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ORIGINAL PAPER

Alon–Tarsi Number and Modulo Alon–Tarsi Number of Signed Graphs

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Abstract
We extend the concept of the Alon–Tarsi number for unsigned graph to signed one. Moreover, we introduce the modulo Alon–Tarsi number for a prime number p. We show that both the Alon–Tarsi number and modulo Alon–Tarsi number of a signed planar graph (G, σ) are at most 5, where the former result generalizes Zhu's result for unsigned case and the latter one implies that (G, σ) is $\mathbb{Z}_5$-colorable.

Keywords Signed graph · Group coloring · $\mathbb{Z}_p$-coloring · Planar graph · List coloring · Combinatorial Nullstellensatz · Alon–Tarsi number

Mathematics Subject Classification 05C15 · 05C22 · 05C10

1 Introduction

In this paper, we only deal with finite and simple graphs. Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. Moreover, let '$<$' be an arbitrary fixed ordering of the vertices of G. The *graph polynomial* of G is defined as

$$P_G(\boldsymbol{x}) = \prod_{u\sim v, u<v} (x_u - x_v),$$

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where $u \sim v$ means that u and v are adjacent, and $\boldsymbol{x} = (x_v)_{v \in V(G)}$ is a vector of $|V(G)|$ variables indexed by the vertices of G. It is easy to see that a mapping $c : V(G) \to \mathbb{N}$ is a proper coloring of G if and only if $P_G(\boldsymbol{c}) \neq 0$, where $\boldsymbol{c} = \big(c(v)\big)_{v \in V(G)}$. Therefore, to find a proper coloring of G is equivalent to find an assignment of $\boldsymbol{x}$ so that $P_G(\boldsymbol{x}) \neq 0$. The following theorem, which was implicit in [2] and appeared in [1], gives sufficient conditions for the existence of such assignments as above.

Lemma 1 [1] (Combinatorial Nullstellensatz) *Let $\mathbb{F}$ be an arbitrary field and let $f = f(x_1, x_2, \ldots, x_n)$ be a polynomial in $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Suppose that the degree $deg(f)$ of f is $\sum_{i=1}^{n} t_i$ where each t_i is a nonnegative integer, and suppose that the coefficient of $\prod_{i=1}^{n} x_i^{t_i}$ of f is nonzero. Then if $S_1, S_2, \ldots, S_n$ are subsets of $\mathbb{F}$ with $|S_i| \geq t_i + 1$, then there are $s_1 \in S_1, s_2 \in S_2, \ldots, s_n \in S_n$ so that $f(s_1, s_2, \ldots, s_n) \neq 0$.*

In particular, a graph polynomial $P_G(\boldsymbol{x})$ is a homogeneous polynomial and $\deg(P_G(\boldsymbol{x}))$ is equal to $|E(G)|$. Therefore, if there exists a monomial $c \prod_{v \in V(G)} {x_v}^{t_v}$ in the expansion of P_G so that $c \neq 0$ and $t_v < k$ for each $v \in V(G)$, then G is k-choosable. The definition of choosability will be described in Sect. 2. Jensen and Toft [6] defined *Alon–Tarsi number* of graph G as follows.

Definition 1 The *Alon–Tarsi number* of G, denoted by $AT(G)$, is the minimum k for which there exists a monomial $c \prod_{v \in V(G)} x_v^{t_v}$ in the expansion of $P_G(\boldsymbol{x})$ such that $c \neq 0$ and $t_v < k$ for all $v \in V(G)$.

Let $\chi(G)$ be the chromatic number of G and $\chi_l(G)$ be the list chromatic number of G. By Lemma 1, we have

$$AT(G) \geq \chi_l(G) \geq \chi(G).$$

Alon and Tarsi [2] found a useful combinatorial interpretation of the coefficient for each monomial in the graph polynomial $P_G(\boldsymbol{x})$ in terms of orientations and Eulerian subgraphs. For an orientation D of G, a subdigraph H of D is called *Eulerian* if $V(H) = V(D)$ and the indegree of every vertex equals its outdegree. We note that an Eulerian subdigraph H defined here is not necessarily connected. In particular, a vertex is called *isolated* in H if it has indegree 0 (and therefore, has outdegree 0) in H. Let $EE(D)$ (resp. $OE(D)$) denote the set of all spanning Eulerian subdigraphs of D with the number of edges even (resp. odd).

Proposition 1 [2] *Let G be a graph, let $P_G(\boldsymbol{x})$ be the graph polynomial and let D be an orientation of G with outdegree sequence $\boldsymbol{d} = (d_v)_{v \in V(G)}$. Then the coefficient of $\prod_{v \in V(G)} x_v^{d_v}$ in the expansion of $P_G(\boldsymbol{x})$ is equal to $\pm(|EE(D)| - |OE(D)|)$.*

By defining hypergraph polynomial and hypergraph orientation, Ramamurthi and West [12] generalized the result of Alon and Tarsi to k-uniform hypergraph for prime k.

Now, let us focus on planar graphs. Thomassen [14] showed that every planar graph is 5-choosable. Moreover, Zhu [15] has recently generalized Thomassen's result as follows.

Theorem 1 [15] *If G is a planar graph, then $AT(G) \leq 5$.*

The notion of the Alon–Tarsi number is ordinary defined for unsigned graph. In this paper, we extend the Alon–Tarsi number of unsigned graph to singed one. Moreover, we extend the $\mathbb{Z}_p$-coloring of unsigned graph to signed one and consider its analogical Alon–Tarsi number when p is a prime, which we call modulo-p Alon–Tarsi number. The main aims of this paper are to extend Theorem 1 to signed graphs (Theorem 2) and to obtain analogical one for modulo-p Alon–Tarsi number (Theorem 4). We show that Theorem 4 is indeed a strengthening of Theorem 2. We organize this paper as follows. In Sect. 2, we introduce signed graphs and define the Alon–Tarsi number for signed graphs. In Sect. 3, we define the $\mathbb{Z}_p$-coloring of signed graphs and consider its Alon–Tarsi number. In Sect. 4, we prove Theorem 4.

