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On the Hodograph Method and Analytic Continuation of Solution in the Theory of Compressible Fluid Flow*

By

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1. Introduction

The hodograph method is very useful for the exact treatment of the two-dimensional stationary flow of a compressible fluid. Difficulties occur however when we deal by this method with flows of practical interest, such as the continuous flow past an obstacle and the flow through a nozzle. This is mainly due to the fact that in such cases of flows the series solution constructed in the hodograph plane converges only in a certain restricted region of the field of flow, and therefore the process of analytic continuation of the solution to the remaining region is necessary in order to cover the whole field of flow. This problem of analytic continuation is in general very difficult and seems to have been left unsolved.

In the present paper the writer gives a method for obtaining such an analytic continuation for the fundamental solution which has a branch-point of the prescribed order in the hodograph plane. As an illustrative example, the flow past a circular cylinder is treated in detail by this method.

The writer wishes to express his cordial thanks to Prof. S. Tomotika and Mr. K. Munakata for their continued interest and valuable advices throughout the present work.

* This work has been done in 1947 and a preliminary report in Japanese has been published in stenciled form (1). Publication in English has been unwillingly delayed however for various reasons. Recently the writer has become aware of the interesting papers by S. Bergman (2), M. J. Lighthill (3), T. M. Cherry (4), and S. Goldstein, M. J. Lighthill, and J. W. Craggs (5), where the same problem of analytic continuation has been treated. Their methods are quite different from that used in the present paper. It is found however that for the special case of flow past a circular cylinder, the result of the present paper is in exact agreement with those obtained in (4) and (5).

2. Fundamental equation

As is well known, the two-dimensional steady irrotational flow of a non-viscous compressible fluid, whose pressure is a function of the density alone, is governed in the hodograph plane by the equation:

$$q^2c^2\psi_{qq} + q(c^2 + q^2)\psi_q + (c^2 - q^2)\psi_{\theta\theta} = 0\,, \tag{1}$$

where ψ is the stream function, q, θ the magnitude and the angle of inclination of the velocity vector respectively, and c is the speed of sound at any point. If we assume, as usual, the adiabatic law for the pressure-density relation, c is given as a function of q in the form:

$$\left(\frac{c}{c_\infty}\right)^2 = 1 - \frac{\gamma - 1}{2}M^2\left\{\left(\frac{q}{q_\infty}\right)^2 - 1\right\}, \tag{2}$$

where q_∞, c_∞ are respectively the velocity magnitude and the speed of sound at a certain standard point suitably chosen in the field of flow, M the Mach number at the standard point, and γ the ratio of the two specific heats of the fluid. For the case of flow past an obstacle, the point in the undisturbed stream at infinity may be conveniently chosen as the standard point. Further, in our future work, we take for convenience the velocity magnitude q_∞ at the standard point as the unit of velocity, so that $q = 1$ there. Inserting (2) in (1), the fundamental equation for the case of adiabatic flow becomes

$$q^2(1 - N^2q^2)\psi_{qq} + q\left(1 + \frac{3-\gamma}{\gamma - 1}N^2q^2\right)\psi_q + \left(1 - \frac{\gamma+1}{\gamma-1}N^2q^2\right)\psi_{\theta\theta} = 0, \tag{3}$$

with

$$N^2 = M^2\left(\frac{2}{\gamma - 1} + M^2\right)^{-1} = q^{-2}_{\max}\,,$$

where $q_{\max}$ is the maximum velocity attainable in the field of flow.

As is well known, the equation (3) has a system of particular solutions of the form:

$$\psi = A_\mu w^\mu F^{(\mu)}(N^2q^2)\,, \qquad w = qe^{i\theta}\,, \tag{4}$$

where μ and A_μ are arbitrary constants, and $F^{(\mu)}(N^2q^2)$ is a hypergeometric function defined as with

$$\left.\begin{aligned} F^{(\mu)}(N^2q^2) &= \sum_{n=0}^{\infty} k_n^{(\mu)}(N^2q^2)^n, \\ k_n^{(\mu)} &= \frac{\Gamma(\mu+1)\,\Gamma(a_\mu+n)\,\Gamma(b_\mu+n)}{\Gamma(n+1)\,\Gamma(\mu+n+1)\,\Gamma(a_\mu)\,\Gamma(b_\mu)}, \\ a_\mu + b_\mu &= \mu - \frac{1}{\gamma-1}, \qquad a_\mu b_\mu = -\frac{\mu(\mu+1)}{2(\gamma-1)}. \end{aligned}\right\} \tag{5}$$

It is evident that the hypergeometric series (5) is convergent for $N^2q^2 < 1$, i. e., for $q < q_{\max}$.

