A Study of the Effect of a Local Wind upon the Sea-Surface and on the Development of the Internal Boundary Wave

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Abstract

In the present paper,[1] from a simple theoretical calculation, the writer first investigates the elevation or depression of the sea-surface and the variation of the total flow of the current generated by a *travelling local* gale of large scale (such as a cyclone), and secondly the behaviour of the generation of the internal long wave of great height.

Introduction

The motion of sea-water produced by a wind action has been investigated by Prof. T. Nomitsu,[2] Dr. K. Hidaka[3] and recently by Mr. G. Nishimura.[4] In this paper the writer will study the elevation or depression of the sea-surface generated by a *local* gale of large scale and of the development of the internal boundary wave (the elevation or depression of the internal surface discontinuous regarding the density).

In Chap. I. we give the fundamental equations and the conditions which are adequate for the present problem. In Chap. II. their solutions will be given, and moreover to give a concrete explanation of the solutions, we will calculate the following special cases: (1). The region, in which the wind-velocity is uniform, *travels* with a velocity V on the *unbounded* ocean of uniform depth. (2). The ocean is bounded by a *coast* and (a) a wind region travels towards the coast from the sea, (b) it travels towards the sea from the coast.

Specially, case (a) of (2) may be one of the causes of the abnormally high water (e. g. at Muroto Sept. 21, 1934) produced by a strong typhoon.

In Chap. III. we will give details of the mechanisms of the development of the internal boundary waves by taking a simple example as wind region.

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1. An outline of this paper was read at the Annual Meeting of Physico-Mathematical Society of Japan in April 1935. 2. These Memoirs, A, **17**, 249 (1934). 3. Geophy. Mag. Cent. Meteo. Obs. Tokyo. **7**, No. 3-4, 1933. 4. 地震研究所彙報第二號(昭和十年五月).

But, here, it must be noticed that the present problems are all treated in two dimensions (vertical and horizontal directions) without consideration of the effect of the earth's rotation, and the assumed boundary conditions are that no friction exists at the bottom or at the boundary surface.

I. Fundamental Equations and Conditions

Take the x-axis on the undisturbed sea-surface and the z-axis vertically upwards. Let us denote

g: acceleration of the gravity;
ρ: density of the sea water (assumed to be unity);
h: depth of ocean;
t: time;
u: velocity of the current in x-direction;
μ: coefficient of the eddy viscosity (assumed to be constant everywhere);
ζ: elevation or depression from the undisturbed surface;
T: wind traction (assumed to exist in a region of breath $2l$ only).

If the small terms may be neglected the Navier equation of motion and the equation of continuity may be written as follows.

$$\frac{\partial u}{\partial t}=\mu\frac{\partial^2 u}{\partial z^2}-g\frac{\partial \zeta}{\partial x} \qquad \text{......} (1),$$

$$\frac{\partial}{\partial x}\int_{-h}^{0} u dz=-\frac{\partial \zeta}{\partial t} \qquad \text{......} (2).$$

The surface and bottom conditions are assumed to be

$$\left.\begin{aligned} \mu\frac{\partial u}{\partial z}&=T \qquad \text{at } z=0 \\ \mu\frac{\partial u}{\partial z}&=0 \qquad \text{at } z=-h \end{aligned}\right\} \qquad \text{......} (3),$$

where $T=f(x, t)$ within the wind region $2l$, $T=0$ outside the region.

Now integrate (1) with respect to z from $-h$ to 0, and take into account eq. (3), then we have

$$\frac{\partial S}{\partial t}=T-gh\frac{\partial \zeta}{\partial x} \qquad \text{......} (1'),$$

where $S=\int_{-h}^{0} u dz$ represents the total flow perpendicular to the vertical section of unit width due to the drift- and slope-current.

By using the total flow S, the equation of continuity (2) may be written as

$$\frac{\partial S}{\partial x}=-\frac{\partial \zeta}{\partial t} \qquad \dots\dots\dots\dots (2').$$

Eliminate ζ from $(1')$ and $(2')$, then we have

$$\frac{\partial^2 S}{\partial t^2}=gh\frac{\partial^2 S}{\partial x^2}+\frac{\partial T}{\partial x} \qquad \dots\dots\dots\dots (4).$$

This is a fundamental equation for the present problem. The initial and boundary conditions are as follows.

$$\left.\begin{array}{l} S=0 \text{ and } \dfrac{\partial S}{\partial t}=T \left(\because \dfrac{\partial \zeta}{\partial x}=0\right) \text{ when } t=0 \\ S \text{ is continuous at the ends of the wind region} \end{array}\right\} \quad \dots(5),$$

and when the ocean is bounded by a coast

$$S=0 \quad \text{at the coast } (x=0). \qquad \dots\dots\dots\dots (6).$$

Now, if the solution S of eq. (4) satisfying conditions (5) and (6) can be obtained, substitute it into $(2')$; then ζ may be obtained by integrating it and using the condition that $\zeta=0$ when $t=0$.

Conversely, if S is eliminated from $(1')$ and $(2')$, the fundamental equation for ζ may be obtained as

$$\frac{\partial^2 \zeta}{\partial t^2}=\frac{\partial}{\partial x}\left(gh\frac{\partial \zeta}{\partial x}\right)-\frac{\partial T}{\partial x} \qquad \dots\dots\dots\dots (4')$$

and the initial and boundary conditions corresponding to (5) and (6) will be

$$\left.\begin{array}{l} \zeta=0 \text{ and } \dfrac{\partial \zeta}{\partial t}=0 \text{ when } t=0 \\ gh\left(\dfrac{\partial \zeta_T}{\partial x}-\dfrac{\partial \zeta_0}{\partial x}\right)=T \text{ at the end of wind region} \end{array}\right\} \quad \dots\dots(5'),$$

where ζ_T and ζ_0 express the elevations corresponding to the regions where T exists and does not exist respectively. When the ocean is bounded by a coast, the condition on the coast is

$$gh\frac{\partial \zeta}{\partial x}=T \qquad \text{at the coast } (x=0) \qquad \dots\dots\dots\dots (6').$$

Now, referring to eq. $(1')$ it must be here noticed that, as long as we use eq. (1), the solutions, which satisfy the condition such that the current u is nil on the coast and is continuous at the place where T changes abruptly, cannot generally be obtained. But if we use eq. (4) or $(4')$ instead of (1), without considering current u, we may obtain solutions appropriate mathematically and also physically, except for the current u in the neighbourhood of the coast. Thus the coast condition should be put $S=0$ or $gh\dfrac{\partial \zeta}{\partial x}=T$, but not $u=0$ or $\dfrac{\partial \zeta}{\partial x}=0$ as assumed by some authors.

Since eq. (4) and (4′) are obtained from the assumption of no bottom-friction and consequently contain no damping terms, the solution will afford only a quasi-steady state but cannot afford a steady state.

For practical purposes, the solution in the case of a bottom-friction is more desirable. If we assume the bottom condition as $\mu\frac{\partial u}{\partial z}=k\underline{u}$, then corresponding to (1′) we get

$$\frac{\partial S}{\partial t}=T-k\underline{u}-gh\frac{\partial \zeta}{\partial x} \qquad \text{......................(1″)},$$

and corresponding to (4)

$$\frac{\partial^2 S}{\partial t^2}+k\frac{\partial \underline{u}}{\partial t}=gh\frac{\partial^2 S}{\partial x^2}+\frac{\partial T}{\partial t} \qquad \text{..................(4″)},$$

is obtained, where $\underline{u}$ and k means the bottom velocity and the proportional constant respectively. If we assume suitably $\underline{u}$ as a function of S, we may obtain an approximate solution of S. For instance, in the case of an infinitely long canal, as $\underline{u}$ may be put to be proportional to S approximately, (4″) becomes

$$\frac{\partial^2 S}{\partial t^2}+2f\frac{\partial S}{\partial t}=gh\frac{\partial^2 S}{\partial x^2}+\frac{\partial T}{\partial t} \qquad \text{...............(4‴)}.$$

The rigorous solution of (4‴) is very troublesome, and we will develop our discussion only with eq. (4) in this paper.

