Absorption at Electron Cyclotron Resonance in Slightly Ionized Gases (II)

Tanaka, Shigetoshi

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ABSORPTION AT ELECTRON CYCLOTRON RESONANCE IN SLIGHTLY IONIZED GASES (II)*

BY

Shigetoshi TANAKA

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ABSTRACT

An expression of the absorption at cyclotron resonance was derived for the microwave propagating through a slightly ionized gas, transversally to the static magnetic field. It was shown that the resonance occurs at $\omega_c=\sqrt{\omega^2-\omega_p{}^2-\nu_c{}^2}$ instead of $\omega_c=\omega$ and the absorption curve is not Lorentzian and the half-width depends not only on collision frequency but also on electron density. Therefore, one can obtain both the electron density and the collision frequency from one measured resonance absorption curve. The same expression was derived by solving the Boltzmann transport equation for the distribution of electrons under the same circumstances.

1. Introduction

In the first part** dealing with the microwave propagating along the static magnetic field ($\boldsymbol{k}_0 /\!\!/ \boldsymbol{B}$), it was shown that the cyclotron resonance occurs at $\omega=\omega_c$, the absorption curve is Lorentzian and the half-width gives a direct measure of the collision frequency.

In this part, we consider the electron cyclotron resonance in a slightly ionized gas, for the microwave propagating tansversally to $\boldsymbol{B}$ ($\boldsymbol{k}_0 \perp \boldsymbol{B}$). Since the wave becomes a so-called hybrid wave with transversal and longitudinal electric vectors in the case of $\boldsymbol{k}_0 \perp \boldsymbol{B}$, the dispersion relation is more complicated in comparison with the purely transversal wave ($\boldsymbol{k}_0 /\!\!/ \boldsymbol{B}$) (**1**).

In a similar way to the case of purely transversal wave, approximate expressions for the attenuation factor and the relative phase shift near the cyclotron resonance were derived from the specific inductive capacity for the hybrid wave. The same absorption expression was obtained by solving the Boltzmann transport equation for the distribution function of electrons and it was shown that there is no power dissipation for the longitudinal electrical component of the hybrid wave.

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** The preceding paper, hereafter referred to as I.

2. Dispersion of the hybrid wave

We consider a microwave propagating through a slightly ionized gas in the direction of $\boldsymbol{k}_0$ transversal to the static magnetic field $\boldsymbol{B}$.

The electron motion parallel to $\boldsymbol{B}$ is unaffected by $\boldsymbol{B}$, so the wave with the electric vector initially parallel to $\boldsymbol{B}$ can remain as purely transversal wave, and the dispersion relation is the same as in the case of no magnetic field. Such a case is left out of consideration. The interesting case to be considered is that in which $\boldsymbol{E}$ is initially perpendicular to $\boldsymbol{B}$. Taking the z-axis parallel to $\boldsymbol{B}$ and the x-axis to $\boldsymbol{k}_0$, the wave, incident with E_y component only on the plasma slab, comes to have both E_x and E_y components inside the plasma. Such a wave is so-called hybrid transversal and longitudinal wave (**1**, **2**).

For the hybrid wave, the effective specific inductive capacity $K_\perp$ is given by (**3**, **4**):

$$K_\perp = 1 - \frac{\eta(1-\eta+j\beta)}{(1-\eta-\gamma^2-\beta^2)+j\beta(2-\eta)}, \tag{1}$$

using the notations given in I. We proceed in a similar way.

If $\eta \ll 1$ and $\beta^2 \ll 1$, by expanding (1) with respect to η, the refractive index $(n_\perp = n'_\perp + jn''_\perp)$ can be approximated by:

$$n'_\perp = 1 - \frac{\eta}{2}\frac{(1-\eta)(1-\eta-\gamma^2)+\beta^2}{(1-\eta-\gamma^2-\beta^2)^2+\beta^2(2-\eta)^2} - \frac{\eta^2}{8}\frac{\{(1-\eta)(1-\eta-\gamma^2)+\beta^2\}^2-\beta^2\{(1-\eta)^2+\gamma^2+\beta^2\}^2}{\{(1-\eta-\gamma^2-\beta^2)^2+\beta^2(2-\eta)^2\}^2}, \tag{2}$$

and

$$n''_\perp = \frac{\beta\eta}{2}\frac{(1-\eta)^2+\gamma^2+\beta^2}{(1-\eta-\gamma^2-\beta^2)^2+\beta^2(2-\eta)^2} \times\left[1+\frac{\eta}{2}\frac{(1-\eta)(1-\eta-\gamma^2)+\beta^2}{(1-\eta-\gamma^2-\beta^2)^2+\beta^2(2-\eta)^2}\right]. \tag{3}$$

