Introduction of the Translational Operator in Frequency Domain and Treatment of Certain Linear Differential Equations (I)

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INTRODUCTION OF THE TRANSLATIONAL OPERATOR IN FREQUENCY DOMAIN AND TREATMENT OF CERTAIN LINEAR DIFFERENTIAL EQUATIONS, I.

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ABSTRACT

In this paper, the author describes a method of solving the ordinary linear differential equation whose coefficients are E-type functions, i. e. sums of exponential functions of independent variable by introducing the translational operator in frequency domain. And further, the Maxwell equations in the medium whose material constants are E-type functions of time, are reformulated in the frequency domain.

1. Introduction

It is well known that the ordinary linear differential equation with constant coefficients can be easily solved by applying the Laplace transform. Pipes (**1**) attempted to solve the ordinary linear differential equation whose coefficients are arbitrary functions of independent variable, by applying the Laplace transform. He tried to obtain the solution in the form of power series of parameter. But he derived only a few initial terms in the series for an example given by him though he obtained the complete solution in the case when the original differential equation is an ordinary linear differential equation of the first order.

The author attempts to obtain the solution of the linear differential equation by applying the Laplace transform, under the restriction that the coefficients are expressed by periodic functions* of independent variable, or more generally, expressed by the sums** of exponentials of independent variable. Such a function as the latter is called E-type function in the following. When the coefficients are restricted to E-type functions, we can introduce the author's translational operator in frequency domain into the transformed equation and derive, in the frequency domain, the general term in the series of the solution that has not been derived by Pipes. Furthermore, without the trouble of directly obtaining the solution, various ideas such as impedance, reflection coefficient, etc. in the linear fixed electrical network*** can be naturally

* It is assumed that they can be expanded in Fourier series.
** The infinite series may be included, if the convergency is assured.
*** A linear electrical network whose circuit elements are constant in time.

extended to the linear varying electrical network*.

In part I, the author considers the method of solving the ordinary linear differential equation whose coefficients are of E-type, by introducing the translational operator. And further, the operator is introduced into the Maxwell equations in the medium where the material constants ε (dielectric constant) and μ (magnetic permeability)** are E-type functions of the time.

And it is discussed how the propagation of the electromagnetic waves in the wave guide filled with a material of fixed constants is extended to the case of a material of varying constants.

In part II, it will be discussed how the various ideas in the linear fixed electrical network are extended to the linear varying electrical network.

2. Introduction of the translational operator into the ordinary linear differential equations with *E*-type coefficients

The E-type function has the form:

$$a_i(t) = \sum_{\alpha=0}^{\nu} \sum_{k=-Ni,\alpha}^{Ni,\alpha} A_k^{(i)}(\alpha)\, e^{ik\omega_\alpha t},$$

and the ordinary linear differential equation with such coefficients as $a_i(t)$ has the following form:

$$\sum_{i=0}^{n} a_i(t) \frac{d^{n-i}}{dt^{n-i}} y(t) = x(t)\,. \tag{2.1}$$

Taking the Laplace transforms of both sides of (2.1), we obtain:

$$L\left(\sum_{i=0}^{n} a_i(t) \frac{d^{n-i}}{dt^{n-i}} y(t)\right) = L(x(t))\,.^{***}$$

Then, putting $p_{k,\alpha} = p - ik\omega_\alpha$, it can be reduced to:

$$\sum_{i=0}^{n} \sum_{\alpha=0}^{\nu} \sum_{k=-Ni,\alpha}^{Ni,\alpha} A_k^{(i)}(\alpha) \left\{ p_{k,\alpha}^{n-i}\, Y(p_{k,\alpha}) - \sum_{j=0}^{n-i-1} p_{k,\alpha}^{n-(i+j+1)}\, y^{(j)}(0) \right\} = X(p)\,, \tag{2.2}$$

where

$$\left.\begin{aligned} X(p) &= L(x(t))\,, \\ Y(p) &= L(y(t))\,, \end{aligned}\right\} \qquad y^{(j)}(0) = \left\{\frac{d^j y(t)}{dt^j}\right\}_{t=0}.$$

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* A linear electrical network whose circuit elements are varying in time. In the present paper, the consideration is restricted only to a network whose circuit elements can be expressed by E-type functions.

** In the dispersive medium, ε and μ cannot be defined in the time domain, but can be defined in the frequency domain (3). They sometimes depend on the external electric and magnetic fields as well as on the frequency as parameters. The time variation of ε and μ means that these parameters are varying in time. The description "ε and μ are E-type functions of time" means that the above-mentioned parameters are so varying as to be expressed by E-type functions of the time.

*** L and L^{-1} denote respectively Laplace transform and its inverse.

Further, putting

$$X'(p) = X(p) + \sum_{i=0}^{n} \sum_{\alpha=0}^{\nu} \sum_{k=-Ni,\alpha}^{Ni,\alpha} A_k^{(i)}(\alpha) \sum_{j=0}^{n-i-1} p_{k,\alpha}^{n-(i+j+1)} y^{(j)}(0) , \quad (2.3)$$

the expression (2. 2) becomes:

$$\sum_{i=0}^{n} \sum_{\alpha=0}^{\nu} \sum_{k=-Ni,\alpha}^{Ni,\alpha} A_k^{(i)}(\alpha) p_{k,\alpha}^{n-i} Y(p_{k,\alpha}) = X'(p) . \quad (2.4)$$

Now, when $f(p)$ is an arbitrary function of p, the author introduces the operator $T(\omega_\alpha)$ defined by the following expression, and calls it the translational operator in frequency domain:

$$T(\omega_\alpha)f(p) = f(p-i\omega_\alpha) . \quad (2.5)$$

Substituting this operator into (2. 4), we obtain:

$$\{\sum_{i=0}^{n} \sum_{\alpha=0}^{\nu} \sum_{k=-Ni,\alpha}^{Ni,\alpha} A_k^{(i)}(\alpha) T^k(\omega_\alpha) p^{n-i}\} Y(p) = X'(p) , \quad (2.6)$$

where

$$X'(p) = X(p) + \sum_{i=0}^{n} \sum_{\alpha=0}^{\nu} \sum_{k=-Ni,\alpha}^{Ni,\alpha} A_k^{(i)}(\alpha) \sum_{j=0}^{n-i-1} T^k(\omega_\alpha) p^{n-(i+j+1)} y^{(j)}(0) . \quad (2.7)$$

When the factor to $Y(p)$ is denoted by $S(T, p)$*, the equation becomes:

$$S(T, p) Y(p) = X'(p) , \quad (2.8)$$

or with an inverse operator $S^{-1}(T, p)$,

$$Y(p) = S^{-1}(T, p) X'(p) . \quad (2.8')$$

Thus, if $S^{-1}(T, p)$ is found, $Y(p)$ can easily be obtained by (2. 8′) and then the solution of the original equation can be derived by taking the inverse Laplace transform of $Y(p)$.