II. Solutions

Case. 1. Unbounded Ocean

Solve eq. (4) using the condition (5), then we have

$$S=\frac{1}{\pi}\int_0^t d\tau\int_0^\infty d\alpha\int_{-\infty}^\infty T(\lambda,\tau)\cos\alpha c(t-\tau)\cos\alpha(\lambda-x)d\lambda \quad \text{......(7)},$$

and from (2′)

$$\zeta=-\frac{1}{c\pi}\int_0^t d\tau\int_0^\infty d\alpha\int_{-\infty}^\infty T(\lambda,\tau)\sin\alpha c(t-\tau)\sin\alpha(\lambda-x)d\lambda \quad \text{...(8)},$$

where $c=\sqrt{gh}$. From these it may be seen that the mode of the motion produced propagates with the long wave velocity towards right and left from the origin.

Now, especially, from the consideration of the following wind region for which

$$\left.\begin{array}{ll} T(x,t)=\text{const.} & \text{for } -l<x-Vt<l \\ T=0 & \text{for } x-Vt>l \text{ and } x-Vt<-l \end{array}\right\} \text{during } 0<t<t_1$$

and $T=0$ all over the sea-surface after $t>t_1$, we will give a concrete example of the effect of the bounded wind region travelling with constant velocity.

For $t_1 > t > 0$, eqs. (7) and (8) give

$$S = \frac{2TV}{\pi(c^2 - V^2)} \int_0^\infty \frac{\sin a(x - Vt) \sin al}{a^2} da + \frac{T}{\pi(c+V)} \int_0^\infty \frac{\sin a(x+ct) \sin al}{a^2} da$$
$$- \frac{T}{\pi(c-V)} \int_0^\infty \frac{\sin a(x-ct) \sin al}{a^2} da \qquad \ldots\ldots\ldots\ldots\ldots\ldots (9),$$

$$\zeta = \frac{2T}{\pi(c^2 - V^2)} \int_0^\infty \frac{\sin a(x - Vt) \sin al}{a^2} da - \frac{T}{\pi c(c+V)} \int_0^\infty \frac{\sin a(x+ct) \sin al}{a^2} da$$
$$- \frac{T}{\pi c(c-V)} \int_0^\infty \frac{\sin a(x-ct) \sin al}{a^2} da \qquad \ldots\ldots\ldots\ldots\ldots\ldots (9'),$$

and for $t > t_1$

$$S = \frac{T}{\pi(c-V)} \int_0^\infty \frac{\sin a(x - Vt_1 - c\overline{t - t_1}) \sin al}{a^2} da$$
$$- \frac{T}{\pi(c+V)} \int_0^\infty \frac{\sin a(x - Vt_1 + c\overline{t - t_1}) \sin al}{a^2} da$$
$$+ \frac{T}{\pi(c-V)} \int_0^\infty \frac{\sin a(x+ct) \sin al}{a^2} da - \frac{T}{\pi(c-V)} \int_0^\infty \frac{\sin a(x-ct) \sin al}{a^2} da$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10),$$

$$\zeta = \frac{T}{\pi c(c-V)} \int_0^\infty \frac{\sin a(x - Vt_1 - c\overline{t - t_1}) \sin al}{a^2} da$$
$$+ \frac{T}{\pi c(c+V)} \int_0^\infty \frac{\sin a(x - Vt_1 + c\overline{t - t_1}) \sin al}{a^2} da$$
$$- \frac{T}{\pi c(c-V)} \int_0^\infty \frac{\sin a(x+ct) \sin al}{a^2} da - \frac{T}{\pi c(c-V)} \int_0^\infty \frac{\sin a(x-ct) \sin al}{a^2} da$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10').$$

Now, carrying out the integration of eqs. (9) and (9′) from a well-known formula $\left(\int_0^\infty \frac{\sin ax \sin bx}{x^2} dx = \frac{\pi}{2} a \text{ for } 0 < a < b, \text{ and } = \frac{\pi}{2} b \text{ for } 0 < b < a\right)$, we have the following results for S and ζ:

1st term of S and $\zeta = \frac{VT}{c^2 - V^2}(x - Vt)$ and $\frac{T}{c^2 - V^2}(x - Vt)$

for $-l < x - Vt < l$,

" " $= \frac{\pm VT}{c^2 - V^2} l$ and $\frac{\pm T}{c^2 - V^2} l$

for $x - Vt > l$, $x - Vt < -l$;

2nd term of these $= \frac{T}{2(c+V)}(x+ct)$ and $\frac{T}{2c(c+V)}(x+ct)$

for $-l < x + ct < l$,

$$\text{,,} \qquad \text{,,} \qquad =\frac{\pm T}{2(c+V)}l \text{ and } \frac{\mp T}{2c(c+V)}l$$

for $x-ct>l,\ x-ct<-\iota$

$$\text{3rd term of these}=\frac{T}{2(c-V)}(x-ct) \text{ and } \frac{T}{2c(c-V)}(x-ct)$$

for $-l<x-ct<l,$

$$\text{,,} \qquad \text{,,} \qquad =\frac{\mp T}{2(c-V)}l \text{ and } \frac{\mp T}{2c(c-V)}l$$

for $x-ct>l,\ x-ct<-l,$

respectively.

From these formulae we know that the 1st term expresses the forced wave (due to the wind action) to be propagated with a velocity V towards the positive direction of x (travelling direction of the wind

Fig. 1a.

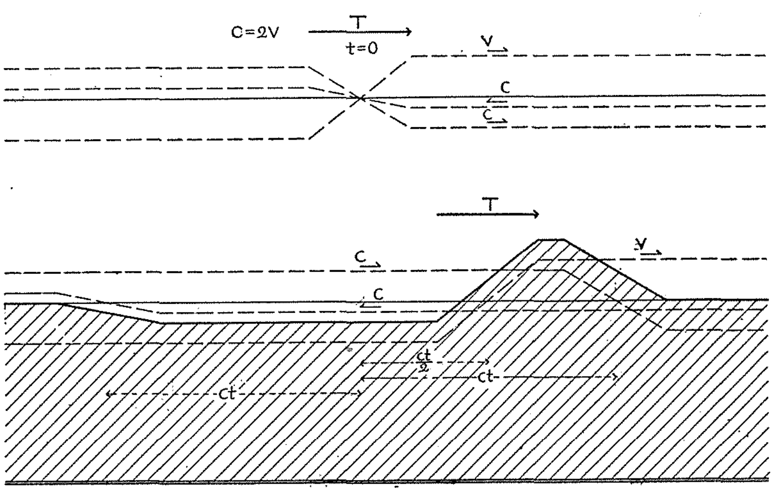
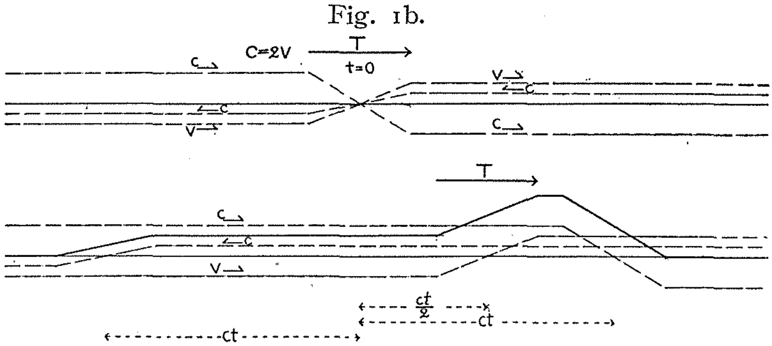


Fig. 1b.



region), and the 2nd and 3rd terms express the increase of the waves of free wave velocity $c=\sqrt{gh}$ towards the positive and the negative direction respectively. The general features of ζ and S are shown in Figs. 1a and 1b.

