A Theory of Tunamis and Seiches produced by Wind and Barometric Gradient

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Abstract

This paper has been prepared from my previous papers on the "Coast effect upon the ocean current and sea level", in order to serve for investigations of the Tunamis caused by the typhoon that attacked the Kwansai District of Japan on Sept. 21 last year. At first, a correction is given to the ordinary formula for the steady value of surface elevation in shallow water, discarding the customary assumption that the elevation of surface is negligibly small compared with the total depth of the water. Next the development of tunamis and seiches under the condition of bottom-slip is dealt with in the case of varying causal action, and also extended to the case of progressive action. The viscosity of water, of course, is taken into account so that even for a perfectly resonancing action we can deduce reasonable results such as that the surface fluctuation cannot be amplified over $\pi/2$ times the steady value during half a period (or one crest) only of the causal force.

Introduction

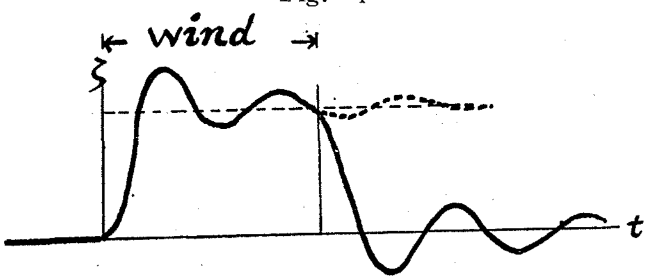
On September 21 last year, a terrible typhoon attacked the Kwansai District and caused disastrous tunamis along the coasts of Ôsaka Bay and the Ki-i Channel, and also produced unprecedented undulations of water surface of Lake Biwa. The writer soon began to inspect and survey these tunami and seiche phenomena with his students and to collect mareo- and limnograms. In order to adjust the results thus obtained and explain them, an adequate theory of meteorological tunamis and seiches is highly desirable. Fortunately the writer's paper[1] on "Coast effect upon the ocean current and sea level" satisfies the demand with slight modifications, and may be taken as a theory of tunamis and seiches produced by wind and barometric gradient. The said paper will show that when a constant wind or barometric gradient begins to act upon a water basin and when after a time t it stops suddenly, then there occurs a rise of mean sea level generally accompanied by oscillations somewhat like those of Fig. 1, which are nothing but seiche oscillations; and if the level of the surface rises to such a degree that the water inundates the coast land, it may be

1. These Memoirs, **17**, 93—141 and 249—280 (1934)

called a tunami. Thus the meteorological tunamis and seiches are not essentially different in nature, and may be discussed together. Similarly we should expect the corresponding fluctuations of the water level whenever the motive forces increase or decrease *suddenly*.

The previous paper, however, was written from the view-point of ocean currents, and as a practical theory of tunamis and seiches

Fig. 1



there remain several points to be modified or supplemented, while in some respects the theory may be simplified. For instance, in the theory of ocean currents the surface elevation may be considered negligibly small with respect to the depth of the sea, but, in the case of tunami it cannot be neglected, nay, it becomes rather greater than the usual depth of the shallow water near the coast. For the ocean current we may be content to consider the motive action as constant, but, when the problem of tunamis and seiches is under consideration, a varying or progressive force should be dealt with, resonance-effect playing an important part in the discussion.

On these problems, Dr. J. Proudman has written two papers. One[1] is very excellent as a theory of meteorological perturbations in a viscous canal with " no bottom-current ", but I believe that a rapid current in a shallow sea such as tunami must have considerable slip-velocity. Proudman's second paper[2] is also very original in dealing with the effect of progression and resonance due to the barometric change. Since the water there, however, is assumed as an ideal fluid (viscosity $\mu \equiv 0$), the method cannot be extended to the case of wind action, and even for the barometric action some[3] of the results will not necessarily coincide with that in the case $\mu \to 0$, especially when the barometric wave has a velocity $\sqrt{gH}$ resonancing with the water depth H.

1. Proc. London Math. Soc., **24**, 140 (1924)
2. Month. Not. Roy. Ast. Soc., Geophys. Suppl. **2**, 197 (1929)
3. Eq. (2. 3) and (2. 4) in his second paper.

The present paper has been prepared from my previous papers on the coast effect, and contains some improvements and supplements so as to cover the investigation of tunami due to wind and barometric change over a lake, a bay or a strait. In Chap. I, the steady value of surface elevation in shallow water is obtained without the ordinary assumption that the elevation is negligibly small compared with the total depth of the water. In Chap. II, the development of tunamis and seiches accompanied by the condition of bottom-slip, is dealt with in the case of varying causal action, and also extended to the case of progressive action. The viscosity of water is taken into account so that even for a perfectly resonancing action we can deduce reasonable results such as that the surface fluctuation cannot be amplified over $\pi/2$ times the steady value during half a period (or one crest) only of the causal force.

We shall here add that the rotation of the earth is neglected in this paper because the problem of meteorological tunami is, generally speaking, important only along the coast of a very shallow sea or in a long narrow bay.

