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AN EXPANSION FORMULA FOR THE LIFT ON A CIRCULAR-ARC AEROFOIL IN A STREAM BOUNDED BY A PLANE WALL

BY

S. TOMOTIKA and H. FUJIKAWA

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SUMMARY

Starting with Green's general exact formula for the lift acting on a circular-arc aerofoil placed in any position in a stream bounded by an infinite plane wall, an expansion formula for the lift is obtained correct to the order of $(l/H)^2$, where l is the chord-length of the arc-aerofoil and H is the distance of the mid-point of the chord from the bounding wall. The result is found to be in complete agreement, up to the order of $(l/H)^2$, with an expansion formula for the lift on a circular-arc aerofoil near a plane wall, which has been obtained recently by the junior writer by extending Green's new method useful for calculating approximate expansion formulae for the lift and moment of an arbitrary two-dimensional aerofoil placed near an infinite plane wall.

1. Introduction

The lift acting on a circular-arc aerofoil placed in any position in a stream bounded by an infinite plane wall has been evaluated by Green (1) in an exact manner, by using suitable conformal transformations. Later on, the senior writer has reinvestigated (2), in conjunction with Tamada and Umemoto, the problem *ab initio* by employing suitable conformal transformations, which are somewhat different from those used by Green. It has been found that except for the difference in notation, our general exact formula for the lift acting on the aerofoil is in perfect agreement with the corresponding formula of Green.

By carrying out detailed numerical calculations for three circular-arc aerofoils, the cambers of which are respectively 0.022, 0.053 and 0.097 approximately, the value of the angle of incidence being taken to be 5° in all cases, we have investigated the manner in which the ground effect upon the lift of an aerofoil is modified by its camber.

Since, however, not only the general exact formula for the lift but also the equations for determining various parameters are all very complicated, it is extremely difficult and almost hopeless to repeat detailed numerical calculations for arc-aerofoils of different cambers and for various values of the angle of incidence. Therefore, it is desirable to derive an approximate expansion formula for the lift useful for numerical computations.

In a recent paper (3) Green has devised an ingenious general method for calculating series expansion formulae for the lift and moment acting on a two-dimensional aerofoil with arbitrary shape in the presence of an infinite plane wall, and applying the method Green himself has obtained the first three terms in the series expansion for the lift acting on a circular-arc aerofoil of arbitrary camber.

Quite recently, the junior writer (4) has extended Green's analysis and obtained an expansion formula correct to the order of $(l/H)^4$ for the lift Y acting on a circular-arc aerofoil in the presence of a plane wall, where l is the chord-length of the aerofoil and H is the distance of the mid-point of the chord from the bounding wall. Thus, omitting, for the sake of simplicity, the third and higher powers of l/H, the result is

$$\frac{Y}{Y_0}=1-\frac{1}{2}\left(\sin\theta+\tan\alpha\cos\theta\right)\frac{l}{H}$$
$$+\frac{1}{64}\left\{(8-6\cos 2\theta)+12\tan\alpha\sin 2\theta\right.$$
$$\left.+\tan^2\alpha(13+8\cos 2\theta)+\frac{2\sin\theta}{\sin\theta+\tan\alpha\cos\theta}\right\}\left(\frac{l}{H}\right)^2, \tag{1.1}$$

where θ is the angle of incidence and 4α is the angle which the arc subtends at its centre. Y_0 denotes the lift acting on the same aerofoil when placed in an unlimited uniform stream of velocity U and is given by the well-known formula:

$$Y_0=\pi\rho l U^2(\sin\theta+\tan\alpha\cos\theta)\,, \tag{1.2}$$

where ρ is the density of the fluid concerned.

It should naturally be expected that such an expansion formula for the lift would have been derived from the general exact formula, by assuming H to be very large and developing various functions in series. The object of the present paper is to derive the expansion formula by starting with Green's general formula for the lift.

2. The conformal transformations

For the sake of reference we shall first reproduce the principal results of Green's analysis which are necessary for our present purpose. For details of the analysis reference should be made to Green's original work (1).

Taking the plane of two-dimensional fluid motion as the z-plane, we consider a steady irrotational continuous flow of an incompressible inviscid fluid past a circular-arc aerofoil AA′ which is placed in any position near an infinite plane wall HH′. We take the x-axis along the wall HH′ and assume that the fluid at infinity flows with a constant velocity U in the positive direction of the x-axis.

Two different cases now arise; in the first case, the circle on which the arc lies intersects the bounding wall at two real points X and Y (Fig. 1), while in the second case the circle on which the arc lies does not intersect the wall at real points (Fig. 2). However, as shown by Green, the results for the second case can be derived from those for the first case so that it suffices to consider only the first case.

