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Existence of positive ground state solutions of Schrödinger–Poisson system involving negative nonlocal term and critical exponent on bounded domain

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Abstract

In this paper, we prove the existence of positive ground state solutions of the Schrödinger–Poisson system involving a negative nonlocal term and critical exponent on a bounded domain. The main tools are the mountain pass theorem and the concentration compactness principle.

Keywords: Negative nonlocal; Critical exponent; Mountain pass theorem; Concentration compactness principle

1 Introduction

In this paper, we consider the following Schrödinger–Poisson system:

$$\begin{cases} -\Delta u - \phi u = \lambda u^{q-1} + u^5 & \text{in } \Omega, \\ -\Delta \phi = u^2 & \text{in } \Omega, \\ u = \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $2 < q < 6$, and Ω is a smooth bounded domain in $\mathbb{R}^3$.

System (1.1) is related to the stationary analogue of the nonlinear parabolic Schrödinger–Poisson system

$$\begin{cases} -i\frac{\partial\psi}{\partial t} = -\Delta\psi + \phi(x)\psi - |\psi|^{p-2}\psi & \text{in } \Omega, \\ -\Delta\phi = |\psi|^2 & \text{in } \Omega, \\ \psi = \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

The first equation in (1.2) is called the Schrödinger–Poisson equation, which describes quantum particles interacting with the electromagnetic field generated by a motion. Similar problems have been widely investigated, and it is well known that they have a strong physical meaning because they appear in quantum mechanics models (see e.g. [3]) and in semiconductor theory [10, 12]. Variational methods and critical point theory are always powerful tools in studying nonlinear differential equations. For more details as regards

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the physical relevance of the Schrödinger–Poisson system, we refer to [1, 13] and some related results [14, 16–18, 20].

The Schrödinger–Poisson system on whole space $\mathbb{R}^N$ has attracted a lot of attention. Few works concern the existence of solutions for the Schrödinger–Poisson system on a bounded domain, particularly, critical nonlinearity except [2, 7, 8]. Up to now, Schrödinger–Poisson system (1.1) has never been studied by variational methods. Lei and Suo [8] studied the following Schrödinger–Poisson system:

$$\begin{cases} -\triangle u + \kappa\phi u = \kappa|u|^{p-2}u + |u|^4 u & \text{in } \Omega, \\ -\triangle\phi = u^2 & \text{in } \Omega, \\ \phi = u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $\kappa > 0$ is a real parameter, and $1 < p < 2$. There exists $\kappa^* > 0$ such that there at least two positive solutions, and one of them is a positive ground state solution for $\kappa \in (0, \kappa^*)$. Zhang [19] considered the negative nonlocal Schrödinger–Poisson system on a bounded domain and obtained thtat there are at least two solutions involving a singularity term by using the Nehari method. Li and Tang [9] obtained at least two positive solutions $(u, \phi_u) \in D^{1,2}(\mathbb{R}^3) \times D^{1,2}(\mathbb{R}^3)$ involving a negative nonlocal term in $\mathbb{R}^3$.

Our paper is motivated by all the results mentioned [2, 7–9, 19]. Up to now, there was no information about system (1.1) on a bounded domain Ω; this is what we are interested in. To deal with our system (1.1), we should estimate the critical value as regards the difficulty caused by the critical exponent.

Now our main results can be stated as follows.

Theorem 1.1 *Let* $2 < q \leq 4$. *Then there exists* $\lambda^* > 0$ *such that system* (1.1) *has at least one positive ground state solution for all* $\lambda > \lambda^*$.

Theorem 1.2 *Let* $4 < q < 6$. *Then system* (1.1) *has at least one positive ground state solution for all* $\lambda > 0$.

2 Preliminaries

Let X be the usual Sobolev space $H_0^1(\Omega)$ with the inner product $(u, v) = \int_\Omega \nabla u \nabla v\,dx$ and norm $\|u\| = \sqrt{(u, u)}$; $|u|_s$ denotes the norm of the space $L^s(\Omega)$, $2 \leq s \leq 6$. For any $r > 0$ and $x \in \Omega$, $B_r(x)$ denotes the ball of radius r centered at x. C and C_i ($i = 1, 2, 3, \dots$) denote various positive constants, which may vary from line to line.