Of course, in the region of a cyclone the wind directions on both sides of the direction in which its centre is travelling are generally opposite to one another, but if we consider only one side, the case seems to belong to the present one.

After the wind traction has become nil (practically, after the force of a cyclone has become very small) S and ζ are given by (10) and (10′) respectively, but here to avoid redundancy we give no concrete explanation for this.

The formulae (9), (9′) and Figs. 1a and 1b show that the nearer the travelling velocity V approaches to the free wave velocity c the larger ζ and S become with time, and the height of the front part of the wave is greater than that of the rear part. But even if V is equal to c they cannot grow infinite instantly[1], as at a first sight. When V is equal to c, from the theory of differential equation the formulae corresponding to (9), (9′), (10) and (10′) will become as follows.

For $0<t<t_1$,

$$S=\frac{T}{\pi}\int_0^\infty\frac{t\cos a(x-ct)\sin al}{a^2}da+\frac{T}{2\pi c}\int_0^\infty\frac{\sin a(x+ct)\sin al}{a^2}da$$
$$-\frac{T}{2\pi c}\int_0^\infty\frac{\sin a(x-ct)\sin al}{a^2}da \qquad \dots\dots(9'')$$,

$$\zeta=\frac{T}{\pi c}\int_0^\infty\frac{t\cos a(x-ct)\sin al}{a^2}da+\frac{T}{2\pi c^2}\int_0^\infty\frac{\sin a(x-ct)\sin al}{a^2}da$$
$$-\frac{T}{2\pi c^2}\int_0^\infty\frac{\sin a(x+ct)\sin al}{a^2}da \qquad \dots\dots(9''')$$,

and for $t>t_1$,

$$S=\frac{T}{\pi}\int_0^\infty\frac{t_1\cos a(x-ct)\sin al}{a^2}da+\frac{T}{2\pi c}\int_0^\infty\frac{\sin a(x+ct)\sin al}{a^2}da$$
$$-\frac{T}{2\pi c}\int_0^\infty\frac{\sin a(x+ct-2ct_1)\sin al}{a^2}da \qquad \dots\dots(10'')$$,

$$\zeta=\frac{T}{\pi c}\int_0^\infty\frac{t_1\cos a(x-ct)\sin al}{a^2}da-\frac{T}{2\pi c^2}\int_0^\infty\frac{\sin a(x+ct)\sin al}{a^2}da$$

1. Moreover, when the friction is considered at the bottom, ζ and S cannot become infinitely large even after a very long time and even if V approaches c. This point has been shown recently by Prof. T. Nomitsu from a simple calculation, These Memoirs, A, **18**, 211 (1935).

$$+\frac{T}{2\pi c^2}\int_0^\infty \frac{\sin\alpha(x+ct-2ct_1)\sin\alpha l}{\alpha^2}d\alpha \qquad \ldots\ldots\ldots\ldots\ldots(10''').$$

These formulae (9''), (9'''), (10'') and (10''') show that ζ and S increase with time from zero, and that, in order to grow very large, T must continue for a fairly long time, namely the modes of the development are similar to the resonance phenomena in Acaustic.[1]

Case 2. Ocean bounded by a Straight Coast

Take the x-axis such that its origin lies on a coast and x is measured from the coast seawards. Solve eq. (4) under the conditions (5) (in this case supposing that the wind blows towards the coast, then (5) becomes $\frac{\partial S}{\partial t}=-T$) and (6), then we have

$$S=\frac{-2}{\pi}\int_0^t d\tau\int_0^\infty d\alpha\int_0^\infty T\sin\alpha x\sin\alpha\lambda\cos\alpha c(t-\tau)d\lambda \quad \ldots(11),$$

$$\zeta=\frac{2}{\pi c}\int_0^t d\tau\int_0^\infty d\alpha\int_0^\infty T\cos\alpha x\sin\alpha\lambda\sin\alpha c(t-\tau)d\lambda \quad \ldots(12).$$

Now, especially, we shall consider the same wind region as in the preceding case except the wind direction is opposite.

(a). *The case of the wind region travelling from the sea towards the coast*

We shall take the case where a wind region of width $2l$ suddenly generates at $x=a$ over the sea and advances towards the coast. Moreover, it seems to be appropriate to take the distance a as equal to the width of the continental shelf, because the action of the wind with regard to the surface elevation is generally large in shallow water only.

To calculate ζ and S for this case, it is convenient to divide the wind traction T into the following two cases. Namely, let $t_1=\frac{a}{V}$ be the time elapsing till the front portion of the wind region reaches the coast after having generated at $x=a$. Then during $0<t<t_1$, the existence domain of T is $a-Vt<\lambda<a+2l-Vt$.

Similarly if $t_2-t_1=\frac{2l}{V}$ be the time elapsing till the whole of this region is ashore, then for during $t_2>t>t_1$ the existence domain of T will be $0<\lambda<2l-V(t-t_1)$.

1. Detailed discussions concerning a similar problem will be found in Chap. III.

Now put $\int_{a-V\tau}^{a+2l-V\tau} T\sin\alpha\lambda d\lambda \equiv T_1(\tau)$, and $\int_0^{2l-V(\tau-t_1)} T\sin\alpha\lambda d\lambda \equiv T_2(\tau)$, and denote the elevations due to the action T_1 as ζ_1 for $t<t_1$ and as ζ_2 for $t>t_1$, then

$$\zeta_1=\frac{2}{\pi c}\int_0^\infty \cos\alpha x d\alpha\int_0^t T_1\sin\alpha c(t-\tau)d\tau$$

$$=-\frac{2T}{\pi(c^2-V^2)}\int_0^\infty\left\{\cos\alpha(a+2l-Vt)-\cos\alpha(a-Vt)\right\}\frac{\cos\alpha x}{\alpha^2}d\alpha$$

$$+\frac{T}{\pi c(c+V)}\int_0^\infty\left\{\cos\alpha(a+2l+ct)-\cos\alpha(a+ct)\right\}\frac{\cos\alpha x}{\alpha^2}d\alpha$$

$$-\frac{T}{\pi c(c-V)}\int_0^\infty\left\{\cos\alpha(a+2l-ct)-\cos\alpha(a-ct)\right\}\frac{\cos\alpha x}{\alpha^2}d\alpha,$$

$$\zeta_2=\frac{2}{\pi c}\int_0^\infty\cos\alpha x d\alpha\int_0^{t_1}T_1\sin\alpha c(t-\tau)d\tau=\frac{2T}{\pi(c^2-V^2)}\int_0^\infty\frac{\cos\alpha c(t-t_1)\cos\alpha x}{\alpha^2}d\alpha$$

$$-\frac{T}{\pi c(c+V)}\int_0^\infty\left\{\cos\alpha(a+ct)+\cos\alpha(2l+c\overline{t-t_1})-\cos\alpha(a+2l+ct)\right\}\times$$

$$\frac{\cos\alpha x}{\alpha^2}d\alpha-\frac{T}{\pi c(c-V)}\int_0^\infty\left\{\cos\alpha(a-ct)+\cos\alpha(2l-c\overline{t-t_1})\right.$$

$$\left.-\cos\alpha(a+2l-ct)\right\}\frac{\cos\alpha x}{\alpha^2}d\alpha.$$

Next also denote the elevations due to the action T_2 as ζ_3 for $t_1<t<t_2$ and as ζ_4 for $t>t_2$, then