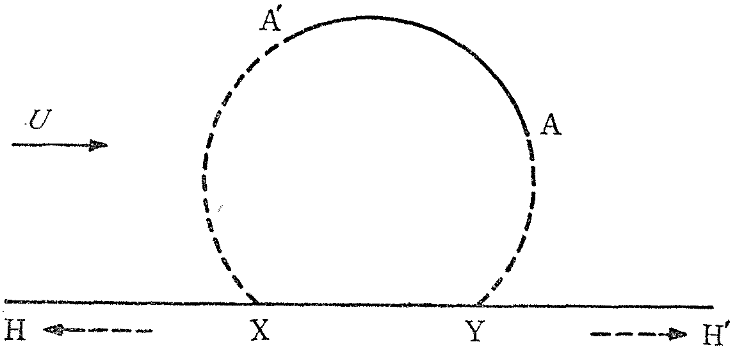

Fig. 1. z-plane.

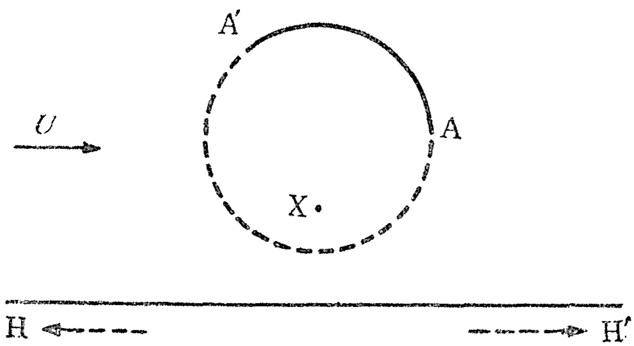

Fig. 2. z-plane.

We take the origin O of the z-plane at the mid-point of the segment XY whose length will be denoted by 2λ. Then, after several transformations, the z-plane is transformed conformally into an annular region in a Z-plane bounded by two concentric circles of radii 1 and $q(<1)$ (Fig. 3), the circular-arc AA′ corresponding to the inner circle and the bounding wall HH′ to the outer circle. The various points are transformed as:

$$\left.\begin{aligned} &\mathrm{A} = qe^{i\theta_1},\ \mathrm{A}' = qe^{i\theta_2},\ \ \mathrm{X} = e^{i\theta_3},\ \ \mathrm{Y} = e^{i\theta_4}, \\ &\mathrm{H} = \mathrm{H}' = -1, \end{aligned}\right\} \tag{2.1}$$

where the arguments $\theta_1, \theta_2, \theta_3, \theta_4$ satisfy the following four relations;

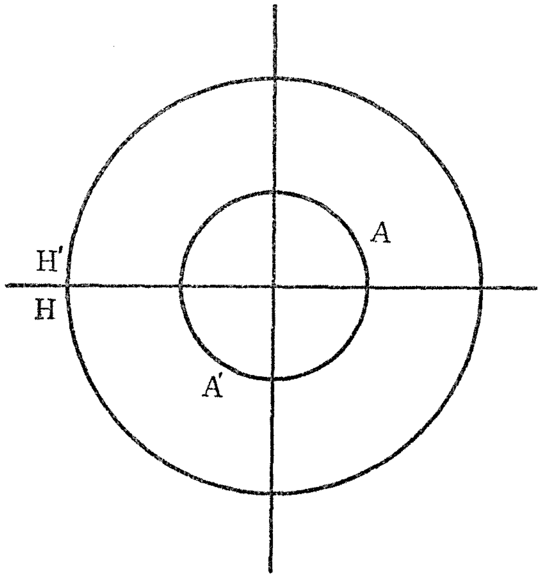

Fig. 3. Z-plane.

$$\frac{\vartheta_4'\{(\theta_1-\theta_3)/2\pi\}}{\vartheta_4\{(\theta_1-\theta_3)/2\pi\}}+\frac{\vartheta_4'\{(\theta_2-\theta_3)/2\pi\}}{\vartheta_4\{(\theta_2-\theta_3)/2\pi\}}=0\,, \tag{2.2}$$

$$\theta_1+\theta_2=\theta_3+\theta_4-2\pi\,, \tag{2.3}$$

$$\theta_4-\theta_3=2(\delta'-\pi)=-2\delta \text{ (say)}, \tag{2.4}$$

$$\pi\geqq\theta_1>\theta_2\geqq-\pi\,. \tag{2.5}$$

Here, $2\delta'$ denotes the angle which the segment XY subtends at the centre O′ of the arc AA′ (Fig. 4).

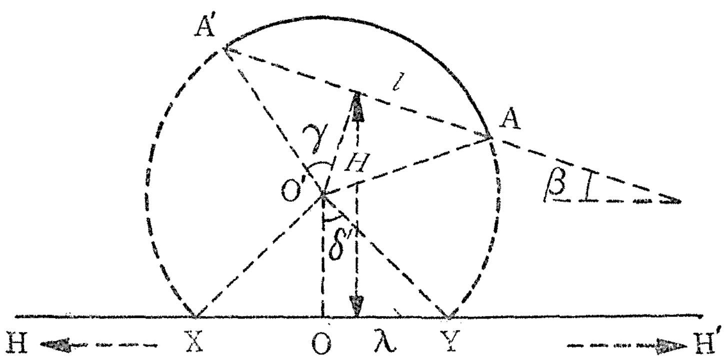

Fig. 4.

