On Transcendental Integral and Transcendental Algebraic Functions and Algebraic Addition Theorems, II.

By

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Chapter III.

ALGEBRAIC ADDITION THEOREMS FOR ANALYTIC FUNCTIONS OF MANY INDEPENDENT VARIABLES.

INTRODUCTION.

Definition. Let $f((z_i)) \equiv f(z_1, z_2, \ldots\ldots, z_n)$ be an analytic function of $z_1, z_2, \ldots\ldots, z_n$ defined all over the Gauss space. Let

$$f_0 \equiv f(z_{11}+z_{12},\ z_{21}+z_{22},\ \ldots\ldots\ldots\ldots,\ z_{n1}+z_{n2}),$$

$$f_{i1} \equiv f(z_{11}+z_{12},\ \ldots\ldots,\ z_{i-11}+z_{i-12},\ z_{i1},\ z_{i+11}+z_{i+12},\ \ldots\ldots,\ z_{n1}+z_{n2}),$$

$$f_{i2} \equiv f(z_{11}+z_{12},\ \ldots\ldots,\ z_{i-11}+z_{i-12},\ z_{i2},\ z_{i+11}+x_{i+12},\ \ldots\ldots,\ z_{n1}+z_{n2}),$$

$$(i=1,\ 2,\ \ldots\ldots,\ n).$$

When among these $(2n+1)$ functions, there exists a relation

$$(1) \qquad F(f_{11}, f_{12}, f_{21}, \ldots\ldots\ldots, f_{n2}, f_0)=0,$$

where F is an irreducible polynominal of $f_{11}, f_{12}, \ldots\ldots\ldots, f_{n2}, f_0$, whose coefficients are independent of z_{ij} $(i=1, 2, \ldots\ldots, n;\ j=1, 2)$ and which contains all the f's expricitly, the function $f((z_i))$ is said to have an *algebraic addition theorem*.

Our object is to find the form of analytic functions which have this property. The main result is:

Theorem. In order that an analytic function $f((z_i))$, defined all over the Gauss space, may have an algebraic addition theorem, it is necessary and sufficient that $f((z_i))$ should be an algebraic fuuction of

$$(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\cdots\cdots\cdots(z_{r_1}+a_{r_1})^{p_{r_1}},\cdots\cdots\cdots\cdots\cdots$$

$(z_{r_{k-1}+1}+a_{r_{k-1}+1})^{p_{r_{k-1}+1}}\cdots\cdots\cdots\cdots\cdots(z_{r_k}+a_{r_k})^{p_{r_k}}$, $e^{\alpha z_{r_k+1}\cdots\cdots z_{r_{k+1}}}$,$\cdots\cdots\cdots\cdots$ $e^{\sigma z_{r_{s-1}+1}\cdots\cdots z_{r_s}}$, $\wp(z_{r_s+1}\cdots\cdots\cdots z_{r_{s+1}}, \omega_1^{(1)}, \omega_2^{(1)})$, $\cdots\cdots\cdots$, *where p's are integers, positive or negative, the coefficients as well as a_1, a_2, $\cdots\cdots$, a_{r_k}, α, $\cdots\cdots$, σ, $\omega_1^{(i)}$, $\omega_2^{(i)}$, $(i=1, 2, \cdots\cdots)$ are independent of the variables, and a_{r_i+1}, a_{r_i+2}, $\cdots\cdots\cdots$, $a_{r_{i+1}}$ $(i=0, 1, \cdots\cdots, k-1; r_0=0)$ are or are not zero simultaneously.*

1. *Lemma 1. Let a function $\varphi(z)$ be analytic in the whole Gauss plane. For any two arguments z_1, z_2, if there exists a relation*

$$G(\varphi(z_1),\ \varphi(z_2),\ z_1+z_2)=0 \tag{2}$$

where $G(u, v, w)$ is an irreducible polynominal of u and v and is analytic with respect to w in the whole plane, then $\varphi(z)$ is either

i) *an algebraic function of z; or*

ii) *an algebraic function of $e^{\alpha z}$; or*

iii) *an algebraic function of $\wp(z, \omega_1, \omega_2)$.*

In virtue of (2), the analytic function $\varphi(z)$ has only a finite number of branch points in any finite part of the plane. Accordingly z_1 and z_2 $(z_1+z_2=\text{constant})$ may describe certain closed circuits, by which $\varphi(z_2)$ returns to its initial element while $\varphi(z_1)$ changes into any other element. As (2) always holds for continuations and G is a polinominal of $\varphi(z_1)$, the number of all branches of $\varphi(z_1)$ is finite, i.e. *$\varphi(z)$ is a finitely many-valued function of z.* Let these branches be $\varphi_1(z)$, $\varphi_2(z)$, $\cdots\cdots\cdots$, $\varphi_n(z)$. Eliminating these values from

$$G(\varphi_i(z_1),\ \varphi(z_2),\ z_1+z_2)=0,\ (i=1,\ 2,\ \cdots\cdots\cdots,\ n)$$

and

$$S_1(\varphi_1(z_1),\ \varphi_2(z_1),\ \cdots\cdots\cdots,\ \varphi_n(z_1))\equiv S_1(\varphi(z_1))$$

where S_1 is any symmetric polinominal of φ_1, φ_2, $\cdots\cdots\cdots$, φ_n, we have

$$G_1(S_1(\varphi(z_1)),\ \ \varphi(z_2),\ \ z_1+z_2)=0,$$

where G_1 is of the same character as G. Again, eliminating $\varphi_1(z_2)$, $\varphi_2(z_2)$, ..., $\varphi_n(z_2)$ from

$$G_1(S_1(\varphi(z_1)),\quad \varphi_i(z_2),\quad z_1+z_2)=0$$

and

$$S_2(\varphi_1(z_2),\ \varphi_2(z_2),\ \dots\dots\dots,\ \varphi_n(z_2))\equiv S_2(\varphi(z_2)),$$

where S_2 is any symmetric polinominal (identical to or different from S_1) of $\varphi_1(z_2)$, $\varphi_2(z_2)$, ..., $\varphi_n(z_2)$, we have

$$(3)\qquad G_2(S_1(\varphi(z_1)),\quad S_2(\varphi(z_2)),\quad z_1+z_2)=0,$$

where G_2 is of the same character as G and G_1. In G_2, $S_1(\varphi(z))$ and $S_2(\varphi(z))$ are evidently uniform functions of z. Putting $S_1\equiv S_2\equiv S$, we have

$$G_2(S(\varphi(z_1)),\quad S(\varphi(z_2)),\quad z_1+z_2)=0.$$

This is an extension of the assumption of Weierstras's theorem[1] and the same result follows by a similar proof. Put

$$(\varphi-\varphi_1)(\varphi-\varphi_2)\dots\dots(\varphi-\varphi_n)\equiv\varphi^n+S^{(1)}\varphi^{n-1}+S^{(2)}\varphi^{n-2}+\dots\dots+S^{(n)}.$$

Then $S^{(i)}$, $(i=1, 2, \dots\dots, n)$, is a symmetric polinominal of $\varphi_1, \varphi_2, \dots\dots, \varphi_n$. Put $S=S^{(i)}$ in (3), and we have the result that $S^{(i)}$ is a rational function of z, or of $e^{\alpha z}$, or of $\wp(z, \omega_1, \omega_2)$ and $\wp'(z, \omega_1, \omega_2)$. Again, putting $S_1=S^{(i)}$ and $S_2=S^{(j)}$, $(i\neq j)$, in (3), it follows easily that $S^{(i)}$ and $S^{(j)}$ are of the same character, i.e. they are rational functions of z, or of $e^{\alpha z}$, or of $\wp(z, \omega_1, \omega_2)$ and $\wp'(z, \omega_1, \omega_2)$ simultaneously. Hence $\varphi(z)$ is

i) an algebraic function of z; or

ii) an algebraic function of $e^{\alpha z}$; or

iii) an algebraic function of $\wp(z, \omega_1, \omega_2)$.

2. *Lemma 2. Let $D(z)$ be a finite domain of z and $E(z, z')$ be a field such that z is any point in D while z' is any finite point in the whole Gauss plane. If $\varphi(z, z')$ be a function which satisfies the conditions:*

i) *regular at any point in E,*

ii) *entire function at most of a finite order p in z' for any fixed value z in D,*

iii) *uniformly increasing function of z' for all z in D,*[2]

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1 Osgood:—Lehrbuch der Funktionentheorie I. Zweite Auflage, p. 492.

2 If, for any prescribed positive value ε, there corresponds a value R such that $|\varphi(z, z')|\leqq e^{r^{\rho+\varepsilon}}$ for $|z'|\geqq R$ and for all z in D, $\varphi(z, z')$ is said to be *uniformly increasing function* of z' for all z in D.

$\frac{\partial^s \varphi(z, z')}{\partial z^s}$ *is an entire function, at most, of the p^{th} order in z' for the same value of z, where s is any positive integer.*

Let

$$(4) \qquad \varphi(z, z') \equiv a_0(z) + a_0(z) + a_1(z)z' + \dots\dots\dots + a_n(z)z'^n + \dots\dots\dots$$

where $a_n(z)$, $(n=0, 1, \dots\dots)$ are regular functions of z in D, and $z=a$ be any point in D. Then since $a_n(z)$ is regular in D, it may be expressed as a power series of $(z-a)$ in the vicinity of $z=a$ so that we have

$$(5) \qquad a_n(z) \equiv \beta_{n0} + \beta_{n1}(z-a) + \dots\dots\dots + \beta_{nm}(z-a)^m + \dots\dots\dots$$

for $|z-a| \leqq R$, where β_{nm}, $(m=0, 1, 2, \dots\dots)$, are constants, and the domain represented by $|z-a| \leqq R$ is entirely in D. Let M be the maximum value of $|\varphi(z, z')|$ for $|z-a| \leqq R$ and $|z'| \leqq R'$. Then M is finite so far as R' is finite, and we have evidently

$$|a_n(z)| \leqq \frac{M}{R'^n} \quad \text{in} \quad |z-a| \leqq R, \quad (n=0, 1, 2, \dots\dots).$$

For any positive value ε, there corresponds a finite positive value N such that

$$|a_n(z)z'^n + a_{n+1}(z)z'^{n+1} + \dots\dots| \leqq \frac{Mr'^n}{R'^n}\left(1 + \frac{r'}{R'} + \dots\dots\right) \frac{Mr'^n}{R'^n\left(1 - \frac{r'}{R'}\right)} \leqq \varepsilon$$

for $n \geqq N$, in which $|z'| = r' < R'$ and $|z-a| \leqq R$.

Consequently, (4) is a uniformly convergent series of z in the domain represented by $|z-a| \leqq R$, for any arbitrarily assigned value of z' such as $|z'| < R'$. As R', is however, arbitrary, (4) is a uniformly convergent series of z in the domain such that $|z-a| \leqq R$, for any arbitrarily assigned value of z'. Moreover $a_n(z)$ is a regular function of z in D, and accordingly $\varphi(z, z')$ is termwise differentiable with respect to z.

$$(6) \qquad \frac{\partial^s \varphi(z, z')}{\partial z} = \frac{\partial_s a_0(z)}{\partial z^s} + \frac{\partial^s a_1(z)}{\partial z^s} z' + \dots\dots + \frac{\partial^s a_n(z)}{\partial z^s} z'^n + \dots\dots\dots.$$

But we have from (5)

$$|\beta_{nm}| \leqq \frac{M}{R'^n} \cdot \frac{1}{R^m} = \frac{M}{R^m R'^n},$$

so that

$$\left|\frac{\partial^s \alpha_n(z)}{\partial z^z}\right| \leqq s(s-1)\ldots\ldots 1\,|\beta_{ns}| + (s+1)s\ldots\ldots 2\,|\beta_{ns+1}||z-a| + \ldots\ldots$$

$$\leqq \frac{M}{R^s R'^n}\cdot\frac{s\,!}{\left(1-\dfrac{r}{R}\right)^{s+1}}$$

$$= \frac{s\,!\,MR}{R'^n(R-r)^{s+1}},$$

where $|z-a| = r < R$. Hence, we have by (6)

$$\left|\frac{\partial^s \varphi(z, z')}{\partial z^s}\right| \leqq \sum_{n=0}^{\infty}\left|\frac{\partial^s \alpha_n(z)}{\partial z^s}\right| |z'^n|$$

$$\leqq \frac{s\,!\,MR}{(R-r)^{s+1}}\sum_{n=0}^{\infty}\frac{r'^n}{R'^n}$$

$$= \frac{s\,!\,MRR'}{(R-r)^{z+1}(R'-r')}.$$

Put $|z'| = r' = R' - \sigma$ where σ is a certain positive constant. Then

$$\left|\frac{\partial^s \varphi(z, z')}{\partial z^s}\right| \leqq \frac{s\,!\,M(r'+\sigma)R}{(R-r)^{s+1}\sigma}.$$

Since $\varphi(z, z')$ is, by assumption, a uniformly increasing entire function of the p^{th} order in z' for all z in D, for any positive value ε, there is a corresponding value G such that $M < e^{R'^{p+\varepsilon}}$[1] for all $R' \geqq G$ and for all z in D. Accordingly, for any fixed value of z in D, we have

$$\left|\frac{\partial^s \varphi(z, z')}{\partial z^s}\right| \leqq e^{R'^{p+\varepsilon}}\cdot\frac{s\,!\,R(r'+\sigma)}{(R-r)^{s+1}\sigma} = \frac{s\,!\,R(r'+\sigma)}{(R-r)^{s+1}\sigma}\cdot e^{r'^{p+\varepsilon}\left(1+\frac{\sigma}{r'}\right)^{p+\varepsilon}} \leqq e^{r'^{p+\varepsilon'}}$$

where ε' is so chosen as to satisfy

$$\frac{s\,!\,R}{(R-r)^{s+1}\sigma}(r'+\sigma) \leqq e^{r'^{p+\varepsilon}\left(r'^{\varepsilon'-\varepsilon} - \left(1+\frac{\sigma}{r'}\right)^{p+\varepsilon}\right)}.$$

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1 Let $M(R')$ be the maximum value of $|\varphi(z, z')|$ for $|z'| = R'$, z being an arbitrarily fixed value in D. Then, as $\varphi(z, z')$ is uniformly increasing, we have $M(R') < e^{R'^{p+\varepsilon}}$ for $R' \geqq G$ and for all z in D. Since $\varphi(z, z')$ is, however, a holomorphic function of z' for any arbitrarily assigned value of z, we have $M = M(R')$, so that $M < e^{R'^{p+\varepsilon}}$ for all $R' \geqq G$ and for all z in D.

By taking $R'=r'+\sigma$ sufficiently great, we may take ε and ε' as small as we please. Hence the proposition is true.