If the last terms in (2) and (3) can be neglected, the relative phase shift $\Delta\theta_\perp$ and the attenuation $\alpha_\perp$ per one free space wavelength λ_0 are :

$$\Delta\theta_\perp = \pi\eta\frac{(1-\eta)(1-\eta-\gamma^2)+\beta^2}{(1-\eta-\gamma^2-\beta^2)^2+\beta^2(2-\eta)^2}\,(\mathrm{rad}/\lambda_0), \tag{4}$$

and

$$\alpha_\perp = \pi\eta\beta\frac{(1-\eta)^2+\gamma^2+\beta^2}{(1-\eta-\gamma^2-\beta^2)^2+\beta^2(2-\eta)^2}\,(\mathrm{nep}/\lambda_0). \tag{5}$$

3. Resonance position and maximum absorption

The cyclotron resonance occurs near the value at which the first term of the denominator of (5) is equal to zero. In the neighbourhood of resonance, $\alpha_\perp$ becomes approximately:

$$\alpha_\perp = \alpha_{\perp m}\frac{\beta^2(2-\eta)^2}{(\gamma_0^2-\gamma^2)^2+\beta^2(2-\eta)^2}, \tag{6}$$

where

$$\alpha_{\perp m} = \frac{1-\eta}{2-\eta}\frac{\pi\eta}{\beta}, \tag{7}$$

and

$$\gamma_0^2 = 1-\eta-\beta^2, \tag{8}$$

or

$$\omega_c^2 = \omega^2-\omega_p^2-\nu_c^2. \tag{8a}$$

Thus, from (6) in the above approximation, the cyclotron resonance occurs at $\gamma=\gamma_0$ which is on the low magnetic field side where $\gamma_0=\omega_c/\omega$ is smaller than unity and the maximum attenuation is $\alpha_{\perp m}(\text{nep}/\lambda_0)$. This is characteristic of the hybrid wave.

Next, we investigate to what extent the approximation of (6) is permissible. The exact resonance position (γ_R) of (5) can be determined by $\partial\alpha_\perp/\partial\gamma=0$, from which follows:

$$\gamma_R^2 = (1-\eta)\left\{1+\frac{\eta}{2}\left(\frac{\beta}{1-\eta}\right)^2\right\}+O(\beta^4), \tag{9}$$

where $O(\beta^4)$ is of the order of magnitude of β^4. Therefore,

$$\gamma_R^2-\gamma_0^2 = \frac{\beta^2}{2}\frac{2-\eta}{1-\eta}+O(\beta^4)>0, \tag{10}$$

which implies that we can regard the position of cyclotron resonance as γ_0, corresponding to the neglect of β^2 against unity.

Substituting (9) into (5), we can obtain the following expression for the exact attenuation at cyclotron resonance ($\alpha_{\perp R}$):

$$\frac{\alpha_{\perp R}}{\alpha_{\perp m}} = 1+\frac{1}{4}\left(\frac{\beta}{1-\eta}\right)^2+O(\beta^4)>1, \tag{11}$$

from which we can regard the maximum attenuation as $\alpha_{\perp m}$, in accordance with the neglect of β^2 against unity.

In Fig. 1, γ_0^2 is plotted against η. In Figs. 2 and 3 $\alpha_{\perp m}$ is plotted against η and β respectively.

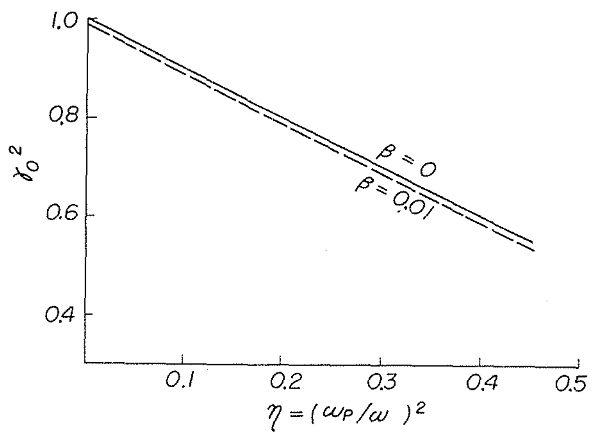

Fig. 1. Resonance position (γ_0^2) *versus* η.