I. Corrected formula for the steady value of the tunami produced by wind along a shallow-water coast

§ 1. Customary formula and the necessity for its correction. Both in the empirical formula of Colding[1] and in the theoretical formula of Ekman[2], the surface elevation ζ in a canal produced by wind may be expressed in the form

$$\frac{\Delta\zeta}{\Delta L}=\gamma=n\frac{T}{g\rho H} \quad \text{......(1)}$$

where γ denotes the surface slope, H the depth of the sea, T the tangential stress of wind, g the acceleration of gravity, ρ the density of water respectively, and n is a coefficient which becomes $3/2$ with Ekman's assumption of no bottom-current.

According to the writer's paper[3] on the coast effect, the coefficient will take the value $n=1$ for no bottom-friction. In the case of finite bottom-friction, putting $\omega=0$ in the said paper, we shall get

$$n=\frac{1+f'\rho H/2\mu}{1+f'\rho H/3\mu} \quad \text{if bottom-friction}=f'\rho u_H, \quad \text{.........(2)}$$

and

$$n=\frac{3}{2}\left\{1+\frac{3\mu^2}{f\rho H^2T}\left(1-\sqrt{1+\frac{2}{9}\frac{f\rho H^2T}{\mu^2}}\right)\right\} \quad \text{.........(2')}$$

1. Kgl. Danske Vidensk. Selsk. Skrifter, **1**, 272 (1880).
2. Ark. f. Mat. Ast. o. Fys. **2**, No. 11 (1905). 3. op. cit.

if bottom-resistance $=f\rho u_H^2$, where u_H denotes the slip-velocity at the bottom. All the above formulae, however, were derived on the supposition that the level rise ζ is negligibly small compared with the depth of the sea, and hence they are obviously not applicable, as they stand, to the investigation of tunamis so high as to invade the land. For instance, if the sea bottom make an inclined plane, formula (1) will give infinite γ and consequently infinite elevation of water at the coast ($H=0$). To avoid this impropriety, as some investigators have done, if we divide the sea into several portions for each of which formula (1) is applied with respective mean depth, and if we take the sum of the elementary rises of level $\Delta\zeta$ thus obtained as the height of tunami ζ at the coast, then we shall get a finite rise, but the value will be indeterminate according to the degree of division of the sea. Really the last landward portion of the sea of inclined plane bottom, however narrow may be taken, will give an identical value

$$\Delta\zeta=n\frac{T\cdot\Delta L}{g\rho\cdot H/2}=n\frac{2T}{g\rho\cdot\tan\theta} \quad (\theta=\text{Angle of inclination of the bottom}),$$

which is independent of the length of that portion of the sea. Thus, for the purpose of rough estimation of the order of tunami we may simply put the mean depth of the whole basin for H in formula (1), but in order to discuss more precisely the tunami problem at the coast we should first attempt to improve the formula, without disregarding the level rise ζ in comparison with H.

It may here be added that the steady value of the surface elevation due to the barometric gradient will not differ from the statical value even for very shallow water.

§ 2. **Corrected formula for a sea of no bottom-current.** If, after Ekman, we assume no bottom-current the motion of water in a one-directional canal in a steady state will be expressed in the ordinary notations, as follows:

$$\mu\frac{\partial^2 u}{\partial z^2}-g\rho\frac{d\zeta}{dx}=0 \quad \text{..........(3)}$$

with the conditions

$$-\mu\partial u/\partial z=T \text{ at the surface } (z=-\zeta) \quad \text{...(4)}$$

$$u=0 \quad \text{at the bottom } (z=H) \quad \text{...(5)}$$

and

$$S\equiv\int_{-\zeta}^{H}udz=0 \quad \text{..........................(6)}$$

where positive z is taken downward from the undisturbed water-surface and the disturbed surface is put as $z=-\zeta$.

Now the first integral of (3) is

$$\mu\frac{\partial u}{\partial z}=g\rho\frac{d\zeta}{dx}z+A \quad \text{.........(7)}$$

which combined with (4) gives

$$T=g\rho\frac{d\zeta}{dx}\cdot\zeta-A. \quad \text{.........(8)}$$

The second integral is $$\mu u = \frac{g\rho}{2}\frac{d\zeta}{dx}z^2 + Az + B \quad \text{......(9)}$$

which becomes by the condition (5)

$$\mu u = \frac{g\rho}{2}\frac{d\zeta}{dx}(z^2 - H^2) + A(z - H). \quad \text{...........(9')}$$

Then the equation of continuity (6) will give

$$0 = g\rho\frac{d\zeta}{dx}\frac{2H^3 + 3H^2\zeta - \zeta^3}{3(H+\zeta)^2} + A. \quad \text{..............(10)}$$

Eliminating A from (8) and (10), we get $$\frac{d\zeta}{dx} = \frac{3}{2}\frac{T}{g\rho(H+\zeta)}. \quad (11)$$

This is the corrected formula for the surface slope due to the wind and shows that precisely we must use the actual depth $H+\zeta$ instead of H in eq. (1).