Lemma 3. Let

$$\Psi(z, z') \equiv \frac{\varphi_1(z, z')}{\varphi_2(z, z')}$$

where $\varphi_1(z, z')$ *aud* $\varphi_2(z, z')$ *are functions which satisfy the conditions* i), ii), *and* iii) *in lemma 2, and are of the orders* p_1 *and* p_2 *in* z' *respectively. Then* $\frac{\partial^s \Psi(z, z')}{\partial z^s}$ (s*: any positive integer*), *is of the order at most equal to the greater of* p_1 *and* p_2.

Supposing the greater of p_1 and p_2 be p, $\Psi(z, z')$ is a meromorphic function, at most, of the p^{th} order in z' for any assigned value of z in D.

$$\frac{\partial \Psi}{\partial z} = \frac{\varphi_2 \frac{\partial \varphi_1}{\partial z} - \varphi_1 \frac{\partial \varphi_2}{\partial z}}{\varphi_2^2}$$

By lemma 2, $\frac{\partial \varphi_1}{\partial z}$ and $\frac{\partial \varphi_2}{\partial z}$ are entire functions, at most, of the p^{th} order in z', and accordingly $\frac{\partial \Psi}{\partial z}$ is a mermorphic function, at most, of the p^{th} order in z'. We may easily prove, by mathematical induction, that $\frac{\partial^s \Psi(z, z')}{\partial z^s}$ is a meromorphic function, at most, of the p^{th} order in z', for any arbitrarily assigned value of z in D, where s is any positive integer.

3. We proceed now to prove the theorem. First, we prove that the condition is sufficient.

Let $f((z_i))$ be an algebraic function of $(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\ldots(z_{r_1}+a_{r_1})^{p_{r_1}}$, ..., $(z_{r_{k-1}+1}+a_{r_{k-1}+1})^{p_{r_{k-1}+1}}$ ....$(z_{r_k}+a_{r_k})^{p_{r_k}}$, $e^{\alpha z_{r_k+1}\ldots\ldots z_{r_{k+1}}}$, ..., $e^{\sigma z_{r_{s-1}+1}\ldots\ldots z_{r_s}}$, $\wp(z_{r_s+1}\ldots\ldots z_{r_{s+1}}, \omega_1^{(1)}, \omega_2^{(1)})$, ..., where p's are integers, positive or negative, the coefficients as well as a_1, a_2, ..., a_{r_k}, α, ..., σ, $\omega_1^{(i)}$, $\omega_2^{(i)}$, $(i=1, 2, \ldots\ldots)$, are independent of the variables, and $a_{r_i+1}, a_{r_i+2}, \ldots\ldots\ldots, a_{r_{i+1}}$, $(i=0, 1, \ldots\ldots, k-1; r_0=0)$, are or are not zero simultaneously.

We denote $z_1, z_2, \ldots\ldots, z_{r_1}$ by Γ_1, $z_{r_1+1}, z_{r_1+2}, \ldots\ldots\ldots, z_{r_2}$ by Γ_2 and so on. Then the number p of all Γ's thus obtained is evidently less than or equal to n. Put

$$(7)\begin{cases}(z_{11}+z_{12}+a_1)^{p_1}(z_{21}+z_{22}+a_2)^{p_2}\cdots\cdots(z_{r_1 1}+z_{r_1 2}+a_{r_1})^{p_{r_1}}\equiv\xi_1\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ (z_{r_{k-1}+11}+z_{r_{k-1}+12}+a_{r_{k-1}+1})^{p_{r_{k-1}+1}}\cdots\cdots(z_{r_k 1}+z_{r_k 2}+a_{r_k})^{p_{r_k}}\equiv\xi_k\\ e^{a(z_{r_k+11}+z_{r_k+12})\cdots\cdots(z_{r_{k+1}1}+z_{r_{k+1}2})}\equiv\xi_{k+1}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ e^{\sigma(z_{r_{s-1}+11}+z_{r_{s-1}+12})\cdots\cdots(z_{r_s 1}+z_{r_s 2})}\equiv\xi_s\\ \wp((z_{r_s+11}+z_{r_s+12})\cdots\cdots(z_{r_{s+1}1}+z_{r_{s+1}2}),\ \omega_1^{(1)},\ \omega_2^{(1)})\equiv\xi_{s+1}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\end{cases}$$

4. First, suppose that $a_1=a_2=\cdots\cdots=a_{r_1}=0$. As f_{ij}, $(i=1, 2, \cdots\cdots, r_1;\ j=1, 2)$, is an algebraic function of $(z_{11}+z_{12})^{p_1}\cdots\cdots\times(z_{i-11}+z_{i-12})^{p_{i-1}}z_{ij}^{p_i}(z_{i+11}+z_{i+12})^{p_{i+1}}\cdots\cdots(z_{r_1 1}+z_{r_1 2})^{p_{r_1}}$, $\xi_2, \xi_3, \cdots\cdots, \xi_p$, we may eliminate z_{i1}, z_{i2} from these two equations for $j=1, 2$ and the first equation in (7) and we have

(8) $\Phi_1(f_{i1}, f_{i2}, \xi_1, \xi_2, \cdots\cdots, \xi_p)=0$, $(i=1, 2, \cdots\cdots, r_1)$, where Φ_1 is an irreducible polynominal of $f_{i1}, f_{i2}, \xi_1, \xi_2, \cdots\cdots, \xi_p$, and contains f_{i1}, f_{i2}, ξ_1 explicitly. For, if it be reducible, we take one of the irreducible factors, which contains at least one of f_{i1}, f_{i2} and ξ_1 explicitly and consider it as Φ_1; if it does not contain one of f_{i1}, f_{i2} and ξ_1, say ξ_1, explicitly, f_{i1} and f_{i2} would be independent of each other which is impossible by (8).

Secondly, we suppose that $a_1\neq 0$. Then we have, by the assumption, $a_i\neq 0$, $(i=1, 2, \cdots\cdots, r_1)$. As f_{ij}, $(i=1, 2, \cdots\cdots, r_1;\ j=1, 2)$, is an algebraic function of $(z_{11}+z_{12}+a_1)^{p_1}\cdots\cdots(z_{i-11}+z_{i-12}+a_{i-1})^{p_{i-1}}(z_{ij}+a_i)^{p_i}\times(z_{i+11}+z_{i+12}+a_{i+1})^{p_{i+1}}\cdots\cdots(z_{r_1 1}+z_{r_1 2}+a_{r_1})^{p_{r_1}}$, $\xi_2, \cdots\cdots\xi_p$, we may eliminate z_{i1} and z_{i2} from these two equations for $j=1, 2$ and the first equation in (7) and we have

(9) $\varphi_i\left(f_{i1}, f_{i2}, \xi_1, \dfrac{a_i^{p_i}\xi_1}{(z_{i1}+z_{i2}+a_i)^{p_i}}, \xi_2, \cdots\cdots, \xi_p\right)=0$, $(i=1, 2, \cdots\cdots, r_1)$, where φ_i ia an irreducible polynominal of $f_{i1}, f_{i2}, \xi_1, \dfrac{a_i^{p_i}\xi_1}{(z_{i1}+z_{i2}+a_i)^{p_i}}$, $\xi_2, \cdots\cdots\cdots, \xi_p$ and contains the first four arguments explicitly.

Eliminating $\dfrac{\xi_1}{(z_{i1}+z_{i2}+a_i)^{p_i}}$, $(i=1, 2, \ldots\ldots\ldots, r_1)$, from (9) and the first equation in (7), we have

$$(10) \qquad \Phi_1(f_{11}, f_{12}, \ldots\ldots\ldots, f_{r_1 2}, \xi_1, \xi_2, \ldots\ldots\ldots, \xi_p)=0$$

where Φ_1 is an irreducible polynominal of the arguments and contains f_{ij}, $(i=1, 2, \ldots\ldots, r_1; j=1, 2)$, and ξ_1 explicitly. The same reasoning holds for $\Gamma_2, \ldots\ldots, \Gamma_k$, and we have

$$(11) \qquad \Phi_i=0, \quad (i=1, 2, \ldots\ldots, k),$$

where Φ_i is of the form (9) or (10).

As f_{ij}, $(i=r_k+1, \ldots\ldots\ldots, r_{k+1}; j=1, 2)$, is an algebraic function of $\xi_1, \xi_2, \ldots\ldots\ldots, \xi_k, \xi_{k+2}, \ldots\ldots\ldots, \xi_p$ and $e^{a(z_{r_k+11}+z_{r_k+12})\ldots\ldots\ldots(z_{i-11}+z_{i-12})z_{ij}(z_{i+11}+z_{i+12})\ldots\ldots(z_{r_{k+1}1}+z_{r_{k+1}2})}$, we may eliminate $e^{Az_{i1}}$ and $e^{Az_{i2}}$ from these two equations for $j=1, 2$ and $e^{Az_{i1}}\cdot e^{Az_{i2}}=e^{A(z_{i1}+z_{i2})}=\xi_{k+1}$, where

$$A\equiv a(z_{r_k+11}+z_{r_k+12})\ldots\ldots(z_{i-11}+z_{i-12})(z_{i+11}+z_{i+12})\ldots\ldots(z_{r_{k+1}1}+z_{r_{k+1}2})$$

and we have

$$(12) \quad \Phi_{k+1}(f_{i1}, f_{i2}, \xi_1, \xi_2, \ldots\ldots, \xi_k, \xi_{k+1}, \xi_{k+2}, \ldots\ldots, \xi_p)=0,$$
$$(i=r_k+1, \ldots\ldots\ldots, r_{k+1}),$$

where Φ_{k+1} is an irreducible polynominal of the arguments and contains $f_{i1}, f_{i2}, \xi_{k+1}$ explicitly. The same reasoning holds for $\Gamma_{k+2}, \ldots\ldots, \Gamma_s$ and we have

$$(13) \quad \Phi_i(f_{j1}, f_{j2}, \xi_1, \xi_2, \ldots\ldots\ldots\ldots, \xi_p)=0,$$
$$(i=k+1, \ldots\ldots, s; j=\text{any one of } r_{i-1}+1, \ldots\ldots, r_i),$$

where Φ_i is, as Φ_{k+1}, an irreducible polynomial of the arguments and contains f_{j1}, f_{j2}, ξ_i explicitly.

Let

$$\Psi(\wp(u), \quad \wp(v), \quad \wp(u+v))=0$$

be an algebraic addition-theorem of $\wp$-function. Then we have

$$(14) \quad \Psi(\wp(Bz_{i1}), \wp(Bz_{i2}), \xi_{s+1})=0$$

where i is any one of r_s+1, ..., r_{s+1} and

$$B\equiv(z_{r_s+11}+z_{r_s+12})\ldots\ldots(z_{i-11}+z_{i-12})(z_{i+11}+z_{i+12})\ldots\ldots(z_{r_{s+1}1}+z_{r_{s+1}2}),$$
$$\xi_{s+1}\equiv\wp((z_{r_s+11}+z_{r_s+12})\ldots\ldots\ldots\ldots(z_{r_{s+1}1}+z_{r_{s+1}2})).$$

As f_{ij}, $(i=r_s+1, \ldots\ldots\ldots, r_{s+1}; j=1, 2)$, is an algebraic function of $\xi_1, \xi_2, \ldots\ldots\ldots, \xi_s, \xi_{s+2}, \ldots\ldots\ldots, \xi_p$ and $\wp((z_{r_s+11}+z_{r_s+12})\ldots\ldots(z_{i-11}+z_{i-12})\times z_{ij}(z_{i+11}+z_{i+12})\ldots\ldots(z_{r_{s+1}1}+z_{r_{s+1}2}))$, we may eliminate $\wp(Bz_{i1})$ and $\wp(Bz_{i2})$ from these two equations for $j=1$, 2 and (14), and we have

$$(15)\quad \Phi_{s+1}(f_{i1}, f_{i2}, \xi_1, \ldots\ldots, \xi_s, \xi_{s+1}, \xi_{s+2}, \ldots\ldots, \xi_p)=0,$$
$$(i=r_s+1, \ldots\ldots, r_{s+1}),$$

where Φ_{s+1} is an irreducible polynominal of the arguments and contains $f_{i1}, f_{i2}, \xi_{s+1}$ explicitly.

The same reasoning holds for $\Gamma_{s+2}, \ldots\ldots\ldots, \Gamma_p$ and we have

$$(16)\quad \Phi_i(f_{j1}, f_{j2}, \xi_1, \xi_2, \ldots\ldots\ldots\ldots, \xi_p)=0,$$
$$(i=s+1, \ldots\ldots, p; j=\text{any one of } r_{i-1}+1, \ldots\ldots, r_i),$$

where Φ_i is, as Φ_{s+1}, an irreducible polynominal of the arguments and contains f_{j1}, f_{j2}, ξ_i explicitly.

As $f_0\equiv f(z_{11}+z_{12}, \ldots\ldots\ldots, z_{n1}+z_{n2})$ is an algebraic function of $\xi_1, \xi_2, \ldots\ldots\ldots, \xi_p$ we have

$$(17)\quad \Phi_{p+1}(f_0, \xi_1, \xi_2, \ldots\ldots\ldots, \xi_p)=0$$

where Φ_{p+1} is an irreducible polynominal of the arguments $f_0, \xi_1, \xi_2, \ldots\ldots\ldots, \xi_p$ and contains all of them explicitly.

5. Eliminating $\xi_1, \xi_2, \ldots\ldots\ldots, \xi_p$ from $p+1$ equations in (11), (13), (16) and (17), we have

$$(18)\quad \Psi_1=0$$

where Ψ_1 is an irreducible polynominal of the arguments. If one of f_{i1} and f_{i2} be not contained in Ψ_1, then the other will not be also. This may be shown as in the preceeding article.

The case where Ψ_1 contains none of f_{i1} and f_{i2}, ($i=$any one of $r_{t-1}+1$, ..., r_t), will occur when and only when ξ_t may be also eliminated by the process of elimination of $\xi_1, \xi_2, \ldots\ldots\ldots, \xi_{t-1}, \xi_{t+1}, \ldots\ldots\ldots, \xi_p$ from $\Phi_1=0, \ldots\ldots\ldots, \Phi_{t-1}=0, \Phi_{t+1}=0, \ldots\ldots\ldots, \Phi_{p+1}=0$ (without using $\Phi_t=0$). In this case we first eliminate ξ_t from $\Phi_t=0$ and $\Phi_{p+1}=0$ and then eliminate the other ξ's. Thus we have, at least, one irreducible factor of the resultant, which contains f_{i1} and f_{i2}, (i=any one of $r_{t-1}+1$, ..., r_t), explicitly. Let one of them be Ψ_2, and we have,

(19) $\Psi_2=0$.

If there is at least one Γ such that all f_{ij} which correspond to it are contained in neither Ψ_1 nor Ψ_2, then we have, by repeating the same reasoning,

(20) $\Psi_i=0$, $(i=1, 2, \ldots\ldots\ldots, q;\ q\leqq p)$,

where Ψ_i, $(i=1, 2, \ldots\ldots\ldots, q)$, is irreducible. Hence, there corresponds, to each Γ_t, at least one Ψ_i which contains some of the f's corresponding to that Γ_t.