It is well known that system (1.1) can be reduced to a nonlinear Schrödinger equation with nonlocal term. Indeed, the Lax–Milgram theorem implies that for all $u \in X$, there exists a unique $\phi_u \in X$ such that

$$-\Delta\phi_u = u^2.$$

It is standard to see that system (1.1) is variational and its solutions are the critical points of the functional defined in X by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_\Omega \phi_u u^2\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \frac{1}{6}\int_\Omega |u|^6\,dx.$$

For simplicity, in many cases, we just say that $u \in X$, instead of $(u, \phi_u) \in X \times X$, is a weak solution of system (1.1). It is easy to see that $I \in C^1(X, \mathbb{R})$ (see [8, 9]) and

$$\langle I'(u), \varphi \rangle = \int_\Omega \nabla u \nabla \varphi \, dx - \int_\Omega \phi_u u \varphi \, dx - \lambda \int_\Omega |u|^{q-2} u \varphi \, dx - \int_\Omega u^5 \varphi \, dx, \quad \forall u, \varphi \in X.$$

Let S be the best Sobolev constant, namely

$$S := \inf_{u \in H_0^1(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, dx}{(\int_\Omega |u|^6 \, dx)^{\frac{1}{3}}}. \tag{2.1}$$

As it is well known, the function

$$U(x) = \frac{(3\epsilon^2)^{\frac{1}{4}}}{(\epsilon^2 + |x|^2)^{\frac{1}{2}}}, \quad x \in \mathbb{R}^3, \tag{2.2}$$

is an extremal function for the minimum problem (2.1), that is, it is a positive solution of the equation

$$-\Delta u = u^5, \quad \forall x \in x \in \mathbb{R}^3, \tag{2.3}$$

and

$$\|U\|^2 = |U|_6^6 = S^{\frac{3}{2}}; \tag{2.4}$$

see [11].

Before proving our Theorem 1.1, we need the following lemma.

Lemma 2.1 (see [6]) *For every $u \in H_0^1(\Omega)$, there exists a unique solution $\phi_u \in H_0^1(\Omega)$ of*

$$\begin{cases} -\triangle \phi = u^2 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

and

(1) $\phi_u \geq 0$;*moreover,* $\phi_u > 0$ *when* $u \neq 0$;
(2) *for each* $t \neq 0$, $\phi_{tu} = t^2 \phi_u$;
(3) $\int_\Omega \phi_u u^2 \, dx = \int_\Omega |\nabla \phi_u|^2 \, dx \leq S^{-1} |u|_{\frac{12}{5}}^4 \leq C\|u\|^4$;
(4) *if* $F(u) = \int_\Omega \phi_u u^2 \, dx$, *then* $F : H_0^1(\Omega) \to H_0^1(\Omega)$ *is* C^1, *and*

$$\langle F'(u), v \rangle = 4 \int_\Omega \phi_u u v \, dx, \quad \forall v \in H_0^1(\Omega).$$

3 The Palais–Smale condition

First, we prove the following mountain-pass geometry of the functional I.

Lemma 3.1 *Let $2 < q < 6$ and $\lambda > 0$. Then the functional I satisfies the following conditions*:

(i) *There exist two constants* $\alpha, \rho > 0$ *such that*

$$I(u) \geq \alpha > 0 \quad \text{with } \|u\| = \rho.$$

(ii) *There exists* $e \in X$ *with* $\|e\| > \rho$ *such that* $I(e) < 0$.

Proof (i). We have

$$\begin{aligned} I(u) &= \frac{1}{2}\|u\|^2 - \frac{1}{4}\int_\Omega \phi_u u^2\,dx - \frac{\lambda}{q}\int_\Omega |u|^q\,dx - \frac{1}{6}\int_\Omega |u|^6\,dx \\ &\geq \frac{1}{2}\|u\|^2 - \frac{1}{4}C\|u\|^4 - C_1\|u\|^q - \frac{1}{6S^3}\|u\|^6. \end{aligned}$$

Therefore, since $q > 2$, there exist $\alpha, \rho > 0$ such that $I(u) \geq \alpha > 0$ with $\|u\| = \rho$.
(ii). For $u \in X \setminus \{0\}$, we have

$$I(tu) \leq \frac{1}{2}t^2\|u\|^2 - \frac{1}{6}t^6\int_\Omega |u|^6\,dx \to -\infty$$

as $t \to +\infty$. Then we can find $e \in X$ such that $\|e\| > \rho$ and $I(e) < 0$. This completes the proof. □

Therefore by using the mountain pass theorem without $(PS)_c$ condition (see [15]) it follows that there exists a $(PS)_c$ sequence $\{u_n\} \subset X$ such that

$$I(u_n) \to c = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I\big(\gamma(t)\big) \quad \text{and} \quad I'(u_n) \to 0,$$

where

$$\Gamma = \big\{\gamma \in C\big([0,1], X\big) : \gamma(0) = 0, \gamma(1) = e\big\}.$$

Lemma 3.2 *Let* $2 < q < 6$ *and* $\lambda > 0$. *Let* $\{u_n\} \subset X$ *be a* $(PS)_c$ *sequence of* I *with* $0 < c < \frac{1}{3}S^{\frac{3}{2}}$. *Then there exists* $u \in X$ *such that* $u_n \to u$ *in* X.

