

only when $x\in\Omega_\lambda$, while $x\in\Omega\setminus\Omega_\lambda$, it vanishes.

If we denote

$$
\Omega_{\lambda 1}=\{x\in\Omega_\lambda:\Delta g+|\nabla g|>0\},
$$

substituting (4.22) into (4.10),we have

$$
\begin{aligned}
\int_\Omega|u(x,s)-v(x,s)|\varphi_\lambda(x)dx\le&\int_\Omega|u(x,\tau)-v(x,\tau)|\varphi_\lambda(x)dx\\
&+\int_\tau^s\frac{1}{\lambda}\int_{\Omega_{\lambda 1}}\nu_\delta|A(u)-A(v)|(\Delta g+|\nabla g|)dxdt\\
&+c\int_\tau^s\int_\Omega\nu_\delta g(x)|u-v|dxdt. \qquad (4.23)
\end{aligned}
$$

According to the definition of the trace, by the partial boundary value condition

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\int_{\Omega_{\lambda 1}}\nu_\delta|A(u)-A(v)|(\Delta g+|\nabla g|)dx$$
$$=\int_{\Sigma_p}\nu_\delta|A(u)-A(v)|(\Delta g+|\nabla g|)d\Sigma=0. \tag{4.24}$$

Letting $\lambda\to 0$ in (4.23), we have

$$\int_\Omega |u(x,s)-v(x,s)|\nu_\delta(x)dx\leq \int_\Omega |u(x,\tau)-v(x,\tau)|\nu_\delta(x)dx$$
$$+c\int_\tau^s\int_\Omega |u-v|\nu_\delta(x)dxdt. \tag{4.25}$$

Then

$$\int_\Omega |u(x,s)-v(x,s)|\nu_\delta(x)dx\leq \int_\Omega |u(x,\tau)-v(x,\tau)|\nu_\delta(x)dx.$$

Let $\tau\to 0$. We have the conclusion. □

Compliance with ethical standards

Conflict of interest The author declares that he has no conflict of interest.

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