On the Density Current in the Ocean
II. The Case of No Bottom-Friction

By

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Abstract

First the "density current" in an open boundless sea with uniform solenoidal field and then the "convection current" influenced by land are discussed under the condition of "no bottom-friction".

The writer thus finds that Ekman's current-diagram put forward by him as the fundamental convection current, represents nothing but the current in an *enclosed sea* with no bottom-*friction* and that all the other current diagrams given by Ekman for convection currents are more or less erroneous. As the source of these errors the writer enumerates four weak points in Ekman's theory.

K. Hidaka's paper on the non-stationary convection current also contains similar weakness.

Although the rising stage of the density current with the condition of "no bottom-friction" can be treated in a similar way as in the previous case[1] of "no bottom-current", it will be omitted for the same reasons as stated before, and only the steady current w_1 will be discussed here.

I. Density Current without any Sloping of Sea-Surface

§ 1. Open Sea of Infinite Extent

Conceive a boundless sea in which the water-density decreases linearly in the horizontal direction of y only and no slope of surface exists.

Using the same notation as before, we know that the solution which satisfies the equation of motion and the surface condition is

1. Nomitsu, These Memoirs, A, **16**, 261 (1933).

$$w_1 = \frac{a}{4k^3}\left[\ 2K\cosh\alpha z + (1-i)e^{-\alpha z} + 2kz\ \right], \qquad (1)$$

where

$$a = -\frac{g}{\mu}\ \frac{\partial \rho}{\partial y}, \quad \alpha = (1+i)k, \quad k = \sqrt{\frac{\rho\omega\sin\lambda}{\mu}},$$

and K is an arbitrary constant.

Now the bottom condition in the present case gives

$$\left|\frac{dw_1}{dz}\right|_{z=H} = 0 = 2K\alpha\sinh\alpha H - (1-i)\alpha e^{-\alpha H} + 2k,$$

i. e.,
$$2K = -(1-i)\frac{1-e^{-\alpha H}}{\sinh\alpha H} = -(1-i)\frac{e^{-\frac{1}{2}\alpha H}}{\cosh\frac{1}{2}\alpha H}. \qquad (2)$$

Hence eq. (1) becomes

$$\left.\begin{aligned} w_1 &= \frac{a}{4k^3}\left[(1-i)\left\{\frac{-e^{-\frac{1}{2}\alpha H}\cosh\alpha z}{\cosh\frac{1}{2}\alpha H} + e^{-\alpha z}\right\} + 2kz\right] \\ &= \frac{a}{4k^3}\left[(1-i)\frac{\sinh\alpha\left(\frac{H}{2}-z\right)}{\cosh\frac{1}{2}\alpha H} + 2kz\right]. \end{aligned}\right\} \qquad (3)$$

From the upper fromula of (3), we have

$$\left.\begin{aligned} u_1 &= \frac{a}{4k^3}\left[A\cosh kz\cos kz + B\sinh kz\sin kz + e^{-kz}(\cos kz - \sin kz) + 2kz\right], \\ v_1 &= \frac{a}{4k^3}\left[\ A\sinh kz\sin kz - B\cosh kz\cos kz - e^{-kz}(\cos kz + \sin kz)\ \right], \end{aligned}\right\} \qquad (4)$$

where

$$\left.\begin{aligned} -A &= 1 - \frac{\sinh kH + \sin kH}{\cosh kH + \cos kH}, \\ -B &= 1 - \frac{\sinh kH - \sin kH}{\cosh kH + \cos kH}. \end{aligned}\right\} \qquad (5)$$

Specially, the surface value is

$$\left.\begin{aligned} \bar{u}_1 &= \frac{a}{4k^3}(A+1) = \frac{a}{4k^3}\cdot\frac{\sinh kH + \sin kH}{\cosh kH + \cos kH}, \\ \bar{v}_1 &= \frac{a}{4k^3}(-B-1) = -\frac{a}{4k^3}\cdot\frac{\sinh kH - \sin kH}{\cosh kH + \cos kH}, \end{aligned}\right\} \qquad (6)$$

and the bottom value is conveniently given by the lower formula of (3)

$$\left.\begin{aligned} \underline{w}_1 &= -\overline{w}_1 + \frac{a}{4k^3}\cdot 2kH, \\ \underline{u}_1 &= \frac{a}{4k^3}\cdot 2kH - \overline{u}_1, \\ \underline{v}_1 &= -\overline{v}_1. \end{aligned}\right\} \qquad (7)$$

