On the Effect of a Conducting Screen with Concentric Aperture in the Circular Wave Guide

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By

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1. General description

When there is a thin conducting screen with a concentric circular aperture in the circular wave guide (see Fig. 1), the screen behaves like a shunt reactance inserted in the transmission line. We shall discuss the behaviour of such a screen in the present paper.

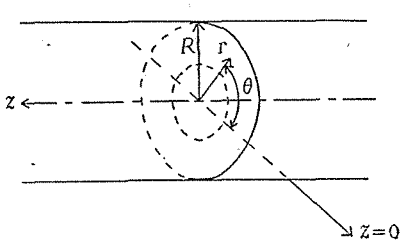

Fig. 1.

Now, the symmetrical E-type wave is taken to be incident waves in the circular wave guide of inner radius R, i. e., we assume that the incident waves are given, in M. K. S. units, by

$$\left.\begin{aligned} E_z^{\pm m} &= J_0(\lambda_m r)\, e^{i(\omega t \mp h_m z)}, \\ E_r^{\pm m} &= \pm i \frac{h_m}{\lambda_m} J_1(\lambda_m r)\, e^{i(\omega t \mp h_m z)}, \\ E_\theta^{\pm m} &= 0, \\ H_z^{\pm m} &= 0, \\ H_r^{\pm m} &= 0, \\ H_\theta &= \mp \frac{\varepsilon_0 \omega}{h_m} E_r^{\pm m}, \end{aligned}\right\} \qquad (1)$$

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where J's are cylinder functions of the first kind and $\lambda_1 = \xi_1/R$, $\lambda_2 = \xi_2/R, \cdots\cdots$, ($\xi_1$, ξ_2, $\xi_3, \cdots\cdots$ being the roots of $J_0(x) = 0$), while $h_m{}^2 = k^2 - \lambda_m{}^2$ with $\omega^2 \varepsilon_0 \mu_0 = k^2$.

In compliance with the mth mode of incident waves and with the axial symmetry of this transmission system, E_{0m} waves are supposed to be excited and therefore E_r depends upon r alone in the aperture. Hence, we can assume in the aperture, which is situated at $z = 0$, that

$$(E_r)_{z=0} = \psi(r), \tag{2}$$

while on the screen outside the aperture, we have

$$E_r = 0 .$$

Therefore we obtain the following equations at $z = 0$; namely, on the incident waves side of the screen ($z < 0$),

$$E_r^{(-)} = E^{+m} + \sum_{n=1}^{\infty} a_n^{(-)} E_r^{-n}, \tag{3}$$

and, beyond the screen ($z > 0$),

$$E_r^{(+)} = \sum_{n=1}^{\infty} a_n^{(+)} E_r^{+n}. \tag{4}$$

Now, making use of the orthogonality of $J_1(\lambda_n r)$, $a_n^{(\pm)}$ are expressed in terms of $\psi(r)$ by (3) and (4). Particularly, when $n = m$,

$$a_m^{(-)} = \frac{\lambda_m}{\gamma_m h_m}\left(\frac{h_m \gamma_m}{\lambda_m} + i\varphi_m\right), \tag{5}$$

and when $n \neq m$,

$$a_n^{(-)} = i\varphi_n \frac{\lambda_n}{\gamma_n h_n}, \tag{6}$$

$$a_n^{(+)} = -\, i\varphi_n \frac{\lambda_n}{\gamma_n h_n}, \tag{7}$$

where

$$\gamma_n = \int_0^R (J_1(\lambda_n r))^2 r\, dr, \qquad \varphi_n = \int_0^\rho \psi(r) J_1(\lambda_n r) r\, dr .$$

Next, $H_\theta^{(+)}$ and $H_\theta^{(-)}$ are expressed in terms of (5), (6) and (7) as follows (in the following, field quantities represent the values at $z = 0$) :

$$H_\theta^{(-)} = -\frac{\varepsilon_0\omega}{h_m}E_r^{+m} + \sum_{n=1}^{\infty} a_n^{(-)}\frac{\varepsilon_0\omega}{h_n}E_r^{-n}$$

$$= -i\varepsilon_0\omega\left\{\frac{J_1(\lambda_m r)}{\lambda_m} + \sum_{n=0}^{\infty}\frac{a_n^{(-)}}{\lambda_n}J_1(\lambda_n r)\right\}$$

$$= -i\varepsilon_0\omega\left\{2\frac{J_1(\lambda_m r)}{\lambda_m} + i\int_0^\rho \psi(r')G(r', r)r'\,dr'\right\}, \tag{8}$$

and

$$H_\theta^{(+)} = -\varepsilon_0\omega\int_0^\rho \psi(r')G(r', r)r'\,dr', \tag{9}$$

with

$$G(r', r) = \sum_{n=1}^{\infty}\left\{J_1(\lambda_n r')J_1(\lambda_n r)\right\}/\gamma_n h_n .$$

Since the screen is thin, it has been assumed in the above that the relation $E_r^{(+)} = E_r^{(-)}$ holds in the region $0 \leqq r \leqq \rho$, where ρ is the radius of the aperture.

