# A Study of the Effect of a Travelling Atmospheric Disturbance upon the Sea-Surface

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**Abstract**

In this paper the writer reports a theoretical study of the fluctuation of sea-surface generated by a pressure disturbance of travelling cyclone with consideration of the effect of a coast. In our investigation, by taking into account not only the forced wave but also the free wave, we have obtained results which may explain tolerably well observed phenomena on the occasion of Kaze-Tunami (the abnormal high water caused by a severe cyclone), and also our solutions thus obtained may be physically reasonable, because they do not give the infinite elevation even when the travelling velocity of the disturbance coincides with the free long wave velocity, resulting in the so-called resonance. And thus we know that our solutions support Nomitsu's opinion.[1]

Moreover, by an example, we have pointed out the possibility that the general solutions can be obtained even if the pressure disturbance varies with the time.

## Introduction

Formerly, from simple mathematical calculations, the writer has investigated the elevation or depression of sea-surface generated by a travelling local gale of a large scale (such as a cyclone), and has published his conclusions in this memoir[2]. In this paper we shall investigate the effect of travelling pressure disturbance on the sea-surface and shall complement the results hitherto obtained by further investigations of the same problem.

Many writers have reported interesting results of their investigations of the effect of travelling pressure disturbance on the sea-surface. Especially Dr. Proudman[3] is a pioneer in the study of this problem and he has given from mathematical calculations the theoretical elevation of sea-surface in the following cases: 1) when a pressure disturbance which is independent of time suddenly generates on a sea-surface of unbounded ocean and travels with a constant velocity in one direction; and 2) when, on the ocean bounded on one side by a

1. Mem. Coll. Sci. Kyoto Imp. Univ. A, Vol. **18** (1935), p. 202, 214.
2. Mem. Coll. Sci. Kyoto Imp. Univ. A, Vol. **19** (1936), p. 109.
3. Month. N. Roy. Ast. Soc.. Geophys. Suppl. **2** (1929), p. 197.

coast, it travels towards the coast. But as has been pointed out by Prof. Nomitsu[1], his solutions of the second case take into consideration only the condition at the coast and disregard the initial condition. It seems probable that he has only taken into account the forced wave and has neglected the generated free wave under the assumption that the free wave, owing to the eddy viscosity, dies away before it reaches the coast. But we can not approve these interpretations, and in Prof. Nomitsu's opinion, between land and the open ocean is a continental shelf, whose depth and width are both small as compared with those of the ocean; therefore the free wave generated on the ocean by the pressure disturbance may die away owing to the eddy viscosity before it reaches the continental shelf, but after the disturbance has reached the continental shelf a different free wave will be again generated on account of the sudden variation of depth. And this free wave, even if it is somewhat dampened by friction, will propagate towards the coast with a tolerable height, because, owing to its large propagating velocity and the narrow width of the continental shelf, the time interval in which the free wave propagates from the fore-end of the continental shelf to the coast is short. To explain the surface elevation at the coast and over the continental shelf, therefore, it seems that not only the forced wave but also the free wave must be considered.

Moreover, Proudman's solutions become physically unreasonable when the travelling velocity of disturbance coincides with the long wave velocity and results in the resonance phenomenon, because in such a case his solutions give the infinite elevation of sea-surface. But it has been pointed out first by Prof. Nomitsu[2] and afterwards also by Mr. H. Yamada[3] that this physical inconsistency can be removed by consideration of the eddy viscosity.

In this paper the writer, starting from the initial state of rest and assuming the water to be non-viscous, treats the same problem with consideration of not only the forced wave but also the free wave and explains that low water sometimes occurs before the abnormal high water on the occasion of Kaze-Tunami. And from the same standpoint we give the solutions which give the heights of high and low water caused by a pressure disturbance, and also show that our solu-

1. *l. c. ante.* p. 55.
2. Mem. Coll. Sci. Kyoto Imp. Univ., A, Vol. 18 (1935), p. 213.
3. 海と空, 第16卷第11號 (1936).

tions do not contain the physical inconsistency noted above even if the so-called resonance phenomenon occurs. Lastly by using Nomitsu's extension[1] of the Duhamel theorem we show, by an example, the possibility that these solutions are in general applicable even if the pressure disturbance varies with the time.

## I. Unbounded Sea.

The present problem is treated in two dimensions without consideration of the effect of the earth's rotation and assuming that the sea-water is incompressible and non-viscous—Prof. Nomitsu has treated this problem on an unbounded sea with consideration of the eddy viscosity, but the effect of a coast makes the problem very complicated so that we disregard the eddy viscosity in this paper—and the density of water is taken as unity.

Then using the usual notations, the eq. of motion for the long wave, the eq. of continuity and the initial condition are given as follows:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}=-g\frac{\partial \zeta}{\partial x}-g\frac{\partial P}{\partial x},\\ &h\frac{\partial u}{\partial x}=-\frac{\partial \zeta}{\partial t}.\end{aligned}\right\}\quad\cdots\cdots\cdots(1)$$

$$\left.\begin{aligned} &\frac{\partial u}{\partial t}=-g\frac{\partial P}{\partial x}\ \text{ or }\ \frac{\partial \zeta}{\partial t}=0 \quad \text{when } t=0,\\ &u=0 \ \text{ or }\ \zeta=0 \quad \text{when } t=0.\end{aligned}\right\}\quad\cdots\cdots(2)$$

Since it is clear that $u$ and $\zeta$ are "additive" as defined by Prof. Nomitsu, from Fourier integral and Nomitsu's extension of the Duhamel theorem, the solutions which satisfy (1) and (2) are easily obtained.

$$\frac{h}{c^2}u=-\frac{1}{\pi}\int_0^t d\tau\int_0^\infty d\alpha\int_{-\infty}^{\infty}\frac{\partial P}{\partial \lambda}\cos\alpha c(t-\tau)\cos\alpha(\lambda-x)d\lambda,\cdots\cdots(3)$$

$$\zeta=\frac{c}{\pi}\int_0^t d\tau\int_0^\infty d\alpha\int_{-\infty}^{\infty}\frac{\partial P}{\partial \lambda}\sin\alpha c(t-\tau)\sin\alpha(\lambda-x)d\lambda\cdots\cdots(4)$$

where $c\equiv gh$ and $P\equiv P(x,\ t)$.