2 Alon–Tarsi Number for Signed Coloring

2.1 Introduction of Signed Graphs

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. A *signed graph* with underling graph G is a pair (G, σ), where σ is a mapping from $E(G)$ to $\{+1, -1\}$. An edge e is *positive* (resp. *negative*) if $\sigma(e) = +1$ (resp. $\sigma(e) = -1$). In particular, we denote by $(G, +)$ (resp. $(G, -)$) the signed graph (G, σ) if every edge is positive (resp. negative). We often identify $(G, +)$ with the (unsigned) underling graph G.

Recently, based on the work of Zaslavsky [16], Máčajová et al. [11] generalized the concept of chromatic number of an unsigned graph to a signed graph. For a signed graph (G, σ) and a color set $C \subset \mathbb{Z}$, a *proper coloring* [16] with color set C is a mapping $\phi\colon V(G) \to C$ such that

$$\phi(u) \neq \sigma(uv)\phi(v) \tag{1}$$

for each edge $uv \in E(G)$. For $k \geq 1$, set $M_k = \{\pm 1, \pm 2, \ldots, \pm k/2\}$ if k is even and $M_k = \{0, \pm 1, \pm 2 \ldots, \pm (k-1)/2\}$ if k is odd. A (proper) *k-coloring* of a signed graph (G, σ) is a proper coloring with color set M_k. A signed graph (G, σ) is *k-colorable* if it admits a k-coloring. The *chromatic number* of (G, σ), denoted by $\chi(G, \sigma)$, is the minimum k for which (G, σ) is k-colorable.

Jin et al. [7] and Schweser et al. [13] further considered the list coloring of signed graphs. For a positive integer k, a *k-list assignment* of (G, σ) is a mapping L which assigns to each vertex v a set $L(v) \subset \mathbb{Z}$ of k permissible colors. For a k-list assignment L of (G, σ), an *L-coloring* is a proper coloring $\phi\colon V(G) \to \bigcup_{v \in V(G)} L(v)$ such that $\phi(v) \in L(v)$ for every vertex $v \in V(G)$. We say that (G, σ) is *L-colorable* if G has an L-coloring. A signed graph (G, σ) is called *k-choosable* if G is L-colorable for any k-list assignment L. The *list chromatic number* (or *choice number*) $\chi_l(G, \sigma)$ is the minimum k for which G is k-choosable. Clearly, $\chi_l(G, \sigma) \geq \chi(G, \sigma)$. We note that when we restrict the signed graphs (G, σ) to $(G, +)$, both the chromatic number and list chromatic number agree with the ordinary chromatic number and list chromatic number of its underlying graph G. This explains why we can identify $(G, +)$ with G.

Let '$<$' be an arbitrary fixed ordering of the vertices of (G, σ). In view of (1), we define the *graph polynomial* of (G, σ) as

$$P_{G,\sigma}(\boldsymbol{x}) = \prod_{u \sim v, u < v} (x_u - \sigma(uv)x_v),$$

where $u \sim v$ means that u and v are adjacent, and $\boldsymbol{x} = (x_v)_{v \in V(G)}$ is a vector of $|V(G)|$ variables indexed by the vertices of G. It is easy to see that a mapping $\phi\colon V(G) \to \mathbb{Z}$ is a proper coloring of (G, σ) if and only if $P_{G,\sigma}((\phi(v))_{v \in V(G)}) \neq 0$.

Note that $P_{G,\sigma}(\boldsymbol{x})$ is also a homogeneous polynomial. It follows from Lemma 1 that if there exists a monomial $c \prod_{v \in V(G)} x_v^{t_v}$ in the expansion of $P_{G,\sigma}(\boldsymbol{x})$ such that $c \neq 0$ and $t_v < k$ for all $v \in V(G)$, then (G, σ) is k-choosable. Thus, the notion of Alon–Tarsi number of unsigned graphs can be naturally extended to signed graphs.

Definition 2 The *Alon–Tarsi number* of (G, σ), denoted by $AT(G, \sigma)$, is the minimum k for which there exists a monomial $c \prod_{v \in V(G)} x_v^{t_v}$ in the expansion of $P_{G,\sigma}(\boldsymbol{x})$ such that $c \neq 0$ and $t_v < k$ for all $v \in V(G)$.

Parallel to the unsigned case, we have

$$AT(G, \sigma) \geq \chi_l(G, \sigma) \geq \chi(G, \sigma).$$

For a subgraph H of G, we use (H, σ) to denote the signed subgraph of (G, σ) restricted on H, i.e., $(H, \sigma) = (H, \sigma|_{E(H)})$. Note that $P_{H,\sigma}(\boldsymbol{x})$ is a factor of $P_{G,\sigma}(\boldsymbol{x})$. From Definition 2, it is clear that $AT(H, \sigma) \leq AT(G, \sigma)$.

For a vertex v in a signed graph (G, σ), a *switching* at v means changing the sign of each edge incident to v. For $X \subseteq V(G)$, a switching at X means switching at every vertex in X one by one. Equivalently, a switching at X means changing the sign of every edge with exactly one end in X. We denote the switched graph by (G, σ^X). In particular, when $X = \{v\}$ we use (G, σ^v) to denote $(G, \sigma^{\{v\}})$. Two signed graphs (G, σ) and (G, σ') are *switching equivalent* if $\sigma' = \sigma^X$ for some $X \subseteq V(G)$.

It is easy to show that two switching equivalent signed graphs have the same chromatic number [11] as well as the same list chromatic number [7,13]. For the Alon–Tarsi numbers, we have the following similar result.