If we put

$$\left.\begin{aligned}\theta_1-\theta_2&=2\alpha\,,\\ \theta_3-\delta&=\theta_4+\delta=\phi\,,\end{aligned}\right\} \tag{2.6}$$

we have, from (2.3) and (2.4),

$$\left.\begin{aligned}\theta_1 &= \phi+\alpha-\pi\,,\\ \theta_2 &= \phi-\alpha-\pi\,,\end{aligned}\right\} \tag{2. 7}$$

and the relation (2. 2) can be rewritten in the form:

$$\frac{\vartheta_3'\{(\delta+\alpha)/2\pi\}}{\vartheta_3\{(\delta+\alpha)/2\pi\}}+\frac{\vartheta_3'\{(\delta-\alpha)/2\pi\}}{\vartheta_3\{(\delta-\alpha)/2\pi\}}=0\,. \tag{2. 8}$$

The same relations as (2. 7) and (2. 8) hold as well for the second case when the circle on which the arc lies does not intersect the wall at real points, provided that we generalise the definitions of λ and δ. In the first case, λ and δ are both real numbers and satisfy the conditions:

$$\lambda \gtrless 0\,,\quad \pi \gtrless \delta \gtrless -\pi\,, \tag{2. 9}$$

while, in the second case, $i\lambda$ and $i(\pi-\delta)$ are real numbers subject to the conditions:

$$i\lambda \gtrless 0\,,\quad i(\pi-\delta) \gtrless 0. \tag{2. 10}$$

With this extension of our definitions, the expressions for the forces acting on the aerofoil as evaluated in the first case can be used, as they stand, in the second case.

3. Geometrical parameters defining the circular-arc aerofoil

We shall now consider the geometrical parameters defining the circular-arc aerofoil AA′. Let l be the length of the chord AA′, a the radius of the arc AA′, 2γ the angle which the arc AA′ subtends at its centre O′ and H the distance of the mid-point of the chord AA′ from the bounding wall HH′. Further, let β be the acute angle which the chord AA′ makes with the wall. This β is nothing but the angle of incidence of the arc-aerofoil.

Then, we have from Fig. 4, the following relations:

$$\frac{l}{a}=2\,\sin\gamma\,,\quad \frac{\lambda}{a}=\sin\delta\,,\quad \frac{H}{a}=\cos\beta\,\cos\gamma-\cos\delta, \tag{3. 1}$$

from which we get

$$\frac{H}{l}=\frac{\cos\beta\,\cos\gamma-\cos\delta}{2\,\sin\gamma}\,. \tag{3. 2}$$

On the other hand, we have, from the transformation equations which are omitted here, the following expressions for $\tan\beta$ and H/l, namely:

$$\tan\beta=\frac{d(1-b^2)\,\sin\delta}{d(1+b^2)\,\cos\delta-b(1+d^2)}\,, \tag{3. 3}$$

$$\frac{H}{l}=\frac{\sin\beta}{2(1-b^2)(1-d^2)}(1+b^2+d^2+b^2d^2-4bd\cos\delta)\,, \tag{3.4}$$

where

$$b=\frac{\vartheta_2\{(\phi+\delta)/2\pi\}}{\vartheta_2\{(\phi-\delta)/2\pi\}}\,,\quad d=\frac{\vartheta_3\{(\delta+\alpha)/2\pi\}}{\vartheta_3\{(\delta-\alpha)/2\pi\}}\,. \tag{3.5}$$

Solving equations (3.2), (3.3) and (3.4) for d and b, we get

$$\left.\begin{aligned} d&=\left\{\frac{\cos(\delta-\gamma)-\cos\beta}{\cos(\delta+\gamma)-\cos\beta}\right\}^{\frac{1}{2}}, \\ b&=-\left\{\frac{\cos(\delta-\beta)-\cos\gamma}{\cos(\delta+\beta)-\cos\gamma}\right\}^{\frac{1}{2}}, \end{aligned}\right\} \tag{3.6}$$

and combining these with (3.5) we have ultimately

$$\left\{\frac{\cos(\delta-\gamma)-\cos\beta}{\cos(\delta+\gamma)-\cos\beta}\right\}^{\frac{1}{2}}=\frac{\vartheta_3\{(\delta+\alpha)/2\pi\}}{\vartheta_3\{(\delta-\alpha)/2\pi\}}\,, \tag{3.7}$$

$$-\left\{\frac{\cos(\delta-\beta)-\cos\gamma}{\cos(\delta+\beta)-\cos\gamma}\right\}^{\frac{1}{2}}=\frac{\vartheta_2\{(\phi+\delta)/2\pi\}}{\vartheta_2\{(\phi-\delta)/2\pi\}}\,. \tag{3.8}$$

Thus, if the values of β, γ and l/H are prescribed, these two equations, together with equations (2.8) and (3.2), determine the values of α, δ, ϕ and q.