§ 3. **Case of bottom-slip.** If some slip-velocity u_H exists at the bottom and the bottom condition is replaced by

$$-|\mu\, \partial u/\partial z|_{H=z} = f'\rho u_H \quad \text{.............................(5')}$$

instead of (5), then the first integral (7) will give

$$T = g\rho\frac{d\zeta}{dx}\cdot\zeta - A \quad \text{...(8)}, \qquad -f'\rho u_H = g\rho\frac{d\zeta}{dx}H + A. \quad \text{...(12)}$$

The bottom-current u_H is also obtained from the second integral (9)

$$\mu u_H = \frac{g\rho}{2}\frac{d\zeta}{dx}H^2 + AH + B \quad \text{.................(13)}$$

Eq. of continuity (6) will give by substitution of (9)

$$\frac{g\rho}{2}\frac{d\zeta}{dx}\frac{H^2 - H\zeta + \zeta^2}{3} + A\frac{H-\zeta}{2} + B = 0 \quad \text{......(14)}$$

The above four equations (8), (12), (13) and (14) serve to determine four unknowns A, B, u_H, $\frac{d\zeta}{dx} = \gamma$. Specially, if there is no bottom-friction and $f' = 0$, the surface slope can be obtained from (8) and (12) only, namely $$\gamma = \frac{d\zeta}{dx} = \frac{T}{g\rho(H+\zeta)}. \quad \text{......}(11_a)$$

When $f' \neq 0$ generally, however, we proceed as follows:—

From (8) and (12) $$-f'\rho u_H = g\rho\gamma(H+\zeta) - T,$$

and from (8) and (13) $$\mu u_H = g\rho\gamma H\left(\frac{H}{2} + \zeta\right) - TH + B.$$

Therefore $$0 = g\rho\gamma\left[\frac{\mu}{f'\rho}(H+\zeta) + H\left(\frac{H}{2} + \zeta\right)\right] - T\left(\frac{\mu}{f'\rho} + H\right) + B.$$

Combining this with the following relation

$$\frac{g\rho}{2}\gamma\left[\frac{H^2-H\zeta+\zeta^2}{3}+(H-\zeta)\zeta\right]-\frac{T}{2}(H-\zeta)+B=0$$

obtained from (8) and (14), we have

$$\gamma=\frac{d\zeta}{dx}=n\frac{T}{g\rho(H+\zeta)} \qquad \text{.....................(15)}$$

where $n=\left(1+\frac{f'\rho(H+\zeta)}{2\mu}\right)\Big/\left(1+\frac{f'\rho(H+\zeta)}{3\mu}\right)$. ....(16)

Thus we see that the more precise formula for the surface slope is obtained by putting $H+\zeta$ instead of all H's in the formula given in § 1. The same may be also said when the bottom-friction is equal to $f\rho u_H^2$.

It is here to be noticed that the coefficient n will reduce to 1 if $f'=0$ (no bottom-friction) and to 3/2 if $f'=\infty$ (no bottom-current), and that, with finite values of f' and μ, the value of n for a very shallow sea ($H\doteqdot 0$) tends to that for "no bottom-friction" and a very deep sea resembles that of "no bottom-current". Morever, since ζ in the expression of n in (16) enters in both numerator and denominator in a similar manner, its effect will be comparatively slight so that we may omit it for practical purposes, while ζ in (15) may occasionally have great importance.

§ 4. **Evaluation of surface elevation ζ.** In order to evaluate the rise of water at the coast we must know the functional relation of H and x. If the relation $H=f(x)$ is complicated, we divide the sea into many portions and calculate $\zeta=\Sigma\Delta\zeta=\Sigma n\frac{T}{g\rho(H+\zeta)}\Delta x$. ....(17)

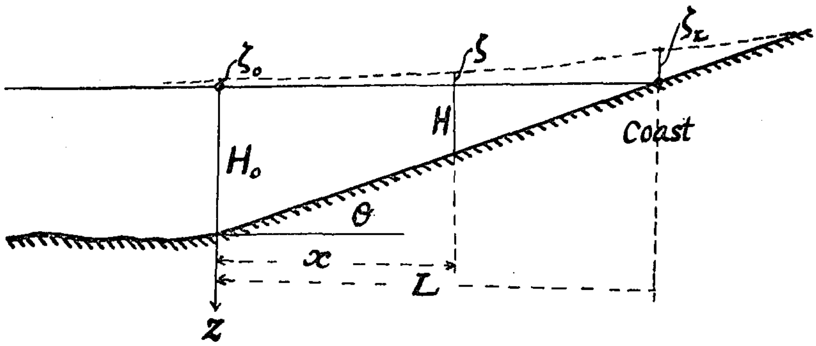
Of course, we begin with those portions in which ζ is negligible compared with H, and when we reach a portion such that the sum of the preceeding $\Delta\zeta$ becomes comparable with H, obtain $H+\zeta$ for the end of that portion and, by using it, calculate $\Delta\zeta$ and hence the next ζ, and so on. When $H=f(x)$ is a simple function, the direct integration of (16) in a finite form is much more convenient. The following two cases are very important in practice.

(Ex. 1) *Uniform shallow water.* If $H=$constant, taking the origin of x at the place of $\zeta=0$, we have $\zeta=H\left\{\sqrt{1+2n\frac{T}{g\rho H^2}x}-1\right\}$.