6. *For suitably chosen constants* $\lambda_2, \lambda_3, \ldots\ldots\ldots, \lambda_q$, *all different from zero,*

$$\Psi_1+\lambda_2\Psi_2+\ldots\ldots\ldots\ldots+\lambda_q\Psi_q$$

is an irreducible polynomial of all f'*s which are contained at least in one of* $\Psi_1, \Psi_2, \ldots\ldots\ldots, \Psi_q$ *and contains all of them explicitly.*

For, let one of the f's, say f_{11}, be contained in either or both of Ψ_1 and Ψ_2 and suppose that Ψ_1 and Ψ_2 are polpnomials of the m_1^{th} and m_2^{th} degrees in f_{11} respectively. Then we may assume, without loss of generality, that $m_1\geqq m_2$. We now assume that there are infinitely many values of μ in the vicinity of $\mu=0$ for which $\Psi_1+\mu\Psi_2$ is reducible, i.e.

$$\Psi_1+\mu\Psi_2\equiv(M_m f_{11}^m+\ldots\ldots\ldots+M_0)(N_{m'}f_{11}^{m'}+\ldots\ldots\ldots+N_0)$$

where m and m' are zero or positive integers such that $m+m'=m_1$, and $M_0, \ldots\ldots\ldots, M_m, N_0, \ldots\ldots\ldots, N_{m'}$ are polynomials of f's except f_{11}, contained in Ψ_1, Ψ_2. Let those values of μ be $\mu_1, \mu_2, \ldots\ldots\ldots, \mu_i, \ldots\ldots\ldots$. Then $\lim\limits_{i=\infty}\mu_i=0$. To each μ_i, there correspond $m(\geqq 0)$ and $m'(\geqq 0)$ such that $m+m'=m_1$, and the number of combinations of (m, m') is finite while the number of the values of μ_i is infinite. Accordingly there are infinitely many values of μ_i which correspond to one and the same combination (m, m'). Considering the limit of these values of μ_i for $i=\infty$, we have

$$\Psi_1\equiv(M_m f_{11}^m+\ldots\ldots\ldots+M_0)(N_{m'}f_{11}^{m'}+\ldots\ldots\ldots+N_0).$$

As Ψ_1 is, however, irreducible, one of m and m', say m, is equal to m_1, and N_0 is an absolute constant. Hence there is at most only a finite number of μ's in the vicinity of $\mu=0$, for which $\Psi_1+\mu\Psi_2$ is reducible, i.e. $\Psi_1+\mu\Psi_2$ becomes irreducible for infinitely many values of μ. There exists at most only a finite number of values of μ, for which $\Psi_1+\mu\Psi_2$ does not contain explicitly at least one of the f's contained in Ψ_1, Ψ_2. Consequently, there are infinitely many values of μ, such that $\Psi_1+\mu\Psi_2$ is irreducible and contains all the f's explicitly, which are contained in Ψ_1, Ψ_2. By repeating the same reasoning, we have the proposition.

7. Put

$$\Psi_1+\lambda_2\Psi_2+\cdots\cdots\cdots\cdots+\lambda_q\Psi_q\equiv X_1.$$

Then X_1 is an irreducible polynomial of all the f's contained in Ψ_1, Ψ_2, ..., Ψ_q and contains all of them explicitly. Accordingly X_1 contains some of the f's explicitly, which correspond to any one of Γ_1, Γ_2, ..., Γ_p.

$$(21)\quad X_1=0.$$

Suppose that f_{i1} and f_{i2} be contained in X_1 and that z_j be any other variable corresponding to the same Γ_t as z_i $(i\neq j)$. X_1 contains f_{j1} and f_{j2} when and only when Γ_t is some one of Γ_1, Γ_2, ..., Γ_k and Φ_t has the form as (10). If X_1 does not contain f_{j1} and f_{j2}, we substitute f_{j1} and f_{j2} instead of f_{i1} and f_{i2} respectively. By all possible such substitutions, we obtain X_1, X_2, ..., X_l, where $l\leqq r_1(r_2-r_1)$ .... $\times(r_{s+1}-r_s)$......... All the f's are now contained in X_1, X_2, ..., X_l.

$$(22)\quad X_i=0,\quad (i=1,\ 2,\ \ldots\ldots\ldots,\ l).$$

By the same reasoning as in the preceding article, there are $(l-1)$ constants μ_2, μ_3, ..., μ_l, all different from zero, such that

$$X_1+\mu_2X_2+\cdots\cdots\cdots\cdots+\mu_lX_l$$

is irreducible and contains all the f's explicitly.

Let

$$X_1+\mu_2X_2+\cdots\cdots\cdots+\mu_lX_l\equiv F(f_{11},\ f_{12},\ f_{21},\ \ldots\ldots\ldots,\ f_{n2},\ f_0)=0.$$

Then F *is an irreducible polynominal of* f_{11}, f_{12}, f_{21}, ..., f_{n2}, *and* f_0 *whose coefficients are independent of* z_{11}, z_{12}, z_{21}, ..., z_{n2} *and contains all the* f'*s explicitly. Hence* $f((z_i))$ *has an algebraic addition-theorem.*

8. We now proceed to prove that the condition is necessary. Since $f((z_i))$ is, by assumption, analytic function of z_1, z_2, ..., z_n, defined

all over the Gauss space, we may choose a set of fixed points $z_{ij}=a_{ij}$, $(i=1, 2, \ldots\ldots, n; j=1, 2)$ such that

$$f_0 \equiv f(z_{11}+z_{12}, \ldots\ldots\ldots\ldots\ldots, z_{n1}+z_{n2}) \qquad \text{and}$$

$$f_{ij} \equiv f(z_{11}+z_{12}, \ldots\ldots, z_{i-11}+z_{i-12}, z_{ij}, z_{i+11}+z_{i+12}, \ldots\ldots, z_{n1}+z_{n2}).$$
$$(i=1, 2, \ldots\ldots, n; j=1, 2),$$

are analytic in the vicinity of $z_{ij}=a_{ij}$ and satisfy the equation

(1) $F(f_{11}, f_{12}, f_{21}, \ldots\ldots\ldots, f_{n2}, f_0)=0$;

that is to say, since $f((z_i))$ is generally a many-valued function, (1) will be satisfied by certain branches of f_0 and of f_{ij}, where the branches, being suitably chosen, are regular functions of z_{ij}'s in the vicinity of $z_{ij}=a_{ij}$. As $f((z_i))$ is an analytic function defined all over the Gauss space, we may easily prove, from (1), that $f((z_i))$ has only a finite number of branch points in any finite part of the plane, with respect to any one of the independent variables, the other ones being considered as arbitrary constants. Moreover, for any assigned values of $z_{21}, z_{22}, \ldots\ldots\ldots, z_{n2}$, we may select closed circuits in the planes of z_{11} and z_{12} respectively, along which the functions are continuated, such that all f's except f_{11} return to their initial elements while f_{11} changes into any other element. Proceeding in the same way as in the proof of lemma 1, we may prove that the number of all branches of f_{11}, considered as a function of z_{11} only, is finite; that is, $f((z_i))$ is a finitely many-valued function of z_1. Similarly for $z_2, z_3, \ldots\ldots\ldots, z_n$. Consequently, $f((z_i))$ is a finitely many-valued function of $z_1, z_2, \ldots\ldots\ldots, z_n$.

For arbitrarily assigned values of z_{ij}, $(i=2, 3, \ldots\ldots\ldots, n; j=1, 2)$, we may write $f_{11}\equiv\varphi(z_{11})$, $f_{12}\equiv\varphi(z_{12})$ and $F\equiv G(\varphi(z_{11}), \varphi(z_{12}), z_{11}+z_{12})=0$, where $\varphi(z)$ is an analytic function of z in the whole Gauss plane, and $G(u, v, w)$ is an irreducible polynominal of u and v and is analytic with respect to w in the whole Gauss plane. Then, by lemma 1 in §1, $f((z_i))$ is an algebraic function of z_1, or of $e^{\alpha z_1}$, or, of $\wp(z_1, \omega_1, \omega_2)$ where the coefficients as well as $\alpha, \omega_1, \omega_2$ are generally functions of $z_2, z_3, \ldots\ldots\ldots, z_n$. We may, however, assume that $f((z_i))$ is an algebraic function of $\wp(z_1, \omega_1, \omega_2)$. For, if the invariants g_2, g_3 satisfy the condition that $\triangle\equiv g_2^3-27g_3^2=0$, then $f((z_i))$ is an algebraic function of $e^{\alpha z_1}$; if $g_2=g_3=0$, then $f((z_i))$ is an algebraic function of z_1. The same reasoning holds for $z_2, z_3, \ldots\ldots\ldots, z_n$. Accordingly $f((z_i))$ is an algebraic function of $\wp(z_1, \omega_1, \omega_2)$ and is, at the same time, an algebraic function of $\wp(z_2, \Omega_1, \Omega_2)$.

9. We assume that at least one of ω_1 and ω_2 is dependent of z_2 and that at least one of Ω_1 and Ω_2 is dependent of z_1. For simplicity, we use the notations z, z' instead of z_1, z_2 respectively and consider $z_3, \ldots\ldots\ldots\ldots, z_n$ as parameters. In the vicinity of the origin suitably chosen, each branch of $f(z, z')$ is expansible as follows.

$$(23)\quad f(z, z') = a_0(z') + a_1(z')z + \ldots\ldots\ldots\ldots + a_r(z')z^r + \ldots\ldots\ldots\ldots$$

$$(24)\quad = A_0(z) + A_1(z)z' + \ldots\ldots\ldots\ldots + A_r(z)z'^r + \ldots\ldots\ldots\ldots$$

where $a_r(z')$ and $A_r(z)$, $(r=0, 1, \ldots\ldots\ldots)$, are regular in the vicinity of the origin. From what has been proved (§8), it follows that $f(z, z')$ may also satisfy the irreducible equations

$$(25)\quad \left\{b_{mm_m}(z')\wp^{m_m}(z, \omega_1(z'), \omega_2(z')) + \ldots\ldots\ldots + b_{m0}(z')\right\}f^m(z, z') + \ldots\ldots$$
$$+ \left\{b_{0m_0}(z')\wp^{m_0}(z, \omega_1(z'), \omega_2(z')) + \ldots\ldots\ldots + b_{00}(z')\right\} = 0,$$
$$(b_{mm_m}(z') \equiv),$$

$$(26)\quad \left\{B_{nn_n}(z)\wp^{n_n}(z', \Omega_1(z), \Omega_2(z)) + \ldots\ldots\ldots B_{n0}(z)\right\}f^n(z, z') + \ldots\ldots\ldots$$
$$+ \left\{B_{0n_0}(z)\wp^{n_0}(z', \Omega_1(z), \Omega_2(z)) + \ldots\ldots\ldots + B_{00}(z)\right\} = 0, (B_{nn_n}(z) \equiv 1),$$

where $b(z')$'s and $B(z)$'s are functions of z' and z respectively. Here we remark that the relation between z and z' is quite reciprocal. If a proposition is made concerning z, z', the same may be directly derived only by interchanging the corresponding functions. Hence, to simplify this relation, we use for the corresponding elements the letters a, A; b, B; m, n.

Substitute the power series in (23) and

$$\frac{1}{z^2} + \frac{g_2(z')}{20}z^2 + \frac{g_3(z')}{28}z^4 + \ldots\ldots\ldots$$

instead of $f(z, z')$ and $\wp(z, \omega_1(z'), \omega_2(z'))$ in (25) respectively, and equate the coefficient of each power of z to zero. Then an infinite number of linear homogeneous equations among $b(z')$'s, whose coefficients are polynomials of $g_2(z')$, $g_3(z')$ and $a_r(z')$, $(r=0, 1, 2, \ldots\ldots\ldots)$, are obtained. Eliminating $b(z')$'s from these equations, we have algebraic relations among $g_2(z')$, $g_3(z')$ and $a(z)$'s. Again eliminating $g_2(z')$ or $g_3(z')$, we may express $g_2(z')$ and $g_3(z')$ as algebraic functions of a finite number of $a(z)$'s. This process may be effected without fail. The only exceptional case to be considered is that in which $g_2(z')$ and $g_3(z')$ are eliminated always at the same time. In this case, one of $g_2(z')$ and $g_3(z')$ and accordingly one of $\omega_1(z')$ and $\omega_2(z')$ may be assigned arbitrarily and the other may be determined so as to satisfy (25). Let $\omega_1^{(1)}(z')$ and $\omega_1^{(2)}(z')$ be two values of $\omega_1(z')$, and $\omega_2^{(1)}(z')$ and $\omega_2^{(2)}(z')$ be their corresponding values of $\omega_2(z')$ respectively. Now we have two equations of the form (25), which will be satisfied by the same value of $f(z, z')$. Hence there exists, between $\wp(z, \omega_1^{(1)}(z'), \omega_2^{(1)}(z'))$ and $\wp(z, \omega_1^{(2)}(z'), \omega_2^{(2)}(z'))$, an algebraic relation and accordingly $\omega_1^{(1)}(z')$, $\omega_2^{(1)}(z')$, $\omega_1^{(2)}(z')$, $\omega_2^{(2)}(z')$ are connected by

$$\begin{cases} p\omega_1^{(2)}=q\omega_1^{(1)}+r\omega_2^{(1)} \\ p'\omega_2^{(2)}=q'\omega_1^{(1)}+r'\omega_2^{(1)} \end{cases}$$

where p, q, r, p', q', r', are integers and $p\neq 0$, $p'\neq 0$. As $\omega_1^{(2)}(z')$ is arbitrary, let

$$\omega_1^{(2)}(z')\equiv k\omega_1^{(1)}(z')$$

where k is real but irrational. We have

$$(pk-q)\omega_1^{(1)}(z')=r\omega_2^{(1)}(z').$$

Consequently $f(z, z')$ would have an infinitesimal period, which is certainly impossible unless it is a constant. The invariants $g_2(z')$ and $g_3(z')$ are, therefore, algebraic functions of a finite number of $a_r(z')$, $(r=0, 1, 2, \ldots\ldots\ldots)$. Since all $a_r(z')$, $(r=0, 1, \ldots\ldots\ldots)$, behave regularly in the vicinity of the origin, $g_2(z')$ and $g_3(z')$ behave algebraically in the same region. By means of linear homogeneous equations of $b(z')$'s, $(b_{mm_m}(z')\equiv 1)$, whose coefficients are polynominals of $g_2(z')$, $g_3(z')$ and $a(z')$'s, $b(z')$'s are also algebraic in the same region. Thus, *if $f(z, z')$ be regular in the vicinity of the origin, then all functions* $g_2(z')$, $g_3(z')$ and $b(z)$'s *behave algebraically in the same region.*

A similar reasoning holds for $G_2(z)$, $G_3(z)$ and $B(z)$'s.