Similarly, we have the relation $H_\theta^{(+)} = H_\theta^{(-)}$ only in the region $0 \leqq r \leqq \rho$. Then, we obtain, in the interval $0 \leqq r \leqq \rho$,

$$\frac{iJ_1(\lambda_m r)}{\lambda_m} = \int_0^\rho \psi(r')G(r', r)r'\,dr'. \tag{10}$$

Multiplying both sides with $\psi(r)$, and intergrating from 0 to ρ, we have

$$\int_0^\rho \frac{i\psi(r)J_1(\lambda_m r)}{\lambda_m}r\,dr = \int_0^\rho\int_0^\rho \psi(r')G(r', r)\psi(r)r'\,dr'r\,dr. \tag{11}$$

If the solution $\psi(r)$ of the integral equation (10) be found, we can compute $a_n^{(-)}$ and $a_n^{(+)}$, and obtain the field quantities exactly. But it is very difficult to find the exact solution $\psi(r)$ and its numerical integration is also troublesome (**1**). Mayer (**2**) has calculated the proper frequency of the nosed-in cavity, by a method somewhat similar to ours. Hansen (**3**) has also treated the similar problem about the cylindrical cavity, his expression corresponding to our expression (10) being however somewhat different from ours. It is because of the fact that we deal with the progressive and reflected waves in the guide, while Hansen considered the stationary waves in the cavity.

2. Behavior of a small aperture

For arbitrary magnitude of ρ, it is difficult to solve (10) either analytically or numerically. We shall therefore discuss the case of $\rho \ll 2\pi/k$. In such a case, it will be seen by analogy in the electrostatic case that $\psi(r)$ will have a singularity like $1/\sqrt{\rho^2 - r^2}$ in the neighbourhood of the periphery of the aperture, and due to the axial symmetry, we have

$$\psi(0) = 0.$$

Hence it will be justified to put

$$\psi(r) = br/\sqrt{\rho^2 - r^2},$$

where b is the unknown constant to be determined by (11).

Substituting this into (11), we obtain

$$b = \frac{i}{\lambda_m}\alpha_m \Big/ \sum_{n=1}^{\infty} \frac{\alpha_n^2}{\gamma_n h_n}, \tag{12}$$

with

$$\alpha_n \equiv \int_0^{\rho} \frac{r^2}{\sqrt{\rho^2 - r^2}} J_1(\lambda_n r)\, dr\,.$$

Hence, if follows that

$$\varphi_n = \frac{i}{\lambda_m}\alpha_m \alpha_n \Big/ \sum_{n'=1}^{\infty} \frac{\alpha^2_{n'}}{\gamma_{n'} h_{n'}}\,. \tag{13}$$

The convergency of series in (12) and (13) will be discussed later.

Now, we assume that the mode of the incident waves is $m = 1$ and other modes in both reflected waves and transmitted waves are cut off. Then, we obtain the reflection coefficient $a_1^{(-)}$ and the transmission coefficient $a_1^{(+)}$ by the expressions:

$$a_1^{(-)} = 1 + i\frac{\lambda_1}{h_1\gamma_1}\varphi_1 = 1 - \left(\frac{\alpha_1^2}{\gamma_1 h_1} \Big/ \sum_{n=1}^{\infty} \frac{\alpha_n^2}{\gamma_n h_n}\right)$$

$$= 1 \Big/ \left\{1 + i\left(-\frac{A}{B}\right)\right\}, \tag{14}$$

$$a_1^{(+)} = i\left(-\frac{A}{B}\right) \Big/ \left\{1 + i\left(-\frac{A}{B}\right)\right\} \tag{14'}$$

respectively, where

$$A = \alpha_1^2/\gamma_1 h_1, \qquad B = \sum_{n=2}^{\infty} \alpha_n^2/h_n'\gamma_n,$$

h_2', h_3',$\cdots\cdots$ being positive numbers and connected with h_2, h_3,$\cdots\cdots$ as $h_2 = -ih_2'$, $h_3 = -ih_3'$,$\cdots\cdots$.