$$\zeta_3=\frac{2}{\pi c}\int_0^\infty\cos\alpha x d\alpha\int_{t_1}^t T_2\sin\alpha c(t-\tau)d\tau$$

$$=\frac{-2T}{\pi c^2}\int_0^\infty\left\{\cos\alpha c(t-t_1)-1\right\}\frac{\cos\alpha x}{\alpha^2}d\alpha-\frac{2T}{\pi(c^2-V^2)}\times$$

$$\int_0^\infty\frac{\cos\alpha(2l+a-Vt)\cos\alpha x}{\alpha^2}d\alpha+\frac{T}{\pi c(c+V)}\int_0^\infty\frac{\cos\alpha(2l+c\overline{t-t_1})\cos\alpha x}{\alpha^2}d\alpha$$

$$+\frac{T}{\pi c(c-V)}\int_0^\infty\frac{\cos\alpha(2l-c\overline{t-t_1})\cos\alpha x}{\alpha^2}d\alpha,$$

$$\zeta_4=\frac{2}{\pi c}\int_0^\infty\cos\alpha x d\alpha\int_{t_1}^{t_2}T_2\sin\alpha c(t-\tau)d\tau$$

$$=-\frac{2T}{\pi c^2}\int_0^\infty\left\{\cos\alpha c(t-t_1)-\cos\alpha c(t-t_2)\right\}\frac{\cos\alpha x}{\alpha^2}d\alpha$$

$$-\frac{2T}{\pi(c^2-V^2)}\int_0^\infty\frac{\cos\alpha c(t-t_2)\cos\alpha x}{\alpha^2}d\alpha+\frac{T}{\pi c(c-V)}\times$$

$$\int_0^\infty\frac{\cos\alpha(2l+c\overline{t-t_1})\cos\alpha x}{\alpha^2}d\alpha+\frac{T}{\pi c(c+V)}\int_0^\infty\frac{\cos\alpha(2l-c\overline{t-t_1})\cos\alpha x}{\alpha^2}d\alpha,$$

by making suitable combinations of ζ_1, ζ_2, ζ_3 and ζ_4, the required elevation becomes

$$\left.\begin{array}{ll}\zeta=\zeta_1 & \text{for } 0<t<t_1 \\ \zeta=\zeta_2+\zeta_3 & \text{for } t_1<t<t_2 \\ \zeta=\zeta_2+\zeta_4 & \text{for } t>t_2\end{array}\right\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(12).$$

Now, put $x=0$ in the above formulae, then we have the time variation of the surface elevation at the coast. But there may be several different cases according to the relative magnitude of the free wave velocity (c) and the travelling velocity (V) of the wind region, i. e.,

$$\text{(a)} \quad \frac{a}{c}<\frac{a+2l}{c}<\frac{a}{V}, \qquad \text{(b)} \quad \frac{a}{c}<\frac{a}{V}<\frac{a+2l}{c},$$

and the cases of $\frac{a}{V}<\frac{a}{c}$. But as generally $V<c$, we will not concern for the last cases.

Now, integrate them by referring to a well known formula $\left(\int_0^\infty \frac{\cos bx}{x^2}dx = -\frac{\pi}{2}b \text{ for } b>0\right)$, then we have the following results.

For the case $\frac{a}{c}<\frac{a+2l}{c}<\frac{a}{V}$

$$\left.\begin{array}{ll}\zeta=0 & \text{for } t<\frac{a}{c} \\ \zeta=\frac{T(ct-a)}{c(c-V)} & \text{for } \frac{a}{c}<t<\frac{a+2l}{c} \\ \zeta=\frac{2Tl}{c(c-V)} & \text{for } \frac{a+2l}{c}<t<\frac{a}{V} \\ \zeta=\frac{T(a+2l-Vt)}{c(c-V)} & \text{for } \frac{a}{V}<t<\frac{a+2l}{V} \\ \zeta=0 & \text{for } \frac{a+2l}{V}<t\end{array}\right\} \qquad \ldots\ldots(13'),$$

and for the case $\frac{a}{c}<\frac{a}{V}<\frac{a+2l}{c}$

$$\left.\begin{array}{ll}\zeta=0 & \text{for } t<\frac{a}{c} \\ \zeta=\frac{T(ct-a)}{c(c-V)} & \text{for } \frac{a}{c}<t<\frac{a}{V} \\ \zeta=\frac{Tct}{c^2} & \text{for } \frac{a}{V}<t<\frac{a+2l}{c}\end{array}\right\} \qquad \ldots\ldots(13''),$$

$$\left.\begin{array}{ll}\zeta=\dfrac{T(a+2l-Vt)}{c(c-V)} & \text{for } \dfrac{a+2l}{c}<t<\dfrac{a+2l}{V} \\ \zeta=0 & \text{for } t>\dfrac{a+2l}{V}\end{array}\right\}$$

and as the extreme case if $V=c$

$$\begin{array}{ll}\zeta=0 & \text{for } t<\dfrac{a}{c} \\ \zeta=\dfrac{T}{c^2}ct & \text{for } \dfrac{a}{c}<t<\dfrac{a+2l}{c} \\ \zeta=0 & \text{for } t>\dfrac{a+2l}{c}.\end{array}$$

Now, we will give the more detailed interpretations about the development of the wave. We may obtain the following diagrams which express the relation between ζ and V, and that between V and the time (t_0) to attain the maximum wave height.

Fig. 2a

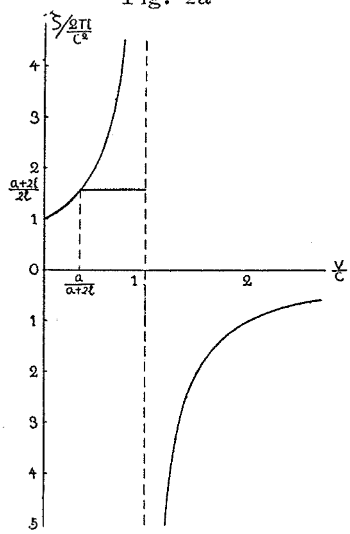
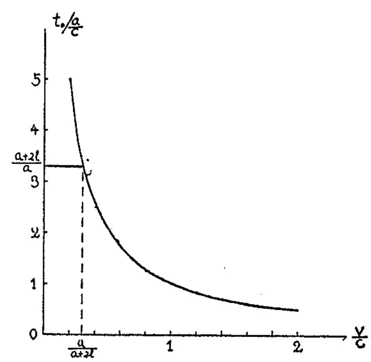


Fig. 2b



Referring to these, we know that if $V<\dfrac{a}{a+2l}c$, ζ increases with V approaching to c, but when $V\geqq\dfrac{a}{a+2l}c$, ζ takes a constant value $\dfrac{T(a+2l)}{c^2}$ (this is a maximum value due to a resonance effect) and increases no more even if V approaches to c still more, and untill

$V<\frac{a}{a+2l}c$, t_0 remains at a constant value $\frac{a+2l}{c}$ but for $V>\frac{a}{a+2l}c$, t_0 decreases inversely proportional to V and if V is equal to c it becomes $\frac{a}{c}$. Thus for an initially motionless sea, ζ can not increase infinitely (unless the width of the continental shelf a is very large), as generally considered, even if V is equal to c. If we don't consider the initial condition and treat the problem as quasisteady, there may exist a solution of ζ which gives an infinite value for $V=c$, yet it must correspond to the case where the width of the continental shelf a is infinitely large or t_0 is infinitely great. But, as such assumption is very far from the actuality, it may be said that the height of wave at the real coast can not be so large even by the resonance effect.

(*b*). *The case in which the region travels from the coast seawards*

We shall consider the case of the region having the width $2l$ beginning to advance from the coast seawards and dying away when its tail end reaches a point at a distance a from the coast (a is taken as the breadth of the continental shelf).