3. Fundamental solutions

The limiting case $M \to 0$ (i. e., $N \to 0$) is evidently the case of an incompressible fluid, and in this case the fundamental equation (3) degenerates into the Laplace equation. Then, as can easily be seen, the solution :

$$\psi_0 = \Im\left\{(1-w)^{-\frac{1}{2}} + (1-w)^{\frac{1}{2}}\right\}$$

(where $\Im$ means imaginary part) represents the flow of incompressible fluid past a circular cylinder without circulation. Whilst the solution :

$$\psi_0 = \Im(1-w)^{\frac{1}{2}}$$

expresses the flow of an incompressible fluid through a converging and diverging nozzle. Thus, it will be of fundamental importance to generalize the solution of the type :

$$\psi_0 = (1-w)^{\lambda} \qquad (\lambda : \text{a real constant}) \tag{6}$$

so as to allow for compressibility.

To this end, we first expand (6) for $|w| = q < 1$ into the series :

$$\psi_0(w) = \sum_{m=0}^{\infty} \binom{\lambda}{m} (-w)^m , \tag{7}$$

where

$$\binom{\lambda}{m} = \frac{\Gamma(\lambda+1)}{\Gamma(m+1)\,\Gamma(\lambda-m+1)}. \tag{7a}$$

Then, following usual procedure, we replace each w^m in this series by the corresponding solution (4) for a compressible fluid. It is clear that the resulting series :

$$\psi_\lambda(q,\ \theta) = \sum_{m=0}^{\infty} A_m \binom{\lambda}{m} (-w)^m F^{(m)}(N^2q^2) \tag{8}$$

is also a solution of the fundamental equation (3) so far as it is convergent. The constants A_m's are now to be determined so that the solution (8) should possess a singularity of similar character to that of the starting function (6) at the same point $w=1$ ($q=1,\ \theta=0$) in the hodograph plane. It is evident here that the character of the said singularity depends upon the remainder after the first m terms of the series (8) for large m. Now, the asymptotic behaviour for large m of the hypergeometric function occurring in (8) has recently been investigated by Z. Hasimoto in his doctoral thesis (**6**). According to him, it is valid for large m and $N^2q^2 < \alpha^{-2}$ (subsonic) that

$$F^{(m)}(N^2q^2) \sim \eta^{-\frac{1}{2}}(1-N^2q^2)^{\frac{1}{2(\gamma-1)}}\omega^m, \tag{9}$$

where

$$\eta = \left(\frac{1-\alpha^2N^2q^2}{1-N^2q^2}\right)^{\frac{1}{2}},\quad \omega = \frac{2}{1+\eta}\left(\frac{\eta+\alpha}{1+\alpha}\right)^{\alpha}(1-N^2q^2)^{\frac{\alpha-1}{2}},\quad \alpha = \sqrt{\frac{\gamma+1}{\gamma-1}}. \tag{9a}$$

Therefore, replacing each $F^{(m)}(N^2q^2)$ in (8) by its asymptotic form, we obtain an asymptotic expression of the solution $\psi_\lambda(q,\theta)$ in the vicinity of the singular point in the form:

$$\psi_\lambda(q,\theta) \sim \eta^{-\frac{1}{2}}(1-N^2q^2)^{\frac{1}{2(\gamma-1)}}\sum_{m=0}^{\infty} A_m\binom{\lambda}{m}(-\omega w)^m + (\text{regular term}).$$

This form suggests that the appropriate values of the constants in (8) would be

$$A_m = \omega_1^{-m},\qquad \omega_1 = [\omega]_{q=1}. \tag{10}$$

Then,

$$\psi_\lambda(q,\theta) \sim \eta^{-\frac{1}{2}}(1-N^2q^2)^{\frac{1}{2(\gamma-1)}}\left(1-\frac{\omega}{\omega_1}w\right)^{\lambda} + (\text{regular term}). \tag{11}^*$$

This $\psi_\lambda(q,\theta)$ has apparently the singularity of the required order at the prescribed point $w=1$, and it degenerates into (6) when N tends

* It may be interesting to note that the asymptotic form (11) is identical with the approximate solution given recently by I. Imai (**7**).

to zero. ($A_m \to 1$, $F^{(m)}(N^2q^2) \to 1$, when $N \to 0$.) Thus, we have shown that the series (8) with the constants A_m's given by (10) may be taken as the fundamental solution for compressible fluid flow, namely:

$$\psi_\lambda(q,\theta) = \sum_{m=0}^{\infty}\binom{\lambda}{m}(-\zeta)^m F^{(m)}(N^2q^2), \quad \zeta = \frac{w}{\omega_1} = \frac{q}{\omega_1}e^{i\theta}, \tag{12}$$

with

$$F^{(m)}(N^2q^2) = \sum_{n=0}^{\infty} k_n^{(m)}(N^2q^2)^n. \tag{12a}$$

It will be seen that (12) is convergent for $q < 1$.