Proof Let $\{u_n\} \subset X$ be a $(PS)_c$ for I, that is,

$$I(u_n) \to c \quad \text{and} \quad I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{3.1}$$

We claim that $\{u_n\}$ is bounded in X. In the case $2 < q \leq 4$, we deduce that

$$\begin{aligned} 1 + c + o\big(\|u_n\|\big) &\geq I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2 + \left(\frac{1}{q} - \frac{1}{4}\right)\int_\Omega \phi_{u_n}{u_n}^2\,dx + \left(\frac{1}{q} - \frac{1}{6}\right)\int_\Omega |u_n|^6\,dx \\ &\geq \left(\frac{1}{2} - \frac{1}{q}\right)\|u_n\|^2; \end{aligned}$$

in the case $4 < q < 6$, we have

$$\begin{aligned} 1 + c + o\big(\|u_n\|\big) &\geq I(u_n) - \frac{1}{4}\big\langle I'(u_n), u_n\big\rangle \\ &= \frac{1}{4}\|u_n\|^2 + \left(\frac{1}{4} - \frac{1}{q}\right)\lambda \int_\Omega |u_n|^q\,dx + \frac{1}{12}\int_\Omega |u_n|^6\,dx \\ &\geq \frac{1}{4}\|u_n\|^2, \end{aligned}$$

which implies that $\{u_n\}$ is bounded in X. Going if necessary to a subsequence, still denoted by $\{u_n\}$, we can assume that for n large enough,

$$\begin{cases} u_n \rightharpoonup u & \text{in } X; \\ u_n \to u & \text{in } L^p(\Omega), p \in [1,6); \\ u_n \to u & \text{for a.e. } x \in \Omega. \end{cases} \tag{3.2}$$

By the concentration compactness principle (see [5, 11]) there exists at most countable set J, points $\{x_j\}_{j\in J} \subset \Omega$, and values $\{\nu_j\}_{j\in J}, \{\mu_j\}_{j\in J} \subset R^+$ such that

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_{j\in J}\mu_j \delta_{x_j}, \tag{3.3}$$

$$|u_n|^6 \rightharpoonup \nu = |u|^6 + \sum_{j\in J}\nu_j \delta_{x_j}, \tag{3.4}$$

where δ_{x_j} is the Dirac mass at x_j. Moreover, we have

$$\mu_j, \nu_j \geq 0, \qquad \mu_j \geq S\nu_j^{\frac{1}{3}}. \tag{3.5}$$

We claim that $J = \emptyset$. Suppose, on the contrary, that $J \neq \emptyset$, that is, there exists $j_0 \in J$ such that $\mu_{j_0} \neq 0$.

On the one hand, for any $\varepsilon > 0$ small, assume that $\psi_{\varepsilon,j}(x) \in C_0^\infty(\mathbb{R}^3)$ is such that $\psi_{\varepsilon,j}(x) \in [0,1]$,

$$\psi_{\varepsilon,j}(x) = 1, \quad \text{in } B\left(x_j, \frac{\varepsilon}{2}\right); \qquad \psi_{\varepsilon,j}(x) = 0, \quad \text{in } X \setminus B(x_j, \varepsilon); \qquad \big|\nabla \psi_{\varepsilon,j}(x)\big| \leq \frac{4}{\varepsilon}.$$

