Memoirs of the College of Science, University of Kyoto, Series A,
Vol. XXVI, No. 3, Article 18, 1950.

# Note on the Drag experienced by a Circular Cylinder moving in a Viscous Fluid at Small Reynolds Numbers.

By

**S. Tomotika** and **T. Aoi**

(Received October 1, 1950)

## 1. Introduction

In two recent papers (**1, 2**) we have obtained the exact analytical solution of Oseen's linearized equations of motion for the case of a circular cylinder moving through an incompressible viscous fluid at small Reynolds numbers, and with the aid of this exact solution we have derived the general expression for the drag experienced by the cylinder, making special reference to the pressure drag and frictional drag separately. The values of the drag coefficient for several small Reynolds numbers have been computed numerically and it has been found that the calculated values are in good agreement with the experimental values. In calculating the values of the drag coefficient, however, a system of simultaneous linear algebraic equations has to be solved numerically so that the work is rather cumbersome. It is therefore desirable to derive, if possible, an approximate formula which may be conveniently used for computing the values of the drag, namely an expansion formula in powers of the Reynolds number*. Such an expansion formula has recently been obtained by us, starting from our general expression for the drag of the circular cylinder, though the result is unpublished yet.

Quite recently Sidrak (**4**) has made, on the basis of Oseen's linearized equations of motion as in our calculation, a similar investigation on the drag on a circular cylinder in a stream of viscous liquid at small Reynolds numbers, and has also derived an expansion formula for the drag. Although his analysis is slightly different from ours, it is naturally expected that the expansion formula derived by him should be

* As is well known, a corresponding expansion formula in powers of the Reynolds number for the drag of a sphere was obtained by Goldstein as early as 1929 (**3**).

coincident with our formula. No agreement can be found between the two formulae, however. We have therefore examined carefully Sidrak's analysis and have found out that there are unfortunately some mistakes in the course of his analysis, which give rise to an erroneous result.

The object of the present note lies in the derivation of a correct expansion formula by starting from the general expression given by Sidrak himself for the drag on a circular cylinder.

## 2. Sidrak's general expression for the drag on a circular cylinder

For convenience of reference we first reproduce briefly, without change in notations, the general expression for the drag on a circular cylinder which has been obtained by Sidrak by applying Filon's general analysis (5) directly to the motion of the circular cylinder.

We assume that a circular cylinder moves with constant velocity $U$ in the positive direction of the axis of $x$, and let $u$ and $v$ denote the rectangular components of the perturbation velocity. In terms of the polar coordinates $(r, \theta)$, these components are given by

$$\left.\begin{aligned} u &= -\delta\frac{\cos\theta}{r} + \sum_{n=1}^{\infty}\frac{nb_n\cos(n+1)\theta}{r^{n+1}} \\ &\quad + e^{kr\cos\theta}\sum_{n=0}^{\infty}a_n\Big[K_n(kr)\cos n\theta + K_{n+1}(kr)\cos(n+1)\theta\Big], \\ v &= -\delta\frac{\sin\theta}{r} + \sum_{n=1}^{\infty}\frac{nb_n\sin(n+1)\theta}{r^{n+1}} \\ &\quad + e^{kr\cos\theta}\sum_{n=0}^{\infty}a_n\Big[K_{n+1}(kr)\sin(n+1)\theta - K_n(kr)\sin n\theta\Big], \end{aligned}\right\} \quad (1)$$

where $k = U/2\nu$, $\nu$ being the kinematic coefficient of viscosity of the fluid, and $K_n$ is the modified Bessel function. The constants of integration $\delta$, $a_n$ and $b_n$ are to be determined by the boundary conditions at the surface of the cylinder. If we denote the radius of the cylinder by $a$, the said conditions are

$$u = -U, \quad v = 0$$

at $r = a$. These conditions give the relations between the constants $\delta$, $a_n$ and $b_n$, and further it is found that the $a_n$'s can be determined by solving the following equations*:

$$\sum_{n=0}^{\infty} a_n \alpha_{n,m} = \begin{cases} -U & (m=0), \\ 0 & (m=1, 2, \cdots\cdots), \end{cases} \tag{2}$$

where

$$\alpha_{n,m} = K_n I_{n-m} + K_{n+1} I_{n+m+1},$$

the argument of both the modified Bessel functions $K_n$ and $I_n$ being $ka$.