Next, we can easily extend the above discussion to the simultaneous ordinary linear differential equations whose coefficients are E-type functions. The simultaneous differential equations are given as follows:

$$\sum_{m=1}^{M} \sum_{i=0}^{n_{lm}} a_{i;lm}(t) \frac{d^{n_{lm}-i}}{dt^{n_{lm}-i}} y_m(t) = x_l(t) , \quad (2.9)$$

$$a_{i;lm}(t) = \sum_{\alpha=0}^{\nu} \sum_{k=-Ni,\alpha;lm}^{Ni,\alpha;lm} A_k^{(i;lm)}(\alpha) e^{ik\omega_\alpha t} , \quad l = 1, 2, \cdots M. \quad (2.10)$$

Taking the Laplace transforms of both sides of (2. 9) and then introducing the translational operator into the resulting expressions, we obtain the following expression:

$$\{\sum_{m=1}^{M} \sum_{i=0}^{n_{lm}} \sum_{\alpha=0}^{\nu} \sum_{k=-Ni,\alpha;lm}^{Ni,\alpha;lm} A_k^{(i;lm)}(\alpha) T^k(\omega_\alpha) p^{n_{lm}-i}\} Y_m(p) = X'_l(p) , \quad (2.11)$$

* The notation $S(T, p)$ is used in order to show that it contains the translational operator as well as p.

with

$$X_l'(p) = X_l(p) + \sum_{m=1}^{M} \sum_{i=0}^{n_{lm}} \sum_{\alpha=0}^{\nu} \sum_{k=-N_{i;lm}}^{N_{i,\alpha;lm}} A_k^{(i;lm)}(\alpha) T^k(\omega_\alpha) \times \sum_{j=0}^{n_{lm}-i-1} p^{n_{lm}-(i+j+1)} y_m^{(j)}(0) , \quad l = 1, 2, \cdots M , \tag{2.12}$$

where

$$Y_m(p) = L(y_m(t)) , \quad X_l(p) = L(x_l(t)) ,$$
$$y_m^{(j)}(0) = \left\{ \frac{d^j y_m(t)}{dt^j} \right\}_{t=0} .$$

If the factor to $Y_m(p)$ in (2.11) is represented by $S_{lm}(T, p)$, we can rewrite (2.11) as follows:

$$\sum_{m=1}^{M} S_{lm}(T, p) Y_m(p) = X_l'(p) . \tag{2.13}$$

The equation (2.13) can be written in a matrix form. Namely,

$$[S(T, p)][Y(p)] = [X_l'(p)] , \tag{2.13'}$$

where

$$[Y(p)] = \begin{pmatrix} Y_1(p) \\ Y_2(p) \\ \vdots \\ Y_M(p) \end{pmatrix} , \quad [X'(p)] = \begin{pmatrix} X_1'(p) \\ X_2'(p) \\ \vdots \\ X_M'(p) \end{pmatrix} , \tag{2.14}$$

$$[S(T, p)] = \begin{pmatrix} S_{11} & S_{12} & \cdots\cdots & S_{1M} \\ S_{21} & S_{22} & \cdots\cdots & S_{2M} \\ \cdots & \cdots & \cdots & \cdots \\ S_{M1} & S_{M2} & \cdots\cdots & S_{MM} \end{pmatrix} . \tag{2.15*}$$

The above equation has the same form as (2.8). And to solve the original simultaneous equations, (2.9) reduces to obtaining the inverse matrix $[S(T, p)]^{-1}$ of $[S(T, p)]$. The circuit equations of the linear varying electrical network are included in the cases when $n_{lm} \leq 2$. They will be discussed in detail later on.

3. Some properties of the translational operator

The translational operator $T(\omega_\alpha)$ has been defined by (2.5) for an arbitrary function $f(p)$ of p. So, the following expressions (3.1) and (3.2) can easily be shown to be true:

$$T^k(\omega_\alpha) f(p) = f(p - ik\omega_\alpha) , \tag{3.1}$$

$$\begin{aligned} T^{k_1}(\omega_\alpha) T^{k_2}(\omega_\alpha) f(p) &= T^{k_2}(\omega_\alpha) T^{k_1}(\omega_\alpha) f(p) \\ &= f(p - i(k_1 + k_2)\omega_\alpha) \\ &= T^{k_1+k_2}(\omega_\alpha) f(p) , \end{aligned} \tag{3.2}$$

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* The matrix element S_{lm} is an abbreviated form of $S_{lm}(T, p)$.

where k_1 and k_2 represent negative or positive integers including zero.

From (3. 2) we obtain:

$$T^{k_1}(\omega_\alpha)T^{k_2}(\omega_\alpha) = T^{k_2}(\omega_\alpha)T^{k_1}(\omega_\alpha) = T^{k_1+k_2}(\omega_\alpha) . \tag{3. 2'}$$

For two functions $f_1(p)$ and $f_2(p)$,

$$\begin{aligned} T^{k_1}(\omega_\alpha)f_1(p)T^{k_2}(\omega_\alpha)f_2(p) &= f_1(p-ik_1\omega_\alpha)T^{k_1+k_2}(\omega_\alpha)f_2(p) \\ &= f_1(p-ik_1\omega_\alpha)f_2(p-i(k_1+k_2)\omega_\alpha) . \end{aligned} \tag{3. 3}$$

Further, if $f(p)$ is analytic at p, the operator $T(\omega_\alpha)$ can be explicitly expressed as follows:

$$T(\omega_\alpha) = \sum_{n=0}^{\infty}(-1)^n\frac{(i\omega_\alpha)^n}{n!}\frac{d^n}{dp^n} = \exp\left(-i\omega_\alpha\frac{d}{dp}\right) .$$

Now, we assume that $F(T(\omega_\alpha))$ is a function of $T(\omega_\alpha)$ and does not include p and that it can be formally expanded into a power series of $T(\omega_\alpha)$, i.e.

$$F(T(\omega_\alpha)) = \sum_n C_n T^n(\omega_\alpha) . \tag{3. 4}$$

Operating $F(T(\omega_\alpha))$ on a function $pf(p)$, we obtain, by (3. 4), the following expression:

$$F(T(\omega_\alpha))pf(p) = \{p(\sum_n C_nT^n(\omega_\alpha)) - i\omega_\alpha(\sum_n nC_nT^n(\omega_\alpha))\}f(p) .$$

From the relation

$$\sum_n nC_nT^n(\omega_\alpha) = T(\omega_\alpha)\frac{\partial}{\partial T(\omega_\alpha)}F(T(\omega_\alpha)) ,$$

we obtain:

$$F(T(\omega_\alpha))pf(p) = \left\{pF(T(\omega_\alpha)) - i\omega_\alpha T(\omega_\alpha)\frac{\partial}{\partial T(\omega_\alpha)}F(T(\omega_\alpha))\right\}f(p) . \tag{3. 5}$$

For the two translational operators $T(\omega_\alpha)$ and $T(\omega_\beta)$, we can easily show

$$\begin{aligned} T^k(\omega_\alpha)T^{k'}(\omega_\beta)f(p) &= T^k(\omega_\alpha)f(p-ik'\omega_\beta) \\ &= f(p-i(k\omega_\alpha+k'\omega_\beta)) \\ &= T^{k'}(\omega_\beta)T^k(\omega_\alpha)f(p) , \end{aligned} \tag{3. 6}$$

i.e.

$$T^k(\omega_\alpha)T^{k'}(\omega_\beta) = T^{k'}(\omega_\beta)T^k(\omega_\alpha) , \tag{3. 7}$$

where both k and k' represent positive or negative integers including zero.

Next, algebraical properties of $T^k(\omega_\alpha)$ are considered, though they will not be used explicitly in later paragraphs.

When k passes through all positive and negative integers including zero, with ω_α fixed, we obtain an ensemble of $T^k(\omega_\alpha)$. If $\mathfrak{M}$ denotes the above ensemble, we can show that the element $T^k(\omega_\alpha)$ of $\mathfrak{M}$ satisfies the following conditions.