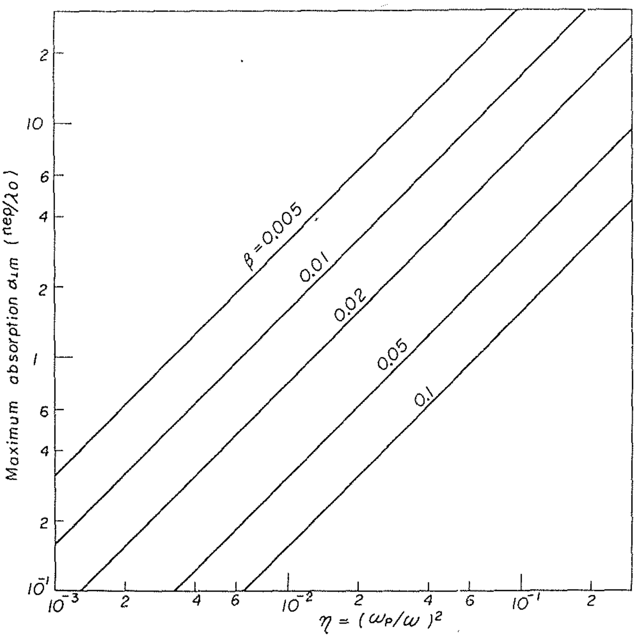

Fig. 2. $\alpha_{\perp m}$ *versus* η.

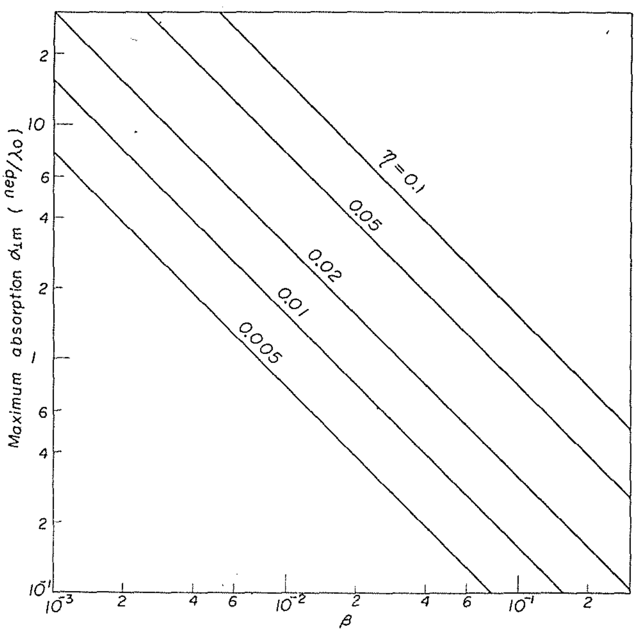

Fig. 3. $\alpha_{\perp m}$ *versus* β.

4. Absorption curve and half-width

It will be seen from (6) that, although the absorption curve plotted against γ^2 is symmetrical with respect to the line $\gamma^2=\gamma_0^2$, the curve plotted against γ is not symmetrical and somewhat steeper on the side of $\gamma>\gamma_0$.

From (6), the half-width ($\Delta\gamma$) of the absorption curve against γ is:

$$\Delta\gamma=\beta\frac{2-\eta}{(1-\eta)^{1/2}}+O(\beta^3)\,, \tag{12}$$

and the measure of asymmetry is given by:

$$\frac{\Delta\gamma_-}{2}-\frac{\Delta\gamma_+}{2}=\frac{\beta^2}{4}\frac{(2-\eta)^2}{(1-\eta)^{3/2}}\,, \tag{13}$$

which implies that the curve is more asymmetric with increasing β. If B_0 and ΔB denote the magnetic field intensity at which cyclotron resonance occurs and the half-width respectively,

$$\frac{\Delta B}{B_0}=\frac{\Delta\gamma}{\gamma_0}=\beta\frac{2-\eta}{1-\eta}+O(\beta^2)\simeq\frac{\nu_c}{\omega}\frac{2-\eta}{1-\eta}\,, \tag{14}$$

which is useful for determining ν_c from the experimental data, since we can determine B_0 and ΔB by experiments.