10. As $f(z, z')$ and $B(z)$'s are algebraic in the vicinity of the origin, the same is true, by (26), of $\wp(z', G_2(z), G_3(z))$. Hence, all branches of $\wp(z', G_2(z), G_3(z))-\frac{1}{z'^2}$ are regular functions of z and z' in the vicinity of the origin which is suitably chosen. Introduce the functions

$$\zeta(z')=-\int \wp(z')dz'+C_1, \qquad \left(\lim_{z'=0}\left(\zeta(z')-\frac{1}{z'}\right)=0\right),$$

$$\sigma(z')=e^{\int\zeta(z')dz'+C_2}, \qquad \left(\lim_{z'=0}\frac{\sigma(z')}{z'}=1\right).$$

Then all branches of $\sigma(z', G_2(z), G_3(z))$ are regular in the same field of variations. Moreover

$$\sigma(z', G_2(z), G_3(z))=z'\prod\left(1-\frac{z'}{m\Omega_1(z)+n\Omega_2(z)}\right)$$
$$\times e^{\frac{z'}{m\Omega_1(z)+n\Omega_2(z)}+\frac{z'^2}{2(m\Omega_1(z)+n\Omega_2(z))^2}}$$

is a uniformly increasing entire function of the 2^{nd} order in z' for all z in the vicinity of the origin. Similarly, the same is true of

$$\Phi(z, z')\equiv\sigma^{2n_i}(z', G_2(z), G_3(z))\{B_{in_i}(z)\wp^{n_i}(z', G_2(z), G_3(z))+\ldots\ldots\ldots\ldots B_{i0}(z)\},$$

$$(i=0, 1, \ldots\ldots\ldots\ldots, n),$$

where

(27) $B_{in_i}(z)\wp^{n_i}(z', G_2(z), G_3(z))+\ldots\ldots\ldots\ldots B_{i0}(z)$

is the coefficient of $f^{(i)}(z, z')$ in (26). Hence, the numerator and the denominator af each branch of $\Phi(z, z')/\sigma^{2n_i}(z, G_2(z), G_3(z))$ are regular functions of z and z' in the assigned field and are uniformly increasing entire functions of the 2^{nd} order in z' for all z in the vicinity of the origin. Hence by lemma 3, §2, the same is true of the derivatives of $\Phi(z, z')/\sigma^{2n_i}(z, G_2(z), G_3(z))$ with respect to z. Substitute the power series in (23) instead $f(z, z')$ in (26). Then, differentiating both members of the equation r-times with respect to z and assigning zero for z, we have $a_r(z')$, $(r=0, 1, 2, \ldots\ldots\ldots)$, as a transcendental algebraic function, at most, of the 2^{nd} order. Since $g_2(z')$, $g_3(z')$ are algebraic functions of a finite number of $a_r(z')$, $r=0, 1, 2, \ldots\ldots\ldots\ldots)$ (§9), *they are also transcendental algebraic functions, at most, of the* 2^{nd} *order. Similarly for* $b(z')$'s $(b_{mm_m}(z')\equiv 1)$.

The same reasoning holds for $G_2(z)$, $G_3(z)$ and $B(z)$'s $(B_{nn_n}(z)\equiv 1)$. We may further conclude that, by

$$\frac{g_3^2(z')}{g_2^3(z')-27g_3^2(z')}\equiv J(z')\equiv\frac{4}{27}\cdot\frac{\{1-\varkappa^2(z')+\varkappa^4(z')\}^3}{\varkappa^4(z')\cdot\{1-\varkappa^2(z')\}^2},^{1}$$

[1] If $g_2(z')\equiv g_3(z')\equiv 0$, $\varkappa^2(z')$ will be indeterminate. But in this case, we have $\omega_1(z')\equiv\omega_2(z')\equiv\infty$, i.e. $\omega_1(z')$ and $\omega_2(z')$ are independent of z', which is against the assumption (§9).

$J(z')$ and $x^2(z')\equiv\zeta(z')$ are transcendental algebraic functions, at most, of the 2^{nd} order.

11. Let u_1, u_2 be a canonic fundamental system of the solutions of

$$(28)\quad \zeta(\zeta-1)\frac{d^2u}{d\zeta^2}+(2\zeta-1)\frac{du}{d\zeta}+\frac{1}{4}u=0.$$

Then they may be expressed as follows. In the vicinity of $\zeta=0$,

$$u_1^{(0)}=F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta\right)$$

$$u_2^{(0)}=F_1\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta\right)+F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta\right)\log\zeta;$$

in the vicinity of $\zeta=1$,

$$u_1^{(1)}=F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ 1-\zeta\right)$$

$$u_2^{(1)}=F_1\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ 1-\zeta\right)+F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ 1-\zeta\right)\log(1-\zeta);$$

in the vicinity of $\zeta=\infty$,

$$u_1^{(\infty)}=\left(\frac{1}{\zeta}\right)^{\frac{1}{2}}F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \frac{1}{\zeta}\right)$$

$$u_2^{(\infty)}=\left(\frac{1}{\zeta}\right)^{\frac{1}{2}}\left\{F_1\left(\frac{1}{2},\frac{1}{2},1,\frac{1}{\zeta}\right)+F\left(\frac{1}{2},\frac{1}{2},1,\frac{1}{\zeta}\right)\log\left(\frac{1}{\zeta}\right)\right\},$$

where $F\left(\frac{1}{2},\frac{1}{2},1,x\right)$ and $F\left(\frac{1}{2},\frac{1}{2},1,x\right)$ are holomorphic functions of x in the vicinity of $x=0$, and $F\left(\frac{1}{2},\frac{1}{2},1,0\right)=1$. In the vicinity of any other point $\zeta=a$, u_1 and u_2 are integral power series of $(\zeta-a)$. Since u_1, u_2 are a fundamental system of the solutions, we may assume that $u_1(a)\neq 0$, $u_2(a)=0$, while $\frac{du_2(a)}{da}\neq 0$. Then all solutions of (28) are expressible by linear combinations of u_1 and u_2.

Now, ζ is a modular function of τ which is a ratio of two linearly independent solutions of (28), so that there is a doubly periodic function $\wp(z, \bar{\omega}_1(z'), \bar{\omega}_2(z'))$ whose periods are solution of (28), the ratio being equal to τ. Hence, we may put

$$(29) \quad \begin{cases} \bar{\omega}_1(z') = Au_1\{\zeta(z')\} + Bu_2\{\zeta(z')\}, \\ \bar{\omega}_2(z') = Cu_1\{\zeta(z')\} + Du_2\{\zeta(z')\}, \end{cases}$$

where A, B, C, D are constants such that the imaginary part of $\dfrac{Cu_1 + Du_2}{Au_1 + Bu_2}$ is always positive and $AD - BC \neq 0$. We have, accordingly,

$$\tau(z') \equiv \frac{\bar{\omega}_2(z')}{\bar{\omega}_1(z')} \equiv \frac{Cu_1\{\zeta(z')\} + Du_2\{\zeta(z')\}}{Au_1\{\zeta(z')\} + Bu_2\{\zeta(z')\}}.$$

Let

$$\bar{g}_2(z') \equiv 60 \sum \frac{1}{\{m\bar{\omega}_1(z') + n\bar{\omega}_2(z')\}^4},$$

$$\bar{g}_3(z') \equiv 140 \sum \frac{1}{\{m\bar{\omega}_1(z') + n\bar{\omega}_2(z')\}^6},$$

where the summations exclude the simultaneous zero values of m and n in the expression of $m\bar{\omega}_1(z') + n\bar{\omega}_2(z')$. Then, as

$$\frac{g_2^3}{g_2^3 - 27g_3^2} = \frac{4}{27} \cdot \frac{(1 - \zeta - \zeta^2)^3}{\zeta^2(1 - \zeta)^2} = \frac{\bar{g}_2^3}{\bar{g}_2^3 - 27\bar{g}_3^2},$$

we have

$$g_2(z')\gamma^4(z') = \bar{g}_2(z')$$

and

$$g_3(z')\gamma^6(z') = \bar{g}_3(z'),$$

where $\gamma(z')$ is a certain function of z'.

Consequently, we may assume, without contradiction, that

$$\bar{\omega}_1(z')\gamma(z') = \omega_1(z'),$$

$$\bar{\omega}_2(z')\gamma(z') = \omega_2(z').$$

12. By (29), we see that if a circuit in the ζ-plane does not contain any one of the points 0, 1 and ∞, ω_1 and ω_2 return to their initial values after a complete rotation along the circuit. If the circuit contains, at least, one of the points 0, 1 and ∞, then at least one of ω_1 and ω_2 will change its value. As ζ is, however, a modular function of τ for group $\Gamma_6' : \begin{pmatrix} p, & q \\ r, & s \end{pmatrix} \equiv \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$ mod 2, $\bar{\omega}_1$ and $\bar{\omega}_2$ will change into $\pm(r\bar{\omega}_2 + s\bar{\omega}_1)$ and $\pm(p\bar{\omega}_2 + q\bar{\omega}_1)$ respectively,[1] where p, q, r, s are integers such that $p \equiv 1 \pmod 2$, $q \equiv 0 \pmod 2$, $r \equiv 0 \pmod 2$, $s \equiv 1 \pmod 2$ and $ps - qr = 1$. As $ps - qr = 1$, $\bar{g}_2$ and $\bar{g}_3$ return to their initial values after a complete rotation of ζ along the circuit. Moreover $\zeta(z')$ is a transcendental algebraic function of z'. Hence, *$\bar{g}_2(z')$ and $\bar{g}_3(z')$, as functions of z', are finitely many-valued functions of z' except in the vicinity of the singular points.*

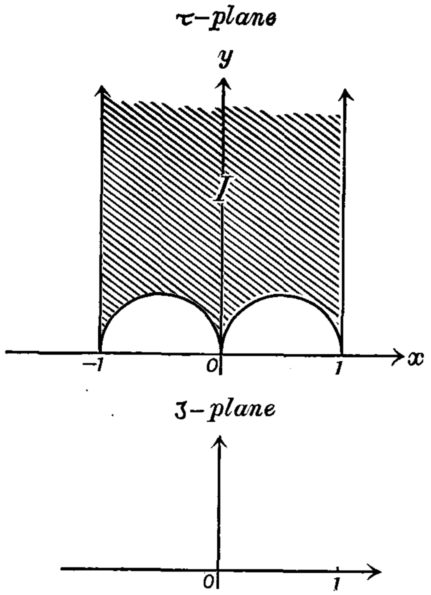

Fig. 2.

As $g_2(z')$ and $g_3(z')$ are, however, transcendental algebraic functions of z' (§10), they are finitely many-valued functions. Accordingly the same is true of $\gamma(z')$ except in the vicinity of the singular points of $\bar{g}_2(z')$ and $\bar{g}_3(z')$.

Since there exists a unique correspondency between the subdomain I of the τ-plane and the total ζ-plane, and since there correspond $\tau = -1$, 0 and ∞ to $\zeta = \infty$, 1 and 0 respectively (Fig. 2), $\tau(z') = \dfrac{\bar{\omega}_2(z')}{\bar{\omega}_1(z')}$ has a finite value from -1 and 0, at any finite point z' where $\zeta(z') \neq 0, 1, \infty$. Also

[1] If may be considered generally that $\bar{\omega}_1$ and $\bar{\omega}_2$ will change into $\bar{\omega}_1^{(1)} = \lambda r\bar{\omega}_2 + \lambda s\bar{\omega}_1$ and and $\bar{\omega}_2^{(1)} = \lambda p\bar{\omega}_2 + \lambda q\bar{\omega}_1$ respectively, where $\begin{pmatrix} p, & q \\ r, & s \end{pmatrix} \equiv \begin{pmatrix} 1, & 0 \\ 0, & 1 \end{pmatrix}$ mod 2, and $ps - qr = 1$. But by the theory of automorphic functions, there correspond $\zeta = 0$, $\zeta = 1$ and $\zeta = \infty$ to $\tau = \infty$, $\tau = 0$ and $\tau = -1$ respectively. Hence in the vicinity of $\zeta = 1$, $\bar{\omega}_1 = AF + B(F_1 + F\log(1-\zeta))$, $\bar{\omega}_2 = CF$, (§11), the constants A, B, C being suitably chosen. Accordingly $\lambda p = 1$, $\lambda q = 0$, $\lambda s = 1$, so that $q = 0$ and $\lambda p \cdot \lambda s = \lambda^2 ps = 1$. We have, however, $ps - qr = ps = 1$, and therefore $\lambda^2 = 1$ or $\lambda = \pm 1$. Consequently, we have $\bar{\omega}_1^{(1)} = \pm(r\bar{\omega}_2 + s\bar{\omega}_1)$ and $\bar{\omega}_2^{(1)} = \pm(p\bar{\omega}_2 + q\bar{\omega}_1)$.

Similarly the proposition is true in the vicinity of $\zeta = 0$ and $\zeta = \infty$.

$\bar{\omega}_1(z')$, $\bar{\omega}_2(z')$ can not be zero simultaneously; for otherwise by (29). $\bar{\omega}_1 = Bu_2$ and $\bar{\omega}_2 = Du_2$ in the vicinity of the common zero point and accordingly $\tau = \frac{\bar{\omega}_2}{\bar{\omega}_1} = \frac{B}{D}$ would be a constant. Thus none of $\bar{\omega}_1$ and $\bar{\omega}_2$ becomes 0 or ∞ at any z' where $\zeta(z') \neq 0, 1, \infty$. Moreover, $\bar{\omega}_1(z')$ and $\bar{\omega}_2(z')$ are, by (29), algebraic in the vicinity of any finite point where $\zeta(z') \neq 0, 1, \infty$, and so also for $\bar{g}_2(z')$ and $\bar{g}_3(z')$. As $g_2(z')$ and $g_3(z)$ are, however, transcendental algebraic functions, $\gamma(z')$ is also algebraic in the vicinity of any finite point where $\zeta(z') \neq 0, 1, \infty$.

13. We assume that there is, at least, one finite point where $\zeta(z') = 0$. Let one of them be $z' = a$. As $\zeta(z')$ is a transcendental algebraic function, we may assign a neighborhood of $z' = a$, such that no other singular point of $\zeta(z')$ and no other point which satisfies $\zeta(z') = 0, 1$, or ∞ is found in it. In this domain, $\bar{g}_2(z')$ is algebraic except at $z' = a$. We have therefore

$$(30) \quad \bar{g}_2(z') = a_0 + a_1(z'-a)^{\frac{1}{\lambda}} + \dots\dots\dots + a_i(z'-a)^{\frac{i}{\lambda}} + \dots\dots\dots$$
$$+ a_{-1}(z'-a)^{-\frac{1}{\lambda}} + \dots\dots\dots + a_{-i}(z'-a)^{-\frac{i}{\lambda}} + \dots\dots\dots,$$

where λ is an integer. But in the vicinity of $\zeta(a) = 0$, $\bar{\omega}_1(z')$ and $\bar{\omega}_2(z')$ take the forms (§11)

$$\bar{\omega}_1(z') = AF\{\zeta(z')\},$$
$$\bar{\omega}_2(z') = DF_1\{\zeta('z)\} + F\{\zeta(z')\} \cdot \{C + D\log\zeta(z')\}.$$

Accordingly

$$\bar{g}_2(z') = 60 \sum \frac{1}{(m\bar{\omega}_1(z') + n\bar{\omega}_2(z'))^4}$$

is continuous at $z' = a$ and take the value

$$\bar{g}_2(a) = 60 \cdot \frac{1}{A^4} \sum \frac{1}{m^4}.$$

This means that, in the expansion (30), the terms of negative powers can not occur. Thus, $\bar{g}_2(z')$ is algebraic at $z' = a$. The same may be said at any finite point, if one exists, where $\zeta(z') = 0, 1$, or ∞. Hence $\bar{g}_2(z')$ is algebraic at any finite point. Moreover, the number of values corresponding to any finite z' can not exceed that of $\zeta(z')$, i.e. a certain integer. Consequently, $\bar{g}_2(z')$ *is a transcendental algebraic function. Similarly for* $\bar{g}_3(z')$ *and* $\gamma(z')\equiv\sqrt[4]{\bar{g}_2(z')/g_2(z)}$.

