Cyclic Equations for O_{III} and Electron Temperature of Gaseous Nebulae

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Introduction

Gaseous nebulae are characterized by the emission of Balmer and of forbidden lines. They are composed mainly of hydrogen with small amount of heavier elements and illuminate themselves being excited by the high temperature radiation from the nuclear star. The Balmer emission is, as is well known, by the Zanstra theory[1], the recombination process, in which the hydrogen atom is ionized fed on the *radiation* from the nuclear star. But the emission of the forbidden lines cannot be explained by such radiative ionization; so I. S. Bowen[2] has put forward a suggestion, that the nebulium elements are excited by *inelastic collision* with free electrons.

In the present paper we have formulated mathematically the Bowen mechanism especially for the system O_{III}. Estimation of physical quantities required for this purpose will be published in detail as a separate paper in these memoirs. Thus we have derived the intensities of the nebular and auroral emissions and then we have proposed a new method to evaluate the electron temperature from the relative intensity of these lines; and lastly the theory has been applied to some planetary nebulae, of which the relative intensity of emission considered has been observed. The result is in good agreement with our previous discussion made from the collision effect on the Balmer emission[3], viz., that the electron temperature is as high as ten or twenty thousand degrees in its order.

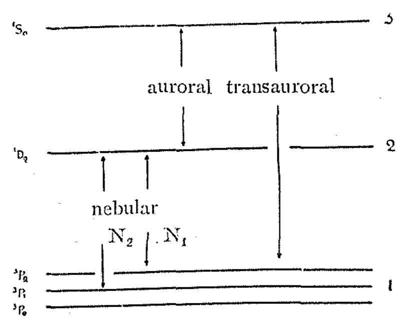
I. Theoretical Consideration

§ 1. *$2p^2$-configuration of O_{III}.*[4] Doubly ionized oxygen has two optical electrons in $2P$ quantum state, forming $2p^2$-configuration, which

1. H. Zanstra, ZS. f. Ap. **2** (1931) 1, 329. Publ. Dom. Ap. Obs. **4** (1931) 209.
2. I. S. Bowen, Ap. J. **67** (1928) 1.
3. These Memoirs, **21** (1938) 173.
4. Cf. Condon and Shortley, The Theory of Atomic Spectra (1935).

is made of two singlet levels ${}^1S_0^0$, ${}^1D_2$ and of a triplet level ${}^3P$. These levels shall be referred to in the following, for the sake of simplicity, as 3, 2, and 1 respectively. (cf. Fig. 1). As they are all of same parity, dipole transitions between them are forbidden.

Fig. 1.



E. U. Condon[1] has shown that there exists a weak transition of quadrupole and of magnetic dipole origin between these levels. As was first identified by Bowen[2], chief nebulium lines N_1(λ 5007 A) and N_2(λ 4959 A) are due to transitions ${}^1D_2 \to {}^3P_2$ and ${}^1D_2 \to {}^3P_1$ respectively and auroral line at λ 4363 A to ${}^1S_0^0 \to {}^1D_2$. Following J. C. Boyce and others[3], we shall call the transitions: 2–1, 3–2, and 3–1 *nebular*, *auroral* and *transauroral* respectively.

§ 2. *Cyclic Equations.* As in the case of the hydrogen atom, we shall now formulate the cyclic equations for O_{III}. Characteristic features of the cycle here may be stated as follows:

(i) Levels of other configurations lie very high above those of $2p^2$, so that the cycle can be approximated with sufficient accuracy as one taking place among the three levels considered.

(ii) The collisional excitation and de-excitation by free electrons play an important rôle, as has been suggested by Bowen.

(iii) As dipole transitions are forbidden between these levels, the absorption coefficients must be also very small as is immediately seen from the thermodynamical relations. The self-absorption and stimulated emission can be neglected compared with the spontaneous transition and collisional excitation and de-excitation as far as the radiation field in the gas is not too strong[4], so that they may be obliterated in our following formulation of the cyclic equations.