As in the preceding case it is convenient to divide the wind action T into the following two parts. Let $t_1=\frac{2l}{V}$ be the time elapsing before the whole region appears over the sea-surface after its front has reached the coast, and let $t_2-t_1=\frac{a}{V}$ be the time elapsing between the region's first appearance over the sea-surface and its almost total loss of action.

Then the existence domain of the wind traction T is obviously $0<\lambda<Vt$ for $0<t<t_1$, and $V(t-t_1)<\lambda<Vt$ for $t_1<t<t_2$.

Now put $\int_0^{V\tau} T\sin a\lambda d\lambda\equiv T_1(\tau)$, and $\int_{V(\tau-t_1)}^{V\tau} T\sin a\lambda d\lambda\equiv T_2(\tau)$,

and let the elevations due to the action T_1 be ζ_1 for $t<t_1$, and ζ_2 for $t>t_1$, then

$$\zeta_1=\frac{2}{\pi c}\int_0^\infty \cos ax da\int_0^t T_1\sin ac(t-\tau)d\tau=-\frac{2T}{\pi c^2}\int_0^\infty\frac{(\cos act-1)\cos ax}{a^2}da$$
$$-\frac{2T}{\pi(c^2-V^2)}\int_0^\infty\frac{(\cos aVt-\cos act)\cos ax}{a^2}da,$$
$$\zeta_2=\frac{2}{\pi c}\int_0^\infty \cos ax da\int_0^{t_1} T_1\sin ac(t-\tau)d\tau$$

$$=-\frac{2T}{\pi c^2}\int_0^\infty \frac{\{\cos act-\cos ac(t-t_1)\}\cos ax}{a^2}da+\frac{2T}{\pi(c^2-V^2)}\times$$
$$\int_0^\infty \frac{\cos act\cos ax}{a^2}da-\frac{T}{\pi c(c-V)}\int_0^\infty \frac{\cos a(ct-\overline{c-Vt_1})\cos ax}{a^2}da$$
$$-\frac{T}{\pi c(c-V)}\int_0^\infty \frac{\cos a(ct-\overline{c+Vt_1})\cos ax}{a^2}da.$$

Also let the elevations due to the action T_2 be ζ_3 for $t_1<t<t_2$ and ζ_4 for $t>t_2$, then

$$\zeta_3=\frac{2}{\pi c}\int_0^\infty \cos ax da\int_{t_1}^t T_2 \sin ac(t-\tau)d\tau=-\frac{2T}{\pi(c^2-V^2)}\times$$
$$\int_0^\infty\left\{\cos aVt-\cos aV(t-t_1)+\cos ac(t-t_1)\right\}\frac{\cos ax}{a^2}da+\frac{T}{\pi c(c-V)}\times$$
$$\int_0^\infty \frac{\cos a(ct-\overline{c-Vt_1})\cos ax}{a^2}da+\frac{T}{\pi c(c+V)}\int_0^\infty \frac{\cos a(ct-\overline{c+Vt_1})\cos ax}{a^2}da,$$

$$\zeta_4=\frac{2}{\pi c}\int_0^\infty \cos ax da\int_{t_1}^{t_2} T_2 \sin ac(t-\tau)d\tau=-\frac{2T}{\pi(c^2-V^2)}\times$$
$$\int_0^\infty \frac{\cos ac(t-t_1)\cos ax}{a^2}da-\frac{T}{\pi c(c+V)}\int_0^\infty\left\{\cos a(a+2l-c\overline{t-t_2})\right.$$
$$\left.-\cos a(2l-c\overline{t-t_1})-\cos a(a-c\overline{t-t_2})\right\}\frac{\cos ax}{a^2}da-\frac{T}{\pi c(c-V)}\times$$
$$\int_0^\infty\left\{\cos a(a+2l+c\overline{t-t_2})-\cos a(2l+c\overline{t-t_1})-\cos a(a+c\overline{t-t_2})\right\}\frac{\cos ax}{a^2}da.$$

By making suitable combinations of ζ_1, ζ_2, ζ_3 and ζ_4, we can get

$$\left.\begin{array}{ll}\zeta=\zeta_1 & \text{for } t<t_1\\ \zeta=\zeta_2+\zeta_3 & \text{for } t_1<t<t_2\\ \zeta=\zeta_2+\zeta_4 & \text{for } t>t_2\end{array}\right\}\quad\ldots\ldots\ldots\ldots(14).$$

Then time relation of the surface elevation at the coast may be given by

$$\left.\begin{array}{ll}\zeta=\dfrac{Vt}{c(c+V)}T & \text{for } t<\dfrac{2l}{V}\\ \zeta=\dfrac{T.2l}{c(c+V)} & \text{for } \dfrac{2l}{V}<t<\dfrac{a}{c}+\dfrac{a+2l}{V}\\ \zeta=\dfrac{T(a+2l-c\overline{t-t_2})}{c(c+V)} & \text{for } \dfrac{a}{c}+\dfrac{a+2l}{V}<t<\left(\dfrac{1}{V}+\dfrac{1}{c}\right)(a+2l)\\ \zeta=0 & \text{for } t>\left(\dfrac{1}{V}+\dfrac{1}{c}\right)(a+2l)\end{array}\right\}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14').$$

As the maximum height of the elevation at the coast is given by $\zeta=\dfrac{T.2l}{c(c+V)}$ ζ decreases with V approaching to c in opposition to the preceding case (a). Namely *in this case the so-called resonance effect does not occur at all,* and the elevation of the water surface at the coast is smaller than that of the case (a) if the other conditions are the same. And the ratio of the elevations for the above two cases is given by $\frac{c+V}{c-V}$ if $\frac{a}{c}<\frac{a+2l}{c}<\frac{a}{V}$, or by $\frac{(a+2l)(c+V)}{2cl}$ if $\frac{a}{c}<\frac{a}{V}<\frac{a+2l}{c}$.

J. Proudman, in the investigations about the effect of the travelling pressure disturbance over the sea, has obtained the theoretical result that the ratio of the wave heights in the above two cases is given by $\frac{c+V}{c-V}$, and this form holds for the all possible value of V. But in the present investigation (about the effect of the travelling local gale) this form holds only when $\frac{a}{c}<\frac{a+2l}{c}<\frac{a}{V}$ as above stated.

Now, we will give the numerical explanation about the result now obtained. We have known already that ζ takes the form (13′) or (13″) according to $\frac{c}{V}$ being greater or smaller than $\frac{a+2l}{a}$. But along the coast of the Japanese island (for an example) the widths of the continental shelves a are generally 20 km～35 km and the length of the wind region may be considered to be about 40 km～60 km, therefore $\frac{a+2l}{a}$ (it may be called the critical value of resonance from its physical meaning) takes the value of 2～4.

Put $a=30$ km, $2l=50$ km, wind velocity $W=30$ m/sec and $h=50$ m, then $\frac{a+2l}{a}\doteqdot 2.7$, $T=0.0025\cdot 0.00129\cdot(30\cdot 10^2)^2$ c.g.s.$\doteqdot 30$ C.G.S. and $c^2=gh=5\cdot 10^6$c.g.s. or $c=80$ km/hour. If $V<\frac{c}{2.8}$ or $V<30$km/hour the surface elevation in the case (a) is given by $\zeta=\frac{24}{\left(1-\frac{V}{c}\right)}$cm, and ζ increases with V; but when V exceeds $\frac{c}{2.7}$ i. e., V is over 30 km/hour, then from (13″) the elevation takes a constant value 42cm and increases no more even if V becomes equal to c. In the case (b),

ζ is given by $\dfrac{24}{\left(1+\dfrac{V}{c}\right)}$ cm. Supposing $V=\dfrac{c}{2.7}$, then ζ is 17 cm and it is about $\dfrac{1}{2.5}$ of that of the case (a).