14. Let there be a finite point, say $z'=a$, such that $\zeta(a)=0$. Then we have, as before (§13), in the vicinity of $z'=a$,

$$\bar{\omega}_1(z')=AF\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta(z)\right)$$

$$\bar{\omega}_2(z')=DF_1\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta(z')\right)$$

$$+\Big(C+D\log(\zeta(z'))\Big)F\left(\frac{1}{2},\ \frac{1}{2},\ 1,\ \zeta(z')\right),$$

and $\lim\limits_{z'=a}\bar{g}_2(z')=\dfrac{60}{A^4}\sum\dfrac{1}{m^4}$, where the summation is excluded $m=0$. As $\bar{g}_2(z')$ is algebraic in the vicinity of $z'=a$, $\bar{g}_2(z')-\dfrac{60}{A^4}\sum\dfrac{1}{m^4}$ has a zero of a finite order (that is, it is not of an infinitesmal order) at $z'=a$. But $\{\log(\zeta(z'))\}^4$ has an infinity of an infinitesimal order at $z'=a$ and

$$\lim_{z'=a}\Big\{\log(\zeta(z'))\Big\}^4\cdot\Big\{\bar{g}_2(z')-\frac{60}{A^4}\sum\frac{1}{m^4}\Big\}\neq 0.$$

This is a contradition, and there exists no finite point which satisfies $\zeta(z')=0$. Similarly for $\zeta(z')=1$ and $\zeta(z')=\infty$. Accordingly, $\zeta(z')$ *acquires none of the values* $\zeta=0$, 1 *and* ∞ *in any finite part of the plane.* Consequently $\bar{g}_2(z')$, $\bar{g}_3(z')$ are finite for all finite values of z'.

Similarly, let $\overline{G}_2(z)$, $\overline{G}_3(z)$, $K^2(z)$, $\Gamma(z)$, $\overline{\Omega}_1(z)$, $\overline{\Omega}_2(z)$ be the corresponding functions to $\bar{g}_2(z')$, $\bar{g}_3(z')$, $\kappa^2(z')$, $\gamma(z')$, $\bar{\omega}_1(z')$, $\bar{\omega}_2(z')$ respectively. Then they have the same properties as the corresponding functions.[1]

15. Suppose that there is a finite point $z'=a$ where $\gamma(a)=0$. Let D be a small region in the z-plane such that all branches of $\Gamma(z)$, $K^2(z)$ and $B(z)$'s are regular in it and that $\lambda\Omega_1(z)+\mu\Omega_2(z)$, ($\lambda$, $\mu=0$, ± 1, ± 2, ...), are different from those points in a small region containing $z'=a$ in it. Then all the branches of $\wp(z', G_2(z), G_3(z))$ are regular in the field of variation such that z is any point in D, while z' is any point in the vicinity of $z'=a$. Similarly for

1 See foot note in §10.

$$(27)\quad B_{in_i}(z)\wp^{n_i}(z',\ G_2(z),\ G_3(z))+\dots\dots+B_{i0}(z),\ (i=0,\ 1,\ \dots\dots,\ n),$$

and its succesive differential quotients with respect to z', where (27) is the coefficient of $f^i(z, z')$ in (26). As the derivatives of (27) with respect to z' do not all vanish identically for $z'=a$, there is a certain rational number s (zero being included) such that at least one of the limiting values of $(z'-a)^s f(z, z')$ for limit $z'=a$ is algebraic in D. Also, as $\gamma(z')$ and $b(z')$'s are determinate for limit $z'=a$, we have, by

$$\frac{b_{mm_m}(z')\dfrac{1}{\gamma^{2m_m}(z')}\wp^{m_m}\left(\dfrac{z}{\gamma(z')},\ \omega_1(z'),\ \omega_2(z')\right)+\dots\dots+b_{m0}(z')}{(z'-a)^{ms}}$$

$$\times\left\{(z'-a)^s f(z,\ z')\right\}^m+\dots\dots+\frac{b_{0m_0}(z')}{\gamma^{2m_0}(z')}\wp^{m_0}\left(\frac{z}{\gamma(z')},\ \omega_1(z'),\ \omega_2(z')\right)+\dots\dots$$

$$+b_{00}(z')=0,\quad (b_{mm_m}(z')\equiv 1),$$

$\lim\limits_{z'=a}\wp\left(\dfrac{z}{\gamma(z')},\ \omega_1(z'),\ \omega_2(z')\right)$ as an algebraic function of z in D. From what has been proved in §§12 and 14, it follows that none of $\bar{\omega}_1(z')$ and $\bar{\omega}_2(z')$ vanishes or becomes infinity for all finite values of z' and that the ratio $\bar{\omega}_2(z')/\bar{\omega}_1(z')$ has always a positive imaginary part. Hence the same is necessarily the case for $\bar{\omega}_1(a)$ and $\bar{\omega}_2(a)$, where $\bar{\omega}_1(a)$ and $\bar{\omega}_2(a)$ are the limiting values of the certain branches of $\bar{\omega}_1(z')$ and $\bar{\omega}_2(z')$ for $z'=a$, which correspond to one and the same branch of $\zeta(z')$. As $\lim\limits_{z'=a}\wp\left(\dfrac{z}{\gamma(z')},\ \bar{\omega}_1(z'),\ \bar{\omega}(z')\right)$ is algebraic, we may suppose that those which correspond to the above assigned branch of $\zeta(z')$ are $\wp(\alpha_i(z,\ a),\ \bar{\omega}_1(a), \bar{\omega}_2(a))$, $(i=1,\ \dots\dots,\ n)$, where, considering that z is an arbitrarily assigned value, $\alpha_i(z, a)$ is a point in the fundamental parallelogram. Therefore $\alpha_i(z,\ a)\not\equiv\alpha_j(z,\ a)$ (mod $\bar{\omega}_1(a),\ \bar{\omega}_2(a)$) for $i\neq j$, $(i, j=1,\ 2,\ \dots\dots,\ n)$. For any prescribed positive value ε, there is a corresponding positive value σ such that

$$\frac{z}{\gamma(z')}=p\bar{\omega}_1(z')+q\bar{\omega}_2(z')+\alpha_i(z,\ z'_2)$$

and

$$|\bar{\omega}_1(z')-\bar{\omega}_1(a)|<\sigma$$

$$|\bar{\omega}_2(z')-\bar{\omega}_2(a)|<\sigma$$

$$|\alpha_i(z,\ z')-\alpha_i(z,\ a)|<\sigma$$

for all values of z', which satisfy $0<|z'-a|<\varepsilon$, where $a_i(z, a)$ is certain one of $a_1(z, a), a_2(z, a), \ldots\ldots\ldots, a_n(z, a)$ and p, q are certain integers. Determine ε so small that 2σ is less than all $|a_i(z, a)-a_j(z, a)|$, $(i, j=1, \ldots\ldots\ldots, n;\ i\neq j)$. Suppose $z'=z_1'$, $z'=z_2'$ be two distinct points in the region $0<|z'-a|<\varepsilon$, such that in the equations

$$\frac{z}{\gamma(z_1')}=p_1\bar{\omega}_1(z_1')+q_1\bar{\omega}_2(z')+a_{i_1}(z,\ z_1')$$

and

$$\frac{z}{\gamma(z_2')}=p_2\bar{\omega}_1(z_1')+q_2\bar{\omega}_2(z')+a_{i_2}(z,\ z_2'),$$

at least one of the inequalities $p_1\neq p_2$, $q_1\neq q_2$, $i_1\neq i_2$ holds. Then as $\dfrac{z}{\gamma(z')}$ takes every value in the vicinity of infinity, there is at least one point $z'=z_0'$ on the straight line joining $z'=z_1'$ and $z'=z_2'$, for which

$$\frac{z}{\gamma(z_0')}=p_1'\bar{\omega}_1(z_0')+q_1'\bar{\omega}_2(z_0')+a_{j_1}(z,\ z_0')=p_2'\bar{\omega}_1(z_0')+q_2'\bar{\omega}_2(z_0')+a_{j_2}(z,\ z_0'),$$

where at least one of the inequalities $p_1'\neq p_2'$, $q_1'\neq q_2'$, $j_1\neq j_2$ holds. Hence we have

$$(31)\quad p\bar{\omega}_1(z_0')+q\bar{\omega}_2(z_0')=a_{j_1}(z,\ z_1')-a_{j_2}(z,\ z_0'),$$

where, in virtue of $\tau(z_0')=\dfrac{\bar{\omega}_2(z_0')}{\bar{\omega}_1(z_0')}\neq\text{real}$, $a_{j_1}(z,\ z_0')\neq a_{j_2}(z,\ z_0')$ or $j_1\neq j_2$. There exist infinitely many points such as $z'=z_0'$ for which the relation (31) holds. But the number of all possible combinations of $1, 2, \ldots\ldots, n$, such as (j_1, j_2), $(j_1\neq j_2)$ is ${}_nC_2=\dfrac{n(n-1)}{2}$ and accordingly there is at least one combination, say (j_1, j_2), for which the relation (31) holds for infinitely many values of z', which converge to $z'=a$. In (31), consider the limit $z'=a$. The right number converges to a finite value $a_{j_1}(z, a)-a_{j_2}(z, a)$. Therefore, as $\dfrac{\bar{\omega}_2(a)}{\bar{\omega}_1(a)}$ has a positive imaginary part, none of p and q becomes infinity for limit $z'=a$, and accordingly, there are infinitely many values of z' which converge to $z'=a$ and for which (31) holds for the same values of p, q, j_1 ann j_2. In the limit $z'=a$, we have

$$p\bar{\omega}_1(a)+q\bar{\omega}_2(a)=a_{j_1}(z,\ a)-a_{j_2}(z,\ a),\quad (j_1\neq j_2),$$

i.e. $$a_{j_1}(z,\ a)=a_{j_2}(z,\ a)\pmod{\bar{\omega}_1(a),\ \bar{\omega}_2(a)},\quad (j_1\neq j_2),$$

which is a contradiction. Similarly for all branches of $\zeta(z')$. Hence, $\gamma(z')$ *never acquires the value zero*

16. Let G be the aggregate of all branch points of at least one of $\gamma(z')$ and $\zeta(z')$. Then there are four cases to be considered.

i) G contains no point.
ii) G contains at least two points which are not branch points of $f(z, z')$ where z is an arbtrary constant.
iii) all points in G, with one exception at most, are branch points of $f(z, z')$ for any assigned value of z, and are infinite in number.
iv) all points in G, with one exception at most, are branch points of $f(z, z')$ for any assigned value of z, but are finite in number.

We now investigate these cases successively.

i) As G contains no branch point of $\gamma(z')$ and of $\zeta(z')$, they are mermorphic functions of z'. $\zeta(z')$ acquires, however, none of the values $\zeta=0$, 1 and ∞ for finite values of z', and hence by Picard's theorem,[1] it is a constant. As $\gamma(z')$ has no zero point at finiteness, $\frac{1}{\gamma(z')}$ is an entire function. As $\zeta(z')$ is a constant, so is $\bar{g}_2(z')$. Therefore $\frac{1}{\gamma(z')}\equiv\sqrt[4]{\frac{g_2(z')}{\bar{g}_2(z')}}$ is, as $g_2(z')$, an entire function, at most, of the 2^{nd} order. We now substitute $\frac{1}{\gamma^2(z')}\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ for $\wp(z, \gamma(z)\bar{\omega}_1, \gamma(z')\bar{\omega}_2)$ in (25). Then, as $b(z')$'s are transcendental algebraic functions, at most of the 2^{nd} order (§10) and $f(z, z')$ is, by (26), a transcendental algebraic function of the 2^{nd} order in z' for any assigned value of z, $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ is, by (25), a transcendental algebraic function, at most, of the 2^{nd} order in z' for any assigned value of $z(\gtrless 0)$. $\frac{1}{\gamma(z')}$ being, however, an entire function, $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ is a mermorphic function at most of the 2^{nd} order in z' for any assigned value of $z(\gtrless 0)$. But $\wp\left(\frac{\varepsilon}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$, $(z\gtrless 0)$, is of the 2^{nd} order in $\frac{1}{\gamma(z')}$. Accordingly for any prescribed positive value ε, there is a corresponding vulue R such that

[1] Borel: *Leçons sur les fonctions mermorphes*, p. 66.

$$\left|\frac{1}{\gamma(z')}\right| \leqq |z'|^{1+\varepsilon} \qquad \text{for all} \qquad |z'| \geqq R.$$

Hence, $\frac{1}{\gamma(z')}$ is a polynomial of the form $\alpha z'+\beta$, where α and β are constants. We conclude, therefore, that *if both $\gamma(z')$ and $\zeta(z')$ have no branch point, then $\zeta(z')$ is a constant and*

$$\gamma(z')=\frac{1}{\alpha z'+\beta}.$$

17. ii) As G contains at least two points which are not branch points of $f(z, z')$, let one of them be $z'=a$. Suppose that $\zeta^{(1)}(z')$, $u_1^{(1)}(z')$, $u_2^{(1)}(z')$, $\bar{\omega}_1^{(1)}(z')$, $\bar{\omega}_2^{(1)}(z')$, $\gamma^{(1)}(z')$ become $\zeta^{(2)}(z')$, $u_1^{(2)}(z')$, $u_2^{(2)}(z')$, $\bar{\omega}_1^{(2)}(z')$, $\bar{\omega}_2^{(2)}(z')$, $\gamma^{(2)}(z')$ respectively after a single description of a closed circuit round the branch point $z'=a$. Then, we have two equations of the form (25), which correspond to $\zeta^{(1)}(z')$ and $\zeta^{(2)}(z')$ respectively. As $z'=a$ is, by assumption, not a branch point of $f(z, z')$, $f(z, z')$ has the same values in both equations. Eliminating $f(z, z')$ from them, we have an algebraic relation between $\wp(z, \omega_1^{(1)}(z'), \omega_2^{(1)}(z'))$ and $\wp(z, \omega_1^{(2)}(z'), \omega_2^{(2)}(z'))$, and accordingly $\omega_1^{(1)}$, $\omega_2^{(1)}$, $\omega_1^{(2)}$, $\omega_2^{(2)}$ are connected by

$$(32) \qquad \begin{cases} p_1\omega_1^{(2)}=q_1\omega_1^{(1)}+r_1\omega_2^{(1)} \\ p_2\omega_2^{(2)}=q_2\omega_1^{(1)}+r_2\omega_2^{(1)} \end{cases}$$

where p_i, q_i, r_i, $(i=1, 2)$, are certain integers and p_1, p_2 are different from zero.