Proposition 2 *If two signed graphs (G, σ) and (G, σ') are switching equivalent then $AT(G, \sigma) = AT(G, \sigma')$.*

Proof It clearly suffices to consider the case that $\sigma' = \sigma^v$, where $v \in V(G)$. For any edge incident with v, say uv, we have $\sigma^v(uv) = -\sigma(uv)$. We use $T(x_u, x_v)$ and $T^v(x_u, x_v)$ to denote the factors corresponding to this edge in $P_{G,\sigma}(\boldsymbol{x})$ and $P_{G,\sigma^v}(\boldsymbol{x})$, respectively. If $u < v$ then $T(x_u, x_v) = x_u - \sigma(uv)x_v$, $T^v(x_u, x_v) = x_u - \sigma^v(uv)x_v$ and hence $T(x_u, x_v) = T^v(x_u, -x_v)$. If $v < u$ then $T(x_u, x_v) = x_v - \sigma(uv)x_u$ and $T^v(x_u, x_v) = x_v - \sigma^v(uv)x_u$ and hence $T(x_u, x_v) = -T^v(x_u, -x_v)$. In either case we have $T(x_u, x_v) = \pm T^v(x_u, -x_v)$. Letting $\boldsymbol{x}^v$ be obtained from $\boldsymbol{x}$ by changing x_v to $-x_v$, we have $P_{G,\sigma}(\boldsymbol{x}) = \pm P_{G,\sigma^v}(\boldsymbol{x}^v)$. Therefore, for each monomial $\prod_{v \in V(G)} x_v^{t_v}$, the coefficients of this monomial in $P_{G,\sigma}(\boldsymbol{x})$ and $P_{G,\sigma^v}(\boldsymbol{x}^v)$ and hence in $P_{G,\sigma^v}(\boldsymbol{x})$ have the same absolute value. This implies that $AT(G, \sigma) = AT(G, \sigma^v)$.

Recently, a few classical results on colorability [4] and choosability [7] of planar graphs were generalized to signed planar graphs. In particular, Jin et al. [7] showed that every signed planar graph is 5-choosable, generalizing the well-known result of Thomassen [14] which states that every (unsigned) planar graph is 5-choosable. Another generalization of Thomassen's result was given by Zhu [15], who showed that $AT(G) \leq 5$ for any planar graph G, which solved an open problem proposed by Hefetz [3]. Considering the above results of Jin et al. [7] and Zhu [15], it is natural to ask whether the Alon–Tarsi number of each signed planar graph is at most 5. We answer this question affirmatively.

Theorem 2 *If (G, σ) is a signed planar graph, then $AT(G, \sigma) \leq 5$.*

In [2], Alon and Tarsi showed that every bipartite planar graph is 3-choosable. The result is sharp as $K_{2,4}$ is a bipartite planar graph and $\chi_l(K_{2,4}) = 3$. The following result is a natural extension of this result for signed planar graphs.

Theorem 3 *For any signed planar graph (G, σ), if (G, σ) is 2-colorable then $AT(G, \sigma) \leq 4$. Moreover, there is a signed planar graph which is 2-colorable but not 3-choosable.*

2.2 Orientation and Alon–Tarsi Number for Signed Graphs

In this section, we consider the signed graphs. Instead of using orientations of signed graphs, we use orientations of the underlying graphs and find that the result of Alon and Tarsi has a natural extension for signed graphs.

Let (G, σ) be a signed graph and '$<$' be an arbitrary fixed ordering of $V(G)$. For an orientation D of the underling graph G, we denote by (v, u) the oriented edge of D with direction from v to u. We call an oriented edge (v, u) *σ-decreasing* if $v > u$ and $\sigma(uv) = +1$, that is, (v, u) is positive and oriented from the larger vertex to the smaller vertex. We note that a negative edge will never be σ-decreasing, no matter how it is oriented. An orientation D of G is called *σ-even* if it has an even number of σ-decreasing edges and called *σ-odd* otherwise. For a nonnegative sequence $\boldsymbol{d} = (d_v)_{v\in V(G)}$, let $\sigma EO(\boldsymbol{d})$ and $\sigma OO(\boldsymbol{d})$ denote the sets of all σ-even and σ-odd orientations of G having outdegree sequence $\boldsymbol{d}$, respectively.

Lemma 2 *$P_{G,\sigma}(\boldsymbol{x}) = \sum(|\sigma EO(\boldsymbol{d})| - |\sigma OO(\boldsymbol{d})|)\prod_{v\in V(G)} x_v^{d_v}$, where $\boldsymbol{d} = (d_v)_{v\in V(G)}$ and the summation is taken over all $\boldsymbol{d}$ such that $d_v \geq 0$ for every vertex v in G and $\sum_{v\in V(G)} d_v = |E(G)|$.*

Proof Let D be an arbitrary orientation of G. For each oriented edge $e = (v, u)$, define

$$w(e) = \begin{cases} -x_v, & \text{if } e \text{ is } \sigma\text{-decreasing} \\ x_v, & \text{otherwise,} \end{cases} \tag{2}$$

and $w(D) = \prod_{e\in E(D)} w(e)$. Let d_v be the outdegree of v in D for each $v \in V(G)$ and let t be the number of σ-decreasing edges in D. It is easy to see that

$$w(D) = (-1)^t \prod_{v\in V(G)} x_v^{d_v}. \tag{3}$$

Recall that

$$P_{G,\sigma}(\boldsymbol{x}) = \prod_{u\sim v, u<v} (x_u - \sigma(uv)x_v).$$

By selecting x_u or $-\sigma(uv)x_v$ from each factor $(x_u - \sigma(uv)x_v)$, we expand $P_{G,\sigma}(\boldsymbol{x})$ and obtain $2^{|E(G)|}$ monomials, each of which has coefficient ± 1. For each monomial, we orient the edge uv of G with direction from u to v if, in the factor $(x_u - \sigma(uv)x_v)$, x_u is selected; or from v to u if $-\sigma(uv)x_v$ is selected. This is clearly a bijection between the $2^{|E(G)|}$ monomials and the $2^{|E(G)|}$ orientations of G. Therefore,

$$P_{G,\sigma}(\boldsymbol{x}) = \sum w(D), \tag{4}$$

where D ranges over all orientations of G.

Let $\boldsymbol{d} = (d_v)_{v\in V(G)}$ be the sequence of outdegrees of some orientation D. Clearly, $d_v \geq 0$ and $\sum_{v\in V(G)} d_v = |E(G)|$. Note that there are exactly $|\sigma EO(\boldsymbol{d})|$ (resp. $|\sigma OO(\boldsymbol{d})|$) σ-even (resp. σ-odd) orientations of G. It follows from (3) and (4) that the coefficient of $\prod_{v\in V(G)} x_v^{d_v}$ in the expansion of $P_{G,\sigma}(\boldsymbol{x})$ is $|\sigma EO(\boldsymbol{d})| - |\sigma OO(\boldsymbol{d})|$. This proves the lemma.

