The Probability of Lightning Strokes

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By

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Abstract

The density distribution of the hits of imitated lightning sparks on a wide circular metallic disc with or without a lightning needle, is examined for the sparks starting from a fixed point above the surface of the disc. The results of the experiment are shown to agree nearly with those obtained by calculation based upon the consideration of the probability.

Some years ago Peek[1] made a laboratory experiment on the distribution of lightning strokes under various conditions in imitating the lightning-strokes from cloud to earth. Recently the writers repeated a similar experiment, and a calculation based upon the consideration of probability has been tried and found to be successful to some extent in explaining the results.

Experiment

The experimental arrangement employed is illustrated schematically in Fig. 1. C is a metal ball of 22 mms. diameter, on whose lower surface is attached a small metallic protuberance, so as to make electric sparks start always from this protuberance. E is an earthed circular metal disc of 31 cms. diameter, which is provided with a small hole of about 1 mm. diameter at its center. This large circular disc, which corresponds to a flat surface of the earth vis à vis the lightning, is placed horizontally in

1 F. W. Peek: J. Franklin Inst., **199**, 14 (1925)

such a position that the central hole of the disc is situated precisely below the protuberance of the spherical electrode C. In examining the effect of the lightning rod, the writers inserted a sharp needle through the central hole of the circular disc at various heights above the surface of the disc. The circular disc E is covered with a thin black paper of 0.1 mm. thickness, so that the hits of the lightning flashes on the circular disc are recorded on this black paper; and the density of the lightning strokes on the disc can be measured by counting the hit-marks on the paper. For the excitation of the sparks between the electrode C and the circular disc electrode E, a small influence machine is employed. The spark-length between the electrode C and the circular disc E, with which the experiment is carried out, is 9 cms. when the electrode C is charged positively. The positively charged electrode C corresponds to the positively charged cloud, as is common in nature. The value of the spark-length above mentioned is of course too short to imitate the actual lightning; but as it seemed to the writers that the essential features of the distribution of the lightning strokes would still be retained with short sparks, they have carried out their experiment with these short sparks and tried to find out some laws governing the distribution of the lightning strokes. When the electrode C is charged negatively the maximum spark length, attainable with the small influence machine employed by the writers, between the electrode C and the disc E is much shorter than when the electrode C is charged positively; and the present experiments were performed only with the electrode C charged positively. With the influence machine employed by the writers the sparks occur in slow succession one after another between the electrodes, so that we could count without any difficulty the number of the sparks which passed by listening to the sound they made. By so counting the number of the sparks which passed, the number of the hit-marks on the black paper placed on the upper surface of the circular disc is checked in every case. When the hit-marks on the black paper are over crowded, the number of the hit-marks is always smaller than that counted by listening to the sound, due to the overlapping of the hit-marks. In examining the density of the hit-marks, this overlapping of the hit-marks must be avoided as much as possible. For this purpose, the writer always confined the total number of the sparks below 200 on one sheet of black paper; and the density determination of the