Fig. 1

Density current with no bottom-friction ($a/4k^3$ being taken as unity)

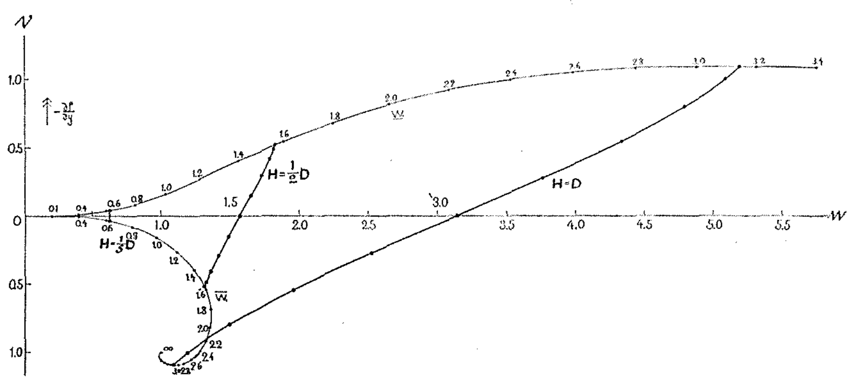

Fig. 2

Density-current with no bottom-friction

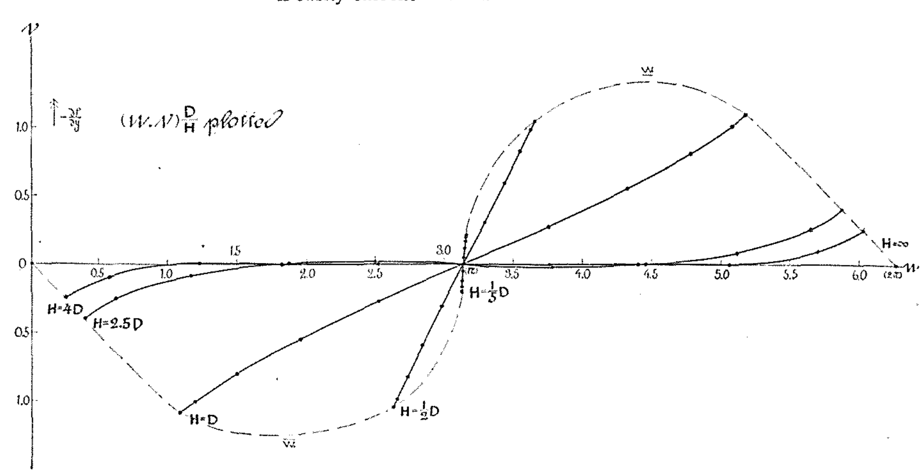

Table 1

Vertical distribution of density current with "no bottom-friction"

	$H=\frac{2}{10}D$		$H=\frac{D}{2}$		$H=D$		$H=2.5D$		$H=4D$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
0	0.626	−0.041	1.316	−0.519	1.090	−1.090	1.000	−0.999	1.000	−1.000
0.1	0.626	−0.039	1.329	−0.489	1.196	−1.002	1.571	−0.644	2.331	−0.359
0.2	0.625	−0.033	1.366	−0.409	1.498	−0.798	2.932	−0.207	4.914	+0.018
0.3	0.626	−0.023	1.422	−0.292	1.957	−0.543	4.575	−0.002	7.535	+0.032
0.4	0.627	−0.010	1.492	−0.152	2.523	−0.272	6.237	+0.034	10.060	+0.005
0.5	0.628	0	1.571	0	3.142	0	7.854	0	12.566	0
0.6	0.629	+0.010	1.649	+0.152	3.760	+0.272	9.471	−0.034	15.072	−0.005
0.7	0.630	+0.023	1.720	+0.292	4.326	+0.543	11.134	+0.002	17.598	−0.032
0.8	0.631	+0.033	1.777	+0.409	4.785	+0.798	12.776	+0.207	20.219	−0.018
0.9	0.632	+0.039	1.813	+0.489	5.087	+1.002	14.138	+0.644	22.802	+0.359
1.0	0.632	+0.041	1.826	+0.519	5.193	+1.090	14.708	+0.999	24.133	+1.000