Now, assume that initially ($t=0$) the pressure disturbance whose geometrical centre is $x=0$, suddenly originates on an unbounded sea and travels towards $x<0$ with a constant velocity $v$. In such a case the pressure disturbance may be given by the form $P(x+vt,\ t)$, then (3) and (4) become

1. Proc. Imp. Acad. Tokyo, **11**, 359 (1935).

$$\frac{h}{c^2}u = -\frac{1}{2}\int_0^t P'(x-ct+\overline{c+v}\tau,\ \tau)d\tau$$
$$-\frac{1}{2}\int_0^t P'(x+ct-\overline{c-v}\tau,\ \tau)d\tau, \quad \ldots\ldots\ldots\ldots (5)$$
$$\zeta = -\frac{c}{2}\int_0^t P'(x-ct+\overline{c+v}\tau,\ \tau)d\tau$$
$$+\frac{c}{2}\int_0^t P'(x+ct-\overline{c-v}\tau,\ \tau)d\tau. \quad \ldots\ldots\ldots\ldots (6)$$

where $P'$s are respectively the differentiations with respect to $x-ct+\overline{c+v}\tau$ and $x+ct-\overline{c-v}\tau$. Especially when the travelling velocity $v$ of the disturbance coincides with the long wave velocity $c$ and therefore the resonance phenomenon occurs, (5) and (6) become as follows:

$$\frac{h}{c^2}u = -\frac{1}{2}\int_0^t P'(x+ct,\ \tau)d\tau$$
$$-\frac{1}{2}\int_0^t P'(x-ct+2c\tau,\ \tau)d\tau, \quad \ldots\ldots\ldots\ldots (5')$$
$$\zeta = \frac{c}{2}\int_0^t P'(x+ct,\ \tau)d\tau$$
$$-\frac{c}{2}\int_0^t P'(x-ct+2c\tau,\ \tau)d\tau. \quad \ldots\ldots\ldots\ldots (6')$$

Now, assume $P=P\,(x+vt)$, namely, that the shape of the pressure disturbance is unchanged with $t$, then

$$\frac{h}{c^2}u = \frac{v}{c^2-v^2}P(x+vt) - \frac{1}{2(c-v)}P(x+ct)$$
$$+\frac{1}{2(c+v)}P(x-ct), \quad \ldots\ldots\ldots\ldots (5'')$$
$$\zeta = -\frac{c^2}{c^2-v^2}P(x+vt) + \frac{c}{2(c-v)}P(x+ct)$$
$$+\frac{c}{2(c+v)}\ P(x-ct). \quad \ldots\ldots\ldots\ldots (6'')$$

and if $c=v$

$$\frac{h}{c^2}u = -\frac{t}{2}P'(x+ct) - \frac{1}{4c}P(x+ct)$$
$$+\frac{1}{4c}P(x-ct), \quad \ldots\ldots\ldots\ldots (5''')$$
$$\zeta = \frac{ct}{2}P'(x+ct) - \frac{1}{4}P(x+ct) + \frac{1}{4}P(x-ct). \quad \ldots (6''')$$

Eqs. (5″) and (6″) are the same as Proudman's solutions. And Dr. Proudman gives solutions resembling (5), (6), (5′) and (6′), but his solutions contain only the effect of time variation of disturbance and not its travelling effect, while our solutions contain both factors, and therefore may be more general than Proudman's solutions. Moreover from these ( 5 ), ( 6 ), ( 5′ ) and ( 6′ ) we may obtain solutions taking into consideration the time variation of disturbance, the effect of a coast and the travelling effect, but these will be stated in detail in the last section.

## II. Sea Bounded by a Straight Coast.

In this case a boundary condition

$$u=0 \text{ or } \frac{\partial \zeta}{\partial x}=-\frac{\partial P}{\partial x} \quad \text{at } x=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2')$$

must be added to ( 2 ).

Now, let the ocean be in $x>0$, and the land be in $x<0$. If we take as the time origin an instant when the fore-end of disturbance arrives at the coast and consider only the forced wave without the free wave, then we have

$$\left.\begin{aligned} \frac{h}{c^2}u&=\frac{v}{c^2-v^2}\left\{P\left(t+\frac{x}{v}\right)-P\left(t-\frac{x}{c}\right)\right\}, \\ \zeta&=-\frac{c^2}{c^2-v^2}\left\{P\left(t+\frac{x}{v}\right)+\frac{v}{c}P\left(t-\frac{x}{c}\right)\right\}. \end{aligned}\right\} \ldots\ldots\ldots (7)$$

These are the original forms first given by Dr. Proudman and later by Mr. H. Yamada. The first term is the incident forced wave and the second term is the reflected wave, but as pointed out by Prof. Nomitsu these solutions do not satisfy the initial condition of rest.

Now assume

$$P(\xi)=P(x+vt-a) \qquad \text{for } -l<\xi<+l,$$
$$P(\xi)=0 \qquad \text{for } \xi<-l \text{ and } \xi>+l.$$

where $a$ is the co-ordinate of the geometrical centre of disturbance at $t=0$ and $2l$ is the width of disturbance. Then the solutions corresponding to ( 7 ) are as follows, but the argument of $P$ is suitably replaced because the free wave is considered.

$$\left.\begin{aligned} \frac{h}{c^2}u&=\frac{v}{c^2-v^2}\left\{P(x+vt-a)-P\left(vt-a-\frac{v}{c}x\right)\right\}, \\ \zeta&=-\frac{c^2}{c^2-v^2}\left\{P(x+vt-a)+\frac{v}{c}P\left(vt-a-\frac{v}{c}x\right)\right\}. \end{aligned}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7')$$

From the suitable combination of ( 7′ ), ( 5″ ) and ( 6″ ) we shall obtain the solutions which satisfy both the initial and the boundary conditions.

