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# On Characteristic Equations of Analytic Functions of Many Independent Variables.

By

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In my previous paper,<sup>1</sup> I investigated the property of an analytic function  $f(z_1, z_2, \dots, z_n)$  of  $n$  independent variables  $z_1, z_2, \dots, z_n$  which has an algebraic addition-theorem :

$$(1) \quad F(f_{11}, f_{12}, f_{21}, \dots, f_{n2}, f_0) = 0,$$

where  $F$  is an irreducible polynomial of  $f_{11}, f_{12}, \dots, f_{n2}, f_0$ , containing all of them explicitly, and

$$f_{ij} \equiv f(z_{11} + z_{12}, \dots, z_{i-11} + z_{i-12}, z_{ij}, z_{i+11} + z_{i+12}, \dots, z_{n1} + z_{n2}),$$

$$f_0 \equiv f(z_{11} + z_{12}, \dots, z_{n1} + z_{n2}).$$

( $i = 1, 2, \dots, n; j = 1, 2$ )

We now define a characteristic equation of an analytic function  $f(z_1, z_2, \dots, z_n)$  of  $n$  independent variables  $z_1, z_2, \dots, z_n$  as follows :—

$$(2) \quad G(f_{11}, f_{12}, f_{21}, \dots, f_{n2}, f_0) = 0,$$

<sup>1</sup> These memoirs, vol. VI, p. 251.

where  $G$  is an irreducible polynomial of  $f_{11}, f_{12}, \dots, f_{n2}, f_0$ , containing, all of them explicitly, and

$$f_{ij} \equiv f\left(\frac{z_{11} + z_{12}}{2}, \dots, \frac{z_{i-11} + z_{i-12}}{2}, z_{ij}, \frac{z_{i+11} + z_{i+12}}{2}, \dots, \frac{z_{n1} + z_{n2}}{2}\right),$$

$$(i = 1, 2, \dots, n; j = 1, 2)$$

$$f_0 \equiv f\left(\frac{z_{11} + z_{12}}{2}, \dots, \frac{z_{n1} + z_{n2}}{2}\right).$$

Then, in the case of *single* independent variable, any analytic function which has an algebraic addition-theorem has also a characteristic equation, and conversely.<sup>1</sup> The following is an outline of the proof given by M. Falk.

I. An analytic function which has an addition-theorem has also a characteristic equation.

Let

$$(3) \quad F(f(u), f(v), f(u+v)) = 0$$

be an algebraic addition-theorem of an analytic function  $f(z)$ . Then the equation (3) holds for certain branches of  $f(z)$ , i.e.

$$(4) \quad F(f_i(u), f_j(v), f_k(u+v)) = 0,$$

where  $f_i, f_j$  and  $f_k$  represent certain branches of  $f$ . By analytic continuations along suitably chosen paths in the  $u$ - and  $v$ - planes, we may obtain, from (4),

$$F(f_i(u'), f_i(v'), f_k(u'+v')) = 0,$$

where  $f_i$  is a branch of  $f$ . Put  $u' = v' = \frac{u+v}{2}$ . Then

$$(5) \quad F\left(f_i\left(\frac{u+v}{2}\right), f_i\left(\frac{u+v}{2}\right), f_k(u+v)\right) = 0.$$

<sup>1</sup> M. Falk: Ueber die Haupteigenschaften der-jenigen analytischen Funktionen eines Arguments, welche Additionstheoreme besitzen. (Nova Acta Regioe Societatis Scientiarum Upsaliensis. Ser. IV., Vol. 1, N. 8).

Eliminating  $f_k(u+v)$  from (4) and (5), we have

$$(6) \quad G(f_i(u), f_j(v), f_i\left(\frac{u+v}{2}\right))=0,$$

where  $G(f_i, f_j, f_i)$  is an irreducible polynomial of  $f_i, f_j$  and  $f_i$ . Thus we have a characteristic equation

$$G\left(f(u), f(v), f\left(\frac{u+v}{2}\right)\right)=0. \quad \text{Q.E.D.}$$

II. An analytic function which has a characteristic equation has also an addition-theorem.

Let

$$(7) \quad G\left(f(u), f(v), f\left(\frac{u+v}{2}\right)\right)=0$$

be a characteristic equation of an analytic function  $f(z)$ .

(i) Suppose that  $z=0$  be a regular point of  $f(z)$ . Let  $u, v, u', v'$  be any four values of  $z$  such that  $u+v=u'+v'$ . Then, by eliminating  $f\left(\frac{u+v}{2}\right) \equiv f\left(\frac{u'+v'}{2}\right)$  from (7) and

$$(8) \quad G\left(f(u'), f(v'), f\left(\frac{u'+v'}{2}\right)\right)=0,$$

we have

$$H(f(u), f(v), f(u'), f(v'))=0,$$

where  $H$  is an irreducible polynomial of its arguments  $f$ 's. As  $z=0$  is a regular point,  $\lim f(v')=b$  is finite and determinate, so that we may write down  $H$  in the form

$$\begin{aligned} H(f(u), f(v), f(u'), f(v')) &\equiv F(f(u), f(v), f(u')) \\ &+ F_1(f(u), f(v), f(u')), (f(v')-b) + \dots\dots\dots \\ &+ F_r(f(u), f(v), f(u')), (f(v')-b)^r = 0, \end{aligned}$$

where  $F$ 's are polynomials of  $f(u)$ ,  $f(v)$ ,  $f(u')$  and  $v'$  is a point in the vicinity of  $v'=0$ . As  $H$  is irreducible,  $F(f(u), f(v), f(u')) \neq 0$ . In the limit  $v'=0$ , we have  $u'=u+v$  and accordingly

$$F(f(u), f(v), f(u+v))=0. \quad \text{Q.E.D.}$$

(ii) Suppose that  $z=0$  be a singular point of  $f(z)$ . It may be easily proved that there is a value  $z=a$  of  $z$  such that  $f(z)$  is regular in the vicinity of  $z=a$  and  $z=2a$ .