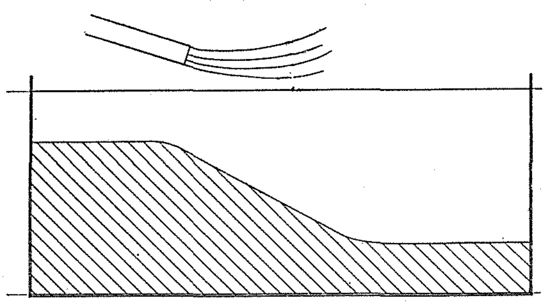
III. Internal Waves of Meteorological Origin

The sea occasionary consists of two horizontal layers, whose densities are different from one another but may be considered uniform within each layer. On the boundary surface of these two layers the phenomena of the so-called internal wave of great height are often observed. These peculiar phenomena were first observed by Dr. Helland-Hansen and Dr. Nansen, namely, during the North Polar expedition from May to July, 1894. They observed that, at a depth 200–300 m in the sea, the density was discontinuous, and that the transition layer was rising and falling to the extent of about 20–30 metres. After then W. J. Sandström endeavoured to explain experimentally the mechanism of this peculiar phenomenon. Fig. 4 shows the vertical section of the water-tank used for the experiment, in which the upper layer is light water and the lower one is heavy water, and is coloured to make the boundary surface distinct. By letting a strong, narrow air-current suddenly act locally, at a certain place on the surface of the light water-layer, this layer was put in motion in the direction of the air-current, and the underlying heavier water at this place and behind it, was lifted to form a great wave, while in front of it the heavier water layer was depressed. If the air-current continues to act on the surface at the same spot for some time, the wave in the underlying heavier water will remain permanent, altering its shape gradually untill new condition of stable equilibrium is attained. If the spot at which the air-current acts on the water-surface moves forward (in the direction of the air-current), the wave will follow in the same direction.

If the air-current suddenly stops, the wave will continue its course as a boundary-wave, untill it reaches the wall of the vessel, or if we think of the sea, untill it reaches the side-slope of the sea-basin. Here it will be partly broken and reflected, and will again pass across the basin, till it reaches the other side.

In the same manner a sudden storm or wind on a sea affects the water strata and may produce a boundary wave of great height. This explanation is an outline of Sandström's experiment, and the present

Fig. 3



Boundary-Wave formed by Local Air-Current (cf. Sandström)

writer will proceed from a theoretical standpoint with his discussion of these phenomena, and especially of the aspect of the development and the propagation of waves produced by a sudden storm or wind.

Now before proceeding with our discussion, we shall make the following assumptions, namely that the ocean consists of two layers whose densities are different through uniform in each layer, that there is no friction at the bottom and on the boundary surface, and that the problem may be considered as of two dimensions.

Now let us denote the density, the thickness, the surface elevation and the current velocity of x-direction in the upper and the lower strata by (ρ_1, ρ_2), (h_1, h_2), (ζ_1, ζ_2) and (u_1, u_2) respectively, then by neglecting the small terms, the equation of motion and the equation of continuity become

$$\rho_1\frac{\partial u_1}{\partial t}=\mu_1\frac{\partial^2 u_1}{\partial z^2}-g\rho_1\frac{\partial \zeta_1}{\partial x} \qquad \text{for upper layer} \quad \ldots(15),$$

$$\rho_2\frac{\partial u_2}{\partial t}=\mu_2\frac{\partial^2 u_2}{\partial z^2}-g\rho_1\frac{\partial \zeta_1}{\partial x}-g(\rho_2-\rho_1)\frac{\partial \zeta_2}{\partial x} \qquad \text{for the lower layer} \ldots\ldots(16),$$

$$\frac{\partial}{\partial x}\int_{-h_1}^{0}\rho_1 u_1 dz=-\rho_1\left(\frac{\partial \zeta_1}{\partial t}-\frac{\partial \zeta_2}{\partial t}\right) \qquad \ldots\ldots(17),$$

$$\frac{\partial}{\partial x}\int_{-(h_1+h_2)}^{-h_1}\rho_2 u_2 dz=-\rho_2\frac{\partial \zeta_2}{\partial t} \qquad \ldots\ldots(18).$$

The conditions at the upper surface, at the intermediate surface and at the bottom are respectively

$$\left.\begin{aligned} &\mu_1\frac{\partial u_1}{\partial z}=T && \text{for } z=0 \\ &\mu_1\frac{\partial u_1}{\partial z}=0,\ \mu_2\frac{\partial u_2}{\partial z}=0 && \text{for } z=-h_1 \\ &\mu_2\frac{\partial u_2}{\partial z}=0 && \text{for } z=-(h_1+h_2) \end{aligned}\right\} \quad \ldots(19),$$

and the initial condition is

$$u_1=u_2=0 \qquad \text{when } t=0 \qquad \text{...........(20)}.$$

Now integrating (15) and (16) with regard to z in the intervals $(0, -h_1)$ and $(-h_1, -(h_1+h_2))$ respectively and putting $S_1 \equiv \int_{-h_1}^{0} \rho_1 u_1 \, dz$, and $S_2 \equiv \int_{-(h_1+h_2)}^{-h_1} \rho_2 u_2 \, dz$, then we have

$$\frac{\partial S_1}{\partial t} = T - g\rho_1 h_1 \frac{\partial \zeta_1}{\partial x} \qquad \text{..............................(15')},$$

$$\frac{\partial S_2}{\partial t} = -g\rho_1 h_2 \frac{\partial \zeta_1}{\partial x} - g(\rho_2-\rho_1)h_2 \frac{\partial \zeta_2}{\partial x} \qquad \text{...........(16')},$$

$$\frac{\partial S_1}{\partial x} = -\rho_1\left(\frac{\partial \zeta_1}{\partial t} - \frac{\partial \zeta_2}{\partial t}\right) \qquad \text{.......................(17')},$$

$$\frac{\partial S_2}{\partial x} = -\rho_2 \frac{\partial \zeta_2}{\partial t} \qquad \text{..................................(18')}.$$

At first, let us assume that T is independent of t, then eliminating ζ_1, ζ_2, and S_2 from the above equations, we get

$$q\frac{\partial^4 S_1}{\partial t^4} - c^2(1+q)\frac{\partial^4 S_1}{\partial t^2 \partial x^2} + c^4(1-p)\frac{\partial^4 S_1}{\partial x^4} = 0 \qquad \text{...(21)},$$

where $q \equiv \dfrac{h_1}{h_2}$, $p \equiv \dfrac{\rho_1}{\rho_2}$, $c^2 \equiv g h_1$.