Let D be an orientation of G. An Eulerian subdigraph H of D is called *σ-even* (resp. *σ-odd*) if H has an even (resp. odd) number of positive edges. Let $\sigma EE(D)$ (resp. $\sigma OE(D)$) denote the set of all σ-even (resp. σ-odd) Eulerian subdigraphs of D.

Lemma 3 *Let (G, σ) be a signed graph and D be an orientation of G with outdegree sequence $\boldsymbol{d} = (d_v)_{v\in V(G)}$. Then the coefficient of $\prod_{v\in V(G)} x_v^{d_v}$ in the expansion of $P_{G,\sigma}(\boldsymbol{x})$ is equal to $\pm(|\sigma EE(D)| - |\sigma OE(D)|)$.*

Proof For any orientation $D' \in \sigma EO(\boldsymbol{d}) \cup \sigma OO(\boldsymbol{d})$, let $D \oplus D'$ denote the set of all oriented edges of D whose orientation in D' is in the opposite direction. Since D and D' have the same outdegree sequence, $D \oplus D'$ is Eulerian. Let (u, v) be an oriented edge in $D \oplus D'$. If uv is positive then exactly one of (u, v) and (v, u) is σ-decreasing. If uv is negative then neither (u, v) nor (v, u) is σ-decreasing. Thus, $D \oplus D'$ contains an even number of positive edges if and only if D and D' are both σ-even or both σ-odd.

Now, the map $\tau: D' \mapsto D \oplus D'$ is clearly a bijection between $\sigma EO(\boldsymbol{d}) \cup \sigma OO(\boldsymbol{d})$ and $\sigma EE(D) \cup \sigma OE(D)$. If D is σ-even, then τ maps $\sigma EO(\boldsymbol{d})$ to $\sigma EE(D)$ and maps $\sigma OO(\boldsymbol{d})$ to $\sigma OE(D)$. In this case $|\sigma EO(\boldsymbol{d})| = |\sigma EE(D)|$ and $|\sigma OO(\boldsymbol{d})| = |\sigma OE(D)|$. Thus, $|\sigma EO(d)| - |\sigma OO(D)| = |\sigma EE(D)| - |\sigma OE(D)|$. It follows from Lemma 2 that the coefficient of $\prod_{v\in V(G)} x_v^{d_v}$ in the expansion of $P_{G,\sigma}(\boldsymbol{x})$ is equal to $|\sigma EE(D)| - |\sigma OE(D)|$. Similarly, if D is σ-odd, then the coefficient of $\prod_{v \in V(G)} x_v^{d_v}$ in the expansion of $P_{G,\sigma}(\boldsymbol{x})$ is equal to $|\sigma OE(D)| - |\sigma EE(D)|$. This proves the lemma.

By Definition 2 and Lemma 3, we have the following characterization of the Alon–Tarsi number $AT(G,\sigma)$.

Corollary 1 *For any signed graph (G,σ), $AT(G,\sigma)$ equals the minimum k for which there exists an orientation D of G such that $|\sigma EE(D)| - |\sigma OE(D)| \neq 0$ and every vertex has outdegree less than k.*

2.3 Proof of Theorem 3

For a graph G, the maximum average degree of G, denoted by $\mathrm{mad}(G)$, is the maximum of $2|E(H)|/|V(H)|$, where H ranges over all subgraphs of G. The following useful criterion on the existence of an orientation with bounded outdegree appeared in [2].

Lemma 4 *A graph G has an orientation D such that every vertex has outdegree at most p if and only if $mad(G) \leq 2p$.*

Corollary 2 *For any graph G,*

$$AT(G,-) = \left\lceil \frac{mad(G)}{2} \right\rceil + 1. \tag{5}$$

Proof Let $p = \lceil \frac{\mathrm{mad}(G)}{2} \rceil$. Then $\mathrm{mad}(G) \leq 2p$ and hence, by Lemma 4, G has an orientation D in which every outdegree is at most p. As each edge in $(G,-)$ is negative, each Eulerian subdigraph of D contains no positive edge and hence is σ-even. Thus $|\sigma OE(D)| = 0$. Since the empty subgraph is a σ-even Eulerian subgraph, we have $|\sigma EE(D)| \geq 1$ and hence $|\sigma EE(D)| \neq |\sigma OE(D)|$. Thus by Corollary 1, $AT(G,-) \leq p+1$.

On the other hand, by Corollary 1, G has an orientation D such that each outdegree is at most $AT(G,-)-1$. Thus, by Lemma 4, $\mathrm{mad}(G) \leq 2(AT(G,-)-1)$, i.e., $AT(G,-) \geq \frac{\mathrm{mad}(G)}{2} + 1$. Therefore, $AT(G,-) \geq p+1$ since $AT(G,-)$ is an integer. This proves the corollary.

Proof of Theorem 3 For a signed graph (G,σ), Schweser and Stiebitz [13] showed that $\chi(G,\sigma) \leq 2$ if and only if (G,σ) is switching equivalent to $(G,-)$. Thus, by Proposition 2, it suffices to consider the case when $(G,\sigma) = (G,-)$, i.e., $\sigma(uv) = -1$ for each $uv \in E(G)$. Let H be any subgraph of a planar graph G. Then by Euler's formula for planar graph we have $2|E(H)|/|V(H)| \leq 6$, i.e., $\mathrm{mad}(G) \leq 6$. By Corollary 2, $AT(G,-) \leq 4$. This proves the first part of Theorem 3.

Let $(G,-)$ be the negative planar graph as shown in Fig. 1. We show that $(G,-)$ is not 3-choosable.

Define a 3-list assignment L as follows:

- $L(a) = L(a') = \{0, -1, -2\}$.