When k_1, k_2 and k_3 are positive or negative integers (zero inclusive), the following four relations are obtained by (3. 2′).

$$T^{k_1}(\omega_\alpha)T^{k_2}(\omega_\alpha) = T^{k_1+k_2}(\omega_\alpha)\ ,$$

i.e.

$$T^{k_1+k_2}(\omega_\alpha) \in \mathfrak{M}\ ; \tag{3.8}$$

$$\begin{aligned} &T^{k_1}(\omega_\alpha)(T^{k_2}(\omega_\alpha)T^{k_3}(\omega_\alpha)) \\ &= T^{k_1}(\omega_\alpha)T^{k_2+k_3}(\omega_\alpha) = T^{k_1+k_2+k_3}(\omega_\alpha)\ ; \\ &(T^{k_1}(\omega_\alpha)T^{k_2}(\omega_\alpha))T^{k_3}(\omega_\alpha) \\ &= T^{k_1+k_2}(\omega_\alpha)T^{k_3}(\omega_\alpha) = T^{k_1+k_2+k_3}(\omega_\alpha)\ , \end{aligned} \tag{3.9}$$

so that

$$T^{k_1}(\omega_\alpha)(T^{k_2}(\omega_\alpha)T^{k_3}(\omega_\alpha)) = (T^{k_1}(\omega_\alpha)T^{k_2}(\omega_\alpha))T^{k_3}(\omega_\alpha)\ .$$

And then

$$T^0(\omega_\alpha)T^k(\omega_\alpha) = T^k(\omega_\alpha)T^0(\omega_\alpha) = T^k(\omega_\alpha)\ , \tag{3.10}$$

i.e. $T^0(\omega_\alpha)$ is the unit element belonging to $\mathfrak{M}$.

$$T^{-k}(\omega_\alpha)T^k(\omega_\alpha) = T^k(\omega_\alpha)T^{-k}(\omega_\alpha) = T^0(\omega_\alpha)\ , \tag{3.11}$$

i.e. $T^{-k}(\omega_\alpha)$ is the inverse of $T^k(\omega_\alpha)$ and belongs to $\mathfrak{M}$.

From the relations (3. 8) to (3. 11), we understand that $\mathfrak{M}$ is a group and that it is an Abelian group by (3. 2′).

4. Solution of the ordinary linear differential equation of the first order with *E*-type coefficients

The ordinary linear differential equation of the first order can be generally solved in the time domain by introducing an integrating factor. Here it is discussed on an example, whether the process corresponding to the integrating factor exists or not. For example, we try to solve the following equation :

$$\frac{dy}{dt}+a_1(t)y = e^{i\omega t}\ , \tag{4.1}$$

$$a_1(t) = A_0^{(1)}+2A_1^{(1)}\cos\omega_0 t\ . \tag{4.2}$$

Corresponding to (2. 8), the following expressions are given :

$$S(T,\ p)Y(p) = X'(p)\ , \tag{4.3}$$

$$\left.\begin{aligned} S(T,\ p) &= p+A_0^{(1)}+A_1^{(1)}(T(\omega_0)+T^{-1}(\omega_0))\ , \\ X'(p) &= (p-i\omega)^{-1}+y(0)\ . \end{aligned}\right\} \tag{4.4}$$

Substituting the expression :

$$F(T(\omega_0)) = \exp\{\beta(T(\omega_0)-T^{-1}(\omega_0))\}$$

into (3. 5), the following relation is obtained :

$$\exp\{\beta(T(\omega_0)-T^{-1}(\omega_0))\}p$$
$$=p\exp\{\beta(T(\omega_0)-T^{-1}(\omega_0))\}-i\omega_0\beta(T(\omega_0)+T^{-1}(\omega_0))\exp\{\beta(T(\omega_0)-T^{-1}(\omega_0))\}\,.$$

Further, putting $\beta=A_1^{(1)}/i\omega_0$ and $U(T(\omega_0))=\exp\left\{\frac{A_1^{(1)}}{i\omega_0}(T(\omega_0)-T^{-1}(\omega_0))\right\}$, we can show from the above relation that the expression (4. 3) becomes:

$$\{U(T(\omega_0))S(T,\,p)U^{-1}(T(\omega_0))\}U(T(\omega_0))Y(p)=U(T(\omega_0))X'(p)\,,$$

where

$$U(T(\omega_0))S(T,\,p)U^{-1}(T(\omega_0))=p+A_0^{(1)}\,.$$

Therefore, we obtain:

$$Y(p)=U^{-1}(T(\omega_0))(p+A_0^{(1)})^{-1}U(T(\omega_0))X'(p)\,.$$

When $U(T(\omega_0))$ and $U^{-1}(T(\omega_0))$ are expressed in such power series of $T(\omega_0)$ as

$$\left.\begin{aligned}U(T(\omega_0))&=\sum_{k'}C_{k'}^{(+)}T^{k'}(\omega_0)\,,\\ U^{-1}(T(\omega_0))&=\sum_{k}C_{k}^{(-)}T^{k}(\omega_0)\,,\end{aligned}\right\}\tag{4. 5}$$

we obtain

$$\begin{aligned}Y(p)=&\sum_{k,k'}C_k^{(-)}C_{k'}^{(+)}T^k(\omega_0)(p+A_0^{(1)})^{-1}T^{k'}(\omega_0)(p-i\omega)^{-1}\\ &+y(0)\sum_{k,k'}C_k^{(-)}C_{k'}^{(+)}T^k(\omega_0)(p+A_0^{(1)})^{-1}\,.\end{aligned}\tag{4. 6}$$

The relations

$$L^{-1}\{T^k(\omega_0)(p+A_0^{(1)})^{-1}T^{k'}(\omega_0)(p-i\omega)^{-1}\}$$
$$=(e^{i\omega_0t})^ke^{-A_0^{(1)}t}\int_0^te^{(A_0^{(1)}+i\omega)t'}(e^{i\omega_0t'})^{k'}dt'\,,$$
$$L^{-1}\{T^k(\omega_0)(p+A_0^{(1)})^{-1}\}=e^{-A_0^{(1)}t}(e^{i\omega_0t})^k$$

can easily be shown to be true.

Taking the inverse Laplace transform of (4. 6), and utilizing the above relations, we obtain:

$$\begin{aligned}y(t)=&\sum_{k,k'}C_k^{(-)}(e^{i\omega_0t})^ke^{-A_0^{(1)}t}\int_0^te^{(A_0^{(1)}+i\omega)t'}C_{k'}^{(+)}(e^{i\omega_0t'})^{k'}dt'\\ &+y(0)\sum_{k,k'}C_k^{(-)}(e^{i\omega_0t})^ke^{-A_0^{(1)}t}C_{k'}^{(+)}\,.\end{aligned}$$

Then, using (4. 5), the above expression becomes:

$$\begin{aligned}y(t)=&U^{-1}(e^{i\omega_0t})e^{-A_0^{(1)}t}\int_0^te^{(A_0^{(1)}+i\omega)t'}U(e^{i\omega_0t'})dt'\\ &+y(0)U^{-1}(e^{i\omega_0t})e^{-A_0^{(1)}t}U(1)\,.\end{aligned}\tag{4. 7}$$

From the above descriptions, it follows that when $a_1(t)$ takes a more general form:

$$a_1(t)=\sum_kA_k^{(1)}e^{ik\omega_0t}\,,$$

the solution of (4.1) is given in the same form as (4.7) in which, however, $U(T(\omega_0))$ defined by the following expressions is used:

$$\left.\begin{aligned} G(T(\omega_0)) &= \sum_k{}' A_k^{(1)} T^k(\omega_0)\,, \\ U(T(\omega_0)) &= \exp\left\{\frac{1}{i\omega_0}\int^{T(\omega_0)} \frac{G(T)}{T}\,dT\right\}, \end{aligned}\right\} \tag{4.8}$$

where the dash in $\sum_k{}'$ means that $k=0$ is excluded from the summation.