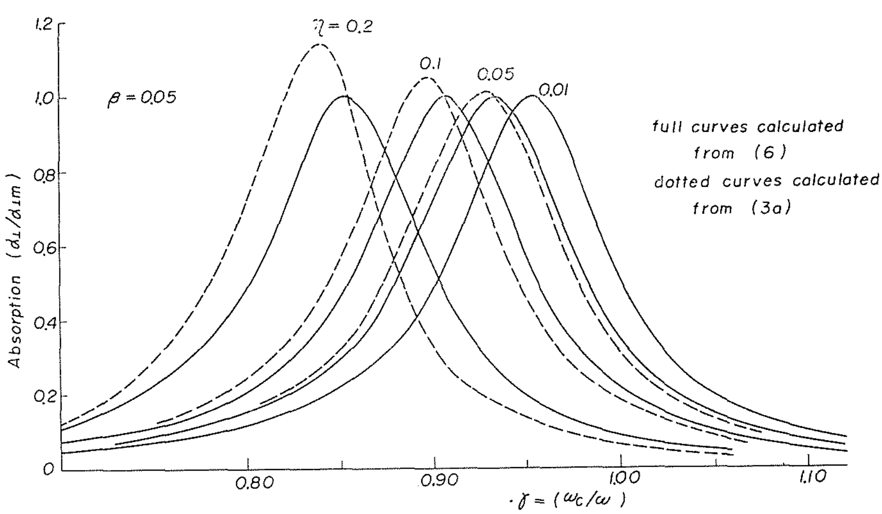

Fig. 4. Absorption curves *versus* γ for various η.

In Figs. 4 and 5 several absorption curves ($\alpha_\perp/\alpha_{\perp m}$) are plotted against γ. In these figures, full curves are calculated by (6), and dotted curves by the following expression, which is derived from (3) and (6):

$$\frac{\alpha_{\perp}}{\alpha_{\perp m}} = \frac{\beta^2(2-\eta)^2}{(\gamma_0^2-\gamma^2)^2+\beta^2(2-\eta)^2}(2-n'_{\perp}) , \tag{3a}$$

with $n'_{\perp}$ approximated from (2) by:

$$n'_{\perp} \simeq 1-\frac{\eta}{2}\frac{(1-\eta)(1-\eta-\gamma^2)+\beta^2}{(\gamma_0^2-\gamma^2)^2+\beta^2(2-\eta)^2} . \tag{2a}$$

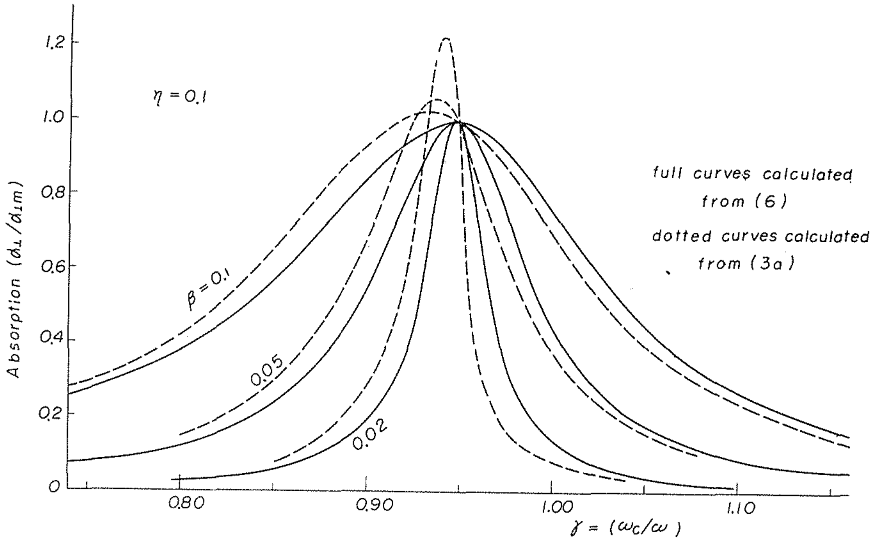

Fig. 5. Absorption curves *versus* γ for various β.

It is to be noted that the dotted curves are more exact than the full curves and that the resonance positions of the former are on the slightly smaller side of γ and the maximum absorptions are slightly larger than those of the latter.