First, we assnme that $\gamma(z')$ has no infinity point at finiteness. As $\bar{\omega}_1(z')$ and $\bar{\omega}_2(z')$ are always finite and different from zero at finiteness (§§ 12 and 14), $\omega_1(z')$ and $\omega_2(z')$ vanish or become infinity simultaneously according as $\gamma(z')$ vanishes or become infinity. $\gamma(z')$ has, however, no zero point (§15) and no infinity point (by assumption), so that we have, at $z'=a$, $\omega_1^{(1)}(a)=\omega_1^{(2)}(a)$ and $\omega_2^{(1)}(a)=\omega_2^{(2)}(a)$, which are both finite. Accordingly, we have, in (32), $r_1=q_2=0$. For, if $r_1\gtrless 0$, then $\tau(a)=\frac{p_1-q_1}{r_1}$, i.e. $\tau(a)$ is a real quantity, which is evidently impossible. Similarly for q_2. Consequently, $p_1=q_1$ and $p_2=r_2$, and we have, therefore,

$$\gamma_1^{(1)}(z')\bar{\omega}_1^{(1)}(z')=\gamma^{(2)}(z')\bar{\omega}_1^{(2)}(z')$$

$$\gamma_1^{(1)}(z')\bar{\omega}_2^{(1)}(z')=\gamma^{(2)}(z')\bar{\omega}_2^{(2)}(z'),$$

or

$$\frac{\bar{\omega}_2^{(1)}(z')}{\bar{\omega}_1^{(1)}(z')}=\frac{\bar{\omega}_2^{(2)}(z')}{\bar{\omega}_1^{(2)}(z')}.$$

We have, however, by (29)

$$\bar{\omega}_1(z')=Au_1\{\zeta(z')\}+Bu_2\{\zeta(z')\}$$
$$\bar{\omega}_2(z')=Cu_1\{\zeta(z')\}+Du_2\{\zeta(z')\},$$

where A, B, C, D are constants such that the imaginary part of $\frac{Cu_1+Du_2}{Au_1+Bu_2}$ is always positive and $AD-BC\neq 0$.

Hence.

$$\frac{Cu_1^{(1)}(z')+Du_2^{(1)}(z')}{Au_1^{(1)}(z')+Bu_2^{(1)}(z')}=\frac{Cu_1^{(2)}(z')+Du_2^{(2)}(z')}{Au_1^{(2)}(z')+Bu_2^{(2)}(z')},$$

or

$$\frac{u_2^{(1)}(z')}{u_1^{(1)}(z')}=\frac{u_2^{(2)}(z')}{u_1^{(2)}(z')},$$

or precisely,

$$(33)\quad u_2\{\zeta^{(1)}(z)\}u_1\{\zeta^{(2)}(z')\}=u_2\{\zeta^{(2)}(z')\}u_1\{\zeta^{(1)}(z')\}.$$

Suppose that $(\lambda-1)$ be the multiplicity of the branch point $z'=a$.

$$(34)\left\{\begin{array}{l}\zeta(z')=\alpha_0+\alpha_1(z'-a)^{\frac{1}{\lambda}}+\dots\dots\dots+\alpha_i(z'-a)^{\frac{i}{\lambda}}+\dots\dots\dots,\ (\alpha_0\neq 0)\\ u_1(\zeta)=a_0+a_1(\zeta-\alpha_0)+\dots\dots\dots+a_i(\zeta-\alpha_0)^i+\dots\dots\dots,\ (a_0\neq 0)\\ u_2(\zeta)=\quad b_1(\zeta-\alpha_0)\quad+\dots\dots\dots+b_i(\zeta-\alpha_0)^i+\dots\dots\dots,\ (b_1\neq 0).\end{array}\right.$$

Let $\sigma=\cos\frac{2\pi}{\lambda}+\sqrt{-1}\sin\frac{2\pi}{\lambda}$ and μ be the least integer such that $\alpha_\mu\neq 0$ and $\mu\not\equiv 0\pmod{\lambda}$. Substituting (34) in (33), we have

$$\left[b_1\left\{\alpha_1(z'-a)^{\frac{1}{\lambda}}+\dots\dots+\alpha_i(z'-a)^{\frac{i}{\lambda}}+\dots\dots\right\}+\dots\dots\right]$$
$$\times\left[a_0+a_1\left\{\alpha_1\sigma(z'-a)^{\frac{1}{\lambda}}+\dots\dots+\alpha_i\sigma^i(z'-a)^{\frac{i}{\lambda}}+\dots\dots\right\}+\dots\dots\right]$$
$$=\left[b_1\left\{\alpha_1\sigma(z'-a)^{\frac{1}{\lambda}}+\dots\dots+\alpha_i\sigma^i(z'-a)^{\frac{i}{\lambda}}+\dots\dots\right\}+\dots\dots\right]$$
$$\times\left[a_0+a_1\left\{\alpha_1(z'-a)^{\frac{1}{\lambda}}+\dots\dots+\alpha_i(z'-a)^{\frac{i}{\lambda}}+\dots\dots\right\}+\dots\dots\dots\right].$$

Comparing the coefficient of $(z'-a)^{\frac{\mu}{\lambda}}$, we have

$$a_0 b_1 a_\mu = a_0 b_1 a_\mu \sigma^\mu,$$

which is impossible unless $\sigma^\mu = 1$. Consequently $\zeta(z')$ can not have any branch point at finiteness, so that it is a constant as before (§16). Accordingly $\bar{\omega}_1$, $\bar{\omega}_2$, $\bar{g}_2$, $\bar{g}_3$ are all constants.

Similarly we may prove that $z'=a$ can not be a branch point of $\gamma(z')$, and accordingly that it can not be a point of G. We conclude, therefore, that *any point of G, which is not a branch point of $f(z, z)$ for any constant value of z, must be an infinity point of $\gamma(z')$.*

18. Let $z'=a$ be any one of such points and transfer the origin to that point. In virtue of (1), we may easily prove that, for suitably assigned value of $z_1, z_2, \ldots\ldots, z_{i-1}, z_{i+1}, \ldots\ldots, z_n$, $f((z_i))$ is a transcendental algebraical function of z_i. Hence, in the vicinity of the new origin, each branch of $f(z, z'+a)$ is expensible as follows.

$$(24)' \quad f(z, z'+a) = \overline{A}_{-\mu}(z)\, z'^{-\frac{\mu}{\lambda}} + \ldots\ldots + \overline{A}_0(z) + \overline{A}_1(z)\, z'^{\frac{1}{\lambda}} + \ldots\ldots\ldots + \overline{A}_i(z)\, z'^{\frac{i}{\lambda}} + \ldots\ldots\ldots,$$

where $\lambda(>0)$ and $\mu(\geqq 0)$ are integers. We have also

$$(26)' \quad \{\overline{B}_{n'n'_{n'}}(z)\wp^{n'_{n'}}(z', \Omega_1(z), \Omega_2(z)) + \ldots\ldots + \overline{B}_{n'0}(z)\} f^{n'}(z, z'+a) + \ldots\ldots$$
$$+ \{\overline{B}_{0n'_0}(z)\wp^{n'_0}(z', \Omega_1(z), \Omega_2(z)) + \ldots\ldots + \overline{B}_{00}(z)\} = 0, (\overline{B}_{n'n'_{n'}}(z) \equiv 1),$$

and

$$(25)' \quad \{b_{mm_m}(z'+a)\wp^{m_m}(z, \omega_1(z'+a), \omega_2(z'+a)) + \ldots\ldots + b_{m0}(z'+a)\}$$
$$\times f^m(z, z'+a) + \ldots\ldots\ldots$$
$$+ \{b_{0m_0}(z'+a)\wp^{m_0}(z, \omega_1(z'+a), \omega_2(z'+a)) + \ldots\ldots\ldots$$
$$+ b_{00}(z'+a)\} = 0, \ (b_{mm_m}(z'+a) \equiv 1).$$

As $g_2(z'+a)$, $g_3(z'+a)$ and $b(z'+a)$'s are algebraic in the vicinity of the new origin (§10), we may determine a positive integer ν, ($\nu \equiv 0 \pmod{\lambda}$) such that all the branches of $g_2(z'+a)$, $g_3(z'+a)$ and $b(z'+a)$'s are uniform functions of $x \equiv z'^{\frac{1}{\nu}}$ in the assigned region, having the new origin as a regular of a non-essential singular point. Hence all branches of $z^2\wp(z, g_2(z'+a), g_3(z'+a))$ are regular functions of z and x in that region and are successively differentiable with respect to x; and so also for all the coefficients of the powers of $f(z, z'+a)$ in (25)'. Supposing that α be the order of infinity of $\gamma(z'+a)$ at $z'=0$, the coefficient of z^{2i}, $(i=1, 2, \dots\dots\dots)$, in the expansion of $\wp(z, g_2(z'+a), g_3(z'+a))$ in power series of z has a zero at least of the $(2i+2)\alpha^{th}$ order at $z'=0$. Accordingly, all the derivatives of $\wp(z, g_2(z'+a), g_3(z'+a))$ with respect to x are polynomials of z for $x=0$; and so also for all the coefficients of the powers of $f(z, z'+a)$ in (25)'. Substitute the fractional power series in (24)' for $f(z, z'+a)$ in (25)'. Then, differentiating successively both the members with respect to x and assigning zero for x, we have $\overline{A}_i(z)$, $(i=-\mu, \dots\dots\dots, 0, 1, 2; \dots\dots\dots)$ as an algebraic function of z. Accordingly $G_2(z)$, $G_3(z)$, $K^2(z)$ and $\overline{B}(z)$'s are all algebraic functions (§9).

We proved that in the region where $\zeta(z')$ is algebraic—even if $\zeta(z')$ acquires some of the values 0, 1 and ∞—$\bar{g}(z')$ is also algebraic (§13), and that in the region where $\bar{g}(z')$ is algebraic, $\zeta(z')$ can not acquire any one of the values 0, 1 and ∞ (§14). By the same reasoning, $K^2(z)$ can not acquire any one of the values 0, 1 and ∞ in the whole Gauss plane (including the point at infinity), which is impossible unless $K^2(z)$ is a constant. Accordingly $\overline{G}_2(z)$ and $\overline{G}_3(z)$ are constants and $\Gamma(z) \equiv \sqrt[4]{\overline{G}_2(z) / G_2(z)}$ is an ordinary algebraic function.

As there are at least two points in G, which are not branch points of $f(z, z'+a)$ for any assigned values of z, let $z'=b(\neq 0)$ be any one of them different from the new origin. Then it must be, by §17, an infinity point of $\gamma(z'+a)$ and accordingly $f(z, a+b)$ is, by (25)', an ordinary algebraic function of z. But $\Gamma(z)$ and $B(z)$'s are also Consequently $\wp\left(\frac{b}{\Gamma(z)}, \overline{\Omega}, \overline{\Omega}_2\right)$ is, by (26)', an ordinary algebraic function of z. This is evidently impossible, and hence *the case* ii) *can never occur.*

19. iii) In this case, G contains infinitely many points which are branch points of $f(z, z')$ for any assigned value of z. Let G' be the aggregate of those points. As all independent variables except $z_2 \equiv z'$ are considered as parameters, (1) reduces to the form

$$F_1(\varphi(z_1'), \varphi(z_2'), z_1'+z_2')=0,$$

where $\varphi(z') \equiv f(z, z')$ and $F_1(u, v, w)$ is an irreducible polynominal of u and v, and is analytic with respect to w in the whole Gauss plane (§8). Considering $z_1'+z_2'=c$ be a constant, let the discriminant of F_1

with respect to $\varphi(z_1')$ be $F_2(\varphi(z_2'), c)$. Then $F_2=0$ at any point in G'' where G'' is an aggregate of infinitely many points, each point z_2' of which corresponds to a certain point z' of G' by the relation $z_1'+z_2'=c$ and inversely. As F_2 is a polynominal of $\varphi_2(z_2')$, we may easily prove that all the points in G'' may be divided into a finite number of systems such that any two points in one and the same system are congruent (mod $\Omega_1(z)$, $\Omega_2(z)$), while any two points in any two different systems are incongruent (mod $\Omega_1(z)$, $\Omega_2(z)$). As G'' contains infinitely many points, there is at least one system which contains infinitely many points. This is, however, impossible unless there exist two integers p, q($p=q=0$ being excluded) such that $p\,\Omega_1(z)+q\Omega_2(z)=$constant. Put $p\Omega_1(z)+q\Omega_2(z)\equiv\Omega_1'$ and determine $\Omega_2'(z)$ such that $\Omega_2'(z')\equiv p'\Omega_1(z)+q'\Omega_2(z)$ and $\dfrac{\Omega_2'(z)}{\Omega_1'(z)}$ has always a positive imaginary part, where p', q' are integers which are relative prime and satisfy $pq'-p'q=1$. Then we may take Ω_1' and $\Omega_2'(z)$ instead of $\Omega_1(z)$ and $\Omega_2(z)$ in (26) without effecting any change on $G_2(z)$, $G_3(z)$, $\Gamma(z)$ and $B(z)$'s. If there be any system which contains infinitely many incongruent points (mod Ω_1'), we have also $\Omega_2'(z)=$constant. Hence both Ω_1' and $\Omega_2'(z)$ are constants, i.e. both $\Omega_1(z)$ and $\Omega_2(z)$ are independent of z, which is against the assumption (§9). Accordingly all points in G'' may be divided into a finite number of systems of congruent points (mod Ω_1').