4. A new form for the fundamental solution

Now, the physical field of flow extends usually beyond the range of convergence of the series solution given above, and consequently it is necessary to find the analytic continuation of the said series in order to cover the whole field of flow. It is this problem that we are concerned with in the following.

We first change the order of summation in the double series (12), and rewrite it in the form:

$$\psi_\lambda(q,\theta) = \sum_{n=0}^{\infty}(N^2q^2)^n f_n(\zeta), \tag{13}$$

with

$$f_n(\zeta) = \sum_{m=0}^{\infty}\binom{\lambda}{m} k_n^{(m)}(-\zeta)^m, \qquad \zeta = \frac{q}{\omega_1}e^{i\theta}. \tag{13a}$$

In the next place, we proceed to sum up the series (13a) for $f_n(\zeta)$. Taking into account that $f_n(\zeta)$ is an analytic function of $\zeta = (q/\omega_1)e^{i\theta}$, we introduce the expression (13) for $\psi_\lambda(q,\theta)$ into the fundamental equation (3), and equating the coefficient of each power of N^2 to zero, we obtain the following recurrence equation for $f_n(\zeta)$:

$$f_n(\zeta) = \frac{1}{2n(\gamma-1)}\Big\{-\zeta\frac{d}{d\zeta}f_{n-1}(\zeta) + (2\gamma-1)(n-1)f_{n-1}(\zeta) - (2\gamma+n)(n-1)\zeta^{-n}\int\zeta^{n-1}f_{n-1}(\zeta)\,d\zeta\Big\}. \tag{14}$$

Now, we have obviously

$$f_0(\zeta) = (1-\zeta)^\lambda, \tag{15}$$

and we can prove by means of the recurrence equation (14) that $f_n(\zeta)$ should have the form :

$$f_n(\zeta) = \zeta^{-n} \sum_{j=-n}^{n} a_j^{(n)} (1-\zeta)^{\lambda+j} + \sum_{m=2}^{n} c_m^{(\lambda)} k_{n-m}^{(m)} \zeta^{-m}, \qquad (16)$$

where $a_j^{(n)}$'s are definite constants satisfying the following recurrence formula :

$$a_j^{(n)} = \frac{1}{2(\gamma-1)} \frac{1}{n(\lambda+j)} \Big\{ (\lambda-n+j)(\lambda-n+j+1) - 2(\gamma-1)(\lambda+j-1)(n-1) + 2(n-1) \Big\} a_{j-1}^{(n-1)}$$
$$+ \frac{1}{n(\gamma-1)} \Big\{ \gamma(n-1) - (\lambda+j) \Big\} a_j^{(n-1)} + \frac{\lambda+j+1}{2n(\gamma-1)} a_{j+1}^{(n-1)}, \qquad (17)$$

with

$$a_0^{(0)} = 1, \quad a_j^{(0)} = 0 \quad (j = \pm 1, \pm 2, \cdots\cdots),$$

and $k_{n-m}^{(m)}$'s are the coefficients defined by (5). While, $c_m^{(\lambda)}$'s are constants of integration to be determined by the condition that (16) should coincide with (13a). Namely, expanding $f_n(\zeta)$ given by (16) in ascending power series of ζ and comparing the result with (13a), we get

$$c_m^{(\lambda)} = (-1)^{n-m+1} (k_{n-m}^{(m)})^{-1} \sum_{j=-n}^{n} \binom{\lambda+j}{n-m} a_j^{(n)}. \qquad (18)$$
$$(m = 2, 3, \cdots\cdots\cdots, n)$$

This relation holds for $n \geqq m$, but it does not involve any singularity for $0 < n < m$. Hence, remembering that $c_m^{(\lambda)}$ is independent of n, we can infer the value of $c_m^{(\lambda)}$ by putting formally $n = 0$ in (18), namely

$$c_m^{(\lambda)} = (-1)^{m+1} \frac{\Gamma(\lambda+1)\,\Gamma(a_m)\,\Gamma(b_m)}{\Gamma(m+1)\,\Gamma(\lambda+m+1)\,\Gamma(a_m-m)\,\Gamma(b_m-m)}. \qquad (19)$$

That this is true can be shown by mathematical induction, with the aid of the recurrence formula (17), though the details of the analysis are omitted here.