Now, divide the time interval into five or three stages until the disturbance has passed into the land, then the solution for the respective stages will be as follows. Here $c>v$ is assumed.

1) $0 \leq t \leq \frac{a-l}{c}$: *Before the pressure disturbance and the generated free wave arrive at the coast.*

$$\frac{h}{c^2}u = \frac{v}{c^2-v^2}P(x+vt-a) - \frac{1}{2(c-v)}P(x+ct-a) + \frac{1}{2(c+v)}P(x-ct-a),$$

$$\zeta = -\frac{c^2}{c^2-v^2}P(x+vt-a) + \frac{c}{2(c-v)}P(x+ct-a) + \frac{c}{2(c+v)}P(x-ct-a).$$

It is obvious that these solutions satisfy (1) and (2); and since in this stage the wave has not arrived at the coast, the boundary condition (2′) is also satisfied.

2) $\frac{a-l}{c} \leq t \leq \frac{a+l}{c}$: *After the free wave has arrived but before the pressure disturbance arrives and the free wave is completely reflected.*

$$\frac{h}{c^2}u = \frac{v}{c^2-v^2}P(x+vt-a) - \frac{1}{2(c-v)}\Big\{P(x+ct-a) - P(ct-a-x)\Big\} + \frac{1}{2(c+v)}P(x-ct-a),$$

$$\zeta = -\frac{c^2}{c^2-v^2}P(x+vt-a) + \frac{c}{2(c-v)}\Big\{P(x+ct-a) + P(ct-a-x)\Big\} + \frac{c}{2(c+v)}P(x-ct-a).$$

In this stage it is not necessary that the initial condition is satisfied. The first and the third term of $u$ have no contribution at the coast from the character of $P$ and the second term of $u$ obviously vanishes at the coast ($x=0$), therefore the boundary condition (2′) is satisfied. Substituting these $u$ and $\zeta$ in (1), it is easily verified that they satisfy eq. (1). Moreover $u$ and $\zeta$ change continuously from stage 1) to stage 2), because, when $t=\frac{a-l}{c}$, the term $P(ct-a-x)$ becomes $P(-l-x)$ which vanishes on the sea everywhere. Similarly we see that the solutions of following stages (3), (4), (5), (3′), (4′) and (5′) satisfy eq. (1), the boundary condition at the coast (2′) and the continuity of the transition from one stage to another.

3) $\dfrac{a+l}{c} \leq t \leq \dfrac{a-l}{v}$: *After the free wave has been reflected completely but before the pressure disturbance arrives.*

$$\frac{h}{c^2}u = \frac{v}{c^2-v^2}P(x+vt-a) + \frac{1}{2(c-v)}P(ct-a-x) + \frac{1}{2(c+v)}P(x-ct-a),$$

$$\zeta = -\frac{c^2}{c^2-v^2}P(x+vt-a) + \frac{c}{2(c-v)}P(ct-a-x) + \frac{c}{2(c+v)}P(x-ct-a).$$

4) $\dfrac{a-l}{v} \leq t \leq \dfrac{a+l}{v}$: *After the pressure disturbance has arrived but before it has completely passed away inland.*

$$\frac{h}{c^2}u = \frac{v}{c^2-v^2}\left\{P(x+vt-a) - P\left(vt-a-\frac{v}{c}x\right)\right\} + \frac{1}{2(c-v)}P(ct-a-x) + \frac{1}{2(c+v)}P(x-ct-a),$$

$$\zeta = -\frac{c^2}{c^2-v^2}\left\{P(x+vt-a) + \frac{v}{c}P\left(vt-a-\frac{v}{c}x\right)\right\} + \frac{c}{2(c-v)}P(ct-a-x) + \frac{c}{2(c+v)}P(x-ct-a).$$

5) $\dfrac{a+l}{v} \leq t$: *After the pressure disturbance has completely passed away inland.*

$$\frac{h}{c^2}u = -\frac{v}{c^2-v^2}P\left(vt-a-\frac{v}{c}x\right) + \frac{1}{2(c-v)}P(ct-a-x) + \frac{1}{2(c+v)}P(x-ct-a),$$

$$\zeta = -\frac{cv}{c^2-v^2}P\left(vt-a-\frac{v}{c}x\right) + \frac{c}{2(c-v)}P(ct-a-x) + \frac{c}{2(c+v)}P(x-ct-a).$$

Moreover, due to the relation of the relative magnitude of $c$ and $v$, another case may be considered as follows:

3′) $\dfrac{a-l}{c} < \dfrac{a-l}{v} \leq t \leq \dfrac{a+l}{c}$: *After the pressure disturbance has arrived but before the free wave has been reflected completely.*

$$\frac{h}{c^2}u=\frac{v}{c^2-v^2}\left\{P(x+vt-a)-P\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$-\frac{1}{2(c-v)}\{P(x+ct-a)-P(ct-a-x)\}$$
$$+\frac{1}{2(c+v)}P(x-ct-a),$$
$$\zeta=-\frac{c^2}{c^2-v^2}\left\{P(x+vt-a)+\frac{v}{c}P\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$+\frac{c}{2(c-v)}\{P(x+ct-a)+P(ct-a-x)\}$$
$$+\frac{c}{2(c+v)}P(x-ct-a).$$

4′) $\frac{a-l}{v}<\frac{a+l}{c}\leq t\leq\frac{a+l}{v}$: *After the free wave has been reflected but before the pressure disturbance has completely passed away inland.*

5′) $\frac{a+l}{v}\leq t$: *After the pressure disturbance has completely passed away inland.*

The solutions of these stages 4′) and 5′) are the same as those of 4) and 5).