(Ex. 2) *Sea of inclined plane bottom.* If the shallow sea has a plane bottom inclined at an angle θ with the horizontal, take the origin of x at the end of the plane bottom where the depth is H_0 and the distance from the coast is L. Then at any x landwards, the depth of the undisturbed sea will be (Fig. 2)

$$H = H_0 - x\tan\theta = H_0(1 - x/L) \qquad \text{.................(18)}$$

Fig. 2



and the integral of (15) will become

$$\zeta + \bar{\zeta}_{H_0} \log\{1 - (H_0 + \zeta - x\tan\theta)/\bar{\zeta}_{H_0}\} = C \qquad \text{.................(19)}$$

where $$\bar{\zeta}_{H_0} = n\frac{T}{g\rho H_0}L = n\frac{T}{g\rho\tan\theta} \qquad \text{...........(20)}$$

and C is the integration constant. In order to determine the constant C, the constancy of water in the whole sea, i. e., $\int \zeta dx = 0$ ......(21) will serve generally. For actual seas we can often easily estimate $\zeta = \zeta_0$ at $x = 0$ ...(21′) so that the constant C can be determined. For instance, a lake with a symmetrical profile will have no ζ at the center, and even for a common sea, the elevation up to the place of depth H_0 (usually moderate depth) may be calculated by the approximate formula (1) starting from the open deep part.

In such cases, eq. (19) may be written

$$\zeta = \zeta_0 + \bar{\zeta}_{H_0} \log\frac{\bar{\zeta}_{H_0} - (H_0 + \zeta_0)}{\bar{\zeta}_{H_0} - (H_0 + \zeta - x\tan\theta)}. \qquad \text{...............(19′)}$$

Let ζ_L represent the elevation at the coast line ($x = L$), then it will be given by

$$\zeta_L = \zeta_0 + \bar{\zeta}_{H_0}\log\frac{H_0 + \zeta_0 - \bar{\zeta}_{H_0}}{\zeta_L - \bar{\zeta}_{H_0}} \qquad \text{.........(22)}$$

Specially, if ζ_0 is 0 or negligibly small,

$$\zeta_L/\bar{\zeta}_{H_0} + \log(\zeta_L/\bar{\zeta}_{H_0} - 1) = \log(H_0/\bar{\zeta}_{H_0} - 1) \qquad \text{...........(22′)}$$

where $\bar{\zeta}_{H_0} = nTL/g\rho H_0$ denotes the rise of wat r at the coast, supposing the sea to be of uniform depth H_0, and $2\bar{\zeta}_{H_0}$ corresponds to that given by eq. (1) for the actual sea under consideration.

Table 1 is prepared for practical convenience, by calculating inversely the corresponding values of H_0 for given values of ζ_L.

Table 1. Relation between ζ_L and H_0 when $\zeta_0 \doteqdot 0$

$\zeta_L/2\xi_{H_0}$	0	0.25	0.5	0.75	1.0	1.5	2.0	2.5	5
ζ_L/ξ_{H_0}	0	0.5	1	1.5	2.0	3.0	4.0	5.0	10
H_0/ξ_{H_0}	0	0.09	1	3.24	8.39	41.2	165	595	198400

II. Changing state in a canal with finite bottom-friction

§ 5. Constant motive action over a lake or an enclosed sea. The changing state dealt with in my previous paper on the coast effect, II, corresponds to the steady value (1) in this paper, namely it stands on the assumption that $\zeta \ll H$, and therefore it might seem necessary to re-investigate the problem carefully, taking the water surface as $z=-\zeta$ but not $z=0$. From the practical viewpoint, however, this will be unnecessary. For, although the final values of the exact solution may be very different from those of approximate solution as explained in Chapter I, the mode of development will not differ greatly. Thus it will be practically sufficient to assume that, even in the accurate solution, the functional relation to the time is nearly the same as before except that the steady value contained in the coefficient must be replaced by the corrected formula given in the preceeding chapter.

Now, for the two extreme cases of " no bottom-current " and " no bottom-friction " the development of surface elevation was discussed in sufficient detail in the previous papers. For " finite bottom-friction ", however, it was only shown that the development can be investigated in an entirely similar manner, the actual calculation in a definite formula being omitted. However, since the last case is obviously most probable in reality, we shall here make definite the formal solution given in the previous paper for level rise in a uniform canal with finite bottom-friction, as a preparation for the investigation of actual tunamis and seiches caused by the recent violent typhoon.

Consider a constant wind T or a barometric gradient γ_0 (measured in water column) begin to act suddenly all over the water initially at rest, and let a surface slope $\gamma=-\partial\zeta/\partial x$ be generated. Then the equation of motion will be represented by

$$\left.\begin{aligned}\frac{\partial u}{\partial t}&=\nu\frac{\partial^2 u}{\partial z^2}+g\gamma \qquad \text{for wind,} \qquad \nu=\mu/\rho,\\ &=\nu\frac{\partial^2 u}{\partial z^2}+g(\gamma+\gamma_0) \quad \text{for barometric gradient,}\end{aligned}\right\}\cdots(23)$$

with the conditions

$$\text{at surface } (z=0), \quad -\mu\partial u/\partial z = T \text{ for wind action}, \quad = 0 \text{ for barom. grad.}, \qquad \ldots(24)$$

$$\text{at bottom } (z=H), \quad -\mu\partial u/\partial z = f'\rho u_H, \qquad \ldots(25)$$

$$\text{initially } (t=0) \quad u=\gamma=0, \qquad \ldots(26)$$

and eq. of continuity

$$\frac{\partial\zeta}{\partial t} = -\frac{\partial S}{\partial x} \quad \text{or} \quad \frac{\partial\gamma}{\partial t} = \frac{\partial^2 S}{\partial x^2}, \quad \text{where} \quad S = \int_0^H u dz. \qquad \ldots(27)$$