Inserting in (13) the expression (16) for $f_n(\zeta)$, and bearing in mind the relation :

$$(N^2q^2)^n\zeta^{-m} = (N^2\omega_1{}^2\bar{\zeta})^m(N^2q^2)^{n-m}$$

(where $\bar{\zeta}$ means conjugate complex of ζ), we obtain a new expression for $\psi_\lambda(q, \theta)$ in the form:

$$\psi_\lambda(q, \theta) = \psi_\lambda^*(q, \theta) + \chi_\lambda(q, \theta), \tag{20}$$

where

$$\psi_\lambda^*(q, \theta) = \sum_{n=0}^{\infty}(N^2\omega_1{}^2\bar{\zeta})^n\sum_{j=-n}^{n}a_j^{(n)}(1-\zeta)^{\lambda+j}, \tag{20a}$$

$$\chi_\lambda(q, \theta) = \sum_{m=:}^{\infty}c_m^{(\lambda)}(N^2\omega_1{}^2\bar{\zeta})^m F^{m}(N^2q^2), \tag{20b}$$

with the constants a_j^n and $c_m^{(\lambda)}$ given by (17) and (19) respectively. It should be noticed here that $\chi_\lambda(q, \theta)$ is independently a solution of the fundamental equation and is regular at the point $q = 1$, $\theta = 0$, as can be shown without difficulty. Consequently, $\psi_\lambda^*(q, \theta)$ given by (20a) must also be a solution which behaves like $\psi_\lambda(q, \theta)$ in the neighbourhood of the singular point $q = 1$, $\theta = 0$, and degenerates as well into (6) when $N \to 0$, i. e., $\psi_\lambda^*(q, \theta)$ may also be taken as a fundamental solution required.

5. Formulae for the analytic continuation

Now, the new series for $\psi_\lambda(q, \theta)$ obtained above proves to be still convergent outside the circle $q = 1$ of convergence of the original series (12).* Hence, we can extend the solution $\psi_\lambda(q, \theta)$ originally defined by the series (12) in the domain $q < 1$ over to the region $q > 1$ by the expression (20).

Further, we can obtain from (20) the analytic continuation of (12) in a form of hypergeometric-trigonometric series similar to (12) in the following way. First, we expand $\psi_\lambda^*(q, \theta)$ as given by (19a) in descending power series of ζ:

$$\psi_\lambda^*(q, \theta) = \sum_{n=0}^{\infty}(N^2q^2)^n f_n^*(\zeta),$$

where

* This form (20a) of the fundamental solution is especially fitted for the purpose of practical calculation. In fact, by carrying out numerical calculations in the case of flow past a cylindrical body the writer has found that a single series of this type can describe the whole field of flow near the surface of the cylinder, and there is no need of analytic continuation even when the flow becomes partly ultrasonic.

$$f_n^*(\zeta) = \sum_{m=0}^{\infty} \binom{\lambda}{m} k_n^{(\lambda-m)} (-\zeta)^{\lambda-m}.$$

Then, interchanging the order of summation, we are led to the result:

$$\psi_\lambda^*(q, \theta) = \sum_{m=0}^{\infty} \binom{\lambda}{m} (-\zeta)^{\lambda-m} F^{(\lambda-m)}(N^2 q^2). \tag{21}$$

It can be shown that this series is convergent in the domain $1 < q < q_{max}$. Eqs. (12), (20) and (20b), together with (21) just obtained, give finally the required formulae for the analytic continuation in the form:

$$\psi_\lambda(q, \theta) = \begin{cases} \sum_{m=0}^{\infty} \binom{\lambda}{m} (-\zeta)^m F^{(m)}(N^2 q^2), & (0 < q < 1) \\ \sum_{m=0}^{\infty} \binom{\lambda}{m} (-\zeta)^{\lambda-m} F^{(\lambda-m)}(N^2 q^2) + \chi_\lambda(q, \theta), & (1 < q < q_{max}) \end{cases} \tag{22}$$

$$\psi_\lambda^*(q, \theta) = \begin{cases} \sum_{m=0}^{\infty} \binom{\lambda}{m} (-\zeta)^m F^{(m)}(N^2 q^2) - \chi_\lambda(q, \theta), & (0 < q < 1) \\ \sum_{m=0}^{\infty} \binom{\lambda}{m} (-\zeta)^{\lambda-m} F^{(\lambda-m)}(N^2 q^2), & (1 < q < q_{max}) \end{cases} \tag{23}$$

where

$$\chi_\lambda(q, \theta) = \sum_{m=2}^{\infty} c_m^{(\lambda)} (N^2 \omega_1{}^2 \bar{\zeta})^m F^{(m)}(N^2 q^2). \tag{24}$$