Table 2

Surface and bottom values of the density current in steady state

kH	surface current		bottom current	
	$\bar{u}_1$	$\bar{v}_1$	$\underline{u}_1$	$\underline{v}_1$
0	0	0	0	0
0.2	0.200	−0.001	0.200	0.001
0.4	0.400	−0.011	0.400	0.011
0.6	0.597	−0.036	0.603	0.036
0.8	0.789	−0.084	0.811	0.084
1.0	0.968	−0.160	1.032	0.160
1.2	1.124	−0.266	1.276	0.266
1.4	1.244	−0.395	1.556	0.395
1.6	1.325	−0.540	1.875	0.540
1.8	1.360	−0.683	2.240	0.683
2.0	1.356	−0.812	2.644	0.812
2.2	1.323	−0.917	3.077	0.917
2.4	1.274	−0.994	3.526	0.994
2.6	1.220	−1.045	3.980	1.045
2.8	1.166	−1.075	4.434	1.075
3.0	1.119	−1.088	4.881	1.088
3.2	1.088	−1.090	5.312	1.090
3.4	1.048	−1.085	5.752	1.085
3.6	1.024	−1.075	6.176	1.075
3.8	1.007	−1.064	6.593	1.064
4.0	0.995	−1.051	7.005	1.051
4.4	0.984	−1.031	7.816	1.031
4.8	0.982	−1.015	8.618	1.015
5.2	0.985	−1.004	9.415	1.004
5.6	0.990	−0.999	10.210	0.999
6.0	0.994	−0.997	11.006	0.997
7.0	0.998	−0.997	13.002	0.997
8.0	1.001	−0.999	14.999	0.999
9.0	1.000	−1.000	17.000	1.000
10.0	1.000	−1.000	19.000	1.000
∞	1.000	−1.000	∞	1.000

Tables 1, 2 and Fig. 1 give some concrete examples, in which $\frac{a}{4k^3}$ is taken as unit velocity. In order to represent the results in a more compact diagram, we may take $\frac{a}{4k^3}\,\frac{H}{D}$ as unity, or plot $\frac{D}{H}$ times the values in Tables 1 and 2. Thus Fig. 2 is obtained.

The following points shown in the diagram should be specially noticed:

1) The current in a shallow sea is entirely different from that in a sea with "no bottom-current".

2) For a very deep sea, the current of the upper part resembles very much that seen in the previous case of "no bottom-current", though in the lowest layer the currents in the two cases differ entirely.

3) The component velocity v_1 is perfectly symmetrical with respect to the middle layer.

4) The component velocity u_1 is always positive, and in the middle layer it is not zero but has the value $kH\left(\text{taking}\ \frac{a}{4k^3}=1\right)$.

§ 2. Total Flow from Surface to Bottom

For later use, we calculate here the total flow from surface to bottom, namely

$$S=\int_0^H w_1 dz=\frac{a}{4k^3}\left|\frac{1-i}{a}\ \frac{\cosh a\left(\frac{H}{2}-z\right)}{\cosh\frac{1}{2}aH}+kz^2\right|_0^H,$$

which gives

$$\left.\begin{aligned} S_x&=\int_0^H u_1 dz=\frac{a}{4k^2}H^2,\\ S_y&=\int_0^H v_1 dz=0. \end{aligned}\right\} \qquad (8)$$

II. Convection Current—Land Effect

§ 3. Long Straight Coast

Let a uniform gradient of density be in the negative direction of y, and suppose a long straight coast in the sea.

a) *If the coast lies parallel to the x-axis*, a surface-slope γ_y may be generated in the positive direction of y and consequently the horizontal isobaric layer must sink to a depth $d=\frac{\rho\gamma_y}{-\frac{\partial\rho}{\partial y}}=\frac{g\rho\gamma_y}{\mu a}$.

Then the equation of motion will become

$$0=\frac{d^2w_1}{dz^2}-2ik^2w_1+ia(d-z), \qquad (9)$$

which is the fundamental equation of Ekman for the convection current.