4. The general expression for the lift acting on the arc-aerofoil

The resultant force acting on the circular-arc aerofoil can be calculated by the use of Blasius's first formula. Referring the detailed calculations to Green's paper, only the final results will be given here. Thus, denoting the x- and y-components of the resultant force by X and Y respectively, we have

$$X=0,\quad YD/l\rho U^2=Y_1+Y_2+Y_3+Y_4+Y_5\,, \tag{4.1}$$

where

$$\begin{aligned} Y_1=&-\left\{\frac{\vartheta_1'^2(0)\vartheta_4^2(\theta_1/2\pi)}{\vartheta_2^2(0)\vartheta_3^2(\theta_1/2\pi)}+\frac{\vartheta_2''(0)}{\vartheta_2(0)}\right\} \\ &\times\left[4E\left\{\frac{\vartheta_1'^2(0)\vartheta_4^2(\theta_1/2\pi)}{\vartheta_2^2(0)\vartheta_3^2(\theta_1/2\pi)}+\frac{\vartheta_2''(0)}{\vartheta_2(0)}-\frac{\vartheta_1'''(0)}{3\vartheta_1'(0)}\right\}\right. \\ &\left.\qquad-\vartheta_1'^3(0)\left\{\frac{\vartheta_1(\theta_1/\pi)}{\vartheta_3^4(\theta_1/2\pi)}-\frac{\vartheta_1(\theta_2/\pi)}{\vartheta_3^4(\theta_2/2\pi)}\right\}\right], \end{aligned} \tag{4.2}$$

$$Y_2=-\frac{2E\vartheta_1'^3(0)\vartheta_1(\theta_1/\pi)}{3\vartheta_3^4(\theta_1/2\pi)}\left\{\frac{\vartheta_1'(\theta_1/\pi)}{\vartheta_1(\theta_1/\pi)}-\frac{2\vartheta_3'(\theta_1/2\pi)}{\vartheta_3(\theta_1/2\pi)}\right\}, \tag{4.3}$$

$$Y_3=\frac{\vartheta_1'^4(0)\vartheta_1^2(\alpha/\pi)\vartheta_1^2\{(\theta_1+\theta_2)/2\pi\}}{\vartheta_3^4(\theta_1/2\pi)\vartheta_3^4(\theta_2/2\pi)}\left\{\frac{\vartheta_1'\{(\theta_1+\theta_2)/2\pi\}}{\vartheta_1\{(\theta_1+\theta_2)/2\pi\}}-\frac{\vartheta_1'(\alpha/\pi)}{\vartheta_1(\alpha/\pi)}\right\}, \tag{4.4}$$

$$Y_4=\frac{\vartheta_1'^4(0)\vartheta_1^2(\alpha/\pi)\vartheta_1^2\{(\theta_1+\theta_2)/2\pi\}}{\vartheta_3^4(\theta_1/2\pi)\vartheta_3^4(\theta_2/2\pi)}\,\frac{2\pi\sin(\beta-\gamma)}{\cos\delta-\cos(\beta-\gamma)}\,, \tag{4.5}$$

$$Y_5 = -\frac{\pi \cot \delta \vartheta_1'^4(0)\vartheta_1(\delta/\pi)\vartheta_1(\alpha/\pi)}{\vartheta_3^2\{(\delta+\alpha)/2\pi\}\vartheta_3^2\{(\delta-\alpha)/2\pi\}}$$

$$\times\left[\frac{\vartheta_1'^2(0)}{\vartheta_2^2(0)}\left\{F+\frac{\vartheta_4^2(\theta_2/2\pi)}{\vartheta_3^2(\theta_2/2\pi)}-\frac{3\vartheta_4^2(\theta_1/2\pi)}{\vartheta_3^2(\theta_1/2\pi)}\right\}\right.$$

$$\left.+\frac{2}{3}E\left\{E-\frac{\vartheta_1'(\alpha/\pi)}{\vartheta_1(\alpha/\pi)}+\frac{\vartheta_3'(\theta_1/2\pi)}{\vartheta_3(\theta_1/2\pi)}-\frac{\vartheta_3'(\theta_2/2\pi)}{\vartheta_3(\theta_2/2\pi)}\right\}\right], \tag{4.6}$$

$$\frac{D\sin\delta}{2\sin\gamma} = -E\left[\frac{\vartheta_2'\{(\phi+\delta)/2\pi\}}{\vartheta_2\{(\phi+\delta)/2\pi\}}-\frac{\vartheta_2'\{(\phi-\delta)/2\pi\}}{\vartheta_2\{(\phi-\delta)/2\pi\}}\right]^3, \tag{4.7}$$

$$E = \frac{\vartheta_3'\{(\delta+\alpha)/2\pi\}}{\vartheta_3\{(\delta+\alpha)/2\pi\}}-\frac{\vartheta_3'\{(\delta-\alpha)/2\pi\}}{\vartheta_3\{(\delta-\alpha)/2\pi\}}+\frac{\vartheta_3'(\theta_2/2\pi)}{\vartheta_3(\theta_2/2\pi)}-\frac{\vartheta_3'(\theta_1/2\pi)}{\vartheta_3(\theta_1/2\pi)}, \tag{4.8}$$

and

$$F = \frac{\vartheta_4^2\{(\delta+\alpha)/2\pi\}}{\vartheta_3^2\{(\delta+\alpha)/2\pi\}}+\frac{\vartheta_4^2\{(\delta-\alpha)/2\pi\}}{\vartheta_3^2\{(\delta-\alpha)/2\pi\}}. \tag{4.9}$$

5. Calculation of α, δ, ϕ and q

Starting from the above general expression for Y, we shall derive an expansion formula for the lift on the arc-aerofoil. To this end, we have to obtain the expansions of α, δ and ϕ in series of ascending q and then to obtain the expansion of q in powers of l/H.