5. Derivation of Hill's determinantal equation in the frequency domain

Hill's determinantal equation has been derived from the consideration in the time domain (**2**). In this paragraph, the derivation in the frequency domain is considered. Hill's equation is given by

$$\left.\begin{aligned} &\frac{d^2y}{dt^2} + \sum_{k=-\infty}^{\infty} A_k^{(2)} e^{ik\omega_0 t} y = 0\,, \\ &A_k^{(2)} = A_{-k}^{(2)}\,. \end{aligned}\right\} \tag{5.1}$$

This is transformed into the frequency domain:

$$S(T,\,p)\,Y(p) = X'(p)\,, \tag{5.2}$$

with

$$\left.\begin{aligned} S(T,\,p) &= p^2 + \sum_k A_k^{(2)} T^k(\omega_0)\,, \\ X'(p) &= py(0) + y'(0)\,. \end{aligned}\right\} \tag{5.3}$$

When

$$Y(p) = \sum_{n=-\infty}^{\infty} \frac{b_n}{p-(\mu+in\,\omega_0)} + \sum_{n=-\infty}^{\infty} \frac{b_n'}{p-(\mu'+in\,\omega_0)} \tag{5.4}$$

is substituted into (5.3), we obtain:

$$\begin{aligned} &p\,(\sum_n b_n + \sum_n b_n') + \{\sum_n (\mu+in\,\omega_0)b_n + \sum_n (\mu'+in\,\omega_0)b_n'\} \\ &+ \sum_n (p-(\mu+in\,\omega_0))^{-1}\{b_n(\mu+in\,\omega_0)^2 + \sum_m A_m^{(2)} b_{n-m}\} \\ &+ \sum_n (p-(\mu'+in\,\omega_0))^{-1}\{b_n'(\mu'+in\,\omega_0)^2 + \sum_m A_m^{(2)} b_{n-m}'\} \\ &= py(0) + y'(0)\,. \end{aligned} \tag{5.3'}$$

Dividing both sides of (5.3′) by p and making $p\to\infty$, we obtain:

$$\sum_n b_n + \sum_n b_n' = y(0)\,, \tag{5.5}$$

and when this is substituted, (5.3′) becomes:

$$\begin{aligned} &\{\sum_n (\mu+in\,\omega_0)b_n + \sum_n (\mu'+in\,\omega_0)b_n'\} \\ &+ \sum_n (p-(\mu+in\,\omega_0))^{-1}\{b_n(\mu+in\,\omega_0)^2 + \sum_m A_m^{(2)} b_{n-m}\} \\ &+ \sum_n (p-(\mu'+in\,\omega_0))^{-1}\{b_n'(\mu'+in\,\omega_0)^2 + \sum_m A_m^{(2)} b_{n-m}'\} \\ &= y'(0)\,. \end{aligned} \tag{5.3''}$$

The above expression becomes with $p \to \infty$:

$$\sum_n (\mu + in\,\omega_0) b_n + \sum_n (\mu' + in\,\omega_0) b'_n = y'(0)\,. \tag{5.6}$$

Further, substituting (5.6) into (5.3″), and multiplying the rest by $p-(\mu+in\,\omega_0)$ or $p-(\mu'+in\,\omega_0)$, and then making $p \to \mu+in\,\omega_0$ or $p \to \mu'+in\,\omega_0$ respectively, we obtain:

$$b_n(\mu+in\,\omega_0)^2 + \sum_m A_m^{(2)} b_{n-m} = 0\,, \tag{5.7}$$

$$b'_n(\mu'+in\,\omega_0)^2 + \sum_m A_m^{(2)} b'_{n-m} = 0\,. \tag{5.7'}$$

Eliminating b_n (or b'_n) from (5.7) (or (5.7′)), we obtain the equation for determining μ (or μ'), i.e. Hill's determinantal equation.

6. Method of series expansion

If the inverse of $S(T, p)$ in (2.8) or that of $[S(T, p)]$ in (2.13′) is obtained, the solution of the original differential equation can easily be obtained, but it is difficult to obtain the inverse in a compact form except in the case of ordinary differential equations of the first order. In such cases, it is useful to obtain the inverse $S^{-1}(T, p)$ or $[S(T, p)]^{-1}$ in a series form by expansion, as will be described in the following lines. However, the validity of the solution in a series form depends on the convergency of the series.

First, we consider the ordinary linear differential equation. Putting

$$\sum_{i=0}^{n} \sum_{\alpha=0}^{\nu} A_0^{(i)}(\alpha) p^{n-i} = H^{(0)}(p)\,, \tag{6.1}$$

$$\sum_{i=0}^{n} \sum_{\alpha=0}^{\nu} \sum_{k=-Ni,\alpha}^{Ni,\alpha}{}' A_0^{(i)}(\alpha) T^k(\omega_\alpha) p^{n-i} = H^{(1)}(T, p)\,, \tag{6.2}$$

(2.6) can be rewritten as follows:

$$\{H^{(0)}(p) + H^{(1)}(T, p)\} Y(p) = X'(p)\,. \tag{6.3}$$

With the notations:

$$(H^{(0)}(p))^{-1} X'(p) = X''(p)\,,$$

$$(H^{(0)}(p))^{-1} H^{(1)}(T, p) = h(T, p)\,,$$

(6.3) is further rewritten:

$$\{1 + h(T, p)\} Y(p) = X''(p)\,,$$

from which we obtain:

$$Y(p) = \sum_{n=0}^{\infty} (-1)^n \{h(T, p)\}^n X''(p)\,. \tag{6.4}$$

The simultaneous ordinary linear differential equations can be treated in a similar way to the above. Namely, we write:

$$\left.\begin{aligned} S_{lm}(T,\ p) &= H^{(0)}_{lm}(p)+H^{(1)}_{lm}(T,\ p)\,,\\ H^{(0)}_{lm}(p) &= \sum_{i=0}^{n_{lm}}\sum_{\alpha=0}^{\nu} A_0^{(i;lm)}(\alpha)p^{n_{lm}-i}\,,\\ H^{(1)}_{lm}(T,\ p) &= \sum_{i=0}^{n_{lm}}\sum_{\alpha=0}^{\nu}\sum_{k=-Ni,\alpha;lm}^{Ni,\alpha;lm}{}' A_k^{(i;lm)}(\alpha)T^k(\omega_\alpha)p^{n_{lm}-i}\,,\end{aligned}\right\} \tag{6.5}$$

$$[H^{(0)}(p)] = \begin{pmatrix} H^{(0)}_{11} & H^{(0)}_{12} & \cdots\cdots & H^{(0)}_{1M}\\ H^{(0)}_{21} & H^{(0)}_{22} & \cdots\cdots & H^{(0)}_{2M}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ H^{(0)}_{M1} & H^{(0)}_{M2} & \cdots\cdots & H^{(0)}_{MM}\end{pmatrix}, \tag{6.6}$$