Now the value of d is to be determined by the condition of continuity which requires the total flow perpendicular to the coast to be zero. Since we have eq. (8) and since moreover we know that the total flow of a slope current with " no bottom-friction " is zero for the direction of the slope, the continuity-condition is identically satisfied, and the quantity d seems to be indeterminate. But, if " no bottom-friction " is considered as the limit of an indefinitely small bottom-friction, we can determine[1] definitely $d=H-\dfrac{\sinh kH+\sin kH}{2k(\cosh kH+\cos kH)}$. The current diagram will be shifted as a whole in the x-direction by some amount corresponding to the value of d.

b) *If the coast lies parallel to y-axis*, the surface slope γ_x must rise in the direction of x, and the equation of motion will become

$$0=\frac{d^2 w_1}{dz^2}-2ik^2 w_1-iaz+\frac{g\rho}{\mu}\gamma_x.$$

In this case, however, the equation of continuity can not satisfied by any value of γ_x, because the slope current due to γ_x is entirely in the direction of y so that it is not able to cancel the total flow of the density current, S_x, in eq. (8). Thus we see that if the coast is parallel to the gradient of density, only the horizontal current can not attain the stationary state in a sea with " no bottom-friction ", and there must arise a large slope and consequently a vertical motion comparable to u and v, so that the eq. of motion itself must be altered.

The same conclusion will be obtained for any straight coast *oblique* to the x-axis.

§ 4. Enclosed Sea

If the sea is enclosed by land, the surface slope may be generated in the direction of x as well as y, and the equation of motion should generally be written

$$0=\frac{d^2 w_1}{dz^2}-2ik^2 w_1+ia(d-z)+\frac{g\rho}{\mu}\gamma_x.$$

The condition of continuity in the y-direction, however, requires the component slope γ_x to be nil, so that the above equation reduces to (9) for an enclosed sea with " no bottom-friction ".

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1. The detail will soon appear in another paper on the effect of land upon the ocean current of various kinds.

On the other hand, the condition of continuity in the x-direction gives

$$\text{Total flow in } x\text{-dir} = \frac{g\gamma_y}{2\omega\sin\lambda}H - \frac{a}{4k^2}H^2 = 0.$$

$$\left.\begin{aligned} \therefore \qquad \gamma_y &= \frac{\omega\sin\lambda}{g}\cdot\frac{a}{2k^2}H = \frac{\mu a}{2g\rho}H, \\ \text{or} \qquad d &= \frac{g\rho\gamma_y}{\mu a} = \frac{H}{2}. \end{aligned}\right\} \qquad (10)$$

Thus, in this case only, the horizontal isobaric layer exists at just half the total depth of the sea.

The convection current will now be given by

$$w_1 = \Big[\text{ slope current due to } \gamma_y \Big] - \Big[\text{ density current } (3)\Big].$$

And since the slope current is

$$\frac{g\gamma_y}{2\omega\sin\lambda} = \frac{a}{2k^2}d = \frac{a}{4k^3}kH,$$

we have

$$\left.\begin{aligned} w_1 &= \frac{a}{4k^3}\left[kH - (1-i)\frac{\sinh a\left(\frac{H}{2}-z\right)}{\cosh\frac{1}{2}aH} - 2kz\right] \\ &= \frac{a}{4k^3}\left[2k(d-z) - (1-i)\frac{\sinh(1+i)k(d-z)}{\cosh(1+i)kd}\right], \end{aligned}\right\} (11)$$

Fig. 3

Convection current in an enclosed sea with no bottom-friction.

$(u, v)\frac{D}{H}$ are plotted.