Solve (21) under the initial conditions $S_1=S_2=\zeta_1=\zeta_2=0$ at $t=0$ or their equivalents

$$S_1=0, \quad \frac{\partial S_1}{\partial t}=T, \quad \frac{\partial^2 S_1}{\partial t^2}=0, \quad \frac{\partial^3 S_1}{\partial t^3}=c^2\frac{\partial^2 T}{\partial x^2} \quad \text{when } t=0 \qquad \text{.....................................(22)}.$$

Since eqs. (15′) to (18′) are obviously "additive" with respect to T in the meaning defined by Prof. T. Nomitsu, we can easily get the solution by his theorem[1] when T is variable with t. Then we have

$$S_1 = \frac{c^2-c_2^2}{\pi(c_1^2-c_2^2)}\int_0^t d\tau \int_0^\infty da \int_{-\infty}^{\infty} T\cos ac_1(t-\tau)\cos a(\lambda-x)d\lambda$$
$$+ \frac{c_1^2-c^2}{\pi(c_1^2-c_2^2)}\int_0^\tau d\tau \int_0^\infty da \int_{-\infty}^{\infty} T\cos ac_2(t-\tau)\cos a(\lambda-x)d\lambda \qquad \text{...(23)},$$

where $c_1^2=g(h_1+h_2)$, $c_2^2=\dfrac{g(\rho_2-\rho_1)h_1h_2}{\rho_2(h_1+h_2)}$,

$$\zeta_1 = -\frac{c_1(c^2-c_2^2)}{\pi\rho_1 c^2(c_1^2-c_2^2)}\int_0^t d\tau \int_0^\infty da \int_{-\infty}^{\infty} T\sin ac_1(t-\tau)\sin a(\lambda-x)d\lambda$$
$$-\frac{c_2(c_1^2-c^2)}{\pi\rho_1 c^2(c_1^2-c_2^2)}\int_0^t d\tau \int_0^\infty da \int_{-\infty}^{\infty} T\sin ac_2(t-\tau)\sin a(\lambda-x)d\lambda \qquad \text{...(24)},$$

1. Proc. Imp. Acad. Tokyo, **11**, 359 (1935).

or $$\zeta_2=\frac{c_1(c^2-c_2^2)}{\pi\rho_1(c_1^2-c_2^2)}\left(\frac{1}{c^2}-\frac{1}{c_1^2}\right)\int_0^t d\tau\int_0^\infty da\int_{-\infty}^\infty T\sin ac_1(t-\tau)\sin a(\lambda-x)d\lambda$$
$$-\frac{c_2(c_1^2-c^2)}{\pi\rho_1(c_1^2-c_2^2)}\left(\frac{1}{c^2}-\frac{1}{c_2^2}\right)\int_0^t d\tau\int_0^\infty da\int_{-\infty}^\infty T\sin ac_2(t-\tau)\sin a(\lambda-x)d\lambda \quad \text{.......(25)},$$

$$S_2=\frac{\rho_2(c^2-c_2^2)(c_1^2-c^2)}{\pi\rho_1 c^2(c_1^2-c_2^2)}\int_0^t d\tau\int_0^\infty da\int_{-\infty}^\infty T\cos a(\lambda-x)\times \{\cos ac_2(t-\tau)-\cos ac_1(t-\tau)\}d\lambda \quad \text{.......(26)}.$$

Eqs. (24) and (25) will give, respectively, the elevations of the upper surface and of the lower intermediate surface. They contain two waves propagating with the external wave velocity c_1 and the internal wave velocity c_2.

Now, in order to obtain a concrete example, we shall consider the same wind region as that of Chap. II, namely $T=\text{const}$ for $-l<x-Vt<l$ and $T=0$ for $x-Vt>l$ and $x-Vt<-l$. Then

$$\zeta_1=\frac{c^2-c_2^2}{\pi\rho_1(c_1^2-c_2^2)c^2}\left\{\frac{2T}{1-\frac{V^2}{c_1^2}}\int_0^\infty\frac{\sin a(x-Vt)\sin al}{a^2}da-\frac{2T}{1-\frac{V}{c_1}}\times\right.$$
$$\left.\int_0^\infty\frac{\sin a(x-c_1t)\sin al}{a^2}da-\frac{T}{1+\frac{V}{c_1}}\int_0^\infty\frac{\sin a(x+c_1t)\sin al}{a^2}da\right\}$$
$$+\frac{c_1^2-c^2}{\pi\rho_1c^2(c_1^2-c_2^2)}\left\{\frac{2T}{1-\frac{V^2}{c_2^2}}\int_0^\infty\frac{\sin a(x-Vt)\sin al}{a^2}da-\frac{T}{1-\frac{V}{c_2}}\times\right.$$
$$\left.\int_0^\infty\frac{\sin a(x-c_2t)\sin al}{a^2}da-\frac{T}{1+\frac{V}{c_2}}\int_0^\infty\frac{\sin a(x+c_2t)\sin al}{a^2}da\right\}\text{...(27)},$$

$$\zeta_2=\frac{c^2-c_2^2}{\pi\rho_1(c_1^2-c_2^2)}\left(\frac{1}{c^2}-\frac{1}{c_1^2}\right)\left\{\frac{2T}{1-\frac{V^2}{c_1^2}}\int_0^\infty\frac{\sin a(x-Vt)\sin al}{a^2}da\right.$$
$$\left.-\frac{T}{1-\frac{V}{c_1}}\int_0^\infty\frac{\sin a(x-c_1t)\sin al}{a^2}da-\frac{T}{1+\frac{V^2}{c_1^2}}\int_0^\infty\frac{\sin a(x+c_1t)\sin al}{a^2}da\right\}$$
$$+\frac{c_1^2-c^2}{\pi\rho_1(c_1^2-c_2^2)}\left(\frac{1}{c^2}-\frac{1}{c_2^2}\right)\left\{\frac{2T}{1-\frac{V^2}{c_2^2}}\int_0^\infty\frac{\sin a(x-Vt)\sin al}{a^2}da\right.$$
$$\left.-\frac{T}{1-\frac{V}{c_2}}\int_0^\infty\frac{\sin a(x-c_2t)\sin al}{a}da-\frac{T}{1+\frac{V}{c_2}}\int_0^\infty\frac{\sin a(x+c_2t)\sin al}{a^2}da\right\}$$
$$\text{.......(27')}.$$

Now, if, for instance, we put $\rho_1 = 1.0275$, $\rho_2 - \rho_1 = 0.0005$, $h_1 = 100$ m, $h_2 = 200$ m and $g = 9.8$ m/sec, we get $c_1 = 450$ m/sec and $c_2 = 0.68$ m/sec. Therefore $\frac{c^2}{c_1^2} = 5.10^{-3}$, $\frac{c_2^2}{c_1^2} = 3.10^{-6}$, $\frac{c_2^2}{c^2} = 5.10^{-4}$ are very small and may be neglected in comparison with unity. Moreover the first two terms become large when V approaches to c_1 very closely, but since here c_1 is 450 m/sec and the value of V, even if it is estimated very highly, is about 30 m/sec, in this deep sea V cannot actually approach to c_1, and consequently the first three terms of ζ_2 (two free waves of the velocity c_1 and the forced wave of the velocity V) are very small. However, as c_2 is proportional to $(\rho_2 - \rho_1)^{\frac{1}{2}}$, the terms containing c_2 in denominators are very large compared with the other terms. Therefore, neglecting the small terms, the main features of the internal waves may be given by the following forms.

$$\zeta_2 = \frac{T}{\pi\rho_1 c_2^2}\left\{\frac{1}{1-\frac{V}{c_2}}\int_0^\infty \frac{\sin a(x-c_2t)\sin al}{a^2}da + \frac{1}{1+\frac{V}{c_2}}\times\right.$$
$$\left.\int_0^\infty \frac{\sin a(x+c_2t)\sin al}{a^2}da - \frac{2}{1-\frac{V^2}{c_2^2}}\int_0^\infty \frac{\sin a(x-Vt)\sin al}{a^2}da\right.$$
$$\qquad\qquad\qquad\qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28')$$

This formula is similar to (9′), therefore it may be written as follows :—

$$\left.\begin{aligned}
&\text{1st term} = \frac{T}{2(c_2-V)c_2\rho_1}(x-c_2t) && \text{for } -l<x-c_2t<l \\
&\quad ,, \quad = \frac{\pm T}{2(c_2-V)c_2\rho_1}l && \text{for } x-c_2t>l \text{ and } x-c_2t<-l \\
&\text{2nd term} = \frac{T}{2(c_2+V)c_2\rho_1}(x+c_2t) && \text{for } -l<x+c_2t<l \\
&\quad ,, \quad = \frac{\pm T}{2(c_2+V)c_2\rho_1}l && \text{for } x+c_2t>l \text{ and } x+c_2t<-l \\
&\text{3rd term} = \frac{T}{(c_2^2-V^2)\rho_1}(x-Vt) && \text{for } -l<x-Vt<l \\
&\quad ,, \quad = \frac{\mp T}{(c_2^2-V^2)\rho_1}l && \text{for } x-Vt>l \text{ and } x-Vt<-l
\end{aligned}\right\}$$
$$\qquad\qquad\qquad\qquad\qquad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(28'')$$

The above formulae are expressed diagramatically in Fig. 4a, 4b and 4c, where the three cases $V=0$ and $V \gtrless c_2$ are given, but among these the cases $V<c_2$ and $V=0$ seem to correspond to those of the Sandström experiment. However, since the travelling velocity of the

Fig. 4a.