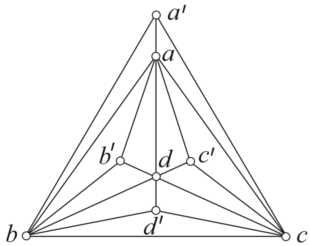

Fig. 1 A non-3-choosable negative planar graph $(G, -)$

- $L(b) = L(b') = \{0, -1, 2\}$.
- $L(c) = L(c') = \{0, 1, -2\}$.
- $L(d) = L(d') = \{0, 1, 2\}$.

It suffices to show that $(G, -)$ is not L-colorable. Suppose to the contrary that ϕ is an L-coloring of $(G, -)$. Let $V = \{a, b, c, d\}$.

Claim 1 *There exists some $x \in V$ such that $\phi(x) = 0$.*

Suppose to the contrary that $\phi(x) \neq 0$ for each $x \in V$. Then $\phi(a) \in \{-1, -2\}$, $\phi(b) \in \{-1, 2\}$, $\phi(c) \in \{1, -2\}$ and $\phi(d) \in \{1, 2\}$. Note that $(G[V], -)$ is a negative complete graph. Thus $\phi(x) \neq -\phi(y)$ for two distinct x, y in V. If $\phi(a) = -1$ then $\phi(c) = -2$ and $\phi(d) = 2$. Now, $\phi(c) = -\phi(d)$, a contradiction. Similarly, if $\phi(a) = -2$ then $\phi(b) = -1$ and $\phi(d) = 1$ and hence $\phi(b) = -\phi(d)$. This is also a contradiction. Thus, Claim 1 follows.

Claim 2 *Let $x \in V$. If $\phi(x) = 0$ then $\phi(N(x')) = -L(x')$.*

We only prove the case that $x = a$ and the other three cases can be settled in the same way. Since $\phi(a) = 0$, we have $\phi(b) \in \{-1, 2\}$, $\phi(c) \in \{1, -2\}$ and $\phi(d) \in \{1, 2\}$. If $\phi(b) = -1$ then $\phi(c) = -2$ and $\phi(d) = 2$. Thus, $\phi(c) = -\phi(d)$, a contradiction. Therefore, $\phi(b) = 2$. As $\phi(c) \neq -\phi(b)$, we have $\phi(c) \neq -2$ and hence $\phi(c) = 1$. Finally, as $N(a') = \{a, b, c\}$ and $L(a') = \{0, -1, -2\}$, we have $\phi(N(a')) = \{\phi(a), \phi(b), \phi(c)\} = \{0, 2, 1\} = -L(a')$. This proves Claim 2.

Now, by Claim 1, let $x \in V$ satisfy $\phi(x) = 0$. Then, $\phi(N(x')) = -L(x')$ by Claim 2. As $\phi(x') \in L(x')$ we have $-\phi(x') \in \phi(N(x'))$, that is, $-\phi(x') = \phi(y)$ for some $y \in N(x')$. Thus, ϕ is not proper since $x'y$ is a negative edge. This is a contradiction and hence completes the proof of Theorem 3.

3 Alon–Tarsi Number of Signed Graphs for $\mathbb{Z}_p$-Coloring

In this section, we extend the $\mathbb{Z}_p$-coloring of unsigned graph to signed one and consider analogical Alon–Tarsi number.

At first, we introduce the $\mathbb{Z}_p$-coloring of unsigned graph. $\mathbb{Z}_p$-coloring is a special case of the group coloring, which was introduced in [5]. Let p be a positive integer and let G be a simple connected graph whose vertices are totally ordered by '<'. Let

W_p be a map from $E(G)$ to $\mathbb{Z}_p$, where $\mathbb{Z}_p$ denotes the cyclic group of order p. We say that G is *modulo p-colorable on* W_p if G has a vertex assignment $c : V(G) \to \mathbb{Z}_p$ such that $c(v) \not\equiv c(u) + W_p(uv) \pmod p$ for every edge $uv \in E(G)$ with $v < u$. Moreover, we say that G is $\mathbb{Z}_p$*-colorable* if G is modulo p-colorable on any map W_p. The *modulo chromatic number* of G, denoted by $\chi_{\text{mod}}(G)$, is the minimum positive integer p such that G is $\mathbb{Z}_p$-colorable.

Next, we extend the $\mathbb{Z}_p$-colorings of unsigned graph to signed one. Let (G, σ) be a signed graph whose vertices are totally ordered by '$<$'. Moreover, let $W_p : E(G) \to \mathbb{Z}_p$. A *modulo p-coloring of* (G, σ) *on* W_p is a mapping c from $V(G)$ to $\mathbb{Z}_p$ such that

$$c(x) \not\equiv \sigma(e)c(y) + W_p(e) \pmod p \tag{6}$$

for any edge $e = xy$ with $x < y$.

We say that (G, σ) is *modulo p-colorable on* W_p if (G, σ, W_p) has a modulo p-coloring on W_p. Moreover, we say that (G, σ) is $\mathbb{Z}_p$*-colorable* if (G, σ) is modulo p-colorable for any map W_p.

If we assume $\sigma(e) = +1$ for every edge e, then the modulo p-coloring of $(G, +)$ on W_p coincides with the modulo p-coloring of G on W_p. On the other hand, if we assume $W_p(e) = 0$ for every edge e, then this mapping is proper coloring with color set $\mathbb{Z}_p$, a notion introduced by Kang and Steffen [8].

Now, we define a modulo-p graph polynomial for a signed graph on a mapping $W_p : E(G) \to \mathbb{Z}_p$. We assign each vertex v to the variable $x_v \in \mathbb{Z}_p$ and define the *modulo-p graph polynomial* of (G, σ) on W_p as

$$P_{G,\sigma,W_p}(\boldsymbol{x}) \equiv \prod_{u\sim v, u<v} (x_u - \sigma(uv)x_v - W_p(uv)) \pmod p,$$

where $u \sim v$ means that u and v are adjacent, and $\boldsymbol{x} = (x_v)_{v\in V(G)}$ is a vector of $|V(G)|$ variables indexed by the vertices of G. Note that $\deg(P_{G,\sigma,W_p}(\boldsymbol{x}))$ always equals $|E(G)|$, independent of the choice of W_p. Moreover, it is not difficult to see that the coefficient of each monomial with degree $|E(G)|$ does not depend on the choice of W_p.