Considering z as a parameter, $f(z, z')$ has a constant period Ω_1': and so has $a_i(z')$, ($i=0, 1, 2, \ldots\ldots\ldots$), in (23). Accordingly so also for $g_2(z')$, $g_3(z')$, $\gamma(z')$, $\zeta(z')$ and $b(z')$'s in (24). Put $e^{\frac{2\pi\sqrt{-1}}{\Omega_1'}z'}\equiv x$. Then these functions are transcendental algebraic functions of x. From what has been proved, it follows that ζ has, as a function of x, at most only a finite number of branch points, and accordingly it reduces, by theorem 8 in Chap. II.,[1] to an ordinary algebraic function of x. By the same reasoning as in §18, $\zeta(z')$ reduces to a constant and accordingly $\bar{g}_2(z')$, $\bar{g}_3(z')$ are also constants. Similarly as in §16, we have, for each branch of $\gamma(z')$,

$$\left|\frac{1}{\gamma(z')}\right| \leqq \left| z' \right|^{1+\varepsilon} \qquad \text{for all} \quad |z'| \geqq R,$$

where ε is any prescribed positive value and R is a corresponding value

[1] My first paper; these Memoirs, vol. VI., no. 3.

to ε. Accordingly $\gamma(z')$ must be an ordinary algebraic function, which has a period Ω_1'. This is evidently impossible and we conclude that *case* iii) *can never occur.*

20. iv). In this case, G contains only a finite number of points, so that the number of branch points of $\zeta(z')$ is finite. Accordingly, by the same reasoning as in §19, $\zeta(z')$, $\bar{g}_2(z')$, $\bar{g}_3(z')$ are constants, and $\gamma(z')$ is an ordinary algebraic function. Also $\gamma(z')$ has no zero point at finiteness (§15) and may be, therefore, expressed by an irreducible equation

$$(35)\quad T_0(z')\gamma^t + T_1(z')\gamma^{t-1} + \dots\dots\dots + 1 = 0,$$

where t is a positive integer and $T_0(z')$, $T_1(z')$, ..., $T_{t-1}(z')$ are polynomials of z.

Similarly, the foregoing reasoning (§§15–20) holds for $\Gamma(z)$ and $K^2(z)$, and we may conclude that $K^2(z)$ is a constant while $\Gamma(z)$ is an ordinary algebraic function which has no zero point at finiteness.

All the derivatives of $\wp(z', \Omega_1(z), \Omega_2(z)) = \frac{1}{\Gamma(z)^2}\wp\left(\frac{z'}{\Gamma(z)}, \bar{\Omega}_1, \bar{\Omega}_2\right)$, $(z' \neq 0)$, with respect to z are algebraic functions of z' and $\wp\left(\frac{z'}{\Gamma(z)}, \bar{\Omega}_1, \bar{\Omega}_2\right)$ where z is considered as a parameter after differentiation. Similarly for

$$(27)\quad B_{in_i}(z)\wp^{n_i}(z', G_2(z), G_3(z)) + \dots\dots\dots + B_{i0}(z), (i=0, 1, \dots\dots\dots, n),$$

which is the coefficient of $f^i(z, z')$, in (26). Substitute the power series (23) instead of $f(z, z')$ in (26). Then differentiating with respect to z and assigning the value zero for z, we have $a_i(z')$, $(i=0, 1, 2, \dots\dots\dots)$, as algebraic functions of z and $\wp(z', \Omega_1(0), \Omega_2(0))$. Similarly for $\gamma(z')$ and $b(z')$'s $(b_{mm_m}(z') \equiv 1)$. By eliminating $f(z, z')$ from (25) and (26), we have

$$\Phi\left(z', \wp(z', \Omega_1(0), \Omega_2(0)), \wp(z', \Omega_1(z), \Omega_2(z)), \wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)\right) = 0,$$

where Φ is a polynomial of z', $\wp(z', \Omega_1(0), \Omega_2(0))$, $\wp(z', \Omega_1(z), \Omega_2(z))$ and $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ whose coefficients are functions of z. Φ contains $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ explicitly. For otherwise, at least one of the values of $f(z, z')$ which are determined by (26) would satisfy (25) for any value of $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$, and accordingly there would be an algebraic relation between $\wp(z, \gamma(z')\bar{\omega}_1, \gamma(z')\bar{\omega}_2)$ and $\wp(z, \gamma(z')\bar{\omega}_1^{(1)}, \gamma(z')\bar{\omega}_2^{(1)})$ for any values of z and z', where $\bar{\omega}_1^{(1)}$, $\bar{\omega}_2^{(1)}$ are arbitrary constants such that $\frac{\bar{\omega}_2^{(1)}}{\bar{\omega}_1^{(1)}}$ has a positive imaginary part. But this is evidently impossible. There are infinitely many values of z whose limiting value is zero and for which $\Gamma(z)/\Gamma(0)$ is a rational number. Let $z=z_0$ be any one of them. Then, we may assume that the function Φ for $z=z_0$ contains $\wp\left(\frac{z_0}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ explicitly. For otherwise, Φ does not contain $\wp\left(\frac{z}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ explicitly for infinitely many values of z whose limiting value is zero: and so also for all values of z in the vicinity of $z=o$, which is impossible. For $z=z_0$, $\Omega_1(0)$, $\Omega_2(0)$, $\Omega_1(z_0)$, $\Omega_2(z_0)$ are connected by the relations (32) and hence between $\wp(z', \Omega_1(0), \Omega_2(0))$ and $\wp(z', \Omega_1(z_0), \Omega_2(z_0))$, there exists an algebraic relation. Accordingly $\Phi=0$ reduces to

$$\Phi_1\left(z', \wp(z', \Omega_1(0), \Omega_2(0)), \wp\left(\frac{z_0}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)\right)=0.$$

As $\wp\left(\frac{z_0}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ is not an ordinary algebraic function, Φ_1 contains $\wp(z', \Omega_1(0), \Omega_2(0))$ explicitly. Accordingly $\wp\left(\frac{z_0}{\gamma(z')}, \bar{\omega}_1, \bar{\omega}_2\right)$ must be a transcendental algebraic function of the 2^{nd} order in z', so that for each branch of $\gamma(z')$, there exist infinitely many values of z', say z'_i, $(i=1, 2, \ldots\ldots\ldots; \lim_{i=\infty} z'_i=\infty)$, for which $\lim_{i=\infty}\frac{\gamma(z'_i)}{z'_i}=1$. Therefore, $T_0(z')$ is a polynomial of the t^{th} degree, and $\gamma(z')$ has at least one infinitely point. Let $z'=a$ be one of them, and transfer the origin to that point. Then we have, as before (§18), (25)′ and (26)′ where $\overline{B}(z)$'s, $(\overline{B}_{nn_n}(z)\equiv 1)$, are algebraic functions of z. If G contains at least one point, say $z'=b$, which is different from the new origin and is a branch point of $f(z, z'+a)$ for any assigned value of z, the discriminant $\mathfrak{D}(\overline{B}(z), \wp(z', \Omega_1(z), \Omega_2(z))$ of (26)′ will vanish identically for $z'=b$ and for all values of z, where $\mathfrak{D}$ is a polynomial of $\wp(z', \Omega_1(z), \Omega_2(z))$ and $\overline{B}(z)$'s. Accordingly, $[\mathfrak{D}]_{z'=b}=0$ shows that $\wp(b, \Omega_1(z), \Omega_2(z))$ or $\wp\left(\frac{b}{\Gamma(z)}, \overline{\Omega}_1, \overline{\Omega}_2\right)$ is an algebraic function of z, which is impossible. Next, if there be a point of

G, which is different from the new origin and is not a branch point of $f(z, z'+a)$, then it must be an infinity point of $\gamma(z'+a)$ (§17). But similarly, as in §18, we may prove that $\gamma(z'+a)$ has at most only one infinity point at finiteness. Consequently in all cases, $\gamma(z'+a)$ has no branch point, except the new origin at most, that is to say, $\frac{1}{\gamma(z'+a)}$ has at most only one singular point—the branch point at the new origin—in the finite part of the z'-plane. Hence, in the vicinity of the new origin, all the branches of $\frac{1}{\gamma(z'+a)}$ may be expressed in the form

$$\frac{1}{\gamma(z'+a)}=y^s(a_0+a_1y+a_2y^2+\dots\dots\dots+a_iy^i+\dots\dots\dots),\ (a_0 \neq 0),$$

where y is a t^{th} root of z' and s is a certain positive integer. Then the above expression is regular for all finite values of y, so that it is a polynomial or an entire function of y. As $\frac{1}{\gamma(z'+a)}$ is, however, an algebraic function of z', it can not be an entire function of $y=z'^{\frac{1}{t}}$ i. e. it is a polynomial of y. Moreover, as $\frac{1}{\gamma(z'+a)}$ has no zero point except the new origin at finiteness, we have

$$\frac{1}{\gamma(z'+a)}=a_0y^s=a_0z'^{\frac{s}{t}}.$$

As $\lim\limits_{i=\infty}\frac{\gamma(z'_i)}{z'_i}=1$ referring to the old origin, we have $s=t$, i.e. $\frac{1}{\gamma(z'+a)}=a_0z'$. Moreover, we have $\zeta(z')=$constant, and hence we conclude that *case* iv) *can never occur.*

21. In short, all the cases except the first one are inadmissible, and we have necessarily $\gamma(z')=\frac{1}{a'z'+\beta'}$ and $\zeta(z')$ constant. Similarly by repeating the same reasoning, we have $\Gamma(z)=\frac{1}{az+\beta}$ and $K^2(z)=$ constant.

Now we transfer the origin to the point $\left(\frac{-\beta}{a}, \frac{-\beta'}{a'}\right)$ and (25) becomes

$$(25)'' \quad \left\{\bar{b}_{m'm'_{m'}}(z')\wp^{m'_{m'}}(z')\,\wp\left(z,\ \frac{\bar{\omega}_1}{a'z'},\ \frac{\bar{\omega}_2}{a'z'}\right)+\cdots\cdots\cdots+\bar{b}_{m'0}(z')\right\}$$

$$\times f^{m'}\left(z-\frac{\beta}{a},\ z'-\frac{\beta'}{a'}\right)+\cdots\cdots\cdots$$

$$+\left\{\bar{b}_{0m'_0}(z')\,\wp^{m'_0}\left(z,\ \frac{\bar{\omega}_1}{a'z'},\ \frac{\bar{\omega}_2}{a'z'}\right)+\cdots\cdots\cdots+\bar{b}_{00}(z')\right)=0,$$

$$\left(\bar{b}_{m'm'_{m'}}(z')\equiv 1\right),$$

where $\bar{b}$'s are algebraic functions of z' (§18). The equation (25)'' may, therefore, by written in the form

$$(36) \quad P_0\left(\wp\left(zz',\ \frac{\bar{\omega}_1}{a'},\ \frac{\bar{\omega}_2}{a'}\right),\ f\left(z-\frac{\beta}{a},\ z'-\frac{\beta'}{a'}\right)\right)z'^{l}+\cdots\cdots\cdots$$

$$+P_l\left(\wp\left(zz'+\frac{\bar{\omega}_1}{a'},\ \frac{\bar{\omega}_1}{a'}\right),\ f\left(z-\frac{\beta}{a},\ z'-\frac{\beta'}{a'}\right)\right)=0,$$

where P_0, P_1, ..., P_l are polynomials of $\wp$ and f. We may determine the field of variation $E(z, z')$ of z and z' such that for any values of z and z' in $E(z, z')$, f and P's are continuous. First, we assume that there exist three integers p, q, $r\,(p \neq 0)$, which satisfy the relation.

$$(37) \quad p\frac{\bar{\Omega}_1}{a}=q\frac{\bar{\omega}_1}{a'}+r\frac{\bar{\omega}_2}{a'}.$$

Substitute $z'+p\frac{\bar{\Omega}_1}{az}i$, $(i=0,\ 1,\ 2,\ \ldots\ldots\ldots\ldots)$, for z' in (36). Then, as

$$f\left(z-\frac{\beta}{a},\ z'-\frac{\beta'}{a'}+\frac{\bar{\Omega}_1}{az}pi\right)=f\left(z-\frac{\beta}{a},\ z'-\frac{\beta'}{a'}\right) \quad ((26)) \quad \text{and}$$

$$\wp\left(z\left(z'+\frac{\bar{\Omega}_1}{az}pi\right)+\frac{\bar{\omega}_1}{a'},\ \frac{\bar{\omega}_2}{a'}\right)=\wp\left(zz'+\frac{\bar{\omega}_1}{a'}qi+\frac{\bar{\omega}_2}{a'}ri,\ \frac{\bar{\omega}_1}{a'},\ \frac{\bar{\omega}_2}{a'}\right)$$

$$=\wp\left(zz',\ \frac{\bar{\omega}_1}{a'},\ \frac{\bar{\omega}_2}{a'}\right),$$

we have $P_0(\wp, f)\left(z'+p'\frac{\overline{\Omega}_1}{az}i\right)^l+\dots\dots\dots+P_l(\wp, f)=0.$

Considering limit $i=\infty$, we have necessarily

$$P_0\left(\wp\left(zz', \frac{\bar{\omega}_1}{a'}, \frac{\bar{\omega}_2}{a'}\right), f\left(z-\frac{\beta}{a}, z'-\frac{\beta'}{a'}\right)\right)\equiv 0.$$

Secondly we suppose that there are no integers which satisfy (37). Then, we may find integers p_i, q_i, r_i, $(i=1, 2, \dots\dots\dots)$, such that

$$p_i\frac{\overline{\Omega}_1}{a}=q_i\frac{\bar{\omega}_1}{a'}+r_i\frac{\bar{\omega}_2}{a'}+\varepsilon_i$$

where $\lim_{i=\infty}\varepsilon_i=0$. For z, z' in $E(z, z')$, we have

$$\lim_{i=\infty} f\left(z-\frac{\beta}{a}, z'-\frac{\beta'}{a'}+p_i\frac{\overline{\Omega}_1}{a}-\varepsilon_i\right)=\lim_{i=\infty} f\left(z-\frac{\beta}{a}, z'-\frac{\beta'}{a'}-\varepsilon_i\right)$$

$$=f\left(z-\frac{\beta}{a}, z'-\frac{\beta'}{a'}\right).$$

Accordingly, substituting $z'+\frac{p_i\overline{\Omega}_1-a\varepsilon_i}{az}=z'+\frac{q_i\bar{\omega}_1}{a'z}+\frac{r_i\bar{\omega}_2}{a'z}$ for z in (36), we have similarly

$$P_l\left(\wp\left(zz', \frac{\bar{\omega}_1}{a'}, \frac{\bar{\omega}_2}{a'}\right), f\left(z-\frac{\beta}{a}, z'-\frac{\beta'}{a'}\right)\right)\equiv 0.$$

Hence, in both cases, $f\left(z-\frac{\beta}{a}, z'-\frac{\beta'}{a'}\right)$ is an algebraic function of $\wp(zz')$ where the periods as well as the coefficients are independent of z and z'. If $\bar{\omega}_1$ or $\bar{\omega}_2$ be a function of some of z_i $(i=3, 4, \dots\dots\dots, n)$, then, we repeat the same reasoning as before and accordingly, referring to the initial origin, we have generally

$$P(\wp((z_1+a_1)(z_2+a_2)\dots\dots\dots(z_r+a_r), \lambda, \mu), f((z_i)))=0,$$

where the periods λ, μ and a_1, a_2, $\dots\dots\dots$, a_r are absolute constants and the coefficients are independent of z_1, z_2, $\dots\dots\dots$, z_r.