Now, equation (2.8) can be put in the form:

$$\vartheta_3'\{(\delta+\alpha)/2\pi\}\vartheta_3\{(\delta-\alpha)/2\pi\}+\vartheta_3'\{(\delta-\alpha)/2\pi\}\vartheta_3\{(\delta+\alpha)/2\pi\}=0, \tag{5.1}$$

which, when the well-known q-expansion formula for the function $\vartheta_3(v)$ is used, becomes (5):

$$\sin\delta\cos\alpha+q\sin 2\delta+2q^3\sin 2\delta\cos 2\alpha$$
$$+q^4(3\sin 3\delta\cos\alpha+\sin\delta\cos 3\alpha)+2q^7\sin 4\delta+\cdots=0. \tag{5.2}$$

From this equation we can obtain the expansion of $\cos\alpha$ in powers of q. The result is

$$\cos\alpha = -2q\cos\delta+4q^3\cos\delta-4q^5\cos\delta(3+2\cos^2\delta)$$
$$+32q^7\cos\delta(1+3\cos^2\delta)+\cdots. \tag{5.3}$$

The relation between α and δ having thus been known, we shall next proceed to express δ in powers of q by means of equation (3.7). As mentioned already, the object of the present paper lies in the derivation of an expansion formula for Y/Y_0 in powers of l/H, which can be adequately used only when l/H is fairly small, i.e. when H is fairly large. Therefore, it suffices to deal with the second case where the circle on which the arc-aerofoil lies does not intersect the bounding wall at real points. In this case, the value of q is sufficiently small, but, owing to the extension of δ as in (2.10), the value of $\cos\delta$ becomes very large.

Thus, for convenience, we shall put $q \cos\delta = \sigma$. Then, equation (5.3) can be written as:

$$\cos\alpha = -2\sigma + 4q^2(\sigma - 2\sigma^3) + O(q^4), \tag{5.4}$$

from which we have

$$\sin\alpha = (1-4\sigma^2)^{\frac{1}{2}}\left\{1 + 8q^2\frac{\sigma^2 - 2\sigma^4}{1-4\sigma^2} + O(q^4)\right\}. \tag{5.5}$$

Next, we have, from the first equation in (3.6),

$$\sigma\cos\gamma - q\cos\beta = \frac{d^2+1}{d^2-1} q \sin\delta \sin\gamma, \tag{5.6}$$

while, after several calculations, we get, from the second equation in (3.5),

$$\frac{d^2+1}{d^2-1} = -\frac{(1-12\sigma^2+32\sigma^4) + 4q^2(1-2\sigma^2-64\sigma^6) + O(q^4)}{4q\sin\delta(1-4\sigma^2)^{\frac{1}{2}}\{(1-4\sigma^2) + 4q^2(\sigma^2+8\sigma^4) + O(q^4)\}}. \tag{5.7}$$

Eliminating $(d^2+1)/(d^2-1)$ from (5.6) and (5.7), we have an equation for determining σ in terms of q. Thus,

$$\begin{aligned} &4(1-4\sigma^2)^{\frac{1}{2}}(\sigma\cos\gamma - q\cos\beta)\{(1-4\sigma^2) + 4q^2(\sigma^2+8\sigma^4) + O(q^4)\} \\ &= -\sin\gamma\{(1-12\sigma^2+32\sigma^4) + 4q^2(1-2\sigma^2-64\sigma^6) + O(q^4)\}. \end{aligned} \tag{5.8}$$

In order to solve this equation, we assume that σ can be developed into a series of ascending q as:

$$\sigma = \sigma_0 + \sigma_1 q + \sigma_2 q^2 + \sigma_3 q^3 + \cdots, \tag{5.9}$$

where $\sigma_0, \sigma_1, \sigma_2, \sigma_3, \ldots$ are certain functions of β and γ. Substituting this series in (5.8) and equating the coefficients of the same powers of q on both sides, we get

$$4(1-4\sigma_0^2)^{\frac{1}{2}}(\sigma_0 - 4\sigma_0^3)\cos\gamma = -\sin\gamma(1-12\sigma_0^2+32\sigma_0^4), \tag{5.10}$$

$$(1-4\sigma_0^2)^{\frac{1}{2}}\{(1-16\sigma_0^2)\sigma_1\cos\gamma - (1-4\sigma_0^2)\cos\beta\} = 2\sigma_0\sigma_1(3-16\sigma_0^2)\sin\gamma, \tag{5.11}$$