Put  $u=u'+a$ ,  $v=v'+a$ ,

and

$$f(u) \equiv f(u'+a) \equiv g(u'), \quad f(v) \equiv g(v').$$

Then the new origin is a regular point of  $g(u')$ , and we have, by (7),

$$G\left(g(u'), g(v'), g\left(\frac{u'+v'}{2}\right)\right)=0,$$

since  $\frac{(u'+a)+(v'+a)}{2} = \frac{u'+v'}{2} + a$ . Accordingly, by (i),

$$F(g(u'), g(v'), g(u'+v'))=0,$$

or as  $g(u') \equiv f(u'+a) \equiv f(u)$ ,  $g(v') \equiv f(v)$ , and  $g(u'+v') \equiv f(u+v-a)$ ,

$$(8) \quad F(f(u), f(v), f(u+v-a))=0,$$

since  $v=2a$  is a regular point of  $f(v)$ , by putting  $v=2a$  and then by replacing  $u$  by  $u+v-a$ , we have

$$(10) \quad F(f(u+v-a), f(2a), f(u+v))=0,$$

where  $f(u+v-a)$  will represent by analytic continuation as in §1, the same branch of  $f$  as that in (9) and accordingly they are identical. Eliminating  $f(u+v-a)$  from (9) and (10), we have

$$F_1(f(u), f(v), f(u+v))=0. \quad \text{Q.E.D.}$$

Now, what is the case of *many* independent variables? This is our present question, and the answer is as follows:—

*Theorem.* In order that an analytic function  $f(z_1, z_2, \dots, z_n)$  of  $n$  independent variables  $z_1, z_2, \dots, z_n$  has a characteristic equation, it is necessary and sufficient that  $f(z_1, z_2, \dots, z_n)$  is an algebraic function of

$$\begin{aligned} & (z_1 + a_1)^{p_1} (z_2 + a_2)^{p_2} \dots (z_{r_1} + a_{r_1})^{p_{r_1}}, \dots, \\ & (z_{r_{s-1}+1} + a_{r_{s-1}+1})^{p_{r_{s-1}+1}} \dots (z_{r_s} + a_{r_s})^{p_{r_s}}, \\ & e^{a_1(z_{r_s+1} + a_{r_s+1}) \dots (z_{r_{s+1}} + a_{r_{s+1}})} \dots, \\ & e^{a_{t-s}(z_{r_{t-1}+1} + a_{r_{t-1}+1}) \dots (z_{r_t} + a_{r_t})}, \\ & \wp((z_{r_t+1} + a_{r_t+1}) \dots (z_{r_{t+1}} + a_{r_{t+1}}), \omega_1^{(1)}, \omega_2^{(1)}), \end{aligned}$$

....., where the coefficients and  $a_1, a_2, \dots, a_{r_{t+1}}, \dots$ , are constants (zero being included),  $p_1, p_2, \dots, p_{r_s}$  are integers positive or negative (zero being excluded), and  $a_1, \dots, a_{t-s}, \omega_1^{(1)}, \omega_2^{(1)}, \dots$  are constants (zero being excluded and  $\frac{\omega_2^{(1)}}{\omega_1^{(1)}}$ , ..... being not real).

As  $\frac{(z_{i1} + a_i) + (z_{i2} + a_i)}{2} = \frac{z_{i1} + z_{i2}}{2} + a_i$ , we may assume, without loss

of generality, that all  $a$ 's are zero and may thus prove the second part of the theorem analogously as in §§3-7 of my paper, *loc. cit.* For the first part of the theorem, the analogous proof to §§1-2, §§8-21 and §§23-24 of the same paper may hold; but since

$$\frac{(z_{i1} + a_i) + (z_{i2} + a_i)}{2} = \frac{z_{i1} + z_{i2}}{2} + a_i, \text{ as before, it is unnecessary that all}$$

$a$ 's  $e^{a_{i-s+1}(z_{r_i+1} + a_{r_i+1}) \dots (z_{r_{i+1}} + a_{r_{i+1}})}$ , ( $i=s, s+1, \dots, t-1$ ), and in  $\wp((z_{r_i+1} + a_{r_i+1}) \dots (z_{r_{i+1}} + a_{r_{i+1}}), \omega_1^{i-t+1}, \omega_2^{i-t+1})$ , ( $i=t, \dots$ ),

are zero and that all  $a$ 's in  $(z_{r_i+1} + a_{r_i+1})^{p_{r_i+1}} \dots (z_{r_{i+1}} + a_{r_{i+1}})^{p_{r_{i+1}}}$ , ( $i=0, 1, \dots, s-1; r_0=0$ ), are or are not zero simultaneously.

Accordingly, an analytic function of many independent variables which has an algebraic addition-theorem has also a characteristic equation, but the converse does not hold.