In Fig. 6, $(n'_{\perp}-1)$ calculated by (2a) is plotted against γ.

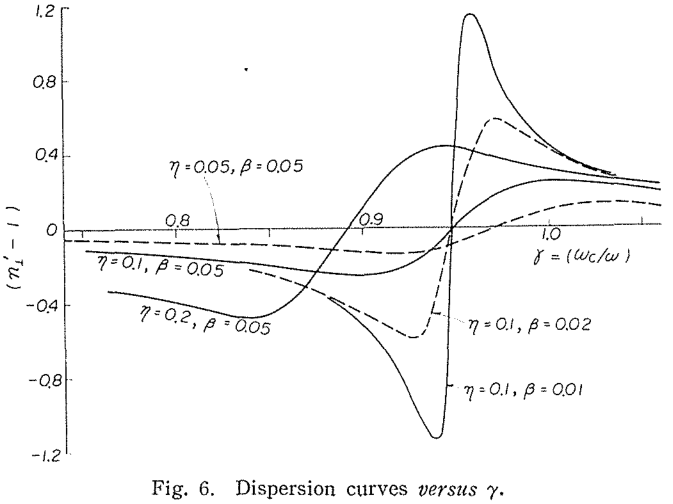

Fig. 6. Dispersion curves *versus* γ.

5. Comparison of cyclotron resonances between various waves

The behaviours of cyclotron resonances are quite different according to the wave types. The characteristics are tabulated in the following table:

Table. Characteristics of cyclotron resonance.

	Right-handed wave	Transversal wave ($k_0 /\!/ B$)	Hybrid wave ($k_0 \perp B$)
Resonance position	$\gamma=1$ independent of N_e, ν_c	$\gamma_0{}^2=1-\beta^2$ independent of N_e	$\gamma_0{}^2=1-\eta-\beta^2$ dependent on N_e, ν_c
Maximum attenuation	$\frac{\pi\eta}{\beta}$	$\frac{1}{2}\frac{\pi\eta}{\beta}$	$\frac{1-\eta}{2-\eta}\frac{\pi\eta}{\beta}$
Half-width	2β	2β	$(2-\eta)\beta$
Absorption curve	symmetrical	Lorentzian	non-Lorentzian
Approximate expression	(8) in I	(20) in I	(6) in the present part

6. Derivation by Boltzmann equation

We can derive the same expression for the absorption curve at cyclotron resonance by solving the Boltzmann equation.

Desloge, Matthysse and Margenau (5) have shown that if one assumes that the effect of high frequency electric field upon an electron in a slightly ionized gas is on the average an addition, in the direction of the field, of a velocity component, the magnitude of which is dependent on the electron velocity, then it is possible to derive very simply the expression for conductivity. Expanding their method in the presence of the magnetic field, Dodo (6) has obtained a tensor specific inductive capacity which is coincident with that derived about uniform distribution of electrons.

Following these methods, we derive the cyclotron resonance absorption curve. The Boltzmann transport equation for a plasma in a high frequency electric field $\boldsymbol{E}=\boldsymbol{E}_0 \exp(-j\omega t)$ and a static magnetic field $\boldsymbol{B}$ may be written, using well-known notations:

$$\frac{\partial f}{\partial t}+\boldsymbol{F}\nabla_v f=\frac{D_c f}{Dt}, \tag{15}$$

where

$$\boldsymbol{F}=\frac{e\boldsymbol{E}}{m}+\frac{e}{m}(\boldsymbol{v}\times\boldsymbol{B}), \tag{16}$$

$$\frac{D_c f}{Dt}=\int f(\boldsymbol{v}', t)W(\boldsymbol{v}', \boldsymbol{v})d\boldsymbol{v}'-f(\boldsymbol{v}, t)\int W(\boldsymbol{v}, \boldsymbol{v}')d\boldsymbol{v}', \tag{17}$$

$$W(\boldsymbol{v}', \boldsymbol{v})=W(\boldsymbol{v}', -\boldsymbol{v}) \tag{18}$$

and

$$\int W(\boldsymbol{v}, \boldsymbol{v}')d\boldsymbol{v}'=\nu(\boldsymbol{v}).$$

Next, if we assume that all of the electrons which are in a given velocity range in the equilibrium distribution are shifted by the Lorentz force to a new range in which the velocity differs from the original velocity by an amount $\boldsymbol{g}$,