$$\begin{aligned} &(1-4\sigma_0^2)^{\frac{1}{2}}\left[12\sigma_0\sigma_1\cos\beta - \cos\gamma\left\{6\sigma_0\sigma_1^2\frac{3-16\sigma_0^2}{1-4\sigma_0^2} - 4\sigma_0^3(1+8\sigma_0^2) - \sigma_2(1-16\sigma_0^2)\right\}\right] \\ &\quad = -\sin\gamma\{(1-2\sigma_0^2-64\sigma_0^6) - 3\sigma_1^2(1-16\sigma_0^2) - 2\sigma_0\sigma_2(3-16\sigma_0^2)\}, \end{aligned} \tag{5.12}$$

$$\begin{aligned} &(1-4\sigma_0^2)^{\frac{1}{2}}\left[2\cos\beta\left\{3\sigma_1^2\frac{1-8\sigma_0^2}{1-4\sigma_0^2} - 2\sigma_0^2(1+8\sigma_0^2) + 6\sigma_0\sigma_2\right\}\right. \\ &\qquad + \cos\gamma\left\{\sigma_3(1-16\sigma_0^2) - 12\sigma_0\sigma_1\sigma_2\frac{3-16\sigma_0^2}{1-4\sigma_0^2}\right. \\ &\qquad\qquad \left.\left. + 16\sigma_0^4\sigma_1\frac{7-40\sigma_0^2}{1-4\sigma_0^2} - 6\sigma_1^3\frac{1-8\sigma_0^2}{1-4\sigma_0^2} + 8\sigma_0^2\sigma_1^3\frac{3-8\sigma_0^2}{(1-4\sigma_0^2)^2} + 4\sigma_0^2\sigma_1(3+8\sigma_0^2)\right\}\right] \\ &= 2\sin\gamma\{-16\sigma_0\sigma_1^3 + 2\sigma_0\sigma_1(1+96\sigma_0^4) + 3\sigma_1\sigma_2(1-16\sigma_0^2) + \sigma_0\sigma_3(3-16\sigma_0^2)\}. \end{aligned} \tag{5.13}$$

From these equations we can determine the coefficients σ_i's successively as functions of β and γ. The values of the first four coefficients will be given below.

$$\left.\begin{aligned}
&\sigma_0 = -\frac{1}{2}\sin\frac{\gamma}{2}, \qquad \sigma_1 = \cos^2\frac{\gamma}{2}\cos\beta, \\
&\sigma_2 = -\frac{1}{2}\sin\frac{\gamma}{2}\left\{\left(1+3\cos^2\frac{\gamma}{2}\right)-6\cos^2\frac{\gamma}{2}\cos^2\beta\right\}, \\
&\sigma_3 = 2\sin^2\frac{\gamma}{2}\cos^2\frac{\gamma}{2}(4\cos^3\beta-3\cos\beta).
\end{aligned}\right\} \tag{5.14}$$

Using (5.9) together with these values of the σ_i's, we have, from (5.4) and (5.5),

$$\begin{aligned}
\cos\alpha = \sin\frac{\gamma}{2} &- 2q\cos^2\frac{\gamma}{2}\cos\beta + 2q^2\sin\frac{\gamma}{2}\cos^2\frac{\gamma}{2}\left(1-3\cos^2\beta\right) \\
&+2q^3\cos^2\frac{\gamma}{2}\left\{\left(2+3\sin^2\frac{\gamma}{2}\right)\cos\beta-8\sin^2\frac{\gamma}{2}\cos^3\beta\right\}+O(q^4),
\end{aligned} \tag{5.15}$$

$$\begin{aligned}
\sin\alpha = \cos\frac{\gamma}{2}\Big[1&+2q\sin\frac{\gamma}{2}\cos\beta-2q^2\left\{\sin^2\frac{\gamma}{2}+\left(1-3\sin^2\frac{\gamma}{2}\right)\cos^2\beta\right\} \\
&-2q^3\sin\frac{\gamma}{2}\left\{3\sin^2\frac{\gamma}{2}\cos\beta+4\left(1-2\sin^2\frac{\gamma}{2}\right)\cos^3\beta\right\}\Big]+O(q^4).
\end{aligned} \tag{5.16}$$

In a similar manner, we get from (3.8) the expansion for $\cos\phi$ in powers of q in the form:

$$\cos\phi = \mu_0+\mu_1 q+\mu_2 q^2+\mu_3 q^3+\cdots, \tag{5.17}$$

where

$$\left.\begin{aligned}
&\mu_0 = -\cos\beta, \qquad \mu_1 = 2\sin\frac{\gamma}{2}\sin^2\beta, \\
&\mu_2 = 4\sin^2\frac{\gamma}{2}\sin^2\beta\cos\beta, \\
&\mu_3 = 2\sin\frac{\gamma}{2}\sin^2\beta\left\{\left(1-2\sin^2\frac{\gamma}{2}\right)+2\left(1+\sin^2\frac{\gamma}{2}\right)\cos^2\beta\right\}.
\end{aligned}\right\} \tag{5.18}$$