Fig. 1

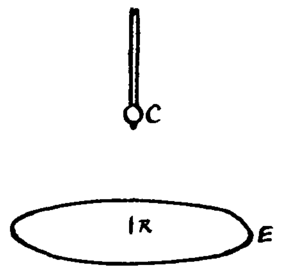


Fig.2

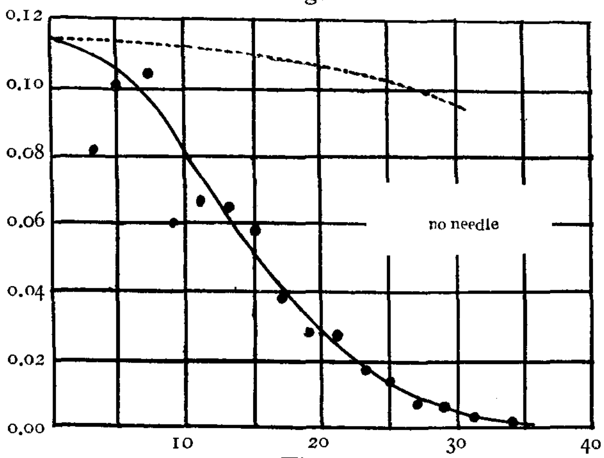


Fig.3

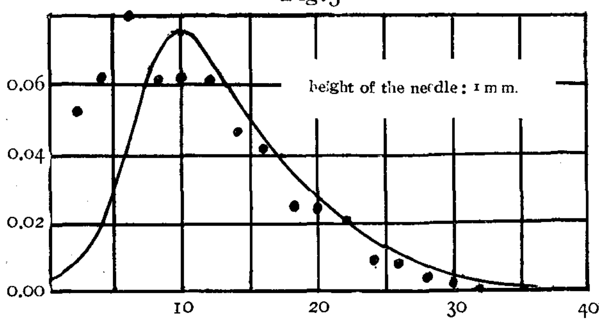


Fig.4

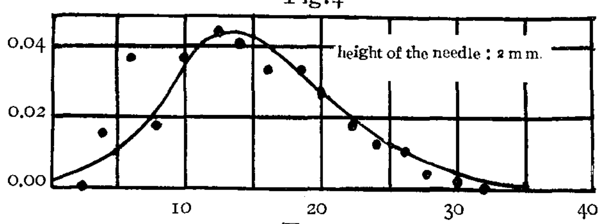


Fig.5

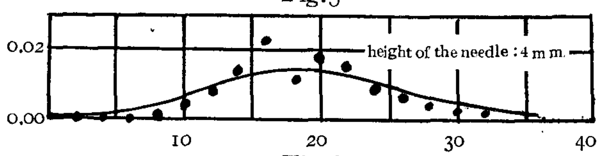
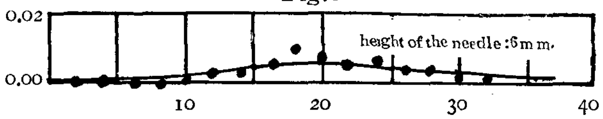


Fig.6



Density of the lightning strokes

Distance in m.m.

lightning strokes was made on about 1000 hit-marks impressed on five or six different sheets of black paper. With sparks less than 200 in number on one sheet of black paper the overlapping of the hit-marks was confirmed to be negligible. The results of the experiments are shown in Figs. 2, 3, 4, 5 and 6. In these figures the abscissa represents the distance of a point on the surface of the circular disc E from the central hole on it in mm. This central hole is situated just below the protuberance on the surface of the electrode C. In examining the effect of the lightning rod, a sharp needle is inserted in this central hole as stated before; consequently the abscissa in the figures of these cases is nothing but the distance between the needle and a point on the surface of the circular disc. The ordinate in the same figures represents the density of the hits of the sparks per square millimetre of the surface of the circular disc expressed in a percentage of the total hits on the whole surface of the circular disc. In the figures, the dots represent the observed densities of the hits of the sparks at various points on the circular disc, and the curves drawn in unbroken and broken lines represent those calculated; the manner of this calculation will be described later. In Fig. 2, the sparks were made to hit the flat surface of the circular disc which was unprovided with any needle; and in Figs. 3, 4, 5 and 6, the sharp needle, which corresponds to the lightning rod, was made to protrude at various heights above the surface of the circular disc; and the heights of this protrusion are written in each figure. As shown in these figures, the density of the lightning strokes, without any lightning rod, is roughly maximum at a point just below the starting point of the spark on the surface of the electrode C. By inserting a lightning rod at this point, the density of the lightning strokes in its vicinity diminishes with the height of the rod as expected.

Theoretical Consideration

If the electrical property of the air were entirely homogeneous the spark would pass along the line of force of the strongest field-intensity; thus with the arrangement described in the preceding chapter the spark should strike the circular disc E at the point just below the protuberance on the surface of the spherical electrode C. But this is not actually so, and the hit-marks of the sparks on the circular disc are scattered in a comparatively wide area. This fact is, of course, to be expected from the irregular zigzag shape of the spark. It was pointed out before by

Mr. H. Hirata and one of the writers[1] that the shape of the spark was different according as the spark occurred along the preliminary negative brush or along the preliminary positive brush. It is also known that the elongation of the irregular positive brush is much longer than that of the rather regular negative brush. Thus unless special care is paid to prevent the appearance or the development of the positive brush, the spark will pass, for the most part of its path, along the positive preliminary brush of mush longer extension. This seems to be the reason why the branches of the lightning from the positively charged could to the earth are toward the earth, and why the irregular zigzag shape of the lightning is similar to that of the positive brush. Considering in such a way it seems not to be unnatural to suppose that the spark in the present experiment occurred, for the most part of its path, along the way formed by the preliminary positive brush starting from the protuberance on the surface of the spherical electrode C. If so the irregular zigzag paths of the sparks as seen in Figs. 1 and 2 in Plate I must be due to the irregular branches of the positive brushes. As to the formation of these irregular branches of the positive brush, an explanation based on the consideration of an irregular and random distribution of the free ions in the air, was given by Mr. Hirata and one of the writers[2]. If such explanation be correct, the irregular and zigzag path of the spark will be also explained by the irregular distribution of the free ions in the air. The calculation, based on the consideration of probability, of the distribution of the density of the lightning strokes, which will be given below, owes its origin to the idea mentioned above.

Considering in the above way, we can suppose that a spark consists of an irregular continuation of a large number of very short and straight sparks, the lengths and the inclinations to the vertical direction of these short sparks being different for every one, due to the irregular distribution of the free ions in the air, and the intensities of the electric fields near the ends of the branches of the preliminary positive brush at the moment of its development. Now let Δz be the mean value of the vertical components of all the short sparks forming a spark, and let Δr be the mean value of the horizontal components of all the short sparks forming the same spark. Then the value of $\Delta r/\Delta z$ will be pretty small compared with unity in the present case, where the vertical component of the intensity of the electric field is strongly predominant; and the value of Δz will be nearly the same as the average value of the lengths of all the

1 U. Yoshida and H. Hirata; These Memoirs, **4**, 89 (1919)

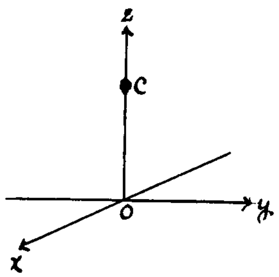
2 ibid.

short sparks forming the same spark. Thus the value of Δz will solely be determined by the density of the free ions in the air and it will nearly be independent from the actual length and the shape of the spark in the present case, where nearly all sparks pass roughly in the vertical direction. Considering in such a way, when a spark passes between a point and a horizontal plane, the total number n of Δz for this spark will be nearly proportional to the height h of the point above the horizontal plane, irrespective of the actual shape and the inclination of the path of the spark, and h may be put equal to $n\Delta z$.