The solutions for the case of $c<v$ have only the difference that the pressure disturbance arrives earlier than the free wave, and contain no other essential difference from the above forms. Therefore we will not state them in detail. But when $c=v$ and the resonance occurs, starting from (5‴) and (6‴) the solutions corresponding to the above forms may be obtained as follows:

1) $0\leq t\leq\frac{a-l}{c}$: *Before the pressure disturbance arrives at the coast.*

$$\frac{h}{c^2}u=-\frac{t}{2}P'(x+ct-a)-\frac{1}{4c}P(x+ct-a)$$
$$+\frac{1}{4c}P(x-ct-a),$$
$$\zeta=\frac{ct}{2}P'(x+ct-a)-\frac{1}{4}P(x+ct-a)+\frac{1}{4}P(x-ct-a).$$

2) $\frac{a-l}{c}\leq t\leq\frac{a+l}{c}$: *After the pressure disturbance has arrived but before it has completely passed away inland.*

$$\frac{h}{c^2}u=-\frac{t}{2}P'(x+ct-a)+\frac{1}{2}\left(t-\frac{x}{c}\right)P'(ct-a-x)$$

$$-\frac{1}{4c}P(x+ct-a)+\frac{1}{4c}P(ct-a-x)$$

$$+\frac{1}{4c}P(x-ct-a),$$

$$\zeta=\frac{ct}{2}P'(x+ct-a)+\frac{c}{2}\left(t-\frac{x}{c}\right)P'(ct-a-x)$$

$$-\frac{1}{4}P(x+ct-a)+\frac{1}{4}P(ct-a-x)$$

$$+\frac{1}{4}P(x-ct-a).$$

3) $\frac{a+l}{c}\leq t$: *After the pressure disturbance has completely passed away inland.*

$$\frac{h}{c^2}u=\frac{1}{2}\left(t-\frac{x}{c}\right)P'(ct-a-x)+\frac{1}{4c}P(ct-a-x)$$

$$+\frac{1}{4c}P(x-ct-a),$$

$$\zeta=\frac{c}{2}\left(t-\frac{x}{c}\right)P'(ct-a-x)+\frac{1}{4}P(ct-a-x)$$

$$+\frac{1}{4}P(x-ct-a).$$

In these formulae it may be noticed that the height of the forced wave is proportional to the time interval after the disturbance occurred, and therefore it does not increase to become infinitely large so long as the time interval is finite. Of course these facts emerged from consideration of the initial condition, but it has already been shown by Prof. Nomitsu that the physical inconsistency in the resonance case—the height of wave increases infinitely if $c=v$—may be removed by taking friction into account even if the time interval is infinite.

When we wish to apply our formulae to a practical case, we must have the actual meteorological record of atmospheric pressure $P$, the presumed travelling velocity of cyclone $v$, the mean depth of continental shelf $h$, the width of disturbance $2l$ and the initial position of the centre of disturbance $a$ ($a$ may be assumed as the sum of the width of the continental shelf and half the width of disturbance, $l$). If these date are known, the dynamic elevation of the sea-surface may be easily calculated. But it must be noticed that, since our solutions are obtained on the assumption of the uniformity of depth, the direct application of our formulae to the boundary between the continental shelf and the ocean may not be rigorously exact. In order to obtain a more reasonable solution which may be applied at the boundary, we must divide the sea-surface into two parts, namely, an ocean and a continental shelf, and suitably combine the solutions which hold in the respective parts and their boundary. But this treatment will make the problem more complicated; therefore we do not touch on it in this paper but shall leave it for another occasion.

Now, for an example, assume that the real pressure record is given in the following form:

$$\left.\begin{aligned} &P=-A\left\{1+\cos\frac{\pi}{l}(x+vt-a)\right\} && \text{for}\quad a-l<x+vt<a+l, \\ &P=0 && \text{for}\quad x+vt<a-l \quad\text{and} \\ & && \qquad x+vt>a+l. \end{aligned}\right\} \qquad \cdots\cdots(8)$$

1) $0\leq t\leq\dfrac{a-l}{c}$:

$$\begin{aligned} \frac{h}{c^2}u=&-\frac{v}{c^2-v^2}A\left\{1+\cos\frac{\pi}{l}(x+vt-a)\right\} \\ &+\frac{7}{2(c-v)}A\left\{1+\cos\frac{\pi}{l}(x+ct-a)\right\} \\ &-\frac{1}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}, \end{aligned}$$

$$\begin{aligned} \zeta=&\frac{c^2}{c^2-v^2}A\left\{1+\cos\frac{\pi}{l}(x+vt-a)\right\} \\ &-\frac{c}{2(c-v)}A\left\{1+\cos\frac{\pi}{l}(x+ct-a)\right\} \\ &-\frac{c}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}. \end{aligned}$$

2) $\dfrac{a-l}{c}\leq t\leq\dfrac{a+l}{c}<\dfrac{a-l}{v}$:

$$\begin{aligned} \frac{h}{c^2}u=&-\frac{v}{c^2-v^2}A\left\{1+\cos\frac{\pi}{l}(x+vt-a)\right\} \\ &+\frac{1}{2(c-v)}A\left\{\left[1+\cos\frac{\pi}{l}(x+ct-a)\right]\right. \\ &\left.-\left[1+\cos\frac{\pi}{l}(ct-a-x)\right]\right\} \\ &-\frac{1}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}, \end{aligned}$$

$$\zeta=\frac{c^2}{c^2-v^2}A\left\{1+\cos\frac{\pi}{l}(x+vt-a)\right\}$$

$$-\frac{c}{2(c-v)}A\left\{\left[1+\cos\frac{\pi}{l}(x+ct-a)\right]\right.$$
$$\left.+\left[1+\cos\frac{\pi}{l}(ct-a-x)\right]\right\}$$
$$-\frac{c}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}.$$