Such u which satisfies eqs. (23) to (26) can be easily obtained by taking the limit when $\omega=0$ in our previous paper, or by solving independently though in a manner entirely analogous to that there used; and in any case we shall get

$$u = F\sum A_n \cos\beta_n z(1-e^{-\nu\beta_n^2 t}) + \sum B_n \cos\beta_n z\int_0^t \gamma(\tau)e^{-\nu\beta_n^2(t-\tau)}d\tau \quad \ldots(28)$$

where $\qquad \beta_n \tan\beta_n H = f'\rho/\mu \equiv h$ (say), $\qquad \ldots(29)$

$$F = T \text{ for wind action}, \quad = \gamma_0 \text{ for barometric gradient}, \qquad \ldots(30)$$

$$A_n = \frac{2(\beta_n^2+h^2)}{H(\beta_n^2+h^2)+h}\cdot\frac{1}{\mu\beta_n^2} \text{ for wind}, \quad = \frac{2(\beta_n^2+h^2)}{H(\beta_n^2+h^2)+h}\cdot\frac{g\sin\beta_n H}{\nu\beta_n^3} \text{ for barom. grad.}, \qquad \ldots(31)$$

and $\quad B_n = g\cdot\dfrac{2(\beta_n^2+h^2)}{H(\beta_n^2+h^2)+h}\cdot\dfrac{\sin\beta_n H}{\beta_n}. \qquad \ldots(32)$

These may also be directly verified by substitution in eqs. (23) to (26), and A_n corresponds to the coefficients when the steady value[1] of pure drift or barometric current is expanded in such a series as $F\sum A_n\cos\beta_n z$. Now, the total flow S becomes

$$S = \int_0^H u dz = F\sum A_n \frac{\sin\beta_n H}{\beta_n}(1-e^{-\nu\beta_n^2 t}) + \sum B_n\frac{\sin\beta_n H}{\beta_n}\int_0^t \gamma(\tau)e^{-\nu\beta_n^2(t-\tau)}d\tau, \qquad \ldots(33)$$

of which, for the reasons mentioned in the previous paper, only terms with $n=0$ may be introduced in eq. of continuity (27). Then we get

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1. Steady drift current $= \dfrac{T}{f'\rho} + \dfrac{T}{\mu}(H-z)$,

Steady barom. current $= g\rho\gamma_0\left[\dfrac{H}{f'\rho} + \dfrac{H^2-z^2}{2\mu}\right]$.

$$\frac{\partial\gamma}{\partial t}=\frac{d^2F}{dx^2}\frac{A_0\sin\beta_0 H}{\beta_0}(1-e^{-\nu\beta_0^2 t})$$
$$+B_0\frac{\sin\beta_0 H}{\beta_0}\int_0^t\frac{\partial^2\gamma(\tau)}{\partial x^2}e^{-\nu\beta_0^2(t-\tau)}d\tau, \quad \ldots\ldots(34)$$

which, combined with its second time-derivative, gives

$$\frac{\partial^2\gamma}{\partial t^2}+\nu\beta_0^2\frac{\partial\gamma}{\partial t}-B_0\frac{\sin\beta_0 H}{\beta_0}\frac{\partial^2\gamma}{\partial x^2}=\frac{d^2F}{dx^2}\frac{A_0\sin\beta_0 H}{\beta_0}\cdot\nu\beta_0^2. \quad \ldots(35)$$

If the canal is of length L, and if we put

$$\left.\begin{aligned}&\gamma=\sum\gamma_m\sin\frac{m\pi}{L}x, \quad \frac{\partial^2\gamma}{\partial x^2}=\sum\gamma_m''\sin\frac{m\pi}{L}x, \quad \gamma_m''=-\left(\frac{m\pi}{L}\right)^2\gamma_m,\\ &F=\sum F_m\sin\frac{m\pi}{L}x, \quad \frac{\partial^2F}{\partial x^2}=\sum F_m''\sin\frac{m\pi}{L}x, \quad F_m''=-\left(\frac{m\pi}{L}\right)^2F_m,\end{aligned}\right\}(36)$$

then (35) will give

$$\frac{d^2\gamma_m}{dt^2}+\nu\beta_0^2\frac{d\gamma_m}{dt}+B_0\frac{\sin\beta_0 H}{\beta_0}\left(\frac{m\pi}{L}\right)^2\gamma_m=-A_0\nu\beta_0\sin\beta_0 H\cdot\left(\frac{m\pi}{L}\right)^2F_m,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(37)$$

whose solution is evidently

$$\left.\begin{aligned}\gamma_m(t)&=\bar{\gamma}_m\left[1-e^{-\frac{1}{2}\nu\beta_0^2 t}\left\{\cos\sigma_m t+\frac{\nu\beta_0^2}{2\sigma_m}\sin\sigma_m t\right\}\right]\\ &=\bar{\gamma}_m\left[1-\frac{\sqrt{4\sigma_m^2+(\nu\beta_0^2)^2}}{2\sigma_m}e^{-\frac{1}{2}\nu\beta_0^2 t}\cos(\sigma_m t-\varepsilon)\right],\end{aligned}\right\}\ldots(38)$$