Comparing (14) and (14′) with the case of transmission line, it will be seen that they correspond to the reflection and transmission coefficients in the transmission line with the shunt reactance iX_p inserted as shown in Fig. 2, where z_0 is its characteristic impedance.

From the comparison, we get

$$-A/B = 2X_p/z_0 = 2\xi_p,$$

and so the relative impedance is

$$\xi_p = -A/2B, \qquad (15)$$

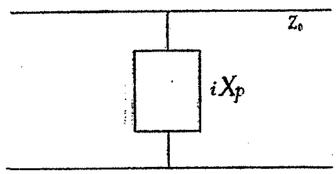

Fig. 2.

or the corresponding relative susceptance is

$$\eta_p = 2B/A. \qquad (15')$$

As A and B are both positive, it follows that $\xi_p < 0$. Therefore, it is found that the small circular aperture behaves like a capacitive reactance.

3. Detailed calculations

We shall estimate the series in the denominators of (12) and (13). The expression α_n is expressed by Sonine's first integral (4), by putting $\nu = -\frac{1}{2}\,\mu = 1$:

$$\alpha_n = \rho^3 \frac{J_{\frac{3}{2}}(\lambda_n\rho)\,\Gamma(\frac{1}{2})}{(\lambda_n\rho)^{\frac{1}{2}}\sqrt{2}} = \rho^3\sqrt{\frac{\pi}{2\lambda_n\rho}}\,J_{\frac{3}{2}}(\lambda_n\rho) = \rho^3 j_1(\lambda_n\rho),$$

where

$$j_1(x) = \sqrt{\pi/2x}\,J_{\frac{3}{2}}(x) = \sin x/x^2 - \cos x/x.$$

Since $h_n^2 = k^2 - (\xi_n/R)^2$ and $\gamma_n = \frac{1}{2}R^2\{J_1(\xi_n)\}^2$, the following approximate formulas are obtained, when $\xi_n \to \infty$,

$$h_n \sim -i\xi_n/R, \qquad (16)$$

and

$$J_1(\xi_n) \sim \sqrt{\frac{2}{\pi\xi_n}}\cos\left(\xi_n - \frac{1}{2}\pi - \frac{1}{4}\pi\right). \qquad (17)$$

Therefore, we get

$$\frac{\alpha_n^2}{\gamma_n h_n} = \frac{2\rho^6}{R^2}\frac{1}{h_n}\left\{\frac{j_1(\xi_n\rho/R)}{J_1(\xi_n)}\right\}^2 \sim \frac{i\pi\rho^4 R}{\xi_n^2}. \tag{18}$$

Since $\xi_n \sim n\pi$ (when $n \to \infty$),

$$\frac{\alpha_n^2}{\gamma_n h_n} \sim i\frac{\rho^4 R}{\pi}\frac{1}{n^2}.$$

From this consideration, it has been proved that the denominator of (13) converges as the series $\sum 1/n^2$.

When we approximate (14) by the sum of finite number of terms, the order of magnitude of the remainder can be estimated by use of the relation $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$, as shown in the following table:

n_0	$\sum_{n=n_0}^{\infty} 1/n^2$	$\sum_{n=1}^{n=n_0-1} 1/n^2$
2	0.64	1.0
3	0.39	1.25
4	0.28	1.36
5	0.24	1.40
6	0.22	1.42
7	0.20	1.44
8	0.18	1.46
9	0.17	1.47
10	0.16	1.48

Next, we shall compute the η_p of (15′), which becomes

$$\frac{1}{2}\eta_p = \frac{B}{A} = \sum_{n=2}^{\infty}\frac{1}{h_n'}\left\{\frac{j_1(\xi_n\rho/R)}{J_1(\xi_n)}\right\}^2 \Big/ \frac{1}{h_1}\left\{\frac{j_1(\xi_1\rho/R)}{J_1(\xi_1)}\right\}^2. \tag{19}$$

We substitute (16) ~ (18) into the above expression, and remembering that $\rho/R \ll 1$, we replace the denominator $\{j_1(\xi_1\rho/R)\}^2$ in the expression (19) for $\frac{1}{2}\eta_p$ by $\{\frac{1}{3}(\xi_1\rho/R)\}^2$. Then, we obtain

$$\frac{1}{2}\eta_p \fallingdotseq \sum_{n=1}^{\infty}\frac{9\pi h_1 R}{2\xi_1^2 x^2}\{J_1(\xi_1)\}^2\left(\frac{\sin\xi_n x}{\xi_n^2 x^2} - \frac{\cos\xi_n x}{\xi_n x}\right)^2, \tag{20}$$

where $\rho/R \equiv x$. To the right-hand side we have added a term with $n = 1$, but this term can give only negligibly small contribution.