Now, if a nebula is in a steady state, the following continuity conditions of quanta must be satisfied between the populations N_i(i=1, 2, 3) in the levels:

1. E. U. Condon, Ap. J. **79** (1934) 217.
2. Bowen, *loc. cit.*
3. J. C. Boyce, D. H. Menzel and C. H. Payne. Proc. Nat. Acad. Sci. **19** (1933) 581.
4. That this is really the case for planetary nebulae and for novae in nebular stage, may be easily verified using the data of § 3.

$$\begin{aligned} N_1(b_{12}+b_{13}) &= N_2(A_{21}+a_{21})+N_3(A_{31}+a_{31}) \\ N_2(A_{21}+a_{21}+b_{23}) &= N_1 b_{12}+N_3(A_{32}+a_{32}) \\ N_3(A_{31}+a_{31}+A_{32}+a_{32}) &= N_1 b_{13}+N_2 b_{23}, \end{aligned} \tag{2.1}$$

in which A_{ij} the Einstein transition probability from level i to j, a_{ij} and b_{ji} the probability of collisional de-excitation for $j\rightarrow i$ and that of excitation for $j\rightarrow i$ respectively; a_{ij} and b_{ji} depending of course not only on the electron density but also on the temperature there. As of these three equations, only two are linearly independent, we shall consider in the following the ratios:

$$\frac{N_2}{N_1}\equiv n_2, \qquad \frac{N_3}{N_1}\equiv n_3. \tag{2.2}$$

Then the solution of the set of equations (2.1) is as follows:

$$\begin{aligned} n_2 &= \frac{b_{12}}{A_{21}+a_{21}}\cdot\frac{1}{D}\left\{1+\frac{b_{13}}{b_{12}}\left(\frac{A_{31}+a_{31}}{A_{32}+a_{32}}\right)\right\} \\ n_3 &= \frac{b_{13}}{A_{32}+a_{32}}\cdot\frac{1}{D}\left\{1+\left(\frac{b_{23}}{A_{21}+a_{21}}\right)\left(1+\frac{b_{12}}{b_{13}}\right)\right\}, \end{aligned} \tag{2.3}$$

where $$D\equiv 1+\left(\frac{A_{31}+a_{31}}{A_{32}+a_{32}}\right)\left(1+\frac{b_{23}}{A_{21}+a_{21}}\right).$$

§ 3. *Evaluation of transition probability.* (i) The probabilities of spontaneous transition A_{ij} have been evaluated by E. U. Condon[1], the values of which are reproduced in Table I. Nebular lines N_1 and N_2 are due to transitions ${}^1D_2\rightarrow{}^3P_2$ and ${}^1D_2\rightarrow{}^3P_1$ respectively (cf. Fig. 1). According to this calculation the ratio of these probabilities are 3 : 1; their sum is given in Table I as A_{21}.

(ii) Probability of collisional excitation b_{ji} is given by

$$b_{ji}=\int_{v_{ji}}^{\infty} Q_{ji}(v)vN_e(v)dv, \tag{3.1}$$

where

v = velocity of colliding electron,

$Q_{ji}(v)$ = effective cross-section for the excitation $j\rightarrow i$ for v-electron,

$N_e(v)dv$ = number of $(v,\ v+dv)$-electrons per c. c.,

v_{ji} = electron velocity corresponding to the limit fo the excitation $j\rightarrow i$, i. e. equivalent to the energy difference between the levels.

For the gas of electron temperature T_e, $N_e(v)dv$ is given by the Maxwell's formula:

$$N_e(v)dv=N_e\cdot 4\pi\left(\frac{m}{2\pi k}\right)^{3/2}\frac{1}{T_e^{3/2}}v^2 e^{-\frac{mv^2}{2kT_e}}dv, \tag{3.2}$$

1. E. U. Condon, *loc. cit.*

where N_e means the total number of free electrons per c. c.:

$$N_e = \int_0^\infty N_e(v)dv.$$

Inserting (3.2) into (3.1), we obtain

$$b_{ji} = N_e 4\pi \left(\frac{m}{2\pi k}\right)^{3/2} \frac{1}{T_e^{3/2}} \int_{v_{ji}}^\infty Q_{ji}(v) v^3 e^{-\frac{mv^2}{2kT_e}} dv. \tag{3.3}$$

Expressing $Q_{ji}(v)$ in atomic unit πa_0^2 (a_0=Bohr radius), T_e in 10,000° and v in 10^8cm/sec., we get

$$b_{ji} \equiv N_e S_{ji}(T_e) = 1.18 \times 10^{-7} \cdot \frac{N_e}{T_e^{3/2}} \int_{v_{ji}}^\infty Q_{ji}(v) v^3 e^{-\frac{3.29v^2}{T_e}} dv. \tag{3.4}$$

As for the numerical estimation of $Q_{ji}(v)$, laborious calculation has been performed, of which details will be published as a separate paper in these memoirs. Using these estimations $S_{ji}(T_e)$ has been evaluated by the numerical integration for the various values of T_e. The results are given in Table I. In the table we have limited the numbers to two figures, for any more accurate tabulation will be meaningless owing to the approximate nature of the evaluation of $Q_{ji}(v)$.