As is well known, Filon (5) has discussed the forces on a cylinder of any shape and has shown that the drag is associated with a particular term in the solution of Oseen's equations, which corresponds to an inward flow along the tail and a compensating outward flow across a large contour surrounding the cylinder. In the case of the circular cylinder now under consideration, the drag on the cylinder is given by†

$$D = -4\pi\mu \sum_{n=0}^{\infty} a_n, \tag{3}$$

where $\mu$ is the coefficient of viscosity of the fluid. The constants $a_n$'s are determined by solving the above system of simultaneous algebraic equations (2) and the general expression for the drag on the circular cylinder can thus be obtained.

## 3. Derivation of an expansion formula for the drag

We now proceed to the derivation of an expansion formula, starting from the general expression (3) for the drag on the cylinder. As mentioned above, the drag is given by (3) in terms of the $a_n$'s which are to be found by solving (2). Theoretically, the solution of (2) is to be achieved by means of infinite determinants; practically, however, to find the solution numerically for any given value of $ka$, we solve a finite number of equations. Thus, for the first approximation, we put $a_1$, $a_2$, $a_3$, and so on, equal to zero, and solve the first equation for $a_0$. For the second approximation, we put $a_2$, $a_3$, and so on, equal to zero, and solve the first two equations for $a_0$ and $a_1$, and so on.

For our present purpose of deriving an expansion formula for the drag, we have first to express the coefficients $\alpha_{n,m}$'s in power series of

---

* These equations should be compared with the similar equations (37) in our previous paper (1).

† It will be seen that this general expression for the drag is of a similar form to ours. See, formula (47) in our previous paper (1).

$ka$ by using the expansions of the modified Bessel functions $K_n$ and $I_n$*.

Writing

$$\Omega = \gamma + \log(\tfrac{1}{2}ka)$$

for the moment where $\gamma = 0.57721\cdots$ is Euler's constant, we get the following results:†

$$\begin{aligned}
\alpha_{0,0} &= K_0 I_0 + K_1 I_1 \\
&= \frac{1}{2} + \frac{3}{16}(ka)^2 + \frac{13}{384}(ka)^4 + \cdots\cdots \\
&\qquad - \Omega\left\{1 + \frac{1}{2}(ka)^2 + \frac{1}{32}(ka)^4 + \cdots\cdots\right\}, \\
\alpha_{0,1} &= K_0 I_1 + K_1 I_2 \\
&= \frac{1}{8}ka + \frac{5}{48}(ka)^3 + \cdots\cdots \\
&\qquad - \Omega\left\{\frac{1}{2}ka + \frac{1}{8}(ka)^3 + \cdots\cdots\right\}, \\
\alpha_{0,2} &= K_0 I_2 + K_1 I_3 \\
&= \frac{1}{48}(ka)^2 + \cdots - \Omega\left\{\frac{1}{8}(ka)^2 + \cdots\cdots\right\}; \\
\alpha_{1,0} &= K_1 I_1 + K_2 I_2 \\
&= \frac{3}{4} - \frac{5}{48}(ka)^2 + \cdots + \Omega\left\{\frac{1}{4}(ka)^2 + \cdots\cdots\right\}, \\
\alpha_{1,1} &= K_1 I_0 + K_2 I_3 \\
&= \frac{1}{ka} + \frac{1}{24}ka + \cdots + \Omega\left\{\frac{1}{2}ka + \cdots\cdots\right\}, \\
\alpha_{1,2} &= K_1 I_1 + K_2 I_4 \\
&= \frac{1}{2} + \cdots\cdots\cdots\cdots; \\
\alpha_{2,0} &= K_2 I_2 + K_3 I_3 \\
&= \frac{5}{12} + \cdots\cdots\cdots\cdots,
\end{aligned}$$

* Throughout our calculations the series definitions for the modified Bessel functions $K_n$, $I_n$ are those adopted by G. N. Watson's '*Theory of Bessel Functions*' (Cambridge, 1922).

† The corresponding expressions are not given in Sidrak's paper (4).

$$\alpha_{2,1} = K_2 I_1 + K_3 I_4$$

$$= \frac{1}{ka} + \cdots\cdots\cdots\cdots ,$$

$$\alpha_{2,2} = K_2 I_0 + K_3 I_5$$

$$= \frac{2}{(ka)^2} + \cdots\cdots\cdots\cdots .$$

Assuming $ka$ to be small we can easily see that the orders of $a_0$, $a_1$, $a_2$, $\cdots$ are 1, $(ka)^2$, $(ka)^4$, $\cdots$ respectively. Therefore our method of approximation is justified.

As usual we introduce here the non-dimensional drag coefficient $C_D$ defined by

$$C_D = \frac{D}{\rho U^2 d}, \tag{4}$$

where $\rho$ is the density of the fluid and $d$ the diameter of the cylinder so that $d = 2a$. Then, the general expression for $C_D$ becomes

$$C_D = -\frac{4\pi}{UR}\sum_{n=0}^{\infty} a_n , \tag{5}$$

where $R = 4ka = Ud/\nu$ is the Reynolds number.