$$[H^{(1)}(T,\ p)] = \begin{pmatrix} H^{(1)}_{11} & H^{(1)}_{12} & \cdots\cdots & H^{(1)}_{1M}\\ H^{(1)}_{21} & H^{(1)}_{22} & \cdots\cdots & H^{(1)}_{2M}\\ \cdots & \cdots & \cdots & \cdots\\ \cdots & \cdots & \cdots & \cdots\\ H^{(1)}_{M1} & H^{(1)}_{M2} & \cdots\cdots & H^{(1)}_{MM}\end{pmatrix}, \tag{6.6'}$$

where $H^{(0)}_{lm}=H^{(0)}_{lm}(p)$ and $H^{(1)}_{lm}=H^{(1)}_{lm}(T,\ p)$. Using (6.6) and (6.6′), we obtain:

$$[Y(p)] = \{\sum_{n=0}^{\infty}(-1)^n[h(T,\ p)]^n\}[X''(p)]\,,$$

where

$$[H^{(0)}(p)]^{-1}[H^{(1)}(T,\ p)] = [h(T,\ p)]\,,$$
$$[H^{(0)}(p)]^{-1}[X'(p)] = [X''(p)]\,.$$

As a first example, we consider the equation:

$$\frac{d^2y}{dt^2}+(A_0^{(1)}+2A_1^{(1)}\cos\omega_0 t)\frac{dy}{dt}+A_0^{(2)}y = e^{i\omega t}\,, \tag{6.7}$$

where it is assumed that $A_0^{(1)}$, $A_1^{(1)}$, $A_0^{(2)}$, ω and ω_0 are all real.

Transforming it to the representation in the frequency domain and taking into account the following expressions:

$$\left.\begin{aligned} X'(p) &= (p-i\omega)^{-1}+\{py(0)+(A_0^{(1)}+2A_1^{(1)})y(0)+y'(0)\}\,,\\ h(T,\ p) &= A_1^{(1)}(p^2+A_0^{(1)}p+A_0^{(2)})^{-1}(T(\omega_0)+T^{-1}(\omega_0))p\,,\end{aligned}\right\} \tag{6.8}$$

we obtain the $(n+1)$th term of (6.4) as follows:

$$\begin{aligned} &\frac{A_1^{(1)n}}{p^2+A_0^{(1)}p+A_0^{(2)}}\left\{(T(\omega_0)+T^{-1}(\omega_0))\frac{p}{p^2+A_0^{(1)}p+A_0^{(2)}}\right\}^n X'(p)\\ &= \frac{A_1^{(1)n}}{p^2+A_0^{(1)}p+A_0^{(2)}}\mathfrak{S}_n\left\{T^{\varepsilon_1}(\omega_0)\frac{p}{p^2+A_0^{(1)}p+A_0^{(2)}}T^{\varepsilon_2}(\omega_0)\frac{p}{p^2+A_0^{(1)}p+A_0^{(2)}}\right.\\ &\qquad\qquad \left.\times\cdots\cdots T^{\varepsilon_n}(\omega_0)\frac{p}{p^2+A_0^{(1)}p+A_0^{(2)}}\right\}X'(p)\,,\end{aligned}$$

where each of ε_1, ε_2, $\cdots$, and ε_n can take only 1 or -1 and $\mathfrak{S}_n$ means the sum taken over all the combinations of ε_1, ε_2, $\cdots$, and ε_n. Further, the above expression can be modified as follows:

$$\frac{A_1^{(1)n}}{p^2+A_0^{(1)}p+A_0^{(2)}}\mathfrak{S}_n\prod_{l=1}^{n}\frac{(p-i\varepsilon_{1,2,\ldots,l}\omega_0)X'(p-i\varepsilon_{1,2,\ldots,l}\omega_0)}{\{(p-i\varepsilon_{1,2,\ldots,l}\omega_0)^2+A_0^{(1)}(p-i\varepsilon_{1,2,\ldots,l}\omega_0)+A_0^{(2)}\}}, \tag{6.9}$$

where $\varepsilon_{1,2,3,\ldots,n}=\varepsilon_1+\varepsilon_2+\cdots\cdots+\varepsilon_n$.

Substituting (6.9) into (6.4) and taking the inverse Laplace transform of the resulting expression, we collect the terms that have the time factor $e^{i\omega t}$. Then, we obtain the following expression which may be called the steady state part of the solution:

$$y_{\text{steady}}(t) = \frac{e^{i\omega t}}{A_0^{(2)}+i\omega A_0^{(1)}-\omega^2} + e^{i\omega t}\sum_{n=1}^{\infty}(-i)^n A_1^{(1)n}\mathfrak{S}_n B(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)e^{i\varepsilon_{1,2,\ldots,n}\omega_0 t}, \tag{6.10}$$

where

$$B(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n) = \omega\{\omega+\varepsilon_{1,2,\ldots,n}\omega_0\}^{-1}\{A_0^{(2)}+i\omega A_0^{(1)}-\omega^2\}^{-1} \times\prod_{l=1}^{n}\frac{(\omega+\varepsilon_{1,2,\ldots,l}\omega_0)}{\{A_0^{(2)}+i(\omega+\varepsilon_{1,2,\ldots,l}\omega_0)A_0^{(1)}-(\omega+\varepsilon_{1,2,\ldots,l}\omega_0)^2\}}. \tag{6.11}$$

We can easily show that $y_{\text{steady}}(t)$ is absolutely convergent, when $|2A_1^{(1)}/A_0^{(1)}|<1$.*

As a second example, we consider the following equation:

$$\frac{d^2y}{dt^2}+(A_0^{(2)}+A_1^{(2)}e^{-\alpha t})y = 0 \quad (\alpha>0). \tag{6.12}$$

Now putting $A_0^{(2)}=\omega_L^2$ and $A_1^{(2)}/A_0^{(2)}=\kappa$, $h(T, p)$ and $X''(p)$ are given as follows:

$$\left.\begin{aligned} h(T, p) &= (\kappa\omega_L^2/(p^2+\omega_L^2))T(i\alpha), \\ X''(p) &= (p^2+\omega_L^2)^{-1}(py(0)+y'(0)), \end{aligned}\right\} \tag{6.13}$$

where we assume that ω_L and κ are real.