Table I.

Transition	$10^{-4}\epsilon_{ij}/k$	A_{ij}	S_{ji} \ $T_e \cdot 10^{-4}$	0.6	0.8	1.0	1.5	2.0	2.5	3 0	4.0	5.0	6.0
I Aurora 2-3	3 27	1.8 sec.^{-1}	$10^8 S_{23}$	0.016	0.08	0.22	0.83	1.65	2.5	3 3	4.8	(6)	(7)
II Tr. Aurora 1-3	6.14	0.102	$10^{11} S_{13}$	0.00013	0.004	0.023	0.39	1.17	2.9	5.5	13.1	(20)	(30)
III Nebula 1-2	2.87	0.024	$10^{10} S_{12}$	0.2	0.8	1.60	4.7	8.1	11.6	15.4	23.	(30)	(40)

Both A_{ij} and b_{ji} are exceedingly large for the auroral transition 2–3.

(iii) Probability of collisional de-excitation a_{ij} is, as well known, connected with b_{ji} by the thermodynamical relation as:

$$N_i^0 a_{ij} = N_j^0 b_{ji}, \tag{3.5}$$

where the notation "o" is used to represent the populations in thermodynamical equilibrium; and in such a case the ratio of the populations in levels i and j is given by Boltzmann's formula, viz.;

$$\frac{N_i^0}{N_j^0} = \frac{\varpi_i}{\varpi_j} e^{-\epsilon_{ji}/kT_e}, \quad (i, j = 1, 2, 3) \tag{3.6}$$

where ϵ_{ji} means the energy difference between levels i and j; k the Boltzmann constant and ϖ_i the statistical weight of the level. For O_{III}, the values of the weight are as follows:

$$\varpi_1 = 9, \ \varpi_2 = 5, \ \varpi_3 = 1. \tag{3.7}$$

§ 4. *Extreme cases.* In the foregoing section we have calculated all the required physical constants of the ion. The probabilities of the collisional transitions are of course functions of the electron temperature; but if the temperature is fixed, they are proportional to the electron density: N_e. Hence, before going further we shall by the way consider two extreme cases.

(i) *Dense gas:* When the gas is sufficiently dense, so that the spontaneous transitions may be neglected compared with the collisional excitation and de-excitation, the general formulae, (2.3) become:

$$n_i = n_i^0 \quad (i=2,\ 3); \tag{4.1}$$

that is, the population n_i coincides with that of the thermodynamical equilibrium. This does not, howerever, necessarily mean that the radiation field in the gas is adjusted at the same time to the very Planck's radiation corresponding to T_e.

(ii) *Dilute gas:* If the gas is sufficiently dilute, then contrary to the previous case, the collisional transitions may be all neglected compared with the spontaneous transitions; thus the general formulae become as follows:

$$n_2 = \frac{b_{12}}{A_{21}}\left\{1 + \left(\frac{b_{13}}{b_{12}}\right)\left(\frac{A_{32}}{A_{32}+A_{31}}\right)\right\},$$
$$n_3 = \frac{b_{13}}{A_{32}+A_{31}}. \tag{4.2}$$

§ 5. *Intensities of the nebular and auroral emissions.* Now we are able to calculate the intensities of the emission lines, i. e., remembering that n_2 and n_3 are both very small quantities, so that $N_1 \sim NO_{III}$, the intensities are given by

$$I_{neb} = I_{12} = h\nu_{12}A_{21}n_2NO_{III}$$
$$I_{aur} = I_{23} = h\nu_{23}A_{32}n_3NO_{III} \tag{5.1}$$

in which ν_{ij} is the frequency of radiation associated with transition $i \rightarrow j$ and h is the Planck's constant as usual. Thus the emission intensity is increased with the electron density N_e provided that the electron temperature T_e is fixed, because the populations n_2 and n_3 are increased with T_e. For example, the values of n_2 and n_3 for $T_e = 20{,}000$ are as follows:

$\log_{10} N_e$	2	3	4	5	6	7	8-∞
$\log_{10} n_2$	−7.5	−6.5	−5.5	−4.5	−3.6	−3.0	−2.9
$\log_{10} n_3$	−11.2	−10.2	−9.0	−7.5	−5.7	−4.6	−4.3

§ 6. *Intensity ratio of the nebular to the auroral line* is given from (2.3), (3.4) and (3.5) by

$$\frac{I_{neb}}{I_{aur}}=\left(\frac{\nu_{12}A_{21}n_2^0}{\nu_{23}A_{32}n_3^0}\right)$$
$$\times\left\{\frac{n_2^0N_e(S_{12}S_{23}+S_{23}S_{13})+N_eS_{13}S_{12}+n_3^0(S_{12}A_{32}+S_{12}A_{31}+S_{13}A_{32})}{n_2^0N_e(S_{12}S_{23}+S_{23}S_{13})+N_eS_{13}S_{12}+n_2^0S_{13}A_{21}}\right\}, \tag{6.1}$$

where n_2^0 and n_3^0 are of course function of T_e and given by (3.6). If the gas is sufficiently dense, the second factor of (6.1) is dropped, while in the case of dilute gas, it is also simplified, as is easily seen from (6.1) or (4.2).

In Table II the results of the computation are given, taking T_e and N_e as arguments.

Table II. Intensity ratio $I_{neb}:I_{aur}$.

$\log_{10} N_e$ / $10^{-4}T_e$	$-\infty$-+2	3	4	5	6	7	8	9-∞
0.6	16000	14000	8000	1400	160	30	15	14
0.8	1900	1800	1100	200	30	6	4	4
1.0	640	600	390	88	11	2.5	1.6	1.6
1.5	140	130	92	23	3.1	0.78	0.54	0.52
2.0	64	61	44	11	1.6	.43	.31	.30
2.5	37	36	26	7.2	1.1	.30	.22	.22
3 0	26	25	19	5.4	0.81	.24	.18	.18
4.0	17	17	13	3 7	.57	.18	.14	.13
5.0	14	13	10	3	.4	.15	.12	.11
6.0	12	12	9	3	.4	.15	.10	.10

§ 7. *A new method for determination of T_e.* The intensity ratio of the nebular to the auroral line is a function of electron temperature and electron density. Its dependence on N_e is, as is clearly seen from the table, very sensitive in the region between ca. $\log_{10} N_e \sim 4$ and 7; whereas, if the gas becomes either dense or dilute, this effect is very weak, and in both the extreme cases the intensity ratio is completely independent of N_e as is easily seen from (6.1).

On the other hand the intensity ratio does depend very sensitively throughout on the electron temperature, unless the temperature is too high. This comes mainly from the fact that $S_{ji}(T_e)$ depends on T_e in a very different manner for the excitation to the different levels; in other words, that the limit of the electron velocity effective to the excitation, which plays an important rôle in the integral of (3.4), is seriously affected by T_e, as can be easily seen from the Maxwell's law.

Thus we can propose a new method of determination of the electron temperature in the celestial body which shows nebular and auroral emissions, no matter what the gas there, either dense or dilute, is; viz., from the intensity ratio of these emission lines, provided we could estimate the electron density N_e.

II. Application to Planetary Nebulae

§ 8. *Electron temperature of planetary nebulae.* For planetary nebula, its density is known to be sufficiently low[1], so that we can determine its electron temperature from observed I_{neb}/I_{aur}, free from its electron density. (cf. § 7. Estimation becomes less accurate if the electron density is as high as 10^4 per c. c.). Until now, this intensity ratio did not draw much attention. Only H. H. Plaskett[2] observed this for some nebulae and has reported that there is no definite correlation between intensities of nebular and auroral lines. From our point of view, this lack of regularity is construed as the evidence for divergent T_e. But his data which are the only observations now available, are too scarce to drow any definite conclusion from them. However that may be, we shall illustrate the method of evaluation of T_e taking N. G. C. 7662 as an example: Plaskett observation gives as nebular N_2 and auroral intensity,