For the mean value Δr of the horizontal components of all the short sparks forming a spark, it will also depend upon the density of the free ions in the air. But for a given value of this density the value $\Delta r/\Delta z$ will not be affected much by the actual path of the spark for the sparking electrodes of given shape and dimensions.

Considering in the above way we may suppose that a spark in our case consists of n short and straight sparks, the vertical and the horizontal components of each of these short sparks being respectively equal to Δz and Δr for every one; and consequently that a spark starting from a point arrives at a horizontal plane, separated at a vertical distance h from the point, by a zigzag continuation of such n short sparks. During the development of the preliminary positive brush, the horizontal components of the electric field in the immediate neighbourhood of the end of every branch of the positive brush may be looked for as nearly symmetrical, on an average, about the vertical line passing through the end point of every branch, for the major part of the the path of spark. In conjunction with this fact, the random distribution of the free ions in the air compels us to consider that the directions of the mean horizontal components Δr of the short sparks forming a spark is at random for all directions in a horizontal plane.

Fig. 7



Let us imagine that a spark starts from the point C in Fig. 7 and arrives at a horizontal plane xoy, which is situated at a distance h below the point C. Let O be the point of intersection of the plane xoy with the vertical line passing through the point C. Then take the direction OC as the z-axis, and the directions ox and oy in the horizontal plane under consideration as the x- and the y- axes respectively of the rectangular coordinates system. Next, resolve the mean value Δr of the horizontal components of the short sparks into x and y components, and let Δx and Δy be respectively the means of the absolute values of these two components, then Δx and Δy will respectively be equal to $\frac{2}{\pi}\Delta r$; because the direction of Δr is considered to be entirely at random in all directions in a horizontal plane.

Now let us consider, for the sake of rough calculation, that a spark passes between the point C and the horizontal plane xoy by the continuation of n short sparks, and roughly that Δx or Δy may have the positive and the negative values with equal probability; then the probability, for a spark, of hitting the plane xoy at a certain distance from the origin O will be calculated by the combination of certain numbers of the positive and the negative values of Δx and Δy, similarly as in the calculation of the probability of errors. Thus if we represent by P the probability per unit area, for a spark, of its hitting a point distant from O by r on the plane xoy, then P will be given by

$$P = \frac{1}{\pi k^2} e^{-\frac{r^2}{k^2}},$$

in which k^2 has been written to represent the quantity $2n\left(\frac{2}{\pi}\right)^2(\Delta r)^2$, and is a constant for a given experimental condition. Thus when N lightning flashes come from the point C to the plane xoy, the density of the lightning strokes at a point on the plane xoy will be

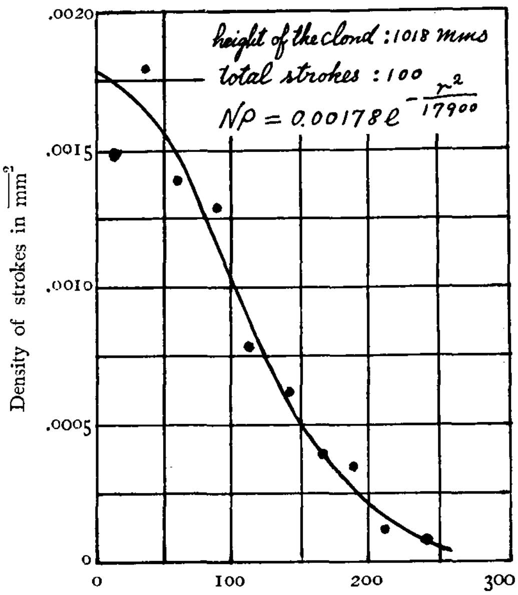
$$NP = \frac{N}{\pi k^2} e^{-\frac{r^2}{k^2}} . \qquad \text{(1)}$$

Lightning strokes on a plane surface. The above consideration may be applied immediately to the case of lightning strokes on a plane surface without any lightning rod, and the experimental result shown in Fig. 2 was seen to be expressed fairly well by equation (1). The unbroken line in that figure is that calculated by equation (1) by expressing r in the unit of mm., and by giving the values of 100 and 278 as the values of N and k^2 respectively, and consequently by letting NP represent the number of the lightning strokes per square millimetre of the plane surface. Though the deviation of the observation-points from the theoretical curve is rather large in the vicinity of the origin; yet this may be ascribed to experimental errors, because the total number of the lightning strokes is not large enough, in this region, to give an accurate mean value of the density of the lightning strokes.