Since $\{u_n\} \subset X$ is bounded and $\{\psi_{\varepsilon,j}u_n\}$ is also bounded, we have

$$\begin{aligned} o(1) &= \big\langle I'(u_n), \psi_{\varepsilon,j}u_n\big\rangle \\ &= \left(\int_\Omega u_n \nabla u_n \nabla \psi_{\varepsilon,j}\,dx + \int_\Omega |\nabla u_n|^2 \psi_{\varepsilon,j}\,dx\right) \\ &\quad - \int_\Omega \phi_{u_n} u_n^2 \psi_{\varepsilon,j}\,dx - \lambda \int_\Omega |u_n|^q \psi_{\varepsilon,j}\,dx - \int_\Omega u_n^6 \psi_{\varepsilon,j}\,dx, \end{aligned} \tag{3.6}$$

and by the Hölder inequality we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left|\int_\Omega u_n\nabla u_n\nabla\psi_{\varepsilon,j}\,dx\right| \\
&\quad\le \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\left(\int_{B_\varepsilon(x_j)}|\nabla u_n|^2\,dx\right)^{\frac{1}{2}}\left(\int_{B_\varepsilon(x_j)}|\nabla\psi_{\varepsilon,j}|^2|u_n|^2\,dx\right)^{\frac{1}{2}} \\
&\quad\le \lim_{\varepsilon\to 0}C_2\left(\int_{B_\varepsilon(x_j)}|\nabla\psi_{\varepsilon,j}|^2|u|^2\,dx\right)^{\frac{1}{2}} \\
&\quad\le \lim_{\varepsilon\to 0}C_2\left(\int_{B_\varepsilon(x_j)}|\nabla\psi_{\varepsilon,j}|^3\,dx\right)^{\frac{1}{3}}\left(\int_{B_\varepsilon(x_j)}|u|^6\,dx\right)^{\frac{1}{6}} \\
&\quad\le \lim_{\varepsilon\to 0}C_3\left(\int_{B_\varepsilon(x_j)}|u|^6\,dx\right)^{\frac{1}{6}} = 0.
\end{aligned}\tag{3.7}$$

From (3.2)–(3.4) we have

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_\Omega |u_n|^6\psi_{\varepsilon,j}\,dx \\
&\quad= \lim_{\varepsilon\to 0}\int_\Omega |u|^6\psi_{\varepsilon,j}\,dx + \nu_j \\
&\quad= \nu_j,
\end{aligned}\tag{3.8}$$

and by Lemma 2.1 and (3.2) we obtain

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_\Omega \phi_{u_n}u_n^2\psi_{\varepsilon,j}\,dx = 0,\tag{3.9}$$

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_\Omega |u_n|^q\psi_{\varepsilon,j}\,dx = 0,\tag{3.10}$$

and

$$\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\int_\Omega |\nabla u_n|^2\psi_{\varepsilon,j}\,dx \ge \lim_{\varepsilon\to 0}\int_\Omega |\nabla u|^2\psi_{\varepsilon,j}\,dx + \mu_j = \mu_j.\tag{3.11}$$

By (3.6)–(3.11) we obtain

$$\nu_j \ge \mu_j,$$

which, combined with $\mu_{j_0}\ne 0$ and (3.5), gives

$$\nu_{j_0} \ge S^{\frac{3}{2}}.\tag{3.12}$$

From (3.3)–(3.5) and (3.12) in the case $2 < q \le 4$, we have

$$\begin{aligned}
c &= \lim_{n\to\infty}\left\{I(u_n) - \frac{1}{q}\langle I'(u_n), u_n\rangle\right\} \\
&= \lim_{n\to\infty}\left\{\left(\frac{1}{2}-\frac{1}{q}\right)\|u_n\|^2 + \left(\frac{1}{q}-\frac{1}{4}\right)\int_\Omega \phi_{u_n}{u_n}^2\,dx + \left(\frac{1}{q}-\frac{1}{6}\right)\int_\Omega |u_n|^6\,dx\right\}
\end{aligned}$$

$$\geq \left(\frac{1}{2}-\frac{1}{q}\right)\left(\|u\|^2+\sum_{j\in J}\mu_j\right)+\left(\frac{1}{q}-\frac{1}{6}\right)\left(\int_\Omega |u|^6\,dx+\sum_{j\in J}\nu_j\right)$$
$$\geq \left(\frac{1}{2}-\frac{1}{q}\right)\mu_{j_0}+\left(\frac{1}{q}-\frac{1}{6}\right)\nu_{j_0}\geq \frac{1}{3}S^{\frac{3}{2}},$$