$$\left.\begin{aligned}\text{where}\quad&\sigma_m=\sqrt{B_0\frac{\sin\beta_0 H}{\beta_0}\left(\frac{m\pi}{L}\right)^2-\left(\frac{\nu\beta_0^2}{2}\right)^2},\\ &\tan\varepsilon=\nu\beta_0^2/2\sigma_m, \quad \text{damping factor}=e^{-\frac{1}{2}\nu\beta_0^2 t},\end{aligned}\right\}\ldots\ldots(39)$$

and $\bar{\gamma}_m$, though equal to $-\nu\beta_0^2 F_m\cdot A_0/B_0$ in mathematical expression, is physically the final value of $\gamma_m(t)$, so that it may be taken as the element in Fourier's series for the already known steady value of slope

$$\left.\begin{aligned}\bar{\gamma}&=nT/g\rho H &&\text{for wind action}\\ &=\gamma_0 &&\text{for barometric action.}\end{aligned}\right\}\ldots\ldots\ldots(40)$$

Eq. (39) shows also that the development of the slope will be oscillatory or non-oscillatory, according as

$$\frac{4B_0\sin\beta_0 H}{\nu^2\beta_0^5}\left(\frac{m\pi}{L}\right)^2>1 \text{ or } <1.$$

Finally it is to be noticed that the above solution has the same form as that found in the case of "no bottom-current" and described in the previous paper, i. e., the mode of development of the phenomena will be entirely similar, the only difference being in the final value $\bar{\gamma}_m$ according to the condition of bottom and the kind of motive force (wind or barometric).

§ **6. Variable action.** If the motive force F is variable with the time, $\bar{\gamma}_m$ proportional to F_m varies also with time such that $\bar{\gamma}_m=\bar{\gamma}_m(t)$. Then, applying the writer's newly established theorem to eq. (38), we easily get

$$\gamma_m(t)=\int_0^t \frac{\partial \bar{\gamma}_m(\tau)}{\partial \tau}\left[1-\frac{\sqrt{4\sigma_m^2+(\nu\beta_0^2)^2}}{2\sigma_m}\cos\{\sigma_m(t-\tau)-\varepsilon\}e^{-\frac{1}{2}\nu\beta_0^2(t-\tau)}\right]d\tau. \qquad \text{(41)}$$

Similarly the elementary surface-elevation ζ_m corresponding to γ_m will be given by

$$\left.\begin{aligned}\zeta_m(t)&=\int_0^t \frac{\partial \bar{\zeta}_m(\tau)}{\partial \tau}\left[1-\frac{\sqrt{4\sigma_m^2+(\nu\beta_0^2)^2}}{2\sigma_m}\cos\{\sigma_m(t-\tau)-\varepsilon\}e^{-\frac{1}{2}\nu\beta_0^2(t-\tau)}\right]d\tau\\ &=\bar{\zeta}_m(t)-\frac{\sqrt{4\sigma_m^2+(\nu\beta_0^2)^2}}{2\sigma_m}\int_0^t \frac{\partial \bar{\zeta}_m(\tau)}{\partial \tau}\cos\{\sigma_m(t-\tau)-\varepsilon\}e^{-\frac{1}{2}\nu\beta_0^2(t-\tau)}d\tau,\end{aligned}\right\} \qquad (42)$$

where $\bar{\zeta}_m$ is the steady value of surface elevation corresponding to $\bar{\gamma}_m$.

Since the level change expressed by (42) is generally of ocillatory nature, which is nothing but seiche motion, and since it may become tunami if the rise attains an exceptional height, we may take eq. (42) as representing *a general theory of meteorological tunamis and seiches.*

If the motive force F is a simple function of time, the above formula may of course be integrated in finite form, as seen at the end of my paper on the drift current in the ocean, I. However complicated the variation of the motive force with the time may be, the mechanical quadrature of eq. (42) will enable us to determine the change of surface level from time to time. The writer has made actual evaluations for the tunami at Ôsaka and the abnormal fluctuation of level of Lake Biwa caused by the recent typhoon, and the results will be described in subsequent papers. Moreover, eqs. (29), (39) and (40) will serve to estimate three unknowns f', ν and β_0, by measuring the amplitudes and periods of fluctuation from the actual mareo- or limnograms. Specially if f'=very small, (29) becomes $\nu\beta_0^2=f'/H$; and f' can be determined by the measurement of the damping factor only.