Some times we hear the vague opinion that the sparks pass roughly along the lines of force. If we interpret this vague opinion as meaning that the density of the sparks is roughly proportional to that of the lines of force, then the density of the lightning strokes on the plane surface must be proportional to the density of the induced charge on it. If we imagine approximately that a point charge is situated in the protuberance on the surface of the spherical electrode C in Fig. 1, then the surface density of the induced charge and consequently the density of the lightning strokes at a point on the plane surface E in that figure will be nearly proportional inversely to the cube of the distance between the point charge and the point on the plane surface E. This relation is shown by the broken line in Fig. 2, and which resulted in a clear disagreement with the experiment.

In the present experiment, the spark-length between the spherical electrode and the circular disc electrode was only 9 cms. as was stated before; while Peek[1] made a similar experiment with a much longer spark-length. His result with a spark-length of about 1 metre is shown by the dots in Fig. 8, and the curve in the same figure is drawn by equation (1) by giving proper values to N and k^2, which are written in the figure. Here again we see a fair agreement between the observation and the calculation.

Fig. 8



Distance from the centre im m.m.
(Peek's observation)

Lightning strokes on a plane surface with a lightning rod. The function of the lightning rod in absorbing the lightning flashes may be roughly understood by the photographs reproduced in Plate I. These are the photographs in profile of an assemblage of many sparks, which passed one after another between the upper spherical electrode C and the lower circular disc electrode E in Fig. 1 when the latter was provided with a needle in a position just below the protuberance on the surface of

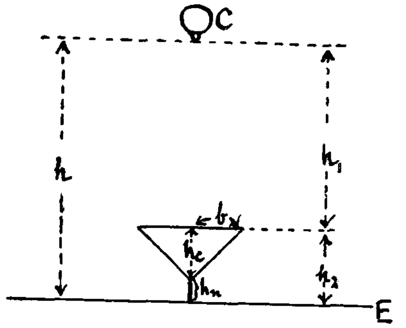
1 Peek: loc. cit.

the spherical electrode C. The height of this needle above the surface of the electrode E are respectively 6 and 8 mms. in Figs. 1 and 2, as is written respectively in these figures; and the vertical distance between the circular disc electrode E and the protuberance on the surface of the upper electrode is 9 cms. in both cases. In either of these figures, the protuberance on the surface of the upper electrode is placed in a position in the upper part of the figure, where the ends of most of the sparks meet together; and the point in the lower part of the figure, where the ends of most of the sparks meet together corresponds to the top of the sharp needle. The circular disc electrode is in a position a little lower than the top of the needle, the presence of which will be imagined in these figures by the somewhat diffuse reflection of light emitted by the sparks.

From the figures in Plate I, we can detect that the outer boundary of the sparks, which are absorbed by the needle, forms nearly a circular conical surface in the region near the top of the needle, by making that top its vertex. Considered thus it seems not to be unnatural that there should exists a conical protecting region just above the top of the needle, its semi-vertical angle and the height of its vertex from the circular base being a constant for a given height of the needle; and that the sparks which arrive at the base of this protecting cone are all absorbed by the needle.

In Fig. 9, let h be the vertical distance between the protuberance on the surface of the spherical electrode C and the circular disc E, which distance is 9 cms. in the present experiment; and let h_n be the height of the needle, h_c the height of the cone and b the radius of the circular base of the cone. If we represent h_n+h_c by h_2, then the vertical distance h_1 between the base of the cone and the protuberance on the surface of the spherical electrode C will be equal to $h-h_2$.

Fig. 9



Now let us roughly suppose that the sparks from the electrode C pass through the horizontal plane passing through the base of the cone with the distribution given by equation (1). Among these sparks, those which pass through the base of the cone will all be absorbed by the needle, and the remaining ones will shoot down upon the circular disc electrode E, according to the same law of probability as was stated before, and which will give rise to the observed distribution of the density of the lightning strokes on the circular disc E. Actually the calculation is made in the following manner. First, the density distribution of the hits of the fulguration on the circular disc electrode, which fulguration have passed through the base of the cone, is calculated on the supposition that there is no needle and that these sparks flash down to the circular disc according to the same law of probability as was stated before; and then the densities of the strokes of these sparks at various points on the circular disc are subtracted respectively from those of the lightning strokes which have occurred in the case of no needle, and which is calculated as was described before.

Here we shall consider about the constant k^2 in equation (1), which has been written to represent the quantity $2n\left(\frac{2}{\pi}\right)^2(\Delta r)^2$. We can express this quantity in another form: i. e.

$$2\left(\frac{2}{\pi}\right)^2 n\Delta z\left(\frac{\Delta r}{\Delta z}\right)\Delta r = 2\left(\frac{2}{\pi}\right)^2 h\left(\frac{\Delta r}{\Delta z}\right)\Delta r;$$

which shows that this quantity is roughly proportional to the vertical distance h between the protuberance on the surface of the spherical electrode and the circular disc electrode, provided that the value of $\frac{\Delta r}{\Delta z}$ and Δr are roughly constant. Thus for the paths of the sparks between the electrode C and the horizontal plane passing through the base of the cone we take $2\left(\frac{2}{\pi}\right)^2 h_1 \left(\frac{\Delta r}{\Delta z}\right)\Delta r$ as the probability constant k_1^2 corresponding to k^2 in equation (1); and for the paths of the sparks between the plane passing through the base of the cone and the circular disc electrode, we take $2\left(\frac{2}{\pi}\right)^2 h_2 \left(\frac{\Delta r}{\Delta z}\right)\Delta r$ as the probability constant k_2^2, for the sake of rough calculation.