3) $\dfrac{a+l}{c}\leq t\leq\dfrac{a-l}{v}$:

$$\frac{h}{c^2}u=-\frac{v}{c^2-v^2}A\left\{1+\cos\frac{\pi}{l}(x+vt-a)\right\}$$
$$-\frac{1}{2(c-v)}A\left\{1+\cos\frac{\pi}{l}(ct-a-x)\right\}$$
$$-\frac{1}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\},$$
$$\zeta=\frac{c^2}{c^2-v^2}A\left\{1+\cos\frac{\pi}{l}(x+vt-a)\right\}$$
$$-\frac{c}{2(c-v)}A\left\{1+\cos\frac{\pi}{l}(ct-a-x)\right\}$$
$$-\frac{c}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}.$$

4) $\dfrac{a-l}{v}\leq t\leq\dfrac{a+l}{v}$:

$$\frac{h}{c^2}u=-\frac{v}{c^2-v^2}A\left\{\left[1+\cos\frac{\pi}{l}(x+vt-a)\right]\right.$$
$$\left.-\left[1+\cos\frac{\pi}{l}\left(vt-a-\frac{v}{c}x\right)\right]\right\}$$
$$-\frac{1}{2(c-v)}A\left\{1+\cos\frac{\pi}{l}(ct-a-x)\right\}$$
$$-\frac{1}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\},$$
$$\zeta=\frac{c^2}{c^2-v^2}A\left\{\left[1+\cos\frac{\pi}{l}(x+vt-a)\right]\right.$$
$$\left.+\frac{v}{c}\left[1+\cos\frac{\pi}{l}\left(vt-a-\frac{v}{c}x\right)\right]\right\}$$
$$-\frac{c}{2(c-v)}A\left\{1+\cos\frac{\pi}{l}(ct-a-x)\right\}$$
$$-\frac{c}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}.$$

5) $\dfrac{a+l}{v}\leq t$:

$$\frac{h}{c^2}u=\frac{v}{c^2-v^2}A\left\{1+\cos\frac{\pi}{l}\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$-\frac{1}{2(c-v)}A\left\{1+\cos\frac{\pi}{l}(ct-a-x)\right\}$$
$$-\frac{1}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\},$$

$$\zeta=\frac{vc}{c^2-v^2}A\left\{1+\cos\frac{\pi}{l}\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$-\frac{c}{2(c-v)}A\left\{1+\cos\frac{\pi}{l}(ct-a-x)\right\}$$
$$-\frac{c}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}.$$

3′) $\dfrac{a-l}{c}<\dfrac{a-l}{v}\leq t\leq\dfrac{a+l}{c}$:

$$\frac{h}{c^2}u=-\frac{v}{c^2-v^2}A\left\{\left[1+\cos\frac{\pi}{l}(x+vt-a)\right]\right.$$
$$\left.-\left[1+\cos\frac{\pi}{l}\left(vt-a-\frac{v}{c}x\right)\right]\right\}$$
$$+\frac{1}{2(c-v)}A\left\{\left[1+\cos\frac{\pi}{l}(x+ct-a)\right]\right.$$
$$\left.-\left[1+\cos\frac{\pi}{l}(ct-a-x)\right]\right\}$$
$$-\frac{1}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\},$$

$$\zeta=\frac{c^2}{c^2-v^2}A\left\{\left[1+\cos\frac{\pi}{l}(x+vt-a)\right]\right.$$
$$\left.+\frac{v}{c}\left[1+\cos\frac{\pi}{l}\left(vt-a-\frac{v}{c}x\right)\right]\right\}$$
$$-\frac{c}{2(c-v)}A\left\{\left[1+\cos\frac{\pi}{l}(x+ct-a)\right]\right.$$
$$\left.+\left[1+\cos\frac{\pi}{l}(ct-a-x)\right]\right\}$$
$$-\frac{c}{2(c+v)}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}.$$

4′) $\dfrac{a-l}{v}<\dfrac{a+l}{c}\leq t\leq\dfrac{a+l}{v}$.

5′) $\dfrac{a+l}{v}\leq t$.

The solutions of stages 4′) and 5′) are respectively the same as those of 4) and 5).

If $v=c$, we have

1) $0 \leq t \leq \frac{a-l}{c}$:

$$\frac{h}{c^2}u = -\frac{\pi t}{2l}A\sin\frac{\pi}{l}(x+ct-a) + \frac{1}{4c}A\left\{1+\cos\frac{\pi}{l}(x+ct-a)\right\} - \frac{1}{4c}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\},$$

$$\zeta = \frac{\pi ct}{2l}A\sin\frac{\pi}{l}(x+ct-a) + \frac{1}{4}A\left\{1+\cos\frac{\pi}{l}(x+ct-a)\right\} - \frac{1}{4}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}.$$

2) $\frac{a-l}{c} \leq t \leq \frac{a+l}{c}$:

$$\frac{h}{c^2}u = -\frac{\pi t}{2l}A\sin\frac{\pi}{l}(x+ct-a) + \frac{\pi}{2l}\left(t-\frac{x}{c}\right)\sin\frac{\pi}{l}(ct-a-x) + \frac{1}{4c}A\left\{\left[1+\cos\frac{\pi}{l}(x+ct-a)\right] - \left[1+\cos\frac{\pi}{l}(ct-a-x)\right]\right\} - \frac{1}{4c}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\},$$

$$\zeta = \frac{\pi ct}{2l}A\sin\frac{\pi}{l}(x+ct-a) + \frac{\pi c}{2l}\left(t-\frac{x}{c}\right)A\sin\frac{\pi}{l}(ct-a-x) + \frac{1}{4}A\left\{\left[1+\cos\frac{\pi}{l}(x+ct-a)\right] - \left[1+\cos\frac{\pi}{l}(ct-a-x)\right]\right\} - \frac{1}{4}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}.$$