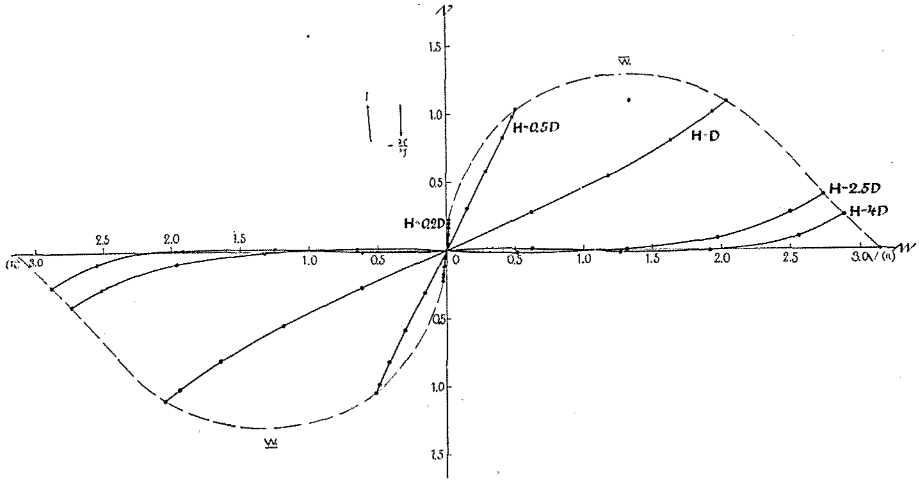


Table 3

Vertical distribution of convection current in an enclosed sea with "no bottom-friction."

	$H=2d=\frac{2}{10}D$		$H=2d=\frac{D}{2}$		$H=2d=D$		$H=2d=2.5D$		$H=2d=4D$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
0	+0.003	+0.041	+0.255	+0.519	+2.051	+1.090	+6.854	+0.999	+11.566	+1.000
0.1	+0.003	+0.039	+0.242	+0.489	+1.945	+1.002	+6.284	+0.644	+10.236	+0.359
0.2	+0.003	+0.033	+0.206	+0.409	+1.643	+0.798	+4.922	+0.207	+ 7.653	−0.018
0.3	+0.002	+0.023	+0.149	+0.292	+1.185	+0.543	+3.280	+0.002	+ 5 032	−0.032
0.4	+0.001	+0.010	+0.078	+0.152	+0.619	+0.272	+1.617	−0.034	+ 2.506	−0.005
0.5	0	0	0	0	0	0	0	0	0	0
0.6	−0.001	−0.010	−0.078	−0.152	−0.619	−0.272	−1.617	+0.034	− 2.506	+0.005
0.7	−0.002	−0.023	−0.149	−0.292	−1.185	−0.543	−3.280	−0.002	− 5.032	+0.032
0.8	−0.003	−0.033	−0.206	−0.409	−1.643	−0.798	−4.922	−0.207	− 7.653	+0.018
0.9	−0.003	−0.039	−0.242	−0.489	−1.945	−1.002	−6.284	−0.644	−10.236	−0.359
1.0	−0.003	−0.041	−0.255	−0.519	−2.051	−1.090	−6.854	−0.999	−11.566	−1.000

which coincides with the formula of Ekman and vanishes at the depth $z=d=H/2$.

Table 3 and Fig. 3 are calculated by eq. (11). In the table a definite unit $(a/4k^3=1)$ is taken for all depths of the sea, but in Fig. 3, for convenience, different units $\left(\frac{a}{4k^3}\frac{H}{D}=1\right)$ are adopted for each sea, so that all may be included in one diagram.

Thus we know now that the convection current given by Ekman represents the current in an *enclosed* sea with no bottom-*friction*, but not the current in an "open boundless sea" or in any sea with "no bottom-current".

Appendix

1. *Convection current in an enclosed sea with "no bottom-current".*

Having found the nature of the fundamental current of Ekman, we shall naturally examine his other current-diagrams which are based on the said current.

Three current diagrams of Ekman[1] are reproduced in Krümmel's Handbuch der Ozeanographie, II, Figs. 138, 139 and 140. According to the explanation given by Krümmel, they were intended to represent the current in "die Wassermasse in horizontal Richtung unendlich ausgedehnt", and were constructed by combining the upper half of the current in our Fig. 3 and a slope current, so as to cancel the total flow in both x and y directions.

In my opinion, however, they can not be the currents for an open boundless sea, because there is no reason why the total flow, particularly its x-component, must be zero. Only an enclosed sea in the steady state will require the total flow to be zero in all directions.

Moreover, even as representing the convection current in an enclosed sea with no bottom-current the diagrams of Ekman are not perfectly right. If he used, as in Krümmel's statement, the upper half of our Fig. 3 for the density current with no bottom-friction, it is obviously wrong. Or he might, I suppose, have constructed his diagrams by adding to Fig. 3 a slope current and a drift current with no bottom-*current*, as he stated in his first paper in 1905 that

> "If the assumption $u=v=0$ for $z=d$ is not true, we have to add to (20) solutions of the form (8) and (15)........"