$$I_{N_2}=32, \quad I_{aur}=1.9.$$

Intensity of N_1 line is three times that of N_2. Therefore, we obtain

$$\frac{I_{neb}}{I_{aur}}=\frac{32\times 4}{1.9}=67.5.$$

Next, we must introduce the correction for space reddening: From the general formula given by L. Berman[3], we obtain $10^{-0.4C_n}$ as the correcting factor to be multiplied to $\left(\frac{I_{neb}}{I_{aur}}\right)$ in order to obtain true intensity ratio $\left(\frac{I_{neb}}{I_{aur}}\right)_0$, where $0.4C_n=0.072r_1$, and the so-called "effective distance" for space absorption, r_1 is given by him as $r_1=0.66$ for the

1. Zanstra, *loc. cit.*; B. Vorontsov-Velyaminov, ZS. f. Ap. **12** (1936) 247; G. G. Cillié, M. N. **92** (1932) 920.

2. H. H. Plaskett, Publ. Dom. Ap. Obs. **4** (1931) 187. Cf. especially his excellent discussions in pp. 204–206.

3. L. Berman, M. N. **96** (1936) 890.

present nebula. Therefore we get finally $\left(\frac{I_{neb}}{I_{aur}}\right)_0 = 60.5$, which corresponds to $T_e = 21{,}000$ as is obtained from the second column of Table II. This is in satisfactory agreement with $T_e = 24{,}000$ derived from Balmer decrement[1].

§ 9. In the same way, the following result has been obtained for other nebulae, in which the auroral as well as the nebular lines are observed.

Table III.

nebula	Group	$T_e \cdot 10^{-3}$	observer		$T_* \cdot 10^{-3}$	r in 10^3 A.U.	remarks
			N_2	λ 4363			
N.G.C. 3242	II	16	Plaskett[1]	Plaskett[1]	55 [4]	18×21, 28×32	double shell
„ 7027	I	21	„	„	86 [4]	10×16	
„ 7662	I	21	„	„	75 [5]	8×10, 16×19	double shell
I.C. 3568	II	20	Berman[2]	„	38 [4]	21×21	
N.G.C. 6790	I	17	„	„	93 [6]	4×44	
I.C. 4997	III	19	„	Page[3]	35 [6]	2×2	

1. H. H. Plaskett, Publ. Dom. Ap. Obs. **4** (1931) 187.
2. L. Berman, M. N. **96** (1936) 890.
3. T. L. Page, M. N. **96** (1936) 604.
4. H. Zanstra, ZS. f. Ap. **2** (1931) 331.
5. These Memoirs, **21** (1938) 173.
6. Vorontsov-Velyaminov, Russ. Ap. J. **9** (1934) 40.

We have employed Plaskett observations as far as possible. But when one or two of them are lacking, we have supplied them by those of other observers, as is seen in the fourth and fifth columns. Group, in the second column, is due to Berman[2] who has classified all nebulae treated by him in three groups: viz. I, II, and III, according to descending brightness of N_2 lines relative to $H\beta$ of hydrogen. Nuclear temperature T_* and radius of the gas nebulae r have been inserted for the sake of completeness; authorities about T_* are given as in the references there, while the values of r have been derived from H. D. Curtis'[3] measurement of apparent diameter combined with the distance obtained by Berman[2].

The table reveals no simple correlation between T_e and other

1. These Memoirs, **21** (1938) 173.
2. L. Berman, *loc. cit.*
3. H. D. Curtis, Publ. Lick Obs. **13** (1918) Part II.

quantities ; but it may be permissible to conclude that the electron temperature is about 10,000–25,000 for most nebulae.

Summary

The well known idea that the forbidden emissions in gaseous nebulae are due to impact excitation by free electron has been developed quantitatively by formulating cyclic equations for the system O_{III} (§ 2) ; and by solving them the population in each quantum level has been given as a function of electron density N_e and temperature T_e. Then it has been possible to express the intensities of the nebular and the auroral emissions as a function of the same arguments (§ 5), of which intensity ratio is given numerically in Table II. In the case of very dilute gas, such as planetary nebulae, this ratio depends only on T_e ; thus a new method of estimation of T_e has been proposed and applied to some nebulae, in which the intensities of the auroral as well as the nebular emissions have been observed. The result of the investigations is that the electron temperature for most nebulae observed lies in the range from 10,000 to 25,000 degrees.

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