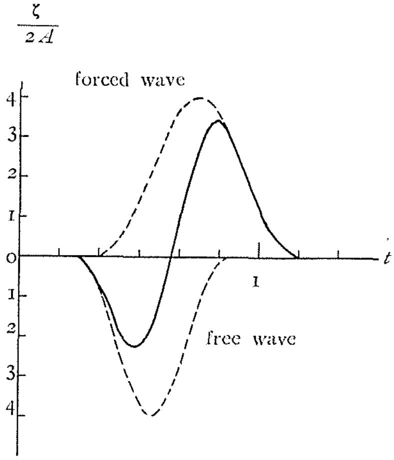
3) $\dfrac{a+l}{c} \leq t$:

$$\frac{h}{c^2}u = \frac{\pi}{2l}\left(t-\frac{x}{c}\right)A\sin\frac{\pi}{l}(ct-a-x)$$
$$-\frac{1}{4c}A\left\{1+\cos\frac{\pi}{l}(ct-a-x)\right\}$$
$$-\frac{1}{4c}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\},$$
$$\zeta = \frac{\pi c}{2l}\left(t-\frac{x}{c}\right)A\sin\frac{\pi}{l}(ct-a-x)$$
$$-\frac{1}{4}A\left\{1+\cos\frac{\pi}{l}(ct-a-x)\right\}$$
$$-\frac{1}{4}A\left\{1+\cos\frac{\pi}{l}(x-ct-a)\right\}.$$

Again for an example, let us assume that the width of continental shelf is 20 km, its mean depth, $h$, is 50 m ($\therefore$ $c=80$ km/hour) and the width of the pressure disturbance $2l$ is 50 km; then the distance $a$ is 45 km. And if we consider three cases in which the travelling velocity $v$ takes 60 km/hour, 20 km/hour and 80 km/hour respectively, then the time curves of surface elevation of the above three cases at the coast will be as shown in Fig. 1, 2 and 3, in which $t$ (unit is hour) is abcissa and $\dfrac{\zeta}{2A}$ ($2A$ is the statical elevation caused by the pressure disturbance) is ordinate.

Fig. 1

Fluctuation of Sea-surface at the coast.




From the above figures it must be noticed that in all the cases low water due to the free wave first arrives at the coast and later high water caused by the forced wave arrives. (If $v>c$ the circumstances are contrary to the above, but $v>c$ scarcely ever happens.) Owing to the eddy viscosity the low water may not be so remarkable as the figures show, but by taking the free wave into account it may be easily explained that the low water sometimes appears at the coast earlier than the high water as a precurser of Kaze-Tunami.

Fig. 2

Fluctuation of Sea-surface at the coast.

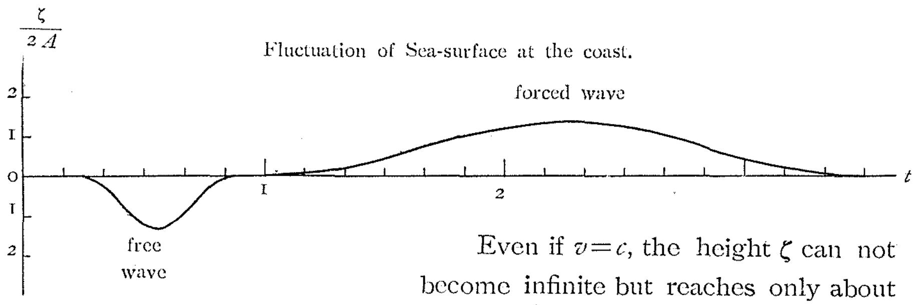
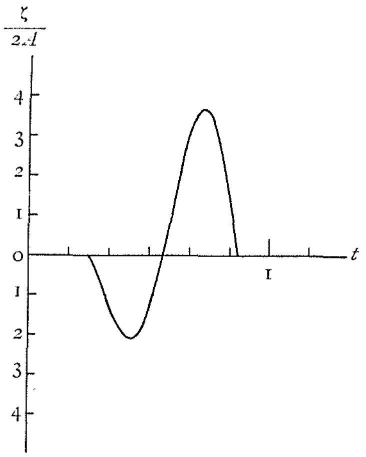



Fig. 3

Fluctuation of Sea-surface at the coast.




Even if $v=c$, the height $\zeta$ can not become infinite but reaches only about four-times the statical elevation, namely these circumstances support Nomitsu's[1] opinion which he expressed in reporting his investigation of Kaze-Tunami.

Now, if we assume $2A=(760-682)\times 13.6$ mm $=106$ cm (682 mm is the minimum reading of barometer at Muroto, Sept. 21, 1934), and calculate the minimum and the maximum value of $\zeta$ for the above three cases, then we have respectively 1) 1.37 m and 3.6 m, 2) 1.42 m and 1.42 m and 3) 2.22 m and 3.9 m. Formerly the writer has given a formula which gives the height of high water generated by the travelling gale of a large scale. If we take 30 m/sec as the wind velocity, and apply that formula to the present case, then the high waters of the three cases are respectively 42 cm, 32 cm and 42 cm. Thus it seems that the wind effect may not be so great as the pressure effect.

The heights of high water originating from wind and pressure become respectively about 4 m, 1.7 m and 4.3 m.

Since $v\doteqdot 60$ km/hour is considered the most appropriate value for the travelling velocity of the Muroto Typhoon of Sept. 21, 1934, 4 m may be taken as the total calculated height of the high water that attacked the southern coast of Shikoku in Japan. And this value will be sufficient to explain the actual height of water reported in the case of the Kaze-Tunami of Sept. 21, 1934.

1. *l. c. ante.* pp. 55, 56.

## III. When the Pressure Disturbance is a Function of Time, for Example, $P=tP(x+vt-a)$

Even when the travelling atmospheric disturbance is a function of time, the solutions for the unbounded sea, ( 5 ), ( 6 ), ( 5′ ) and ( 6′ ) can be used. If the effect of coast is considered, the solutions may be also obtained by the suitable combination of the reflected wave with the incident wave corresponding to ( 5 ), ( 6 ). But since to obtain the solutions for the general form of $P$ is difficult in respect to the integration, we shall show its possibility only with an example in which $P$ is a linear function of $t$.