Also, we have the expansion for $\sin\phi$ in the form:

$$\begin{aligned}
\sin\phi = \sin\beta\Big[1&+2q\sin\frac{\gamma}{2}\cos\beta+2q^2\sin^2\frac{\gamma}{2}\left(2\cos^2\beta-1\right) \\
&+2q^3\sin\frac{\gamma}{2}\left\{\left(1-4\sin^2\frac{\gamma}{2}\right)\cos\beta+2\left(1+\sin^2\frac{\gamma}{2}\right)\cos^3\beta\right\}\Big]+O(q^4).
\end{aligned} \tag{5.19}$$

Lastly, we get from (3.2) the expansion for l/H in ascending powers of q in the form:

$$\frac{l}{H} = 8q\cos\frac{\gamma}{2}\left\{1+2q\sin\frac{\gamma}{2}\cos\beta+O(q^2)\right\}, \tag{5.20}$$

and solving this equation for q, we have

$$q\cos\frac{\gamma}{2} = \frac{1}{8}\frac{l}{H}-\frac{1}{32}\tan\frac{\gamma}{2}\cos\beta\left(\frac{l}{H}\right)^2+O\left[\left(\frac{l}{H}\right)^3\right]. \tag{5.21}$$

6. An expansion formula for the lift

Making use of the preceding various expansions we have obtained, after lengthy calculations, expansions in powers of q for various quantities such as Y_1, Y_2, Y_3, Y_4, Y_5 and D as defined in (4.2)–(4.9). The results are

$$\begin{aligned}Y_1+Y_2 = &-256\pi^5 q^2 \cos^2\frac{\gamma}{2}\\ &\times\left\{\sin\beta\cos\beta+\tan\frac{\gamma}{2}(3-\cos^2\beta)+4\tan^2\frac{\gamma}{2}\sin\beta\cos\beta+\tan^3\frac{\gamma}{2}(1+2\cos^2\beta)\right\}\\ &+512\pi^5 q^3\cos^3\frac{\gamma}{2}\left\{\sin^2\beta\cos\beta+\tan\frac{\gamma}{2}\sin\beta(1+4\cos^2\beta)\right.\\ &\qquad+\tan^2\frac{\gamma}{2}(12\cos\beta-7\cos^3\beta)+16\tan^3\frac{\gamma}{2}\sin\beta\cos^2\beta\\ &\qquad\left.+\tan^4\frac{\gamma}{2}\left(\cos\beta+6\cos^3\beta\right)\right\}+O(q^4), \qquad (6.1)\end{aligned}$$

$$\begin{aligned}Y_3 = &-256\pi^5 q^2\cos^2\frac{\gamma}{2}\left(\sin\beta\cos\beta+\tan\frac{\gamma}{2}\sin^2\beta\right)\\ &+512\pi^5 q^3\cos^3\frac{\gamma}{2}\left\{\sin^2\beta\cos\beta+\tan\frac{\gamma}{2}\sin\beta(1+4\cos^2\beta)\right.\\ &\qquad\left.+5\tan^2\frac{\gamma}{2}\sin^2\beta\cos\beta\right\}+O(q^4), \qquad (6.2)\end{aligned}$$

$$Y_4 = 512\pi^5 q^3\cos^3\frac{\gamma}{2}\left(4\sin^2\beta\cos\beta+2\tan\frac{\gamma}{2}\sin^3\beta-2\cot\frac{\gamma}{2}\sin^3\beta\right)+O(q^4), \qquad (6.3)$$

$$\begin{aligned}Y_5 = &\,256\pi^5 q\cos\frac{\gamma}{2}\left(\tan\frac{\gamma}{2}\sin\beta+\tan^2\frac{\gamma}{2}\cos\beta\right)\\ &-256\pi^5 q^2\cos^2\frac{\gamma}{2}\left\{2\sin\beta\cos\beta+2\tan\frac{\gamma}{2}\cos^2\beta\right.\\ &\qquad\left.+6\tan^2\frac{\gamma}{2}\sin\beta\cos\beta-\tan^3\frac{\gamma}{2}\left(1-4\cos^2\beta\right)\right\}\\ &+512\pi^5 q^3\cos^3\frac{\gamma}{2}\left\{\tan\frac{\gamma}{2}\sin\beta(6+5\cos^2\beta)+\tan^2\frac{\gamma}{2}\cos\beta(4+3\cos^2\beta)\right.\\ &\qquad+4\tan^3\frac{\gamma}{2}\sin\beta(1+\cos^2\beta)\\ &\qquad\left.+\tan^4\frac{\gamma}{2}\cos\beta(3+2\cos^2\beta)\right\}+O(q^4), \qquad (6.4)\end{aligned}$$

and

$$\begin{aligned}D = &\,256\pi^4\cot\frac{\gamma}{2}\left\{q\cos\frac{\gamma}{2}\tan^2\frac{\gamma}{2}-2q^2\cos^2\frac{\gamma}{2}\tan\frac{\gamma}{2}\left(2+\tan^2\frac{\gamma}{2}\right)\cos\beta\right.\\ &\qquad\left.-q^3\cos^3\frac{\gamma}{2}\left(2+\tan^2\frac{\gamma}{2}\right)\left(2\sin^2\beta-3\tan^2\frac{\gamma}{2}\right)+O(q^4)\right\}. \qquad (6.5)\end{aligned}$$