§ **7. Resonance effect.** Eq. (42) includes of course the resonance effect when the motive force varies with a period nearly equal to the proper period of the sea. For example, let us consider here a simple case in which the motive force, and consequently $\bar{\zeta}_m$, is a sine function of time, i. e.,

$$\left.\begin{aligned}\bar{\zeta}_m&=Z\sin(\sigma' t+a) && \text{for } 0\leqq t\leqq t_1\\ &=0 && \text{for } t<0 \text{ and } t>t_1.\end{aligned}\right\} \qquad \text{(43)}$$

Then eq. (42) gives: For $t<t_1$ (during the action of the motive force),

$$\zeta_m(t)=Z.\frac{\sigma_m^2+\left(\frac{1}{2}\nu\beta_0^2\right)^2}{2\sigma_m}\times$$

$$\left[\frac{1}{\left(\frac{1}{2}\nu\beta_0^2\right)^2+(\sigma_m+\sigma')^2}\left\{\begin{array}{l}\frac{1}{2}\nu\beta_0^2\cos(\sigma't+a)+(\sigma_m+\sigma')\sin(\sigma't+a)\\ -e^{-\frac{1}{2}\nu\beta_0^2t}\Big[\frac{1}{2}\nu\beta_0^2\cos(\sigma_mt-a)\\ \qquad -(\sigma_m+\sigma')\sin(\sigma_mt-a)\Big]\end{array}\right\}\right.$$

$$\left.-\frac{1}{\left(\frac{1}{2}\nu\beta_0^2\right)^2+(\sigma_m-\sigma')^2}\left\{\begin{array}{l}\frac{1}{2}\nu\beta_0^2\cos(\sigma't+a)-(\sigma_m-\sigma')\sin(\sigma't+a)\\ -e^{-\frac{1}{2}\nu\beta_0^2t}\Big[\frac{1}{2}\nu\beta_0^2\cos(\sigma_mt+a)\\ \qquad -(\sigma_m-\sigma')\sin(\sigma_mt+a)\Big]\end{array}\right\}\right] \quad (44a)$$

For $t>t_1$ (after the motive force has ceased to act), since

$$\int_0^t \bar{\zeta}_m(\tau)\phi(t-\tau)d\tau=\int_0^{t_1}\bar{\zeta}_m(\tau)\phi(t-\tau)d\tau+\int_{t_1}^t 0\times\phi(t-\tau)d\tau$$
$$=\int_0^{t_1}\bar{\zeta}_m(\tau)\phi(t-\tau)d\tau,$$

$$\zeta_m(t)=Z\cdot\frac{\sigma_m^2+\left(\frac{1}{2}\nu\beta_0^2\right)^2}{2\sigma_m}\times$$

$$\left\{\frac{1}{\left(\frac{1}{2}\nu\beta_0^2\right)^2+(\sigma_m+\sigma')^2}\left\{\begin{array}{l}e^{-\frac{1}{2}\nu\beta_0^2(t-t_1)}\Big[\frac{1}{2}\nu\beta_0^2\cos\{\sigma_mt-(\sigma_m+\sigma')t_1-a\}\\ \qquad -(\sigma_m+\sigma')\sin\{\sigma_mt-(\sigma_m+\sigma')t_1-a\}\Big]\\ -e^{-\frac{1}{2}\nu\beta_0^2t}\Big[\frac{1}{2}\nu\beta_0^2\cos(\sigma_mt-a)\\ \qquad -(\sigma_m+\sigma')\sin(\sigma_mt-a)\Big]\end{array}\right\}\right.$$

$$\left.-\frac{1}{\left(\frac{1}{2}\nu\beta_0^2\right)^2+(\sigma_m-\sigma')^2}\left\{\begin{array}{l}e^{-\frac{1}{2}\nu\beta_0^2(t-t_1)}\Big[\frac{1}{2}\nu\beta_0^2\cos\{\sigma_mt-(\sigma_m-\sigma')t_1+a\}\\ \qquad -(\sigma_m-\sigma')\sin\{\sigma_mt-(\sigma_m-\sigma')t_1+a\}\Big]\\ -e^{-\frac{1}{2}\nu\beta_0^2t}\Big[\frac{1}{2}\nu\beta_0^2\cos(\sigma_mt+a)\\ \qquad -(\sigma_m-\sigma')\sin(\sigma_mt+a)\Big]\end{array}\right\}\right] \quad \ldots\ldots(44b)$$

If $\nu\to 0$ and $t<t_1$, (44a) gives

$$\zeta_m=\frac{Z}{2}\Big[\frac{\sigma_m}{\sigma_m+\sigma'}\{\sin(\sigma't+a)+\sin(\sigma_mt-a)\}$$
$$+\frac{\sigma_m}{\sigma_m-\sigma'}\Big\{\sin(\sigma't+a)-\sin(\sigma_mt+a)\Big\}\Big] \quad \ldots\ldots(44'a)$$

which, when $\sigma' \to \sigma_m$, becomes

$$\zeta_m = \frac{Z}{2}[\cos\alpha\ \sin\sigma' t - \sigma' t\cos(\sigma' t + \alpha)]. \qquad \text{.........(45a)}$$

For $\infty > t > t_1$, the corresponding formula will be

$$\zeta_m = \frac{Z}{2}\Big[\sin\sigma' t_1\cos\{\sigma'(t-t_1)-\alpha\} - \sigma' t_1\cos(\sigma' t+\alpha)\Big]. \qquad \text{.........(45b)}$$

Thus, if t_1 is large, the amplitude will increase greatly with time; but when t_1 is small, the amplitude will not be so much enlarged even for a period of perfect resonance. For example, if t_1= half a period i. e. $\sigma' t_1 = \pi$ (only one crest of force), the maximum value of ζ_m cannot exceed $\frac{1}{2}\pi Z$, and when t_1= a complete period, the possible maximum value is $\zeta_m = \pi Z$.