and in the case $4 < q < 6$, we have

$$c=\lim_{n\to\infty}\left\{I(u_n)-\frac{1}{4}\langle I'(u_n),u_n\rangle\right\}$$
$$=\lim_{n\to\infty}\left\{\frac{1}{4}\|u_n\|^2+\left(\frac{1}{4}-\frac{1}{q}\right)\lambda\int_\Omega |u_n|^q\,dx+\frac{1}{12}\int_\Omega |u_n|^6\,dx\right\}$$
$$\geq \frac{1}{4}\left(\|u\|^2+\sum_{j\in J}\mu_j\right)+\frac{1}{12}\left(\int_\Omega |u|^6\,dx+\sum_{j\in J}\nu_j\right)$$
$$\geq \frac{1}{4}\mu_{j_0}+\frac{1}{12}\nu_{j_0}\geq \frac{1}{3}S^{\frac{3}{2}},$$

where we use $\nu_j \geq \mu_j$ and $\nu_j \geq S^{\frac{3}{2}}$. Therefore by $c < \frac{1}{3}S^{\frac{3}{2}}$ it is a contradiction. This implies that J is empty, which means that $\int_\Omega |u_n|^6\,dx \to \int_\Omega |u|^6\,dx$. We can also get $u_n \to u$ in X (see Lemma 2.2 in [8]). So Lemma 3.2 holds. □

Lemma 3.3 *If $2 < q \leq 4$, then there exist $\lambda^* > 0$ and $\overline{v}_0 \in H_0^1(\Omega)$ such that*

$$\sup_{s\geq 0} I(s\overline{v}_0) < \frac{1}{3}S^{\frac{3}{2}} \quad \textit{for all } \lambda > \lambda^*.$$

If $4 < q < 6$, then there exists $\overline{v}_1 \in H_0^1(\Omega)$ such that

$$\sup_{s\geq 0} I(s\overline{v}_1) < \frac{1}{3}S^{\frac{3}{2}} \quad \textit{for all } \lambda > 0.$$

Proof We choose a function $\eta \in C_0^\infty(\Omega)$ such that $0 \leq \eta(x) \leq 1$, $|\nabla\eta| \leq C$ in Ω. $\eta(x) = 1$ for $|x| < 2r_0$, and $\eta(x) = 0$ for $|x| > 3r_0$. Define

$$u_\epsilon(x) = \eta(x)U(x).$$

It is known (see [15]) that

$$\begin{aligned}
&|u_\epsilon|_6^6 = S^{\frac{3}{2}} + O(\epsilon^3),\\
&\|u_\epsilon\|^2 = S^{\frac{3}{2}} + O(\epsilon),\\
&\|u_\epsilon\|^4 \leq S^3 + O(\epsilon),\\
&C_4\epsilon^{\frac{p}{2}} \leq \int_\Omega u_\epsilon^p\,dx \leq C_5\epsilon^{\frac{p}{2}}, \quad 1 \leq p < 3,\\
&C_6\epsilon^{\frac{p}{2}}|\ln\epsilon| \leq \int_\Omega u_\epsilon^p\,dx \leq C_7\epsilon^{\frac{p}{2}}|\ln\epsilon|, \quad p = 3,\\
&C_8\epsilon^{\frac{6-p}{2}} \leq \int_\Omega u_\epsilon^p\,dx \leq C_9\epsilon^{\frac{6-p}{2}}, \quad 3 < p < 6.
\end{aligned} \tag{3.13}$$

Set

$$h(su_\epsilon) = \frac{s^2}{2}\|u_\epsilon\|^2 - \frac{s^q\lambda}{q}\int_\Omega |u_\epsilon|^q\,dx - \frac{s^6}{6}\int_\Omega |u_\epsilon|^6\,dx.$$

We can also prove that $\max_{s\geq 0} h(su_\epsilon)$ is attained at s_0 for $0 < s_1 < s_0 < s_2$, that is,