Thus, by (4.1) we have an expansion in powers of q for the total lift Y acting on the arc-aerofoil in the form:

$$
\begin{aligned}
YD/256\pi^5 l\rho U^2 = {} & q \cos\frac{\gamma}{2}\left(\tan\frac{\gamma}{2}\sin\beta+\tan^2\frac{\gamma}{2}\cos\beta\right) \\
& -q^2\cos^2\frac{\gamma}{2}\left(4\sin\beta\cos\beta+4\tan\frac{\gamma}{2}+10\tan^2\frac{\gamma}{2}\sin\beta\cos\beta\right. \\
& \qquad \left.+6\tan^3\frac{\gamma}{2}\cos^2\beta\right) \\
& +2q^3\cos^3\frac{\gamma}{2}\left\{6\sin^2\beta\cos\beta+\tan\frac{\gamma}{2}\sin\beta(10+11\cos^2\beta)\right. \\
& \qquad +\tan^2\frac{\gamma}{2}\cos\beta(21-9\cos^2\beta)+\tan^3\frac{\gamma}{2}\sin\beta(4+20\cos^2\beta) \\
& \qquad \left.+\tan^4\frac{\gamma}{2}\cos\beta(4+8\cos^2\beta)-2\cot\frac{\gamma}{2}\sin^3\beta\right\}+O(q^4), \qquad (6.6)
\end{aligned}
$$

or, dividing the both sides by D as given by (6.5),

$$
\begin{aligned}
Y/\pi l\rho U^2 = {} & \left(\sin\beta+\tan\frac{\gamma}{2}\cos\beta\right)-4q\cos\frac{\gamma}{2}\left(\sin\beta+\tan\frac{\gamma}{2}\cos\beta\right)^2 \\
& +q^2\cos^2\frac{\gamma}{2}\left\{\sin\beta(16-12\cos^2\beta)+\tan\frac{\gamma}{2}\cos\beta(30-28\cos^2\beta)\right. \\
& \qquad +\tan^2\frac{\gamma}{2}\sin\beta(5+24\cos^2\beta) \\
& \qquad \left.+\tan^3\frac{\gamma}{2}\cos\beta(5+8\cos^2\beta)\right\}+O(q^3). \qquad (6.7)
\end{aligned}
$$

Now, it is naturally expected that in the limit when the distance H of the mid-point of the chord from the wall becomes infinitely large, the above expression (6.7) would degenerate into the well-known formula for the lift Y_0 acting on a circular-arc aerofoil placed in an unbounded stream, the length as well as the angle of incidence of the aerofoil being considered to be of the same values in both cases. In effect, we easily find that

$$
Y_0=\lim_{q\to 0} Y=\pi l\rho U^2\left(\sin\beta+\tan\frac{\gamma}{2}\cos\beta\right), \qquad (6.8)
$$

which is nothing else than the well-known formula for the lift acting on an arc-aerofoil placed in an unlimited stream.

Combining (6.7) with (6.8) we have

$$
\begin{aligned}
\frac{Y}{Y_0} = {} & 1-4q\cos\frac{\gamma}{2}\left(\sin\beta+\tan\frac{\gamma}{2}\cos\beta\right) \\
& +q^2\cos^2\frac{\gamma}{2}\left\{(14-12\cos^2\beta)+16\tan\frac{\gamma}{2}\sin\beta\cos\beta\right. \\
& \qquad \left.+\tan^2\frac{\gamma}{2}(5+8\cos^2\beta)+\frac{2\sin\beta}{\sin\beta+\tan\frac{\gamma}{2}\cos\beta}\right\}+O(q^3), \qquad (6.9)
\end{aligned}
$$

and if use is made of (5.21), we obtain ultimately an expansion formula for Y/Y_0 in powers of l/H in the form:

$$\frac{Y}{Y_0} = 1-\frac{1}{2}\left(\sin\beta+\tan\frac{\gamma}{2}\cos\beta\right)\frac{l}{H}$$
$$+\frac{1}{64}\left\{\left(8-6\cos 2\beta\right)+12\tan\frac{\gamma}{2}\sin 2\beta\right.$$
$$\left.+\tan^2\frac{\gamma}{2}(13+8\cos 2\beta)+\frac{2\sin\beta}{\sin\beta+\tan\frac{\gamma}{2}\cos\beta}\right\}\left(\frac{l}{H}\right)^2, \qquad (6.10)$$

where the third and higher powers of l/H have been neglected. Except for the difference in notation this expansion formula is in perfect agreement with the formula (1.1), which, as mentioned in the introduction, has been obtained by the junior writer by extending Green's new method (**3**).

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