In the following, we always assume that p is a prime number and hence $\mathbb{Z}_p$ is a field. It is easy to see that a mapping $c\colon\ V(G) \to \mathbb{Z}_p$ is a modulo p-coloring of (G, σ) on W_p if and only if $P_{G,\sigma,W_p}(\boldsymbol{c}) \not\equiv 0 \pmod p$, where $\boldsymbol{c} = \big(c(v)\big)_{v\in V(G)}$.

Note that $\mathbb{Z}_p$ is a field. It follows from Lemma 1 that if there exists a monomial $c\prod_{v\in V(G)} x_v^{t_v}$, with degree $|E(G)|$, in the expansion of $P_{G,\sigma,W_p}(\boldsymbol{x})$ (or equivalently, of $P_{G,\sigma}(\boldsymbol{x})$) such that $c \not\equiv 0 \pmod p$ and $t_v < p$ for all $v \in V(G)$, then (G, σ) is modulo p-colorable on W_p. Note that the condition '$t_v < p$ for all $v \in V(G)$' is necessary since $\mathbb{Z}_p$ has only p elements.

Definition 3 Let p be a prime. The *modulo-p Alon–Tarsi number* of (G, σ), denoted by $AT_p(G, \sigma)$, is the minimum k for which there exists a monomial $c\prod_{v\in V(G)} x_v^{t_v}$ in the expansion of $P_{G,\sigma}(\boldsymbol{x})$ such that $c \not\equiv 0 \pmod p$, and $t_v < k$ for all $v \in V(G)$.

By Definition 3 and above discussions, the following holds.

Proposition 3 *Let p be a prime and let (G, σ) be a signed graph. If $AT_p(G, \sigma) \leq p$, then (G, σ) is $\mathbb{Z}_p$-colorable, and in particular, (G, σ) has a proper coloring with color set $\mathbb{Z}_p$.*

Comparing Definition 3 with Definition 2, one easily finds that $AT_p(G, \sigma) \geq AT(G, \sigma)$ for any signed graph (G, σ) and prime p. For a subgraph H of G, since $P_{H,\sigma,W_p}(\boldsymbol{x})$ is a factor of $P_{G,\sigma,W_p}(\boldsymbol{x})$, we have $AT_p(H, \sigma) \leq AT_p(G, \sigma)$. Moreover, using Lemma 3, we obtain a similar characterization of modulo-p Alon–Tarsi number as in Corollary 1.

Corollary 3 *For any signed graph (G, σ) and prime p, $AT_p(G, \sigma)$ equals the minimum k for which there exists an orientation D of G such that $|\sigma EE(D)| - |\sigma OE(D)| \not\equiv 0 (mod\ p)$ and every vertex has outdegree less than k.*

Lai et al. [10] showed that every K_5-minor free graph is $\mathbb{Z}_5$-colorable. This implies that every planar graph is $\mathbb{Z}_5$-colorable. Moreover, Král' et al. [9] showed that there exists a plane graph which is not $\mathbb{Z}_4$-colorable. We show that every signed planar graph is $\mathbb{Z}_5$-colorable. Indeed, we prove a stronger result.

Theorem 4 *If (G, σ) is a signed planar graph and p is a prime, then $AT_p(G, \sigma) \leq 5$. In particular, $AT_5(G, \sigma) \leq 5$ and hence (G, σ) is $\mathbb{Z}_5$-colorable.*

As $AT_p(G, \sigma) \geq AT(G, \sigma)$, Theorem 2 follows from Theorem 4. The next section is devoted to the proof of Theorem 4.

4 Proof of Theorem 4

Let p be a fixed prime.

Definition 4 Let (G, σ) be a signed graph where G is a near triangulation with outer facial cycle $v_1 v_2 \ldots v_k$ and let $e = v_1 v_2$. An orientation D of $G - e$ is *p-nice* for $G - e$ if both of the followings hold.

- $|\sigma EE(D)| - |\sigma OE(D)| \not\equiv 0 (\text{mod } p)$, and
- $d_D{}^+(v_1) = d_D{}^+(v_2) = 0$, $d_D{}^+(v_i) \leq 2$ for each $i \in \{3, \ldots, k\}$ and $d_D{}^+(u) \leq 4$ for each interior vertex $u \in V(G)$.

Theorem 5 *Let (G, σ) be a signed graph where G is a near triangulation with outer facial cycle $C = v_1 v_2 \ldots v_k$ and let $e = v_1 v_2$. Then $G - e$ has a p-nice orientation.*

Proof We prove the theorem by induction on $|V(G)|$. If $|V(G)| = 3$, then $G - e$ is a path $v_2 v_3 v_1$. Let D be the orientation of $G - e$ such that $E(D) = \{(v_3, v_2), (v_3, v_1)\}$. Since $|\sigma EE(D)| = 1$ and $|\sigma OE(D)| = 0$, D is a p-nice orientation. Thus we assume that $|V(G)| > 3$.

First we consider the case that C has a chord $e' = v_k v_j$ where $2 \leq j \leq k - 2$ (see Fig. 2a). In this case $C_1 = v_1 v_2 \cdots v_j v_k$ and $C_2 = v_k v_j v_{j+1} \cdots v_{k-1}$ are two cycles of G. For $i \in \{1, 2\}$, let G_i be the subgraph of G formed by C_i and its interior part. By the induction hypothesis, $G_1 - e$ has a p-nice orientation D_1, and $G_2 - e'$ has a