$$f(\boldsymbol{v}, t) = f_0(\boldsymbol{v}-\boldsymbol{g})\left(1-\frac{\partial g_x}{\partial v_x}\right)\left(1-\frac{\partial g_y}{\partial v_y}\right)\left(1-\frac{\partial g_z}{\partial v_z}\right), \tag{19}$$

where $f_0(\boldsymbol{v})$ is the equilibrium distribution function and $g_i=g_i(\boldsymbol{v}, t)$ $(i=x, y, z)$. If $f(\boldsymbol{v}, t)$ of (19) is expanded in powers of g_i and one assumes rapid convergence of the resulting series,

$$f(\boldsymbol{v}, t) = f_0-\boldsymbol{v}\cdot\boldsymbol{G}, \tag{20}$$

where

$$\boldsymbol{G} = \frac{1}{v}\frac{\partial}{\partial v}(f_0\boldsymbol{g}), \tag{21}$$

which is even with respect to $\boldsymbol{v}$, assuming $\boldsymbol{g}=\boldsymbol{g}(v, t)$.

Substituting (20) and (16) into (15), we obtain:

$$\begin{aligned}\boldsymbol{v}\left\{\frac{\partial \boldsymbol{G}}{\partial t}-\frac{e\boldsymbol{E}}{m}\frac{1}{v}\frac{\partial f_0}{\partial v}-\frac{e}{m}(\boldsymbol{G}\times\boldsymbol{B})\right\}+\frac{e\boldsymbol{E}}{m}\left(\boldsymbol{G}+v\frac{\partial \boldsymbol{G}}{\partial v}\right)\\ = \nu f(\boldsymbol{v}, t)-\int f(\boldsymbol{v}', t)\,W(\boldsymbol{v}', \boldsymbol{v})d\boldsymbol{v}'.\end{aligned} \tag{22}$$

Replacing $\boldsymbol{v}$ by $-\boldsymbol{v}$ in (22) and subtracting the resulting equation from (22), one finds

$$\frac{\partial \boldsymbol{G}}{\partial t} = \frac{e\boldsymbol{E}}{m}\frac{1}{v}\frac{\partial f_0}{\partial v}+\frac{e}{m}(\boldsymbol{G}\times\boldsymbol{B})-\nu\boldsymbol{G}, \tag{23}$$

which is coincident with Eq. (14) in Kelly-Margenau-Brown's paper (7), in which $f(\boldsymbol{v}, t)$ is expanded in spherical harmonics and $\boldsymbol{b}$ is used such as:

$$\boldsymbol{b} = -v\boldsymbol{G}. \tag{24}$$

If $\boldsymbol{G}$ is obtained by (23), the current density $\boldsymbol{J}$ is given by:

$$\boldsymbol{J} = Ne\int \boldsymbol{v}f(\boldsymbol{v}, t)\,d\boldsymbol{v} = -Ne\int \boldsymbol{v}(\boldsymbol{v}\cdot\boldsymbol{G})d\boldsymbol{v} \tag{25}$$

and the average power $(\bar{P})$ absorbed by electrons per unit volume is

$$\bar{P} = \frac{1}{2}\,\mathrm{Re}\,(\boldsymbol{J}\cdot\boldsymbol{E}^*). \tag{26}$$

Now we calculate for the two interesting cases.

(i) Transversal wave

Since the wave is purely transversal, in the above-mentioned coordinates $\boldsymbol{J}$ exists only in the xy plane. In quite the same way as Brown *et al.* we obtain for the constant mean free time:

$$\bar{P}=\frac{E_0^2\varepsilon_0\omega\eta}{4}\frac{}{\beta}\left\{\frac{\beta^2}{(1-\gamma)^2+\beta^2}+\frac{\beta^2}{(1+\gamma)^2+\beta^2}\right\},\tag{27}$$

which is identical with (22) in I.