$$\max_{s\geq 0} h(su_\epsilon) = h(s_0 u_\epsilon). \tag{3.14}$$

Combining (3.13) with (3.14), $4 < q < 6$, we deduce

$$\begin{aligned}
\sup_{t\geq 0} I(su_\epsilon) &= \frac{s^2}{2}\|u_\epsilon\|^2 - \frac{s^4}{4}\int_\Omega \phi_{u_\epsilon} {u_\epsilon}^2\,dx - \frac{s^q\lambda}{q}\int_\Omega |u_\epsilon|^q\,dx - \frac{s^6}{6}\int_\Omega |u_\epsilon|^6\,dx \\
&\leq \frac{s^2}{2}\|u_\epsilon\|^2 - \frac{s^q\lambda}{q}\int_\Omega |u_\epsilon|^q\,dx - \frac{s^6}{6}\int_\Omega |u_\epsilon|^6\,dx \\
&\leq \frac{s_0^2}{2}\|u_\epsilon\|^2 - \frac{s_0^q\lambda}{q}\int_\Omega |u_\epsilon|^q\,dx - \frac{s_0^6}{6}\int_\Omega |u_\epsilon|^6\,dx \\
&\leq \frac{1}{2}s_0^2\bigl(S^{\frac{3}{2}} + O(\epsilon)\bigr) - C_{10}\epsilon^{\frac{6-q}{2}} - \frac{s_0^6}{6}\bigl(S^{\frac{3}{2}} + O(\epsilon^3)\bigr) \\
&= \frac{1}{2}s_0^2 S^{\frac{3}{2}} - \frac{s_0^6}{6}S^{\frac{3}{2}} + O(\epsilon) - O(\epsilon^3) - C_{10}\epsilon^{\frac{6-q}{2}} \\
&\leq \sup_{k\geq 0}\left\{\frac{k^2}{2}S^{\frac{3}{2}} - \frac{k^6}{6}S^{\frac{3}{2}}\right\} + O(\epsilon) - C_{10}\epsilon^{\frac{6-q}{2}} < \frac{1}{3}S^{\frac{3}{2}} \quad \text{as } \epsilon \to 0^+.
\end{aligned} \tag{3.15}$$

Similarly, in the case $2 < q \leq 4$, by (3.13) and (3.15) we have

$$\begin{aligned}
\sup_{t\geq 0} I(su_\epsilon) &\leq \frac{s_0^2}{2}\|u_\epsilon\|^2 - \frac{s_0^q\lambda}{q}\int_\Omega |u_\epsilon|^q\,dx - \frac{s_0^6}{6}\int_\Omega |u_\epsilon|^6\,dx \\
&\leq \frac{1}{2}s_0^2\bigl(S^{\frac{3}{2}} + O(\epsilon)\bigr) - C_{11}\lambda\epsilon^{\frac{6-q}{2}} - \frac{s_0^6}{6}\bigl(S^{\frac{3}{2}} + O(\epsilon^3)\bigr) \\
&\leq \frac{1}{3}S^{\frac{3}{2}} + O(\epsilon) - C_{11}\lambda\epsilon^{\frac{6-q}{2}} < \frac{1}{3}S^{\frac{3}{2}} \quad \text{as } \epsilon \to 0^+,
\end{aligned} \tag{3.16}$$

provided that λ is large enough. Thus there exists $\lambda^* > 0$ such that $I(su_\epsilon) < \frac{1}{3}S^{\frac{3}{2}}$ for all $\lambda > \lambda^*$. This completes the proof. □

4 Proof of theorems

Proof of Theorems 1.1 and 1.2 Due to Lemma 3.1, $I(u)$ satisfies the mountain pass geometry. From Lemmas 3.2 and 3.3 we obtain the $(PS)_c$ condition with $0 < c < \frac{1}{3}S^{\frac{3}{2}}$. Therefore system (1.1) has a nontrivial solution u_0, and $I(u_0) = c > 0$, which is a mountain pass solution. Since $I(|u|) = I(u)$, by a result due to Brézis and Nirenberg (Theorem 10 in [4]) we conclude that $u_0 \geq 0$. By the strong maximum principle we have $u_0 > 0$ in Ω. Therefore u_0 is a positive solution of system (1.1) with $I(u_0) > 0$.

Next, we show that system (1.1) has a positive ground state solution in X when $2 < p \leq 4$ or $4 < p < 6$.

Define

$$m := \inf_{v \in \mathcal{M}} I(v), \mathcal{M} = \{v \in X \setminus \{0\} \mid I'(v) = 0\}.$$

There exists $\{u_n\} \subset X$ such that $u_n \neq 0$. Since u_0 is a solution of system (1.1), by the definition of m we have

$$I(u_n) \to m, m < \frac{1}{3} S^{\frac{3}{2}}, \qquad I'(u_n) \to 0 \quad \text{as } n \to \infty. \tag{4.1}$$

Obviously, from Lemma 3.2 we can easily deduce that $\{u_n\}$ is bounded in X. Then there exist a nonnegative subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$) and $u_1 \in X$ such that $u_n \rightharpoonup u_1$ in X. We can obtain that $u_n \to u_1$ in X and $I(u_1) = m$ with $u_1 > 0$ by the last section in [8], that is, u_1 is a positive ground state solution to system (1.1). This completes the proof. □

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