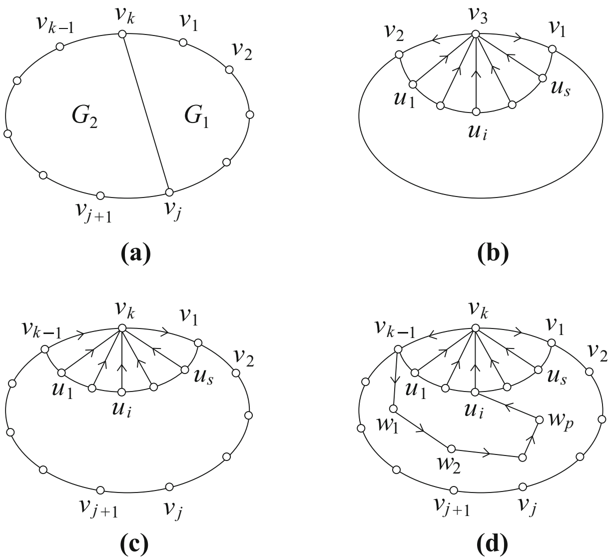

Fig. 2 Proof of Theorem 5

p-nice orientation D_2. We notice that D_1 and D_2 are edge disjoint. Let $D = D_1 \cup D_2$. It is clear that D is an orientation of $G - e$. We will show that D is p-nice for $G - e$. It can be easily checked that D satisfies the outdegree condition in Definition 4. Thus we will show that $|\sigma EE(D)| - |\sigma OE(D)| \not\equiv 0 \pmod p$. Note that both v_k and v_j have outdegree 0 in D_2. This implies that no edge in D_2 incident with v_k or v_j is contained in any Eulerian subdigraph of D. Therefore, any Eulerian subdigraph H of D has an edge-disjoint decomposition $H = H_1 \cup H_2$, where H_1 and H_2 are Eulerian subdigraphs in D_1 and D_2, respectively. Thus, the map $\tau\colon\ H \mapsto (H_1, H_2)$ is a bijection satisfying that

- $\tau(\sigma EE(D)) = (\sigma EE(D_1) \times \sigma EE(D_2)) \cup (\sigma OE(D_1) \times \sigma OE(D_2))$ and
- $\tau(\sigma OE(D)) = (\sigma EE(D_1) \times \sigma OE(D_2)) \cup (\sigma OE(D_1) \times \sigma EE(D_2))$.

Thus, we have

$$\begin{aligned}
&|\sigma EE(D)| - |\sigma OE(D)| \\
&\quad = (|\sigma EE(D_1)| \times |\sigma EE(D_2)| + |\sigma OE(D_1)| \times |\sigma OE(D_2)|) \\
&\qquad - (|\sigma EE(D_1)| \times |\sigma OE(D_2)| + |\sigma OE(D_1)| \times |\sigma EE(D_2)|) \\
&\quad = (|\sigma EE(D_1)| - |\sigma OE(D_1)|) \cdot (|\sigma EE(D_2)| - |\sigma OE(D_2)|) \\
&\quad \not\equiv 0 \pmod p,
\end{aligned}$$

where the last inequality holds since D_1 and D_2 are p-nice and since $\mathbb{Z}_p$ does not have any zero divisors. This proves that D is a p-nice orientation of $G - e$.

Secondly, we assume that C has no chord of the form $v_k v_j$ where $2 \le j \le k - 2$. Let $v_{k-1}, u_1, \ldots, u_s, v_1$ be the neighbors of v_k and be ordered so that

$v_k v_{k-1} u_1, \ldots, v_k u_s v_1$ are inner facial cycles of G. Let $G' = G - v_k$. It is clear that G' is a near triangulation with outer facial cycle $v_1 v_2 \ldots v_{k-1} u_1 \ldots u_s$. Therefore, by the induction hypothesis, $G' - e$ has a p-nice orientation D'.

If $k = 3$ (i.e., C is a triangle), then let D be the orientation of $G - e$ obtained from D' by adding the vertex v_3 and oriented edges (v_3, v_1), (v_3, v_2) and (u_i, v_3) for $i \in \{1, 2, \ldots, s\}$, as shown in Fig. 2b. It is easy to verify that D satisfies the outdegree condition in Definition 4. In particular, both v_1 and v_2 have outdegree 0. Thus, v_1 and v_2 are both isolated in any Eulerian subdigraph of D and therefore, by the definition of D, v_3 is also isolated in any Eulerian subdigraph of D. This means that each Eulerian subdigraph of D is an Eulerian subdigraph of D' by ignoring the isolated vertex v_3. Thus, $|\sigma EE(D)| = |\sigma EE(D')|$ and $|\sigma OE(D)| = |\sigma OE(D')|$. Since D' is a p-nice orientation, we have $|\sigma EE(D)| - |\sigma OE(D)| = |\sigma EE(D')| - |\sigma OE(D')| \not\equiv 0 \pmod p$. This proves that D is a p-nice orientation of $G - e$.

Next, we assume $k \geq 4$. We call an orientation D of $G' - e$ *special* if both of the followings hold:

- v_1 and v_2 have outdegree 0, v_{k-1} has outdegree at most 1, each of $v_3, v_4, \ldots, v_{k-2}$ has outdegree at most 2, and each of $u_1, u_2, \ldots, u_s$ has outdegree at most 3.
- Every interior vertex has outdegree at most 4.

To show that $G - e$ has a p-nice orientation, we consider two cases:

Case 1. $G' - e$ has a special orientation D'' with $|\sigma EE(D'')| - |\sigma OE(D'')| \not\equiv 0 \pmod p$.

Let D be the orientation of $G-e$ obtained from D'' by adding the vertex v_k and $s+2$ oriented edges (v_k, v_1),(v_{k-1}, v_k) and (u_i, v_k) for $i \in \{1, 2, \ldots, s\}$, see Fig. 2c. Then D satisfies the outdegree condition of a p-nice orientation. Since v_1 has outdegree 0 in D, by a similar discussion as above, v_k is isolated in any Eulerian subdigraph of D. Therefore, each Eulerian subdigraph of D is an Eulerian subdigraph of D'' by ignoring the isolated vertex v_k, i.e., $|\sigma EE(D)| = |\sigma EE(D'')|$ and $|\sigma OE(D)| = |\sigma OE(D'')|$. This yields that $|\sigma EE(D)| - |\sigma OE(D)| \not\equiv 0 (\mathrm{mod}\ p)$ by the condition of this case. Thus, D is a p-nice orientation of $G - e$, as desired.

Case 2. For any special orientation D'' (if exists), $|\sigma EE(D'')| - |\sigma OE(D'')| \equiv 0$ (mod p).