### 1 ) Unbounded Sea.

In this case the disturbance is given by $tP(x+vt)$. Substituting it in ( 5 ), ( 6 ), (5′ ) and ( 6′ ) we have the following solutions :

$$\frac{h}{c^2}u=-\frac{1}{2}\int_0^t \tau P'(x-ct+\overline{c+v}\tau)d\tau-\frac{1}{2}\int_0^t \tau P'(x+ct-\overline{c-v}\tau)d\tau$$

$$=\frac{v}{c^2-v^2}tP(x+vt)+\frac{c^2+v^2}{(c^2-v^2)^2}\phi(x+vt)-\frac{1}{2(c-v)^2}\phi(x+ct)$$

$$-\frac{1}{2(c+v)^2}\phi(x-ct),$$

$$\zeta=-\frac{c}{2}\int_0^t \tau P'(x-ct+\overline{c+v}\tau)d\tau+\frac{c}{2}\int_0^t \tau P'(x+ct-\overline{c-v}\tau)d\tau$$

$$=-\frac{c^2}{c^2-v^2}tP(x+vt)-\frac{2vc^2}{(c^2-v^2)^2}\phi(x+vt)+\frac{c}{2(c-v)^2}\phi(x+ct)$$

$$-\frac{c}{2(c+v)^2}\phi(x-ct).$$

If $v=c$

$$\frac{h}{c^2}u=-\frac{1}{2}\int_0^t \tau P'(x+ct)d\tau-\frac{1}{2}\int_0^t \tau P'(x-ct+2c\tau)d\tau$$

$$=-\frac{1}{4}t^2P'(x+ct)-\frac{1}{4c}tP(x+ct)$$

$$+\frac{1}{8c^2}\{\phi(x+ct)-\phi(x-ct)\},$$

$$\zeta=\frac{c}{2}\int_0^t \tau P'(x+ct)d\tau-\frac{c}{2}\int_0^t \tau P'(x-ct+2c\tau)d\tau$$

$$=\frac{c}{4}t^2P'(x+ct)-\frac{1}{4}tP(x+ct)$$

$$+\frac{1}{8c}\{\phi(x+ct)-\phi(x-ct)\}.$$

where $\phi'=P$.

2 ) **Bounded Sea.**

In this case the disturbance is given by $tP(x+vt-a)$. By the suitable combination of the incident wave and the reflected wave, we get the required solutions as follows:

1 ) $0 \leq t \leq \frac{a-l}{c}$:

$$\frac{h}{c^2}u=\frac{v}{c^2-v^2}tP(x+vt-a)+\frac{c^2+v^2}{(c^2-v^2)^2}\phi(x+vt-a)$$
$$-\frac{1}{2(c-v)^2}\phi(x+ct-a)-\frac{1}{2(c+v)^2}\phi(x-ct-a),$$

$$\zeta=-\frac{c^2}{c^2-v^2}tP(x+vt-a)-\frac{2vc^2}{(c^2-v^2)^2}\phi(x+vt-a)$$
$$+\frac{c}{2(c-v)^2}\phi(x+ct-a)-\frac{c}{2(c+v)^2}\phi(x-ct-a).$$

2 ) $\frac{a-l}{c} \leq t \leq \frac{a+l}{c}$:

$$\frac{h}{c^2}u=\frac{v}{c^2-v^2}tP(x+vt-a)+\frac{c^2+v^2}{(c^2-v^2)^2}\phi(x+vt-a)$$
$$-\frac{1}{2(c-v)^2}\{\phi(x+ct-a)-\phi(ct-a-x)\}$$
$$-\frac{1}{2(c+v)^2}\phi(x-ct-a),$$

$$\zeta=-\frac{c^2}{c^2-v^2}tP(x+vt-a)-\frac{2vc^2}{(c^2-v^2)^2}\phi(x+vt-a)$$
$$+\frac{c}{2(c-v)^2}\{\phi(x+ct-a)+\phi(ct-a-x)\}$$
$$-\frac{c}{2(c+v)^2}\phi(x-ct-a).$$

3 ) $\frac{a+l}{c} \leq t \leq \frac{a-l}{v}$:

$$\frac{h}{c^2}u=\frac{v}{c^2-v^2}tP(x+vt-a)+\frac{c^2+v^2}{(c^2-v^2)^2}\phi(x+vt-a)$$
$$+\frac{1}{2(c-v)^2}\phi(ct-a-x)-\frac{1}{2(c+v)^2}\phi(x-ct-a),$$

$$\zeta=-\frac{c^2}{c^2-v^2}tP(x+vt-a)-\frac{2vc^2}{(c^2-v^2)^2}\phi(x+vt-a)$$
$$+\frac{c}{2(c-v)^2}\phi(ct-a-x)-\frac{c}{2(c+v)^2}\phi(x-ct-a).$$

4) $\dfrac{a-l}{v} \leq t \leq \dfrac{a+l}{v}$:

$$\frac{h}{c^2}u=\frac{v}{c^2-v^2}\left\{tP(x+vt-a)-\left(t-\frac{x}{c}\right)P\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$+\frac{c^2+v^2}{(c^2-v^2)^2}\left\{\phi(x+vt-a)-\phi\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$+\frac{1}{2(c-v)^2}\phi(ct-a-x)-\frac{1}{2(c+v)^2}\phi(x-ct-a),$$

$$\zeta=-\frac{c^2}{c^2-v^2}\left\{tP(x+vt-a)+\frac{v}{c}\left(t-\frac{x}{c}\right)P\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$-\frac{c^2}{(c^2-v^2)^2}\left\{2v\phi(x+vt-a)+\frac{c^2+v^2}{c}\phi\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$+\frac{c}{2(c-v)^2}\phi(ct-a-x)-\frac{c}{2(c+v)^2}\phi(x-ct-a).$$