1. Ekman, Beiträge zur Theorie der Meeresströmungen, Ann. d. Hydr. 566 (1906), Figs. 26, 27, 28.

His eq. (20) corresponds to our eq. (11) and his eqs. (15) and (8) are the formulae for a slope current and a drift current respectively with "no bottom-*current*". But I can not agree with that process. In order to get the current with "no bottom-current" from the current with "no bottom-friction" of the same kind, the bottom-effect may be replaced by a current in the form of a drift-current with "no bottom-*friction*" but not with "no bottom-current"; because, in the comparison of the bottom-effect with a wind-action, the supposed bottom must be the actual sea-surface where no friction exists. The solution of the drift current with "no bottom-friction" was given first in my second paper[1] on the drift current in the ocean.

At any rate, Ekman's diagrams can not be proper representations of currents in an enclosed sea with "no bottom-current", and therefore I will here give my own current-diagram of the problem in question.

In the previous paper[2] I gave the mathematical solution for it, namely we can determine the slope-components γ_x and γ_y (or d) by the following formulae.

$$k(H-d)=\frac{FQ-GP}{F^2+G^2},$$

$$\gamma_x=-\frac{\mu a}{g\rho^2 k}\cdot\frac{FP+GQ}{F^2+G^2},$$

where
$$F=\sinh 2kH-\sin 2kH,$$
$$G=\sinh 2kH+\sin 2kH-2kH(\cosh 2kH+\cos 2kH)$$
$$P=-2\sinh kH\sin kH+k^2H^2(\cosh 2kH+\cos 2kH),$$
$$Q=\cosh 2kH+\cos 2kH-2\cosh kH\cos kH.$$

Table 4 shows the results thus obtained.

Table 4

H	$D/4$	$D/2$	D	$1\frac{1}{2}D$	$2D$
d/D	0.094	0.189	0.414	0.540	0.922
γ_x	0.011	0.123	0.564	0.701	0.827

Then, calculating the composite current due to the slope γ_x, γ_y and the

1. Nomitsu, These Memoirs, A, **16**, 275 (1933).
2. Nomitsu, These Memoirs, A. **16**, 261 (1933).

Table 5

Vertical distribution of convection current in an enclosed sea with "no bottom-current."

	$H=\frac{1}{4}D$		$H=\frac{D}{2}$		$H=D$		$H=1\frac{1}{4}D$		$H=2D$	
z/H	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1	u_1	v_1
0	0.003	0.041	0.074	0.305	1.286	1.302	2.277	1.484	4.817	1.412
0.1	0.002	0.037	0.068	0.280	1.187	1.192	2.106	1.348	4.443	1.157
0.2	0.002	0.029	0.052	0.215	0.916	0.884	1.631	1.049	3.477	0.809
0.3	0.001	0.016	+0.028	0.125	0.507	0.567	0.943	0.691	2.192	0.590
0.4	0.001	0.003	−0.002	+0.023	+0.032	+0.158	+0.152	+0.292	+0.757	0.494
0.5	−0.001	−0.001	−0.022	−0.075	−0.381	−0.231	−0.621	−0.132	−0.741	0.388
0.6	−0.002	−0.020	−0.043	−0.155	−0.714	−0.603	−1.243	−0.576	−2.207	+0.083
0.7	−0.002	−0.025	−0.049	−0.203	−0.875	−0.882	−1.580	−0.996	−3.393	−0.562
0.8	−0.002	−0.026	−0 041	−0.205	−0.810	−0.972	−1.507	−1.239	−3.379	−1.454
0.9	−0.001	−0.018	−0.013	−0.142	−0.501	−0.731	−0.954	−1.043	−2.725	−1.894
1.0	0	0	0	0	0	0	0	0	0	0

density gradient with no bottom-current, we obtain Table 5 and Fig. 4.

Here we observe that:

1) This diagram is similar to Ekman's in the main feature, but different from the latter in the relative proportion of various parts and in detail.