By (6.4) we obtain:

$$Y(p) = X''(p)+y(0)\sum_{n=1}^{\infty}(-1)^n(\omega_L^2\kappa)^n(p+n\alpha)\prod_{l=0}^{n}\{(p+l\alpha)^2+\omega_L^2\}^{-1} + y'(0)\sum_{n=1}^{\infty}(-1)^n(\omega_L^2\kappa)^n\prod_{l=0}^{n}\{(p+l\alpha)^2+\omega_L^2\}^{-1}.$$

Then, defining $A_n(p)$ and $B_n(p)$ by the following expression:

$$Y(p) = X''(p)+y(0)\sum_{n=1}^{\infty}(-1)^n\left(\frac{\omega_L^2\kappa}{\alpha^2}\right)^n A_n(p) + y'(0)\sum_{n=1}^{\infty}(-1)^n\left(\frac{\omega_L^2\kappa}{\alpha^2}\right)^n B_n(p), \tag{6.14}$$

and denoting the inverse Laplace transform of $A_n(p)$ and $B_n(p)$ by $A_n(t)$ and $B_n(t)$ respectively, we obtain:

* Reference should be made to Appendix.

$$y(t) = y(0)\cos\omega_L t + \omega_L^{-1} y'(0)\sin\omega_L t + y(0)\sum_{n=1}^{\infty}(-1)^n\left(\frac{\omega_L^2\kappa}{\alpha^2}\right)^n A_n(t) + y'(0)\sum_{n=1}^{\infty}(-1)^n\left(\frac{\omega_L^2\kappa}{\alpha^2}\right)^n B_n(t)\,, \tag{6.15}$$

where $A_n(t)$ and $B_n(t)$ are given by the following expressions:

$$\left.\begin{aligned}
A_n(t) &= \mathrm{Im}\left\{ie^{i\omega_L t}\sum_{k=0}^{n} e^{-\alpha k t}\frac{\left|\Gamma\left(1+\frac{i}{x^2}\right)\right|^2}{\Gamma(k+1)\Gamma(n-k+1)\Gamma\left(k+1-\frac{i}{x^2}\right)\Gamma\left(n-k+1+\frac{i}{x^2}\right)}\right. \\
&\quad \left. -e^{i\omega_L t}\left(\frac{\alpha}{\omega_L}\right)\sum_{k=0}^{n-1} e^{-\alpha k t}\frac{\left|\Gamma\left(1+\frac{i}{x^2}\right)\right|^2}{\Gamma(k+1)\Gamma(n-k)\Gamma\left(k+1-\frac{i}{x^2}\right)\Gamma\left(n-k+1+\frac{i}{x^2}\right)}\right\}, \\
B_n(t) &= \mathrm{Im}\left\{\frac{e^{i\omega_L t}}{\omega_L}\sum_{k=0}^{n} e^{-\alpha k t}\frac{\left|\Gamma\left(1+\frac{i}{x^2}\right)\right|^2}{\Gamma(k+1)\Gamma(n-k+1)\Gamma\left(k+1-\frac{i}{x^2}\right)\Gamma\left(n-k+1+\frac{i}{x^2}\right)}\right\}. \\
x &= \left(\frac{\alpha}{2\omega_L}\right)^{1/2} \qquad n \geq 1\,.
\end{aligned}\right\} \tag{6.16}$$

We can also easily show that (6.5) is absolutely convergent*.

7. Formulation of Maxwell equations in the medium whose material constants ε and μ are E-type functions of time

In the frequency domain, we will formulate Maxwell equations in such a medium that the material constants (dielectric constant ε and magnetic permeability μ) are E-type functions of time t, by introducing the translational operator. We can write the phenomenological electromagnetic field equations as follows, in M. K. S. units:

$$\left.\begin{aligned}
&\nabla\times\mathbf{E}(\mathbf{r},\,t)+\frac{\partial\mathbf{B}(\mathbf{r},\,t)}{\partial t}=0\,, \\
&\nabla\times\mathbf{H}(\mathbf{r},\,t)-\frac{\partial\mathbf{D}(\mathbf{r},\,t)}{\partial t}=\mathbf{J}(\mathbf{r},\,t)\,, \\
&\nabla\cdot\mathbf{D}(\mathbf{r},\,t)=\rho(\mathbf{r},\,t)\,, \\
&\nabla\cdot\mathbf{B}(\mathbf{r},\,t)=0\,.
\end{aligned}\right\} \tag{7.1}$$

Taking the Laplace transforms of equations (7.1), we obtain:

$$\left.\begin{aligned}
&\nabla\times\mathbf{E}(\mathbf{r},\,p)+p\mathbf{B}(\mathbf{r},\,p)=\mathbf{B}(\mathbf{r},\,0)\,, \\
&\nabla\times\mathbf{H}(\mathbf{r},\,p)-p\mathbf{D}(\mathbf{r},\,p)=\mathbf{J}(\mathbf{r},\,p)-\mathbf{D}(\mathbf{r},\,0)\,, \\
&\nabla\cdot\mathbf{D}(\mathbf{r},\,p)=\rho(\mathbf{r},\,p)\,, \\
&\nabla\cdot\mathbf{B}(\mathbf{r},\,p)=0\,.
\end{aligned}\right\} \tag{7.2}$$

* Reference should be made to Appendix.

The Laplace transforms of the quantities which appear in (7.1), are represented by the corresponding symbols in which t is replaced by p. Thus, for instance,

$$L(\mathbf{E}(\mathbf{r},\ t)) = \mathbf{E}(\mathbf{r},\ p)\ ,$$
$$L(\mathbf{H}(\mathbf{r},\ t)) = \mathbf{H}(\mathbf{r},\ p)\ , \quad \text{etc.}$$

$\mathbf{B}(\mathbf{r}, 0)$ and $\mathbf{D}(\mathbf{r}, 0)$ mean $\lim_{t\to 0}\mathbf{B}(\mathbf{r}, t)$ and $\lim_{t\to 0}\mathbf{D}(\mathbf{r}, t)$ respectively. We will derive connections between $\mathbf{D}(\mathbf{r}, p)$ and $\mathbf{E}(\mathbf{r}, p)$, and $\mathbf{B}(\mathbf{r}, p)$ and $\mathbf{H}(\mathbf{r}, p)$ by introducing the material constants and the electric and the magnetic polarization $\mathbf{P}(\mathbf{r}, t)$ and $\mathbf{M}(\mathbf{r}, t)$. From

$$\left.\begin{aligned}\mathbf{D}(\mathbf{r},\ t) &= \varepsilon_0\mathbf{E}(\mathbf{r},\ t)+\mathbf{P}(\mathbf{r},\ t)\ ,\\ \mathbf{B}(\mathbf{r},\ t) &= \mu_0\{\mathbf{H}(\mathbf{r},\ t)+\mathbf{M}(\mathbf{r},\ t)\}\ ,\end{aligned}\right\} \tag{7.3}$$

we obtain, by taking Laplace transforms:

$$\left.\begin{aligned}\mathbf{D}(\mathbf{r},\ p) &= \varepsilon_0\mathbf{E}(\mathbf{r},\ p)+\mathbf{P}(\mathbf{r},\ p)\ ,\\ \mathbf{B}(\mathbf{r},\ p) &= \mu_0\{\mathbf{H}(\mathbf{r},\ p)+\mathbf{M}(\mathbf{r},\ p)\}\ ,\end{aligned}\right\} \tag{7.4}$$

and

$$\left.\begin{aligned}\mathbf{P}(\mathbf{r},\ p) &= \varepsilon_0\vec{\chi}_e(T,\ p)\cdot\mathbf{E}(\mathbf{r},\ p)\ ,\\ \mathbf{M}(\mathbf{r},\ p) &= \vec{\chi}_m(T,\ p)\cdot\mathbf{H}(\mathbf{r},\ p)\ ,\end{aligned}\right\} \tag{7.5)*$$

where $\vec{\chi}_e(T, p)$ and $\vec{\chi}_m(T, p)$ are tensors of the second rank and may be called the electric susceptibility operator and the magnetic susceptibility operator. The ij-components of $\vec{\chi}_e(T, p)$ and $\vec{\chi}_m(T, p)$ can be written as follows:

$$\left.\begin{aligned}(\vec{\chi}_e)_{ij} &= \sum_{\alpha=0}^{\nu}\sum_{k=-N_{ij;\alpha e}}^{N_{ij;\alpha e}}\chi_{e,ij}^{(k)}(p\,;\alpha)T^k(\omega_\alpha)\ ,\\ (\vec{\chi}_m)_{ij} &= \sum_{\alpha=0}^{\nu}\sum_{k=-N_{ij;\alpha m}}^{N_{ij;\alpha m}}\chi_{m,ij}^{(k)}(p\,;\alpha)T^k(\omega_\alpha)\ ,\end{aligned}\right\} \tag{7.6}$$

where i and j represent x, y and z.