Since $\xi_n \sim n\pi$, we may write

$$S = \sum_{n=1}^{\infty}\left(\frac{\sin\xi_n x}{(\xi_n x)^2} - \frac{\cos\xi_n x}{\xi_n x}\right)^2 = \sum_{n=1}^{\infty}\left(\frac{\sin n\pi x}{(n\pi x)^2} - \frac{\cos n\pi x}{n\pi x}\right)^2.$$

Splitting S up into three parts S_{I}, S_{II} and S_{III} as

$$S_{\text{I}} = \sum_{n=1}^{\infty}\frac{\sin^2 n\pi x}{(n\pi)^4 x^4} = \frac{1}{(\pi x)^4}\sum_{n=1}^{\infty}\frac{\sin^2 n\pi x}{n^4},$$

$$S_{\text{II}} = \sum_{n=1}^{\infty}\frac{\sin n\pi x\cos n\pi x}{(n\pi)^3 x^3} = \frac{1}{(\pi x)^3}\sum_{n=1}^{\infty}\frac{\sin n\pi x\cos n\pi x}{n^3},$$

$$S_{\text{III}} = \sum_{n=1}^{\infty}\frac{\cos^2 n\pi x}{(n\pi)^2 x^2} = \frac{1}{(\pi x)^2}\sum_{n=1}^{\infty}\frac{\cos^2 n\pi x}{n^2},$$

and modifying them a little, we get

$$S_{\text{I}} = \frac{1}{2(\pi x)^4}\left\{\sum_{n=1}^{\infty}\frac{1}{n^4} - \sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^4}\right\},$$

$$S_{\text{II}} = \frac{1}{2(\pi x)^3}\left\{\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n^3}\right\},$$

$$S_{\text{III}} = \frac{1}{2(\pi x)^2}\left\{\sum_{n=1}^{\infty}\frac{1}{n^2} + \sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{n^2}\right\}.$$

Using Bernoulli's polynomials

$$B_{2m}(x) = (-1)^{m+1}2(2m)!\sum_{n=1}^{\infty}\frac{\cos 2n\pi x}{(2n\pi)^{2m}},$$

$$B_{2m+1}(x) = (-1)^{m+1}2(2m+1)!\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{(2n\pi)^{2m+1}},$$

we obtain

$$S_{\text{I}} = \frac{1}{6x^4}(x^4 - 2x^3 + x^2),$$

$$S_{II} = \frac{1}{3x^3}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{6}x\right),$$

$$S_{III} = \frac{1}{2x^2}(x - x^2).$$

Since $x \ll 1$, the predominating terms in S_I and S_{II} are both of the order of $1/x^2$, and that in S_{III} is of the order of $1/x$. Hence,

$$S_I \fallingdotseq \frac{1}{6x^2}, \qquad S_{II} \fallingdotseq \frac{1}{6x^2}, \qquad S_{III} \fallingdotseq \frac{1}{2x}.$$

Therefore, we get $S \fallingdotseq 1/(3x^2)$ and consequently

$$\frac{1}{2}\eta_p \fallingdotseq \frac{3\pi h_1 R}{2\xi_1^2}\left\{J_1(\xi_1)\right\}^2 \frac{1}{x^4}.$$

If we let λ_g and λ_c represent the guide wave-length and the cut off wave-length of the incident waves E_{01} respectively, i. e., $h_1 = 2\pi/\lambda_g$ and $\xi_1 = 2\pi R/\lambda_c$, the above expression can be written as follows:

$$\eta_p \fallingdotseq \frac{3}{2}\frac{\{J_1(\xi_1)\}^2\lambda_c}{\lambda_g R}\frac{1}{x^4}.$$

Since $J_1(\xi_1) \fallingdotseq 1/2$,

$$\eta_p \fallingdotseq \frac{3}{8}\frac{\lambda_c^2}{\lambda_g R}\frac{1}{x^4}.$$

Thus, it is clearly seen that η_p tends to infinite as $1/x^4$.

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