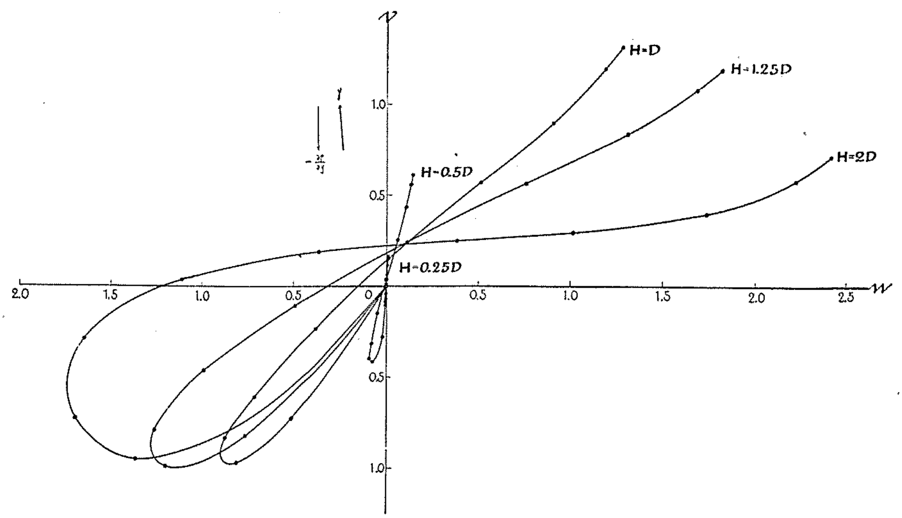
2) The current is almost the same as that in the sea of "no bottom-friction", Fig. 3, except the lowest layers.

I here recollect that in connection with the current in a two-layer ocean also, the same conclusion as the above (1) was obtained with

Fig. 4

Convection current in an enclosed sea with no bottom-current.

$(u, v)\frac{D}{H}$ are plotted.



regard to our[1] current-diagram and Ekman's[2]. Perhaps the same can be said for the current due to the coast-water wedge[3] reproduced in Krümmel's Handbuch, II. Fig. 142.

2. *Weak points in Ekman's theory of convection currents*

In the foregoing discussion, we observed that all the current-diagrams of Ekman for convection currents are more or less erroneous. As the source of errors, I shall enumerate the following four weak points in his theory of convection currents:

1. Nomitsu and Takegami, These Memoirs, A, **16**, 139 (1933), Fig. 1.
2. Ekman, loc. cit. Fig. 34; Krümmel, Handbuch der Ozeanographie, II (1911), S. 468, Fig. 143.
3. Ekman, loc. cit. Fig. 33.

1) Ekman does not consider such a "density current" as would be produced by a difference of density alone and not attended by any sloping of the sea surface, while he separates a "drift current" from a "wind current".

2) Consequently he does not know how to determine the depth of the horizontal isobaric layer in the convection current, and assumes it "about half the height of the layer within which the circulation takes place".

3) He also presumes wrongly that the current at the horizontal isobaric layer always vanishes.

4) He replaces the bottom-effect in the case of "no bottom-current" by a drift current with "no bottom-current", while it should be properly a drift current with "no bottom-friction".

Next, K. Hidaka's paper[1] on the non-stationary convection current also contains similar weakness, and especially the fatal mistake is that he assumed an invariable pressure-gradient[2] $\mu a(H-z)$, i. e., he put $d=H$, during the non-stationary state. Such condition, however, can hold good only for a sea in statical equilibrium, but not for a sea in motion. Indeed, if the sea had innumerable vertical partition-walls and were communicated only at the bottom, the water might be in statical equilibrium with a surface slope corresponding to the local difference in density, so that the pressure gradient would become $\mu a(H-z)$ and the horizontal isobaric layer lay at the bottom. But, as soon as the partition-walls are removed, the water must circulate so as to diminish the surface slope and consequently to raise the horizontal isobaric layer from depth H up to $\frac{3}{8}H$. Thus, during the rising stage of convection current, it is obviously wrong to assume $d=$constant; and much more to put $d=H$. The formulae deduced from such assumption are entirely erroneous and ought to be erased.

In conclusion, the writer's thanks are due to the Hattori Hōkōkwai for the subsidy given to him with which he was able to get an assistant to help him in the numerical calculations.

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1. Mem. Imp. Mar. Observ. Kobe, **5**, 231 (1933).
2. Ditto, eq. (149).