(ii) Hybrid wave

$\boldsymbol{E}$ and $\boldsymbol{J}$ have the x- and y- components. Solving (23), we obtain:

$$\begin{pmatrix}G_x\\G_y\end{pmatrix}=-\frac{e}{m}\frac{1}{v}\frac{\partial f_0}{\partial v}\frac{1}{(j\omega-\nu)^2+\omega_c^2}\begin{pmatrix}j\omega-\nu & \omega_c\\-\omega_c & j\omega-\nu\end{pmatrix}\begin{pmatrix}E_x\\E_y\end{pmatrix}.\tag{28}$$

Therefore, from (25) J_x is given by:

$$J_x=\frac{e^2N}{3m}\int\frac{(j\omega-\nu)E_x+\omega_cE_y}{(j\omega-\nu)^2+\omega_c^2}4\pi v^3\frac{\partial f_0}{\partial v}dv.\tag{29}$$

Integration is easily carried out for the constant mean free time and J_x, J_y are given by:

$$\begin{pmatrix}J_x\\J_y\end{pmatrix}=\frac{\eta\omega}{4\pi}\frac{1}{(1+j\beta)^2-\gamma^2}\begin{pmatrix}j(1+j\beta) & \gamma\\-\gamma & j(1+j\beta)\end{pmatrix}\begin{pmatrix}E_x\\E_y\end{pmatrix}.\tag{30}$$

On the other hand, using the specific inductive capacity tensor (K) for this configuration, the electric field vector should fulfil the following equation:

$$\nabla^2\boldsymbol{E}-\nabla\nabla\cdot\boldsymbol{E}=\frac{1}{c^2}(K)\frac{\partial\boldsymbol{E}}{\partial t}.\tag{31}$$

Therefore, the longitudinal component E_x can be expressed by E_y as:

$$E_x=-\frac{K_{xy}}{K_{xx}}E_y=-\frac{j\eta\gamma}{(1-\eta-\gamma^2-\beta^2)+j\beta(2-\eta)}E_y.\tag{32}$$

The average power absorbed from this hybrid wave $(\bar{P})$ is the sum of the powers absorbed from the longitudinal wave $(\bar{P}_x)$ and from the transversal wave $(\bar{P}_y)$. Substituting (30) and (32) into (26), we obtain:

$$\bar{P}_x=\frac{1}{2}\operatorname{Re}(J_xE_x^*)=0,$$

and

$$\bar{P}_y=\frac{|E_y|^2\omega\eta}{8\pi}\frac{}{\beta}\frac{\beta^2\{(1-\eta)^2+\gamma^2+\beta^2\}}{(1-\eta-\gamma^2-\beta^2)^2+\beta^2(2-\eta)^2}=\bar{P},\tag{33}$$

which is coincident with (5). It is concluded from (33) that as there is no power dissipation in the longitudinal component, the current component J_x is out-of-phase with the field E_x.

It has been shown by Watanabe (8) that the effect of the longitudinal component of our hybrid wave can be neglected, and K is expressed by (1) to a good approximation under the following condition:

$$\frac{\gamma_e\kappa T_e}{mc^2}\ll 1,\tag{34}$$

where γ_e is the ratio of specific heat at constant pressure to that at constant volume of electron gas. The above condition is fully permissible in the cold plasma.

If we express the ratio of the longitudinal field component to the transversal one as:

$$E_x/E_y = A \exp(j\theta), \tag{35}$$

we obtain, by (32),

$$A = \eta\gamma/\{(1-\eta-\gamma^2-\beta^2)^2+\beta^2(2-\eta)^2\}^{1/2}, \tag{36}$$

and

$$\theta = \tan^{-1}\left\{\frac{1-\eta-\gamma^2-\beta^2}{\beta(2-\eta)}\right\}. \tag{36a}$$

Therefore, at cyclotron resonance,

$$A = \eta\gamma_0/\beta(2-\eta), \qquad \theta = 0. \tag{37}$$

7. Conclusion

Using the specific inductive capacity, we have derived the expression for the absorption curve at the electron cyclotron resonance of a slightly ionized gas for the hybrid wave. It was shown from the analyses that the behaviour of the resonance absorption by the hybrid wave is very different from that by the transversal wave.

The main features are as follows:

(1) The resonance occurs not at $\omega_c=\omega$ but at $\omega_c=\sqrt{\omega^2-\omega_p^2-\nu_c^2}$;

(2) The absorption curve is not Lorentzian;

(3) The half-width depends not only on collision frequency but also on electron density.

Although the expression for the absorption is more complicated, there is the following advantage in this case. As shown above, one can calculate both the electron density and the collision frequency from the resonance position and the half-width, by measuring one resonance absorption curve. In addition an experimental apparatus becomes simpler. We shall report the experimental data and analyses based on the above discussion in the next part.

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