When the medium is isotropic, $\vec{\chi}_e(T, p)$ and $\vec{\chi}_m(T, p)$ reduce to the simple scalar operators, which we will denote by $\chi_e(T, p)$ and $\chi_m(T, p)$, respectively and we obtain:

$$\left.\begin{aligned}\mathbf{D}(\mathbf{r},\ p) &= \varepsilon_0\{1+\chi_e(T,\ p)\}\mathbf{E}(\mathbf{r},\ p)\ ,\\ \mathbf{B}(\mathbf{r},\ p) &= \mu_0\{1+\chi_m(T,\ p)\}\mathbf{H}(\mathbf{r},\ p)\ .\end{aligned}\right\} \tag{7.7}$$

Now, putting

$$\left.\begin{aligned}\varepsilon(T,\ p) &= \varepsilon_0(1+\chi_e(T,\ p))\ ,\\ \mu(T,\ p) &= \mu_0(1+\chi_m(T,\ p))\ ,\end{aligned}\right\} \tag{7.8}$$

---

* Here are introduced the relations between $\mathbf{P}(\mathbf{r}, p)$ and $\mathbf{E}(\mathbf{r}, p)$, and $\mathbf{M}(\mathbf{r}, p)$ and $\mathbf{H}(\mathbf{r}, p)$, since ε and μ, and accordingly χ_e and χ_m are generally dependent on the frequency. Reference should be made to the footnote to the Introduction.

and assuming that $\mathbf{B}(\mathbf{r}, 0)=0$, $\mathbf{D}(\mathbf{r}, 0)=0$, and $\rho(\mathbf{r}, t)=0$, $\mathbf{J}(\mathbf{r}, t)=0$, the equations (7.2) become as follows:

$$\left.\begin{aligned} \nabla\times\mathbf{E}(\mathbf{r}, p)+p\mu(T, p)\mathbf{H}(\mathbf{r}, p) &= 0\,, \\ \nabla\times\mathbf{H}(\mathbf{r}, p)-p\varepsilon(T, p)\mathbf{E}(\mathbf{r}, p) &= 0\,, \\ \nabla\cdot\mathbf{E}(\mathbf{r}, p) &= 0\,, \\ \nabla\cdot\mathbf{H}(\mathbf{r}, p) &= 0\,. \end{aligned}\right\} \tag{7.9}$$

Next we consider the propagation of the electromagnetic waves in the hollow metallic (perfectly conducting) tube filled with such a medium that the equations (7.9) are valid. When we take the z-axis along the pipe axis and the x, y axes in a plane perpendicular to the pipe axis, we can express in the frequency domain the propagation modes as follows:

For the TE waves ($E_z(\mathbf{r}, t)=0$), $H_z(\mathbf{r}, p)$ is given by

$$H_z(\mathbf{r}, p) = \psi_n(x, y)e^{\mp\gamma_z(T,p)z}F(p)\,, \tag{7.10}$$

where $\psi_n(x, y)$ is defined by

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_n^2\right)\psi_n(x, y) = 0\,,$$

subject to the boundary condition $\frac{\partial\psi_n}{\partial n}=0$ at the wall and

$$\left.\begin{aligned} \gamma^2(T, p) &= \gamma_z^2(T, p)-k_n^2\,, \\ \gamma^2(T, p) &= p\varepsilon(T, p)p\mu(T, p)\,. \end{aligned}\right\} \tag{7.10'}$$

The other field components are expressed as follows:

$$\left.\begin{aligned} H_x(\mathbf{r}, p) &= \mp k_n^{-2}\gamma_z(T, p)\frac{\partial}{\partial x}H_z(\mathbf{r}, p)\,, \\ H_y(\mathbf{r}, p) &= \mp k_n^{-2}\gamma_z(T, p)\frac{\partial}{\partial y}H_z(\mathbf{r}, p)\,, \\ E_x(\mathbf{r}, p) &= -k_n^{-2}p\mu(T, p)\frac{\partial}{\partial y}H_z(\mathbf{r}, p)\,, \\ E_y(\mathbf{r}, p) &= k_n^{-2}p\mu(T, p)\frac{\partial}{\partial x}H_z(\mathbf{r}, p)\,, \end{aligned}\right\} \tag{7.11}$$

where $F(p)$ is an arbitrary function of p.

For TM waves ($H_z(\mathbf{r}, t)=0$), $E_z(\mathbf{r}, p)$ is given by

$$E_z(\mathbf{r}, p) = \phi_n(x, y)e^{\mp\gamma_z'(T,p)z}G(p)\,, \tag{7.12}$$

where $\phi_n(x, y)$ is defined by

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+k_n'^2\right)\phi_n(x, y) = 0$$

subject to the boundary condition $\phi_n(x, y)=0$ at the wall and

$$\left.\begin{aligned} \gamma'^2(T,\ p) &= \gamma_z'^2(T,\ p) - k_n'^2\ , \\ \gamma'^2(T,\ p) &= p\mu(T,\ p)p\varepsilon(T,\ p)\ . \end{aligned}\right\} \tag{7.12'}$$

The other field components are given by the following expressions:

$$\left.\begin{aligned} H_x(\mathbf{r},\ p) &= k_n'^{-2}p\varepsilon(T,\ p)\frac{\partial}{\partial y}E_z(\mathbf{r},\ p)\ , \\ H_y(\mathbf{r},\ p) &= -k_n'^{-2}p\varepsilon(T,\ p)\frac{\partial}{\partial x}E_z(\mathbf{r},\ p)\ , \\ E_x(\mathbf{r},\ p) &= \mp k_n'^{-2}\gamma_z'(T,\ p)\ \frac{\partial}{\partial x}E_z(\mathbf{r},\ p)\ , \\ E_y(\mathbf{r},\ p) &= \mp k_n'^{-2}\gamma_z'(T,\ p)\ \frac{\partial}{\partial y}E_z(\mathbf{r},\ p)\ , \end{aligned}\right\} \tag{7.13}$$

where $G(p)$ is an arbitrary function of p. From (7.10)$\sim$(7.13), it is evident that in the frequency domain, the propagation mode takes the same form as that in the medium of fixed material constants, except that the propagation constant operator $\gamma_z(T, p)$ or $\gamma_z'(T, p)$ and the material constant operator $\mu(T, p)$ or $\varepsilon(T, p)$ include the translational operator.

8. Conclusion

Summarizing we may say that the descriptions in the preceding paragraphs could be an answer for the following problem. How can the method of solving the ordinary linear differential equation with constant coefficients be extended by the Laplace transform to the case of the ordinary linear differential equation with E-type coefficients? As shown in §2, to solve the ordinary linear differential equations (2.1) and (2.9) is reduced to obtaining the inverses of $S(T, p)$ and $[S(T, p)]$.

In the special cases shown in §4, the inverse of $S(T, p)$ was easily obtained in a compact form, but in the general case it will be difficult to obtain the inverse in a compact form. Then, in such a case, the method of the series expansion should be used. Though in §6 the geometrical series expansion was utilized, it is hoped that the more powerful expansion method will be devised.