We have obtained the first, second and third approximate expressions for $C_D$ respectively. The first approximation gives

$$C_D = \frac{4\pi}{RA}, \tag{6}$$

where we have written $A$ for $\frac{1}{2} - \Omega$ so that

$$A = \frac{1}{2} - \gamma - \log\frac{R}{8} = 2.0022 - \log R . \tag{7}$$

Formula (6) is nothing but the well-known result obtained by Lamb.

Next, the second approximation yields an expansion formula of the form:

$$C_D = \frac{4\pi}{RA}\left[1 - \frac{1}{A}\left(A^2 - \frac{1}{2}A + \frac{5}{16}\right)\frac{R^2}{32}\right]. \tag{8}$$

Further, the third approximation gives*

* Sidrak has gone as far as the fourth approximation, but it will be seen in the subsequent section that the third approximation is quite sufficient for computing the values of $C_D$ provided that the Reynolds number $R$ is less than 4.

$$C_D = \frac{4\pi}{RA}\left[1 - \frac{1}{A}\left(A^2 - \frac{1}{2}A + \frac{5}{16}\right)\frac{R^2}{32} - \frac{1}{A^2}\left(A^4 - \frac{1}{3}A^3 + \frac{7}{72}A - \frac{25}{256}\right)\frac{R^4}{32^2}\right]. \qquad (9)$$

It is found that this formula is in accordance with that obtained by us previously from our general expression (47) for the drag in our former paper (**1**).*

## 4. Numerical computations

It will be important to estimate the range of values of the Reynolds number in which our expansion formula (9) just obtained is applicable. Such an estimation can be made by examining the convergency of the expansion formula and also by comparing the values of the drag coefficient calculated by the expansion formula (9) with those calculated numerically with the aid of the general expression (5).

As for the convergency of our formula, it is first seen that in the neighbourhood of $R = 7.4$ at which $A$ becomes zero, our formula is evidently divergent. In Table I the numerical values of the three terms in the square brackets in the formula (9) are given for various values of the Reynolds number $R$ less than 7. The third column gives the values of the second term:

$$-\frac{1}{A}\left(A^2 - \frac{1}{2}A + \frac{5}{16}\right)\frac{R^2}{32},$$

while the fourth column gives the values of the third term:

$$-\frac{1}{A^2}\left(A^4 - \frac{1}{3}A^3 + \frac{7}{72}A - \frac{25}{256}\right)\frac{R^4}{32^2}.$$

* The corresponding expression obtained by Sidrak will be given here, for comparison, though it is not correct. Thus,

$$C_D = \frac{4\pi}{RA}\left[1 - \left(\frac{R}{8}\right)^2\left(2A + \frac{9}{8A} - \frac{\gamma}{A} - 1\right) - \left(\frac{R}{8}\right)^4\left(4A^2 + 4\gamma A - \frac{9}{32A^2} + \frac{9\gamma}{4A^2} - \frac{\gamma^2}{A^2} - \gamma - \frac{\gamma}{A} + \frac{8}{9A} - \frac{10}{3A} + \frac{1}{2}\right) - \cdots\cdots\right],$$

$\gamma = 0.57721\cdots$ being as before Euler's constant.

It will be seen that our formula (9) begins to diverge at $R = 4$, below which, however, our formula converges rather rapidly.

TABLE I

| $R$ | 1st term | 2nd term | 3rd term |
|---|---|---|---|
| 0.4 | 1 | − 0.01 | − 0.00 |
| 0.6 | 1 | − 0.02 | − 0.00 |
| 0.8 | 1 | − 0.04 | − 0.00 |
| 1 | 1 | − 0.05 | − 0.00 |
| 2 | 1 | − 0.13 | − 0.02 |
| 3 | 1 | − 0.21 | − 0.04 |
| 4 | 1 | − 0.31 | − 0.02 |
| 5 | 1 | − 0.54 | 0.22 |
| 6 | 1 | − 1.34 | 2.24 |
| 7 | 1 | − 7.82 | 68.21 |

The values of the drag coefficient calculated respectively by the first, second and third approximate formulae are given in Table II and are shown graphically in Fig. 1. In the last column of this table are also shown the values of the drag coefficient calculated numerically by the general formula (5).* It will be seen that for the values of the Reynolds number less than 4, our third approximate expansion formula (9) gives, with sufficient accuracy, the values of the drag coefficient of the circular cylinder.†