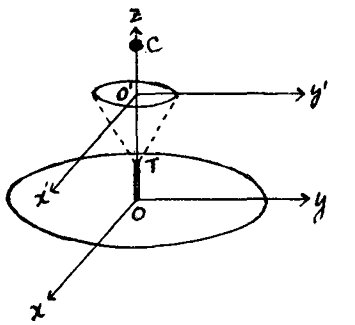
Let D_c be the density of the paths of the sparks at the plane passing through the base of the cone, when N sparks start from the upper electrode C; then D_c will be given by

$$D_c = \frac{N}{\pi k_1^2} e^{-\frac{r^2}{k_1^2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Among these sparks, those which pass through the base of the cone will all be absorbed by the lightning needle. If this lightning needle should be absent, then these sparks will hit the circular disc; and the distribution of the strokes of these sparks will be governed also by the law of probability as was stated before.

Now take the vertical direction passing through the lightning needle as z-axis, as shown in Fig. 10; and take the x- and y-axes, and x'- and y'-axes of the rectangular coordinates-system respectively in the plane of the circular disc and in that of the base of the cone, by making respectively x- and x'-axes and y- and y'-axes parallel to each other. Let us consider a surface element $dx'dy'$ at a point (x', y') on the surface of the base of the cone, then the number of the paths of the sparks which pass through this surface element will be

Fig. 10



$$D_c dx'dy' = \frac{N}{\pi k_1^2} e^{-\frac{r'^2}{k_1^2}} dx'dy', \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

where $r'^2 = x'^2 + y'^2$. Now consider a surface element at a point (x, o) on the surface of the circular disc, then the surface density of the sparks which have passed through the surface element $dx'dy'$ and arrive at this surface element is

$$\frac{N}{\pi^2 k_1^2 k_2^2} e^{-\frac{r'^2}{k_1^2}} e^{-\frac{(x-x')+y'^2}{k_2^1}} dx'dy'.$$

By putting $x' = r' \cos\theta'$, and $y' = r' \sin\theta'$, the above expression becomes to

$$\frac{N}{\pi^2 k_1^2 k_2^2} e^{-\frac{r'^2}{k_1^2}} e^{-\frac{r'^2 - 2xr'\cos\theta' + x^2}{k_2^2}} r'dr'd\theta' = \frac{N}{\pi^2 k_1^2 k_2^2} e^{-\frac{x^2}{k_2^2}} \times$$

$$e^{-\left(\frac{1}{k_2^2} + \frac{1}{k_1^2}\right) r'^2} e^{\frac{2xr'\cos\theta'}{k_2^2}} r'dr'd\theta'. \quad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

This expression, when integrated over the whole surface of the base of the cone, will give the density of the hits of the sparks at the point (x, o) on the surface of the circular disc, if the lightning needle should be absent. Thus if we represent this density by D_E, it will be

$$D_E = \frac{N}{\pi^2 k_1^2 k_2^2} e^{-\frac{x^2}{k_2^2}} \int_0^b e^{-\left(\frac{1}{k_1^2} + \frac{1}{k_2^2}\right) r'^2} r'dr' \int_0^{2\pi} e^{\frac{2xr'\cos\theta'}{k_2^2}} d\theta'.$$

The integration with respect to θ' is made by expanding $e^{\frac{2xr'\cos\theta'}{k_2^2}}$ into a power series of $\left(\frac{2xr'\cos\theta'}{k_2^2}\right)$, and this gives

$$\int_0^{2\pi} e^{\frac{2xr'\cos\theta'}{k_2^2}} d\theta' = 2\pi\left\{1 + \frac{1}{(1!)^2}\frac{x^2r'^2}{(k_2^2)^2} + \frac{1}{(2!)^2}\frac{x^4r'^4}{(k_2^2)^4} + \frac{1}{(3!)^2}\frac{x^6r'^6}{(k_2^2)^6} + \cdots\cdots\right\},$$

and consequently, by putting $k_i = \frac{1}{k_1^2} + \frac{1}{k_2^2}$,

$$D_E = \frac{2N}{\pi k_1^2 k_2^2} e^{-\frac{x^2}{k_2^2}} \left\{\int_0^b e^{-k_i r'^2} r'dr' + \frac{1}{(1!)^2}\left(\frac{x}{k_2^2}\right)^2 \int_0^b e^{-k_i r'^2} r'^3 dr' + \frac{1}{(2!)^2}\left(\frac{x}{k_2^2}\right)^4 \int_0^b e^{-k_i r'^2} r'^5 dr' + \cdots\cdots\right\}$$