22. Consider z_{ij}, $(i=r+1, \dots\dots\dots, n; j=1, 2)$, as parameters and put $z_{i1}=z_{i2}=z_i$, $(i=1, 2, \dots\dots\dots, r)$. Then we have $f_{i1}=f_{i2}=f_i$, $(i=1, 2, \dots\dots\dots, r)$, and (1) becomes

$$(1)' \quad F(f_1, f_2, \dots\dots\dots, f_r, f_{r+11}, f_{r+12}, \dots\dots\dots, f_{n2}, f_0)=0.$$

Supposing that $a_2 \gtrless 0$, we assign suitably chosen values to z_i, $(i=2, 3, \dots\dots\dots, r)$, such that $\frac{2z_i+a_i}{z_i+a_i}$, $(i=3, \dots\dots\dots, r)$, is rational while $\frac{2z_2+a_2}{z_2+a_2}$ is real and irrational. In $(1)'$, f_1 is a doubly periodic function of z_1 whose periods are

$$\frac{\lambda}{(2z_2+a_2)\dots\dots\dots(2z_r+a_r)} \quad \text{and} \quad \frac{\mu}{(2z_2+a_2)\dots\dots\dots(2z_r+a_r)}:$$

f_i, $(i=2, 3, \dots\dots\dots, r)$, is a doubly periodic function of z_1 whose periods are

$$\frac{\lambda}{2(2z_2+a)\dots\dots\dots(2z_{i-1}+a_{i-1})(z_i+a_i)(2z_{i+1}+a_{i+1})\dots\dots\dots(2z_r+a_r)}$$

and

$$\frac{\mu}{2(2z_2+a_2)\dots\dots\dots(2z_{i-1}+a_{i-1})(z_i+a_i)(2z_{i+1}+a_{i+1})\dots\dots\dots(2z_r+a_r)}:$$

f_0 and f_{ij}, $(i=r+1, \dots\dots\dots, n; j=1, 2)$, are doubly periodic functions of z_1 whose periods are

$$\frac{\lambda}{2(2z_2+a_2)\dots\dots\dots(2z_r+a_r)} \quad \text{and} \quad \frac{\mu}{2(2z_2+a_2)\dots\dots\dots(2z_r+a_r)}.$$

Each period of any one of all f's except f_2 is in a commensurable ratio to the corresponding one of f_1 and those f's are therefore algebraic functions of f_1. Hence, in virtue of $(1)'$, f_2 must also be an algebraic function of f_1 and the relation (37) subsists among the periods of f_1 and of f_2. This is, however, impossible, since $\frac{2z_2+a_2}{z_2+a_2}$ is irrational. Consequently, we have $a_2=0$. Similarly $a_1=a_2=\dots\dots\dots=a_r=0$.

As $f((z_i))$ is a doubly periodic function of z_j, $(j=1, 2, \dots\dots\dots, r)$, we have

$$g_2^{(j)}(z_1,\ldots\ldots z_{j-1}z_{j+1}\ldots\ldots z_r)^4\bar{g}_2,\ g_3^{(j)}=(z_1\ldots\ldots z_{j-1}z_{j+1}\ldots\ldots z_r)^6\bar{g}_3,$$

and

$$\triangle^{(j)}=\{(g_2^{(j)})^3-27(g_3^{(j)})^2\}=(z_1\ldots\ldots z_{j-1}z_{j+1}\ldots\ldots z_r)^{12}(\bar{g}_2^3-27\bar{g}_3^2),$$

where $\bar{g}_2$ and $\bar{g}_3$ are absolute constants. If $f((z_i))$ be a simply periodic function of z_j, $\triangle^{(j)}\equiv 0$, or $\bar{g}_2^3-27\bar{g}_3^2=0$ and accordingly $f((z_i))$ is a simply periodic fuuction of $z_1z_2\ldots\ldots\ldots z_r$, i.e. it is an algebraic fnnction of $e^{Kz_1z_2\ldots\ldots z_r}$, where K is a certain constant different from zero.

23. We have hitherto discussed this problem under the assumption that at least one of ω_1 and ω_2 is dependent of z' and that at least one of Ω_1 and Ω_2 is dependent of z (§9). We now assume that at least one of ω_1 and ω_2 is dependent of z' while both Ω_1 and Ω_2 are independent of z. As Ω_1 and Ω_2 are independent z, we may conclude, from (23) and (26), that all $a(z')$'s are algebraic functions of $\wp(z',\ \Omega_1,\ \Omega_2,)$ and accordingly the same is true of $b(z')$'s, $(b_{mm_m}(z')\equiv 1)$, $g_2(z')$, $g_3(z')$ and $\zeta(z')$. Thus, $\zeta(z')$ is an algebraic function of $\wp(z',\ \Omega_1,\ \Omega_2)$, of $e^{Kz'}$, or of z', and in all cases, it must be a constant.[1] We have accordingly $\gamma(z')=\dfrac{1}{\alpha'z'+\beta'}$(§§16–20). But $\gamma(z')=\sqrt[4]{\bar{g}_2/g_2(z')}$ is an algebraic function of $\wp(z',\ \Omega_1,\ \Omega_2)$ and we have consequently $\alpha'=0$, i.e., $\gamma(z')=$constant or $\Omega_1=\Omega_2=\infty$. If $\Omega_1=\Omega_2=\infty$, $f(z,\ z')$ and $b(z')$'s will be algebraic functions of z' for any assigned value of z, and so will

$$\wp(z,\ \omega_1(z'),\ \omega_2(z'))=(\alpha'z'+\beta')^2\wp(z(\alpha'z'+\beta'),\ \bar{\omega}_1,\ \bar{\omega}_2).$$

This is however impossible, and $\gamma(z')$ must be a constant, i.e., both $\omega_1(z')$ and $\omega_2(z')$ are independent of z'. Hence, *if one of the conditions*

i) *both* $\omega_1(z')$ *and* $\omega_2(z')$ *are independent of* z',

ii) *both* $\Omega_1(z)$ *and* $\Omega_2(z)$ *are independent of* z

subsists, then the other will follow necessarily.

24. From §§22 and 23, we may conclude that if $f((z_i))$ be an algebraic function of z_i, it can neither be an algebraic function of $\wp(z_j,\ \omega_1,\ \omega_2)$, $(i\neq j)$, nor an algebraic function of $e^{\alpha z_j}$, ω_1, ω_2, or α being dependent of z_i. For suitably chosen functions $a'(z')$ and $a(z)$, if $f(z,\ z')$ be an algebraic function of $a'(z')(z+a)$ and be, at the same time, an algebraic function of $a(z)(z'+a')$, in each of which the coefficients are

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[1] As $\zeta(z')$ acquires none of the value 0, 1 and ∞, it can not be an algebraic function of $\wp(z',\ \Omega_1,\ \Omega_2)$. By the same reasoning as in the proof that $K^2(z)$ is constant (§18), $\zeta(z')$ can not be an algebraic function of z' or of $x\equiv e^{Kz'}$.

independent of z and z', then $a(z)$ and $a'(z')$ must be algebraic functions of z and of z' respectively and $a'(z')(z+a)$ is an algebraic function of $a(z)(z'+a')$ whose coefficients are independent of z and z'.

$$Q_0(a(z)(z'+a'))\{a'(z')(z+a)\}^q + \dots\dots\dots + Q_q(a(z)(z'+a'))=0,$$

where Q_0, ..., Q_q are polynomials of $a(z)(z'+a')$ whose coefficients are independent of z and z'. If $a'(z')$ has a zero point at finiteness, it must be a zero point of $Q_q(a(z)(z'+a'))$ which depends on z unless it is $z'=-a'$. Similarly, if $a'(z')$ has an infinity point at finiteness, it must also be the point $z'=-a'$. By the same reasoning, we may prove that if $a'(z')$ has a branch point at finiteness, it must also be the point $z'=-a'$. Hence we may easily conclude that $a'(z')\equiv A(z'+a')^{\frac{p'}{p}}$ where A is a constant different from zero and p, p' are integers, positive or negative; and accordingly $f(z, z')$ is an algebraic function of $(z+a)^p(z'+a')^{p'}$.

In general, $f((z_i))$ is an algebraic function of $(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\dots\dots\dots \times (z_r+a_r)^{p_r}$ where p_1, p_2, ..., p_r are integers, positive or negative, and a_1, a_2, ..., a_r are constants.

Consequently $f((z_i))$ is an algebraic function of $(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\dots\dots \times (z_{r_1}+a_{r_1})^{r_1}$, ..., $(z_{r_{k-1}+1}+a_{r_{k-1}+1})^{p_{r_{k-1}+1}}\dots\dots\dots\dots\dots(z_{r_k}+a_{r_k})^{p_{r_k}}$, $e^{\alpha z_{r_k+1}\dots\dots z_{r_{k+1}}}$, ..., $e^{\sigma z_{r_{s-1}+1}\dots\dots z_s}$, $\wp(z_{r_s+1}\dots\dots\dots\dots z_{r_{s+1}}, \omega_1^{(1)}, \omega_2^{(1)})$, ..., where p's are integers, positive or negative, and the coefficients, as well as a_1, a_2, ..., a_{r_k}, α, ..., σ, $\omega_1^{(i)}$, $\omega_2^{(i)}$, $(i=1, 2, \dots\dots\dots)$, are independent of the variables.

25. We now prove that if one of a_1, a_2, ..., a_{r_1} be zero, then all the others will be also. For the sake of brevity, we first consider that $r_1=2$. Suppose that $a_1 \neq 0$ while $a_2=0$, and eliminate z_{i1} and z_{i2}, $(i=1, 2, \dots\dots\dots, n)$, from

$$f_{i1}\equiv f(z_{11}+z_{12}, \dots\dots\dots, z_{i-11}+z_{i-12}, z_{i1}, z_{i+12}+z_{i+11}, \dots\dots\dots, z_{n1}+z_{n2}),$$

$$f_{i2}\equiv f(z_{11}+z_{12}, \dots\dots\dots, z_{i-11}+z_{i-12}, z_{i2}, z_{i+11}+z_{i+12}, \dots\dots\dots, z_{n1}+z_{n2}),$$

and

$$z_{i1}+z_{i2}\equiv Z_i, \quad (i=1, 2, \dots\dots\dots, n).$$

This may be done as in §4, and we have

$$(38)\quad \varphi_i(f_{i1}, f_{i2}, (Z_1+a_1)Z_2^{\frac{p_2}{p_2}}, Z_3, \ldots\ldots\ldots, Z_n)=0, \ (i=2, \ldots\ldots\ldots, n),$$

since f_{21}, f_{22} and f_{ij}, $(i=3, \ldots\ldots\ldots, n\,;\, j=1, 2)$, are algebraic functions of $(Z_1+a_1)z_{21}^{\frac{p_2}{p_1}}$, $(Z_1+a_1)z_{22}^{\frac{p_2}{p_1}}$ and $(Z_1+a_1)Z_2^{\frac{p_2}{p_1}}$ respectively. We have however for $i=1$

$$(39)\quad \varphi_1(f_{11}, f_{12}, (Z_1+2a_1)Z_2^{\frac{p_2}{p_1}}, Z_3, \ldots\ldots\ldots, Z_n)=0.$$

We may eliminate, as in §§5–7, $(Z_1+a_1)Z_2^{\frac{p_2}{p_1}}$, $Z_3, \ldots\ldots\ldots, Z_n$ from (38) and $f_0\equiv f((Z_1+a_1)Z_2^{\frac{p_2}{p_1}}, Z_3, \ldots\ldots\ldots, Z_n)$, we have

$$(40)\quad \Psi_1(f_{21}, f_{22}, f_{31}, \ldots\ldots\ldots, f_{n2}, f_0)=0,$$

where Ψ_1 is an irreducible polynomial of the arguments and contains f_{21}, f_{22} explicitly.

Eliminating f_{21} from (1) and (40), we have

$$(41)\quad \Psi_2(f_{11}, f_{12}, f_{22}, f_{31}, \ldots\ldots\ldots, f_{n2}, f_0)=0,$$

which does not contain f_{22} explicitly. For otherwise, considering z_{22} as a variable and $z_{11}, z_{12}, Z_2, z_{31}, \ldots\ldots\ldots, z_{n2}$ as constants, all f's in Ψ_2 except f_{22} are constants while f_{22} is a variable. This is however impossible by (41). As Ψ_2 is a function of $f_{11}, f_{12}, f_{31}, \ldots\ldots\ldots, f_{n2}, f_0$ and does not contain f_{21}, f_{22} and z's explicitly, (41) shows that we may eliminate $Z_1, Z_2, \ldots\ldots\ldots, Z_n$ from $\varphi_i=0$, $(i=1, 3, \ldots\ldots\ldots, n)$, and $f_0\equiv f((Z_1+a_1)Z_2^{\frac{p_2}{p_1}}, Z_3, \ldots\ldots\ldots, Z_n)$. This is however impossible since φ_i, $(i=3, \ldots\ldots\ldots, n)$, are functions of $(n-1)$ independent variables $(Z_1+a_1)Z_2^{\frac{p_2}{p_1}}, Z_3, \ldots\ldots\ldots, Z_n$ while φ_1 contains another variable Z_2 which is independent of the former ones. Accordingly if $a_2=0$, we have also $a_1=0$. We may similarly prove that if $f((z_i))$ be an algebraic function of $(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\ldots\ldots\ldots\ldots(z_{r_1}+a_{r_1})^{p_{r_1}}$, $a_1, a_2, \ldots\ldots\ldots, a_{r_1}$ are or are not zero simultaneously. Similarly for $a_{r_1+1}, a_{r_1+2}, \ldots\ldots\ldots, a_{r_2}$; $a_{r_2+1}, \ldots\ldots\ldots, a_{r_3}$; $\ldots\ldots\ldots\ldots\ldots$

Consequently $f((z_i))$ *is an algebraic function of* $(z_1+a_1)^{p_1}(z_2+a_2)^{p_2}\ldots\ldots\ldots\ldots$

$\times\,(z_{r_1}+a_{r_1})^{p_{r_1}}$.................. , $(z_{r_{k-1}+1}+a_{r_{k-1}+1})^{p_{r_{k-1}+1}}$............$(z_{r_k}+a_{r_k})^{p_{r_k}}$,

$e^{\alpha z_{r_k+1}\cdots\cdots z_{r_{k+1}}}$, ..., $e^{\sigma z_{r_{s-1}+1}\cdots\cdots z_{r_s}}$, $\wp(z_{r_s+1}\cdots\cdots\cdots z_{r_{s+1}},\ \omega_1^{(1)},\ \omega_2^{(1)})$, ...

where p's are integers, positive or negative, the coefficients as well as a_1, a_2, ..., a_{r_k}, α, ..., σ, $\omega_1^{(i)}$, $\omega_2^{(i)}$, $(i=1, 2, \ldots\ldots\ldots)$, *are independent of the variables and* a_{r_i+1}, a_{r_i+2}, ..., $a_{r_{i+1}}$, $(i=0, 1, \ldots\ldots\ldots, k-1;\ r_0=0)$, *are or are not zero simultaneously.*

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