$\psi_\lambda(q, \theta)$ and $\psi_\lambda^*(q, \theta)$ are connected by the relation:

$$\psi_\lambda(q, \theta) = \psi_\lambda^*(q, \theta) + \chi_\lambda(q, \theta). \tag{25}$$

It may be added here that $\chi_\lambda(q, \theta)$ can be shown to be convergent in the whole domain $0 < q < q_{max}$ provided that $M < 1$.

6. Flow past a circular cylinder

As an illustrative example, we now apply the results obtained above to the case of flow past a circular cylinder without circulation. In this case, the solution for incompressible flow is, as mentioned previously, given by

$$\psi_0 = \Im\left\{(1-w)^{-\frac{1}{2}} + (1-w)^{\frac{1}{2}}\right\}. \tag{26}$$

The corresponding solution for compressible flow may be put in the form:

$$\psi^* = \Im\left\{\psi^*_{-\frac{1}{2}} + A_1\psi^*_{\frac{1}{2}} + A_3\psi^*_{\frac{3}{2}} + \cdots\cdots\right\}, \tag{27}$$

where $\psi^*_{-\frac{1}{2}}$ etc. are given by (23) and $A_1, A_3, \cdots\cdots$ are constants to be determined by the boundary condition. The fundamental solutions $\psi^*_{-\frac{1}{2}}$ etc. are used here instead of $\psi_{-\frac{1}{2}}$ etc., because of the symmetry of flow about the axis $\theta = 0$. In the following, we take only the first two terms in (27). Then the constant A_1 is determined to be equal to 1 by the condition that the cylinder should be blunt-nosed at the stagnation points. Hence,

$$\psi^* = \Im\left\{\psi^*_{-\frac{1}{2}} + \psi^*_{\frac{1}{2}}\right\}. \tag{28}$$

After some calculations, we obtain finally the result that

$$\psi^* = \begin{cases} -\sum_{m=2}^{\infty}\left\{\dfrac{(2m-2)!}{2^{2m-2}m!\,(m-2)!}\left(\dfrac{q}{\omega_1}\right)^m + 2(m+1)c_m^{(\frac{1}{2})}N^{2m}(\omega_1 q)^m\right\} \\ \qquad\qquad \times F^{(m)}(N^2q^2)\left|\sin m\theta\right|, \qquad (q<1) \\ \left(\dfrac{q}{\omega_1}\right)^{\frac{1}{2}}F^{(\frac{1}{2})}(N^2q^2)\cos\dfrac{\theta}{2} - \sum_{m=0}^{\infty}\dfrac{(2m+3)(2m)!}{2^{2m+1}m!\,(m+1)!}\left(\dfrac{q}{\omega_1}\right)^{-\frac{1}{2}-m} \\ \qquad\qquad \times F^{(-\frac{1}{2}-m)}(N^2q^2)\cos\dfrac{2m+1}{2}\theta, \quad (q>1) \end{cases} \tag{29}$$

where we have used an obvious relation:

$$c_m^{(\lambda-1)} = \left(\frac{m}{\lambda} + 1\right)c_m^{(\lambda)}. \tag{30}$$

It will be seen that this result is in exact agreement with those obtained by T. M. Cherry (4) and S. Goldstein, M. J. Lighthill, and J. W. Craggs (5). Cherry has carried out numerical computation of the flow-field corresponding to (29). According to him, the solution (29) represents in the physical plane the flow past a nearly circular cylinder which has two axes of symmetry, one parallel and the other perpendicular to the direction of the undisturbed stream. The shape of the cylinder gradually deforms as the Mach number of the stream is increased, and the deformation becomes so conspicuous at high Mach numbers that no definite conclusion can be obtained as to the state of flow past a cylinder of exactly circular shape.

If, however, we take at least the first three or four terms in (27), we can get certainly a closer approximation to the shape of the boundary, and it may become possible to discuss more completely the state of affairs in the field of flow at high Mach numbers. The results of calculations developed in this direction will be published in the near future.

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