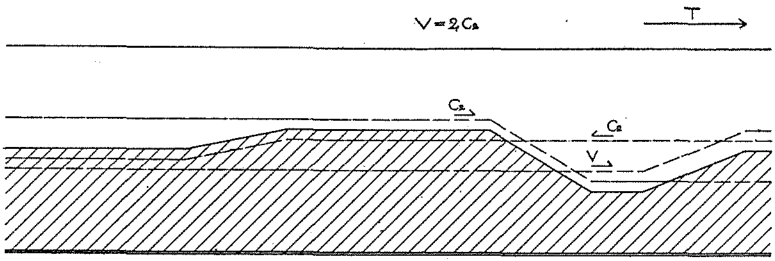


Fig. 4b.

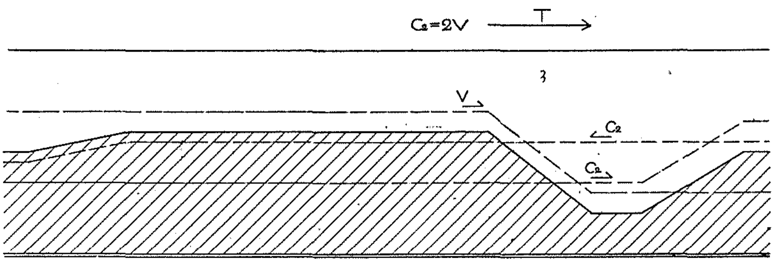
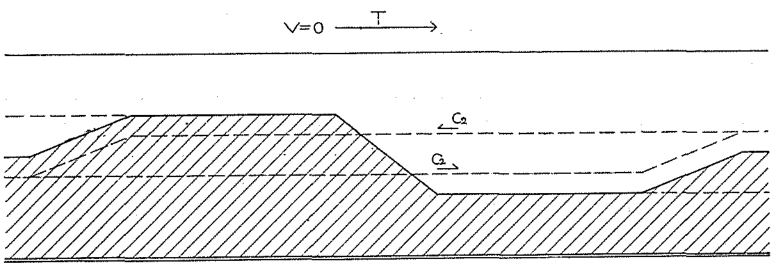


Fig. 4c.

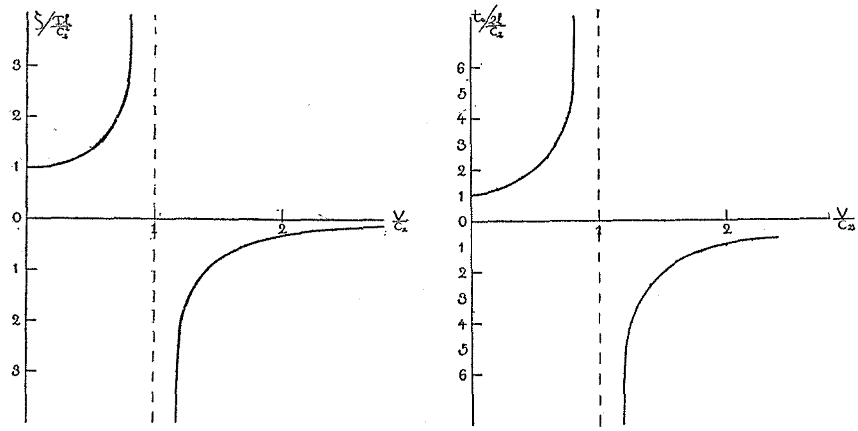


wind region is generally greater than the internal wave velocity c_2, the circumstances are not generally like those of the Sandström experiment.

Now let us discuss the results of (28′) in detail. The maximum height of the wave is given by $\zeta_2=\dfrac{Tl}{(c_2^2-V^2)\rho_1}$, where $c_2=\left(\dfrac{g(\rho_2-\rho_1)h_1h_2}{\rho_2(h_1+h_2)}\right)^{\frac{1}{2}}$ and it is about 1 m/sec for a deep sea such as here considered. And since the time that the wave height becomes to maximum is given by $t_0=\dfrac{2l}{c_2-V}$ and it is related to the travelling velocity V like that of ζ_2, to discuss the development of the large wave heights we must, in addition to the values T and l, take into account the effect of the value V for them.

For the above purpose, in Fig. 5a and 5b we shall give the relations of $\frac{V}{c_2}$ to ζ_2 or t_0, where $\frac{\zeta}{Tl/c_2^2}$ and $\frac{t_0}{2l/c_2}$ are taken as the ordinate and $\frac{V}{c_2}$ as the abscissa. Now if, as a very usual example, we take $l=20$ km, $c_2=0.68$ m/sec, wind velocity $W=20$ m/sec and use $T=0.0025\,\rho_{air}W^2$, then we get $\frac{Tl}{c_2^2} \doteqdot 40$ m and $t_0 \doteqdot 16$ hours.

Fig. 5a and 5b.



By referring to the above two figures we obtain the following results, namely that the waves usually observed (about 40 metres in height) correspond to the neighbourhood $V=0$, that, for the generation of such a large wave of about 200 m height, the wind velocity and the area of the wind region must be, of course, very large, but also it is necessary that the wind region travelling with a velocity nearly equal to c_2 continues to act for a long time and consequently the so-called resonance phenomena occurs, and on the other hand that, if the travelling velocity of the wind region is much larger than c_2, the wave cannot become large even if the wind velocity and the width of the region are great.

From the above discussion it has become clear that large internal waves can be produced only under suitable conditions as above stated, even under the effect of a local gale of large scale.

We shall next touch the effect of the pressure disturbance for the development of the internal wave briefly. It is clear that the formulae giving the elevations of the upper surface and of the lower boundary surface under the effect of the travelling atmospheric pressure disturbance are

$$\zeta_1=\frac{c_1}{2}\int_0^t\left\{\gamma(x-c_1t+\overline{c_1-V}\tau)-\gamma(x+c_1t-\overline{c_1+V}\tau)\right\}d\tau,$$

$$\zeta_2=\frac{c^2-c_1^2}{2c_1}\int_0^t\left\{\gamma(x+c_1t-\overline{c_1+V}\tau)-\gamma(x-c_1t+\overline{c_1-V}\tau)\right\}d\tau,$$

where γ is $\dfrac{\partial\bar{\zeta}}{\partial x}$ and $\bar{\zeta}$ is the atmospheric pressure measured by the water column.

Now it must be noticed that, since the above ζ_1 and ζ_2 contain no terms inversely proportional to $\rho_2-\rho_1$, the *waves cannot be developed greatly by pressure disturbance as by the wind action*, and also that, since the wave velocity is in all cases the external free wave velocity and is not the internal one, in such a deep sea as that under consideration the so-called *resonance phenomena due to the travelling pressure disturbance can scarcely occur in practice.*

Therefore the great internal wave cannot develop from the pressure disturbance, even if the travelling effect is considered. The above results have been deduced from the most idealized case, but we infer the general features about the development of the internal boundary waves of the large height from them.

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