TABLE II Values of $C_D$

| $R$ | 1st approx. | 2nd approx. | 3rd approx. | Calculated by (5) |
|---|---|---|---|---|
| 0.4 | 10.76 | 10.63 | 10.63 | 10.63 |
| 0.6 | 8.33 | 8.13 | 8.13 | 8.13 |
| 0.8 | 7.06 | 6.80 | 6.78 | 6.78 |
| 1 | 6.28 | 5.95 | 5.93 | 5.93 |
| 2 | 4.80 | 4.17 | 4.07 | 4.04 |
| 3 | 4.64 | 3.66 | 3.47 | 3.39 |
| 4 | — | 3.51 | 3.42 | 2.92 |

* The values of the drag coefficient calculated numerically by the general formula (5) are exactly the same as those calculated by formula (48) in our previous paper (**1**). See Table I there.

† It is merely accidental that the values of the drag coefficient computed by Sidrak by his own expansion formula differ only slightly from those calculated by our third approximate formula. See Table I in his paper (4).

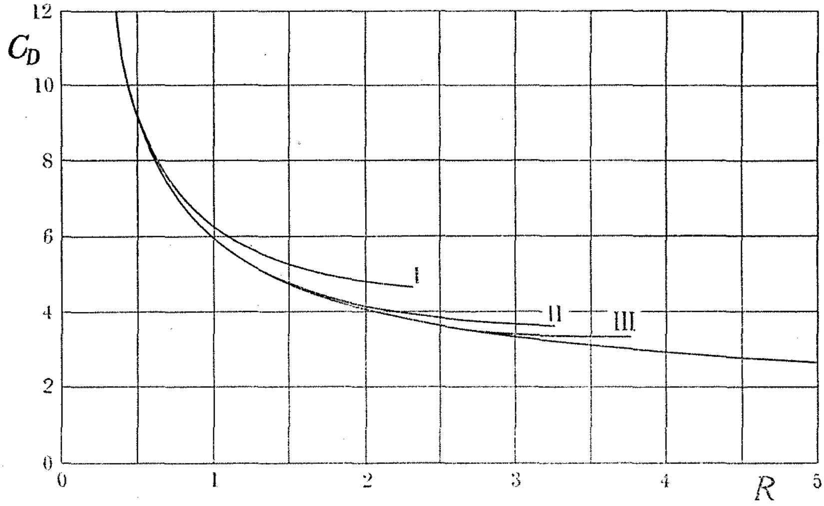


Fig. 1.

## 5. Summary and conclusion

In the present note an expansion formula correct to the fourth power of the Reynolds number $R$ is derived by starting from the general expression for the drag on a circular cylinder obtained recently by Sidrak, and it is found that the formula thus obtained agrees with that obtained by us previously, which is not yet published.

Sidrak himself has also derived a similar expansion formula for the drag coefficient by making use of his general formula. Notwithstanding that our analysis is developed along the same lines as Sidrak's, no agreement is found between our expansion formula and the corresponding Sidrak's formula. By careful examination of Sidrak's analysis, it has been found that his result is unfortunately erroneous. It seems that the source of error lies in his use of incorrect expansions for the modified Bessel function $K_n$.

Some numerical discussions are made on the values of the drag coefficient by using our expansion formula, and the range of values of the Reynolds number $R$ is estimated in which our expansion formula is applicable. It is thus found that our expansion formula correct to the order of $R^4$ can be used with sufficient accuracy, provided that the Reynolds number assumes the values less than 4.

## 6. Acknowledgements

The writers wish to express their cordial thanks to the Ministry of Education for a grant in aid for fundamental scientific research.

REFERENCES

**1.** S. Tomotika and T. Aoi, The steady flow of viscous fluid past a sphere and circular cylinder at small Reynolds numbers. Quart. Journ. Mech. and Applied Math., Oxford, **3** (1950), 140—161.

**2.** S. Tomotika and T. Aoi, The pressure distributions on the surface of an obstacle in a running viscous fluid at small Reynolds numbers. This Memoirs, **26** (1950), 9—19.

**3.** S. Goldstein, The steady flow of viscous fluid past a fixed spherical obstacle at small Reynolds numbers. Proc. Roy. Soc. London, A **123** (1929), 225—235.

**4.** S. Sidrak, The drag on a circular cylinder in a stream of viscous liquid at small Reynolds numbers. Proc. Roy. Irish Acad. Dublin, **53** (1950), 17—30.

**5.** L. N. G. Filon, The forces on a cylinder in a stream of viscous fluid. Proc. Roy. Soc. London, A **113** (1926), 7—27.

Physical Institute,
Faculty of Science,
University of Kyoto.