5) $\dfrac{a+l}{v} \leq t$:

$$\frac{h}{c^2}u=-\frac{v}{c^2-v^2}\left(t-\frac{x}{c}\right)P\left(vt-a-\frac{v}{c}x\right)$$
$$-\frac{c^2+v^2}{(c^2-v^2)^2}\phi\left(vt-a-\frac{v}{c}x\right)+\frac{1}{2(c-v)^2}\phi(ct-a-x)$$
$$-\frac{1}{2(c+v)^2}\phi(x-ct-a),$$

$$\zeta=-\frac{vc}{c^2-v^2}\cdot\left(t-\frac{x}{c}\right)P\left(vt-a-\frac{v}{c}x\right)-\frac{c(c^2+v^2)}{(c^2-v^2)^2}\phi\left(vt-a-\frac{v}{c}x\right)$$
$$+\frac{c}{2(c-v)^2}\phi(ct-a-x)-\frac{c}{2(c+v)^2}\phi(x-ct-a).$$

3′) $\dfrac{a-l}{c} < \dfrac{a-l}{v} \leq t \leq \dfrac{a+l}{c}$:

$$\frac{h}{c^2}u=\frac{v}{c^2-v^2}\left\{tP(x+vt-a)-\left(t-\frac{x}{c}\right)P\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$+\frac{c^2+v^2}{(c^2-v^2)^2}\left\{\phi(x+vt-a)-\phi\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$-\frac{1}{2(c-v)^2}\{\phi(x+ct-a)-\phi(ct-a-x)\}$$
$$-\frac{1}{2(c+v)^2}\phi(x-ct-a),$$

$$\zeta=-\frac{c^2}{c^2-v^2}\left\{tP(x+vt-a)+\frac{v}{c}\left(t-\frac{x}{c}\right)P\left(vt-a-\frac{v}{c}x\right)\right\}$$

$$-\frac{c^2}{(c^2-v^2)^2}\left\{2v\phi(x+vt-a)+\frac{c^2+v^2}{c}\phi\left(vt-a-\frac{v}{c}x\right)\right\}$$
$$+\frac{c}{2(c-v)^2}\{\phi(x+ct-a)+\phi(ct-a-x)\}-\frac{c}{2(c+v)^2}\phi(x-ct-a).$$

4′) $\frac{a-l}{v}<\frac{a+l}{c}\leq t\leq\frac{a+l}{v}.$

5′) $\frac{a+l}{v}\leq t.$

The solutions of the stages 4′) and 5′) are the same as those of 4) and 5).

If $v=c$,

1) $0\leq t\leq\frac{a-l}{c}$:

$$\frac{h}{c^2}u=-\frac{1}{4}t^2P'(x+ct-a)-\frac{1}{4c}tP(x+ct-a)$$
$$+\frac{1}{8c^2}\{\phi(x+ct-a)-\phi(x-ct-a)\},$$

$$\zeta=\frac{c}{4}t^2P'(x+ct-a)-\frac{1}{4}tP(x+ct-a)$$
$$+\frac{1}{8c}\{\phi(x+ct-a)-\phi(x-ct-a).$$

2) $\frac{a-l}{c}\leq t\leq\frac{a+l}{c}$ :

$$\frac{h}{c^2}u=-\frac{1}{4}\left\{t^2P'(x+ct-a)-\left(t-\frac{x}{c}\right)^2P'(ct-a-x)\right\}$$
$$-\frac{1}{4c}\left\{tP(x+ct-a)-\left(t-\frac{x}{c}\right)P(ct-a-x)\right\}$$
$$+\frac{1}{8c^2}\{\phi(x+ct-a)-\phi(ct-a-x)-\phi(x-ct-a)\},$$

$$\zeta=\frac{c}{4}\left\{t^2P'(x+ct-a)+\left(t-\frac{x}{c}\right)^2P'(ct-a-x)\right\}$$
$$-\frac{1}{4}\left\{tP(x+ct-a)-\left(t-\frac{x}{c}\right)P(ct-a-x)\right\}$$
$$+\frac{1}{8c}\{\phi(x+ct-a)-\phi(ct-a-x)-\phi(x-ct-a)\}.$$

3) $\frac{a+l}{c}\leq t$ :

$$\frac{h}{c^2}u=\frac{1}{4}\left(t-\frac{x}{c}\right)^2P'(ct-a-x)+\frac{1}{4c}\left(t-\frac{x}{c}\right)P(ct-a-x)$$

$$-\frac{1}{8c^2}\{\phi(ct-a-x)+\phi(x-ct-a)\},$$

$$\zeta=\frac{c}{4}\left(t-\frac{x}{c}\right)^2 P'(ct-a-x)+\frac{1}{4}\left(t-\frac{x}{c}\right)P(ct-a-x)$$

$$-\frac{1}{8c}\{\phi(ct-a-x)+\phi(x-ct-a)\}.$$

By the direct substitutions of the above formulae in ( 1 ), ( 2 ) and ( 2′ ) we can verify the required solutions.

If the form of $P$ is more general, we may obtain the solutions by similar combinations of the incident wave and the reflected wave. Here we point out only the solutions of an example and the possibility that the solutions may be given for the general form of $P$, but give no numerical calculation.

Summary

1 ) In the unbounded sea the generated free wave may die away in a long time owing to the eddy viscosity. But in a sea bounded by a straight coast, after the disturbance has arrived at the fore-end of the continental shelf, there can be no question that the free wave will be newly generated on account of the sudden variation of depth. Therefore if we wish to treat the problem of the elevation of sea-surface in the continental shelf, we must start, as Prof. Nomitsu has pointed out, from the initial state of rest and consequently take into account the free wave. Thus we have obtained the appropriate solutions in the general form and explained the real phenomena on the occasion of Kaze-Tunami.

2 ) By considering the initial state, we have the solution which is physically reasonable even if $v=c$.

3 ) And we have shown, by an example, the possibility that the more general solutions will be obtained even if the travelling disturbance varies with the time and when the effect of a straight coast and the initial condition are considered. And thus we may say that we have advanced a step beyond the studies of these problems which many investigators have made hitherto.

In conclusion the writer wishes to express his sincere thanks to Prof. T. Nomitsu and Dr. T. Namekawa for their kind advice and encouragement during the study.