which becomes, on integration and by putting $\frac{k_1^2}{k_1^2+k_2^2} = g$, to

$$D_E = \frac{N}{\pi(k_1^2+k_2^2)} e^{-\frac{x^2}{k_2^2}} \left\{B_0 + \frac{1}{1!}\left(\frac{x^2}{k_2^2}\right) g B_1 + \frac{1}{2!}\left(\frac{x^2}{k_2^2}\right)^2 g^2 B_2 + \frac{1}{3!}\left(\frac{x^2}{k_2^2}\right)^3 g^3 B_3 + \cdots\cdots\right\}, \quad \cdots\cdots(5)$$

where B_0, B_1, B_2, etc. represent respectively

$$\left.\begin{aligned}
B_0 &= 1 - e^{-k_i b^2}, \\
B_1 &= 1 - e^{-k_i b^2}\left\{1 + (k_i b^2)^1\right\}, \\
B_2 &= 1 - e^{-k_i b^2}\left\{1 + (k_i b^2)^1 + \frac{1}{2!}(k_i b^2)^2\right\}, \\
B_3 &= 1 - e^{-k_i b^2}\left\{1 + (k_i b^2)^1 + \frac{1}{2!}(k_i b^2)^2 + \frac{1}{3!}(k_i b^2)^3\right\},
\end{aligned}\right\} \quad \cdots\cdots(6)$$

etc.

If we represent by r the distance between a point on the plane xoy and the origin O, then we can write r for x in equation (5), because the direction of the x-axis has nothing to do with the density-distribution of the strokes of sparks in the present case.

Fig. 11

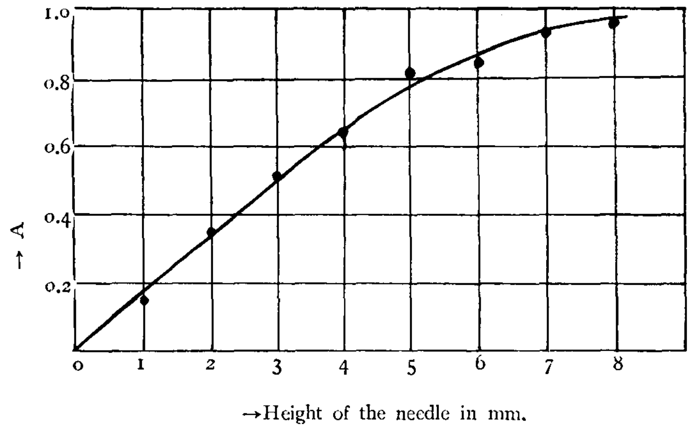
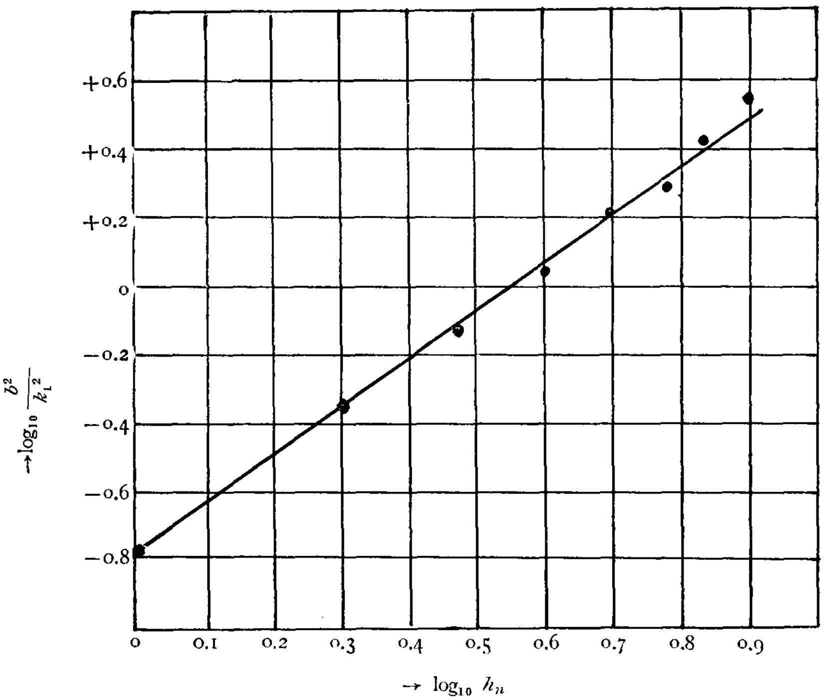


Fig. 12



For the numerical calculation of D_E in equation (5), we must know the values of b and k_1^2 for various heights of the needle. For this purpose the ratio of the number of sparks absorbed by the needle to the total number of sparks passed between the electrodes, serves in giving us the values of b^2/k_1^2. If we denote this ratio by A, then we have the following relation :

$$A=\frac{\int_0^b \frac{N}{\pi k_1^2} e^{-\frac{r'^2}{k_1^2}} 2\pi r' dr'}{\int_0^\infty \frac{N}{\pi k_1^2} e^{-\frac{r'^2}{k_1^2}} 2\pi r' dr'}=1-e^{-\frac{b^2}{k_1^2}} \quad \text{...........(7)}$$

Thus when A is known from the observation, the value of b^2/k_1^2 will be obtained readily by equation (7). The observed values of A for various heights of the needle is shown in Fig. 11. The relation between the value of b^2/k_1^2 thus obtained and the height h_n of the needle is shown in Fig. 12, by taking the logarithms of these quantities in ordinate and abscissa. As seen in this figure the observed points denoted by the dots lie nearly on a straight line; and from this linear relation the writers obtained the following empirical formula representing the relation between b^2/k_1^2 and h_n :

$$\frac{b^2}{k_1^2}=0.162\, h_n^{1.4}, \quad \text{....................................(8)}$$

where h_n represents the height of the needle.