§ 8. **Seiches and tunamis in a bay, a shelf sea or a strait.** All the above investigations have been restricted to an enclosed sea or a lake of length L, but they can very easily be extended to a bay, a continental shelf sea and a strait. For a bay, it is unnecessary to say that the phenomena will be the same as for a symmetrical lake half of which coincides with the given bay. The so-called continental shelf sea, whose depth diminishes sharply from the continuing ocean depth, will act like a bay in this respect, because both seas are equally characterised by an abrupt contraction of cross-section at the entrance, where level fluctuation must have a node. Thus the seiches and tunamis of meteorological origin in a bay or a continental shelf sea can be discussed by the foregoing formula, in which L must be taken as twice the length of the bay or the shelf sea.

Lastly for a strait of length L, we expand the motive force F and consequently $\bar{\gamma}$ and $\bar{\zeta}$ in forms

$$F = \sum F_m \cos\frac{m\pi}{L}x, \quad \bar{\gamma} = \sum \bar{\gamma}_m \cos\frac{m\pi}{L}x, \quad \bar{\zeta} = \sum \bar{\zeta}_m \sin\frac{m\pi}{L}x, \qquad (36')$$

instead of (36). Then exactly the same equations as (38) to (45) are obtained.

§ 9. **Progressive motive force in an endless canal.** In order to estimate the effect of advance of a motive force, let us consider the case in which

$$\left.\begin{aligned} \bar{\zeta} &= Z\cdot\sin(\sigma' t - lx) && 0 \leqq t \leqq t_1 \\ &= 0 && t < 0 \text{ and } t > t_1, \end{aligned}\right\} \qquad \text{........(46)}$$

with initial condition $\qquad \zeta = u = 0 \quad \text{for} \quad t \leqq 0.$

Since $\qquad \bar{\zeta} = Z[\sin\sigma' t\ \cos lx + \sin(\sigma' t - \pi/2)\sin lx],$

ζ_m=that in a bay (with $\alpha=0$) + that in a strait (with $\alpha=-\pi/2$)
$=\{\cos lx \times$ eq. (44) with $\alpha=0\}+\{\sin lx \times$ eq. (44) with $\alpha=-\pi/2\}$.

i. e.
$$\zeta_m = Z\frac{\sigma_m^2+\left(\frac{1}{2}\nu\beta_0^2\right)^2}{2\sigma_m}\left[\frac{1}{\left(\frac{1}{2}\nu\beta_0^2\right)^2+(\sigma_m+\sigma')^2}\times\left\{\begin{array}{l}\frac{1}{2}\nu\beta_0^2\cos(\sigma't-lx)+(\sigma_m+\sigma')\sin(\sigma't-lx)\\ -e^{-\frac{1}{2}\nu\beta_0^2 t}\left[\frac{1}{2}\nu\beta_0^2\cos(\sigma_m t+lx)-(\sigma_m+\sigma')\sin(\sigma_m t+lx)\right]\end{array}\right\}\right.$$
$$\left.-\frac{1}{\left(\frac{1}{2}\nu\beta_0^2\right)^2+(\sigma_m-\sigma')^2}\left\{\begin{array}{l}\frac{1}{2}\nu\beta_0^2\cos(\sigma't-lx)\\ \quad-(\sigma_m-\sigma')\sin(\sigma't-lx)\\ \quad-e^{-\frac{1}{2}\nu\beta_0^2 t}\left[\frac{1}{2}\nu\beta_0^2\cos(\sigma_m t-lx)\right.\\ \quad\left.-(\sigma_m-\sigma')\sin(\sigma_m t-lx)\right]\end{array}\right\}\right] \qquad (47a)$$

for $t<t_1$. Specially if $\nu\to 0$, the above eq. becomes

$$\zeta_m=\frac{Z}{2}\left[\frac{1}{1+\sigma'/\sigma_m}\{\sin(\sigma't-lx)+\sin(\sigma_m t+lx)\}\right.$$
$$\left.+\frac{1}{1-\sigma'/\sigma_m}\{\sin(\sigma't-lx)-\sin(\sigma_m t-lx)\}\right], \quad \text{.........}(47'a)$$

Farther when $\sigma'\to\sigma_m$,

$$\zeta_m \doteqdot \frac{Z}{2}\left[\cos lx \sin\sigma' t-\sigma' t\cos(\sigma' t-lx)\right]. \quad \text{.........}(47''a)$$

For $\infty>t>t_1$, also, a similar procedure can be used.

Here we notice that all these equations in the present paragraph will be obtained by putting $-lx$ instead of α in eqs. (44) and (45); and eq. (47″a) shows that the resonance effect will be such as explained in the preceding article. Eq. (47′a) coincides with that given by Proudman, (2.4), and is valid only when $\sigma_m \sim \sigma' \gg \frac{1}{2}\nu\beta_0^2$ and $t \neq \infty$. When $t\to\infty$, however, the free waves with σ_m will die away entirely.

Proudman's eq. (2.3) for semi-infinite canal does not correspond a motion started from rest a finite time before. It has a form similar to our eq. (44′a), which, when $t\to\infty$, holds good only for $\nu\equiv 0$, and if $\nu\neq 0$ however small, free oscillations will vanish for $t\to\infty$, as seen from (44a).