Recall that D' is a p-nice orientation of $G' - e$. Let D be the orientation of $G - e$ obtained from D' by adding the vertex v_k and $s + 2$ oriented edges (v_k, v_1), (v_k, v_{k-1}) and (u_i, v_k) for $i \in \{1, 2, \ldots, s\}$, as shown in Fig. 2d. Clearly, D satisfies the outdegree condition of a p-nice orientation. To show that D is p-nice for $G - e$, it remains to show that $|\sigma EE(D)| - |\sigma OE(D)| \not\equiv 0 \pmod p$.

Notice that v_1 has outdegree 0 in D and therefore, is isolated in any Eulerian subdigraph of D. Thus, if H is an Eulerian subdigraph of D and v_k is non-isolated in H then H contains the oriented edge (v_k, v_{k-1}) and exactly one of the s oriented edges $(u_1, v_k), (u_2, v_k), \ldots, (u_s, v_k)$. For $i \in \{1, 2, \ldots, s\}$, let

$$\sigma EE_i(D) = \{H \in \sigma EE(D) \colon\ (u_i, v_k) \in E(H)\},$$
$$\sigma OE_i(D) = \{H \in \sigma OE(D) \colon\ (u_i, v_k) \in E(H)\}.$$

For an Eulerian subdigraph of D', we regard it as an Eulerian subdigraph of D by adding v_k as an isolated vertex. Then we have

$$\sigma EE(D) = \sigma EE(D') \cup \bigcup_{i=1}^{s} \sigma EE_i(D),$$

$$\sigma OE(D) = \sigma OE(D') \cup \bigcup_{i=1}^{s} \sigma OE_i(D).$$

Since D' is p-nice, $|\sigma EE(D')| - |\sigma OE(D')| \not\equiv 0 (\text{mod } p)$. Therefore, in order to complete the proof in this case, it suffices to show the following claim.

Claim $|\sigma EE_i(D)| - |\sigma OE_i(D)| \equiv 0$ (mod p) for any $i \in \{1, \ldots, s\}$.

Let i be any integer in $\{1, 2, \ldots, s\}$. If $\sigma EE_i(D) \cup \sigma OE_i(D) = \emptyset$ then $|\sigma EE_i(D)| = |\sigma OE_i(D)| = 0$, as desired. Thus, we may assume that $\sigma EE_i(D) \cup \sigma OE_i(D) \neq \emptyset$. Therefore, D has an Eulerian subdigraph and hence a directed cycle containing (u_i, v_k). Let $C_i = u_i v_k v_{k-1} w_1 w_2 \cdots w_p$ be such a directed cycle and let D'_i be the orientation of $G' - e$ obtained from D' by reversing the direction of edges in the path $v_{k-1} w_1 w_2 \cdots w_p u_i$. The reversing operation decreases the outdegree of v_{k-1} by 1, increases the outdegree of u_i by 1, and leaves the outdegrees of other vertices in $G' - e$ unchanged. Since D' is p-nice for $G' - e$, the outdegree condition of D' implies that D'_i is special. Hence, $|\sigma EE(D'_i)| - |\sigma OE(D'_i)| \equiv 0$ (mod p) by the condition of this case.

Let C_i^{-1} be the reverse of C_i, i.e., $C_i^{-1} = w_p w_{p-1} \cdots w_1 v_{k-1} v_k u_i$. For each Eulerian subdigraph $H \in \sigma EE_i(D) \cup \sigma OE_i(D)$, let $H \triangle C_i^{-1}$ be the symmetric difference of the edge sets of H and C_i^{-1}, that is, the set obtained from the edge union $H \cup C_i^{-1}$ of H and C_i^{-1} by deleting the directed 2-cycles. One may verify that $H \triangle C_i^{-1}$ is an Eulerian subdigraph of D'_i and the map $\tau : \ H \mapsto H \triangle C_i^{-1}$ is a bijection between $\sigma EE_i(D) \cup \sigma OE_i(D)$ and $\sigma EE(D'_i) \cup \sigma OE(D'_i)$. For a set S of some oriented edges in an orientation of (G, σ), we use $N(S)$ to denote the number of positive edges in S. It is easy to see that $N(H \triangle {C_i}^{-1}) = N(H) + N({C_i}^{-1}) - 2N(H \cap C_i)$. Therefore, if $N({C_i}^{-1})$ is even, then $N(H \triangle {C_i}^{-1})$ and $N(H)$ are the same parity and hence τ maps $\sigma EE_i(D)$ to $\sigma EE(D'_i)$ and $\sigma OE_i(D)$ to $\sigma OE(D'_i)$. Similarly, if $N({C_i}^{-1})$ is odd, then it maps $\sigma EE_i(D)$ to $\sigma OE(D'_i)$ and $\sigma OE_i(D)$ to $\sigma EE(D'_i)$. In either case, we have

$$\Big||\sigma EE_i(D)| - |\sigma OE_i(D)|\Big| = \Big||\sigma EE({D'}_i)| - |\sigma OE({D'}_i)|\Big| \equiv 0 \ (\text{mod } p),$$

and hence complete the proof.

Proof of Theorem 4 Since $AT_p(H, \sigma) \leq AT_p(G, \sigma)$ for any subgraph H of G, it suffices to consider the case when G is a near triangulation. Let $v_1 v_2 \cdots v_k$ be the outer facial cycle of G and $e = v_1 v_2$. By Theorem 5, $G - e$ has a p-nice orientation D. Let D' be obtained from D by adding the oriented edge (v_1, v_2). Clearly, each vertex has outdegree at most 4 in D'. Moreover, as v_2 has outdegree 0 in D', the orientated edge (v_1, v_2) will never appear in any Eulerian subgraph of D'. Thus,

$|\sigma EE(D')| = |\sigma EE(D)|$ and $|\sigma OE(D')| = |\sigma OE(D)|$. As D is p-nice, we have $|\sigma EE(D')| - |\sigma OE(D')| = |\sigma EE(D)| - |\sigma OE(D)| \not\equiv 0 (\mathrm{mod}\ p)$. Therefore, by Corollary 3, we have $AT_p(G, \sigma) \leq 5$.

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