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Appendix

It will be shown that (6.10) and (6.15) are absolutely convergent. First, the convergency of (6.10) is examined. By (6.11), we obtain the following inequality

$$|B(\varepsilon_1, \varepsilon, \cdots, \varepsilon_n)| < \left|\frac{\omega}{A_0^{(2)}+i\omega A_0^{(1)}-\omega^2}\right| |A_0^{(1)}|^{-(n-1)}$$
$$\times |A_0^{(2)}+iA_0^{(1)}(\omega+\varepsilon_{1,2,\ldots,n}\omega_0)-(\omega+\varepsilon_{1,2,\ldots,n}\omega_0)^2|^{-1}.$$

We represent the minimum of $|\omega+\varepsilon_{1,2,\ldots,n}\omega_0|$ for all the values of n, by ω_m.

Now putting

$$\left.\begin{aligned} &\omega_m|A_0^{(1)}| = m, && \text{when} \quad \omega_m \neq 0, \\ &\min. (|A_0^{(2)}|, |\omega_0 A_0^{(1)}|) = m, && \text{when} \quad \omega_m = 0, \end{aligned}\right\}$$

we obtain:

$$|A_0^{(2)}+i(\omega+\varepsilon_{1,2,\ldots,n}\omega_0)A_0^{(1)}-(\omega+\varepsilon_{1,2,\ldots,n}\omega_0)^2|^{-1} < \frac{1}{m},$$

$$\therefore \quad |B(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)| < \frac{1}{m}|A_0^{(1)}|^{-n}.$$

Since in $\mathfrak{S}_n B(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$ are included 2^n of $B(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)$'s,

$$|\mathfrak{S}_n B(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)| < \frac{1}{m}\left|\frac{2}{A_0^{(1)}}\right|^n.$$

Therefore $\quad |(-i)^n A_1^{(1)n}\mathfrak{S}_n B(\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n)\, e^{i\varepsilon_{1,2,\cdots,n}\omega_0 t}| < \dfrac{1}{m}\left|\dfrac{2A_1^{(1)}}{A_0^{(1)}}\right|^n,$

$$\therefore \quad \left|y_{\text{steady}}(t) - \frac{e^{i\omega t}}{A_0^{(2)}+i\omega A_0^{(1)}-\omega^2}\right| < \frac{1}{m}\sum_{n=1}^{\infty}\left|\frac{2A_1^{(1)}}{A_0^{(1)}}\right|^n.$$

From the above inequalities, we see that $y_{\text{steady}}(t)$ is absolutely convergent when $|2A_1^{(1)}/A_0^{(1)}|<1$ i.e. $|2A_1^{(1)}|<|A_0^{(1)}|$, under the assumption that $A_0^{(1)}$, $A_0^{(2)}$ and ω_0 are so taken that $m \neq 0$.

Next the convergency of (6.5) is examined. Putting $L^{-1}(X''(p))=x''(t)$ and $\Delta y(t)=y(t)-x''(t)$, we consider the convergency of $\Delta y(t)$. We can easily show the following inequalities:

$$\left|\frac{\Gamma\left(1-\frac{i}{x^2}\right)}{\Gamma\left(k+1-\frac{i}{x^2}\right)}\right| < (x^2)^k = \left(\frac{\alpha}{2\omega_L}\right)^k \quad (\omega_L > 0)$$

and

$$\left|\frac{\Gamma\left(1+\frac{i}{x^2}\right)}{\Gamma\left(n-k+1+\frac{i}{x^2}\right)}\right| < \left(\frac{\alpha}{2\omega_L}\right)^{n-k},$$

$$\therefore \quad \left|\sum_{k=0}^{n} e^{-\alpha kt}\frac{\left|\Gamma\left(1+\frac{i}{x^2}\right)\right|^2}{\Gamma(k+1)\Gamma(n-k+1)\Gamma\left(k+1-\frac{i}{x^2}\right)\Gamma\left(n-k+1+\frac{i}{x^2}\right)}\right|$$
$$< \left(\frac{\alpha}{2\omega_L}\right)^n \frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}e^{-\alpha kt} = \left(\frac{\alpha}{2\omega_L}\right)^n \frac{1}{n!}(1+e^{-\alpha t})^n.$$

Similarly

$$\left|\left(\frac{\alpha}{\omega_L}\right)\sum_{k=1}^{n-1} e^{-\alpha kt}\frac{\left|\Gamma\left(1+\frac{i}{x^2}\right)\right|^2}{\Gamma(k+1)\Gamma(n-k)\Gamma\left(k+1-\frac{i}{x^2}\right)\Gamma\left(n-k+1+\frac{i}{x^2}\right)}\right|$$

$$<\frac{\alpha}{\omega_L}\left(\frac{\alpha}{2\omega_L}\right)^n\frac{1}{(n-1)!}(1+e^{-\alpha t})^{n-1}.$$

Therefore, the following inequalities are established:

$$|(-1)^n\left(\frac{\omega_L^2\kappa}{\alpha^2}\right)^2 A_n(t)|<\frac{1}{n!}\left|\frac{\omega_L\kappa}{2\alpha}\right|^n(1+e^{-\alpha t})^n$$

$$+\frac{\alpha}{\omega_L}\frac{1}{(n-1)!}\left|\frac{\omega_L\kappa}{2\alpha}\right|^n(1+e^{-\alpha t})^{n-1},$$

$$|(-1)^n\left(\frac{\omega_L^2\kappa}{\alpha^2}\right)^2 B_n(t)|<\frac{1}{\omega_L}\frac{1}{n!}\left|\frac{\omega_L\kappa}{2\alpha}\right|^n(1+e^{-\alpha t})^n.$$

Finally

$$|\Delta y(t)|<|y(0)|\sum_{n=1}^{\infty}\frac{1}{n!}\left\{\left|\frac{\omega_L\kappa}{2\alpha}\right|(1+e^{-\alpha t})\right\}^n$$

$$+\frac{\alpha}{\omega_L}|y(0)|\sum_{n=1}^{\infty}\frac{1}{(n-1)!}\left\{\left|\frac{\omega_L\kappa}{2\alpha}\right|(1+e^{-\alpha t})\right\}^{n-1}$$

$$+\frac{1}{\omega_L}|y'(0)|\sum_{n=1}^{\infty}\frac{1}{n!}\left\{\left|\frac{\omega_L\kappa}{2\alpha}\right|(1+e^{-\alpha t})\right\}^n$$

$$<|y(0)|e^{\left|\frac{\omega_L\kappa}{2\alpha}\right|(1+e^{-\alpha t})}+\frac{\alpha}{\omega_L}|y(0)|e^{\left|\frac{\omega_L\kappa}{2\alpha}\right|(1+e^{-\alpha t})}$$

$$+\frac{|y'(0)|}{\omega_L}e^{\left|\frac{\omega_L\kappa}{2\alpha}\right|(1+e^{-\alpha t})}$$

$$=\left(|y(0)|+\frac{\alpha}{\omega_L}|y(0)|+\frac{|y'(0)|}{\omega_L}\right)e^{\left|\frac{\omega_L\kappa}{2\alpha}\right|(1+e^{-\alpha t})}.$$

Thus it has been shown that $\Delta y(t)$ is absolutely convergent and so is also $y(t)$.

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