Table I

h_n in mm.	$b_{calc.}$	$b_{obs.}$ on curves	$b_{obs.}$ on photos.	k_1^2	k_2^2
0	0	—	—	278	0
1	6.56	7	—	266.2	11.8
2	10.34	11	—	249.3	28.7
3	13.21	15	—	231.0	47.0
4	15.51	17	9	213.7	64.3
5	17.42	18	12	197.0	81.0
6	18.92	19	15	182.3	95.7
7	20.35	20	18	167.5	110.5
8	21.45	23	19.	155.0	123.0

For the values of b we can make a rough estimate in two ways: one, by measuring the conical outer boundary of the sparks absorbed by the needle, as seen in Figs. 1 and 2 in Plate I; and the other, by measuring the distance between the centre and the point of maximum density of the strokes of sparks in the curves as shown in Figs. 3, 4, 5 and 6. Of these two estimates, the former will give a somewhat smaller value, because the bulging of the conical outer boundary of the sparks absorbed by the needle is still to be expected by increasing the number of sparks; and the latter estimate will give us a somewhat greater value, because the point of the maximum density of the strokes of sparks was found, by actual calculation, to be displaced to a point a little farther apart, from the vertical axis passing through the needle, than the circle forming the outer boundary of the protecting cone considered before. The values of b estimated in such a way are tabulated in the 3rd and 4th column in Table I.

Taking the circumstances above mentioned into consideration, the writers tried to get an empirical formula representing the relation between b and h_n, and the formula was so selected as to satisfy the following three conditions:

1) b must be zero when $h_n=0$.

2) b must tend to a certain finite value when h_n becomes very large.

3) k_1^2 must be equal to 278 when h_n is equal to zero; this value of k_1^2 is that found for the circular disc electrode without any needle. The first two conditions are satisfied by the formula

$$b^2=\frac{c\,h_n^{1.4}}{a+d\,h_n^{1.4}}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(9)$$

when a, d, and c are constants; and these constants can be chosen, so as to fulfil the 3rd condition. From equations (8) and (9) we have

$$k_1^2=\frac{b^2}{0.162h_n^{1.4}}=\frac{ch_n^{1.4}}{0.162h_n^{1.4}(a+dh_n^{1.4})}=\frac{c}{0.162\,a\left(1+\frac{d}{a}h_n^{1.4}\right)}.$$

By putting $h_n=0$ and $k_1^2=278$, the above equation becomes

$$\frac{c}{a}=0.162\times 278=45\,;$$

and by putting this value into equation (9), we get

$$b^2 = \frac{45\, h_n^{1.4}}{1+\dfrac{d}{a}\, h_n^{1.4}}\,. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (10)$$

For the value of d/a in the above equation, the writers took the value of 0.0433 by trial, so as to fit the observation most closely. By using this value of d/a, equation (10) becomes

$$b^2 = \frac{45\, h_n^{1.4}}{1+0.0433\, h_n^{1.4}}\,. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (11)$$

This is the empirical formula used by the writers for the calculation of the radius b of the base of the cone. The values of b thus calculated for various heights of the needle are tabulated in the second column of Table I. By comparing the values of b thus calculated with those observed we know that the calculated ones are not unreasonable; and consequently the writers utilized the values calculated by equation (11) for further calculation.

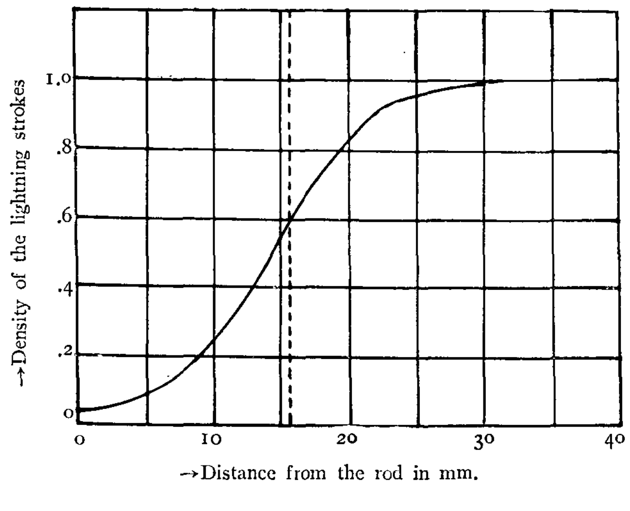
Next, the values of k_1^2 are calculated with equation (8) by using the values of b calculated, and they are given in the 5[th] column of Table I. The values of k_2^2 given in the last column of the same table are obtained from the values of k_1^2 by making use of the relation $k_1^2+k_2^2=k^2=278$.

With the values of b, k_1^2 and k_2^2 obtained in the manner stated above, we can get the values of g and k_i in equations (5) and (6) immediately; and thus the values of D_E for various points on the surface of the circular disc electrode and for various heights of the needle will be calculated by equations (5) and (6). If D_E thus found is subtracted from NP in equation (1), which is the density of the strokes of sparks without any needle, then the remainder represents the density of the strokes of sparks, when the lightning needle is present. The curves drawn in unbroken lines in Figs. 3, 4, 5 and 6 are calculated in this way. As shown in these figures, the general features of the density distribution of the strokes of sparks seem to be well represented by the calculated curves. This fact seems to support conversely our consideration, described before, on the mechanism of the development of the spark, and on the protecting cone of the lightning rod.

Here it must be noted that the convergency of the infinite series in equation (5) is not good, and consequently that very many terms (sometimes about 20 terms) have to be taken into account; which fact accompanies a laborious numerical calculation.

Up to now we have treated the case of the lightning starting at a point just above the lightning needle. But this is not the case with actual lightning in nature; and the random distribution of the starting-points of the lightning in a high altitude will be rather suited to the case of the natural lightning. This random distribution of the starting-points of the lightning in a high altitude will give a uniform density of the paths of the lightning in the region above the lightning rod; and this uniform density of the paths of the lightning will be given also by a fixed starting point of the lightning when its altitude becomes immense. With this consideration the writers calculated the density of the lightning strokes on the circular disc, in the neighbourhood of a lightning needle of 4 mms. in height. The calculation was made with equations (5) and (6) by putting $h_1=\infty$, and consequently by putting $k_1^2=\infty$; from which follows that $k_i=\frac{1}{k_1^2}+\frac{1}{k_2^2}=\frac{1}{k_2^2}$, and $g=\frac{k_1^2}{k_1^2+k_2^2}=1$. As to the constants b and k_2^2 of the lightning needle the writers employed the values given in Table I for the needle of 4 mms. height; and for the value $\frac{N}{\pi(k_1^2+k_2^2)}$ in equation (5) it was taken to be unity, which means that the density of the lightning strokes is one per mm^2. on the circular disc if the needle be absent. The density of the lightning hits thus calculated for various points on the circular disc are shown by the curve in Fig. 13, where the value of the radius of the circular base of the protecting cone is represented by the position of the broken line.

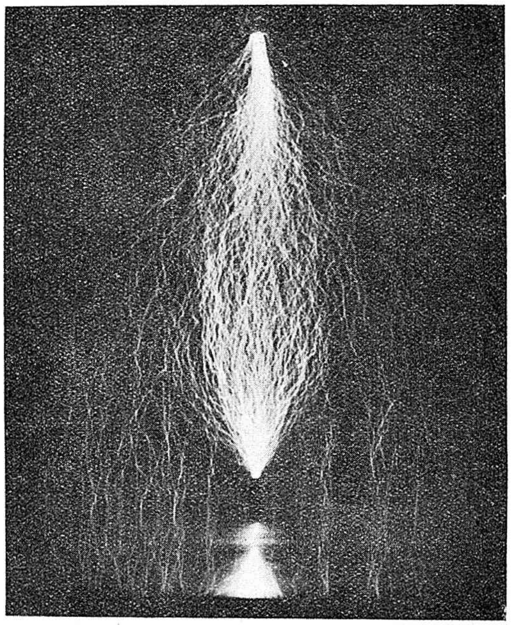
Fig. 13



The scale of the experiments made by the writers is too small to extend the results obtained to the case of the natural lightning and of the real lightning-rod. A further experiment with a longer spark length and a longer lightning needle is expected to be carried out in the near future.

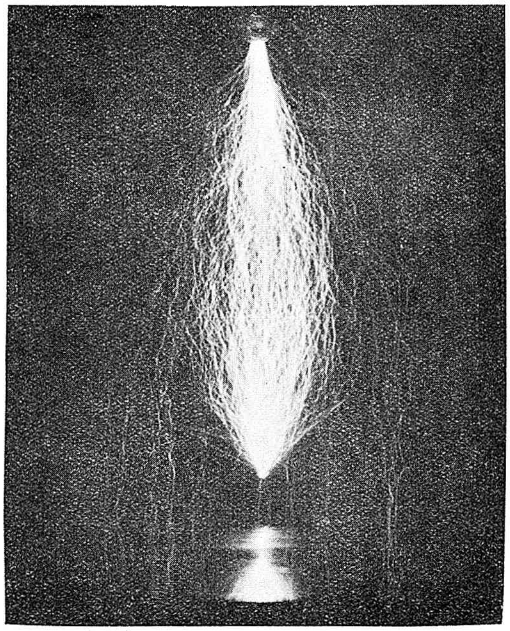
Plate I

Fig. 1



Height of the needle: 6 mm.s

Fig. 2



Height of the needle: 8 mm.s