On Transcendental Integral and Transcendental Algebraic Functions and Algebraic Addition Theorems, I

Yasuda, Ryo

On Transcendental Integral and Transcendental Algebraic Functions and Algebraic Addition Theorems, I.

By

Ryô Yasuda.

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Our main problem is the study of the analytic functions of many independent variables, which have algebraic addition theorems. For that purpose we shall first discuss some properties of transcendental integral and transcendental algebraic functions, which are of fundamental importance for our later investigations.

Chapter I.

TRANSCENDENTAL INTEGRAL FUNCTIONS OF TRANSFINITE ORDERS.

INTRODUCTION.

Suppose that $F(z)$ be a transcendental integral function and $M(r)$ be the maximum value of its modulus for $|z|=r$. Then there exist two finite numbers α, β such that $e^{r^\alpha}<M(r)<e^{r^\beta}$, or $\dfrac{\log\log M(r)}{\log r}$ is a number in the interval (α, β). In the case that β is limited for all values of r, we define, following Prof. Borel,[1] that when $\lim\limits_{r=\infty}\dfrac{\log\log M(r)}{\log r}$ is determinate, $F(z)$ is said to be *a regularly increasing function* (la

[1] Borel, *Leçons sur les fonctions entières*, p. 107.

fonction à croissance régulière), and in the other case, *an irregularly increasing function* (la fonction à croissance irrégulière). The upper limit ρ of $\dfrac{\log\log M(r)}{\log r}$ for limit $r=\infty$ is called the *order* of $F(z)$. In this case the order of infinitude of the moduli of the zero points of $F(z)$ is determinate and is generally equal to the inverse of its order; and conversely.

We may exdend this conception to the case where ρ is transfinite.

Definition. Let $\log_2 = \log\log, \ldots\ldots\ldots, \log_p = \log\log\ldots\ldots\log$; p: any positive integer. When $\overline{\lim\limits_{r=\infty}}\dfrac{\log_{p-1}M(r)}{\log r}=\infty$, and $\overline{\lim\limits_{r=\infty}}\dfrac{\log_p M(r)}{\log r}=\rho$ is finite, we say that the *order* of $F(z)$ is $\omega^{p-2}\cdot\rho$ (e.g. the orders of e^{e^z} and $e^{e^{e^{z^2}}}$ are ω and $\omega\cdot 2$ respectively). We define also that $F(z)$ is *regularly* or *irregularly increasing* according as $\lim\limits_{r=\infty}\dfrac{\log_p M(r)}{\log r}$ is or is not determinate.

In the following, we deal with only those transcendental integral functions whose orders are less than $\Omega=\omega^{\omega}$.

1. Let $F(z)$ be an entire function whose zero points are the origin (of multiplicity λ) and $z=a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$. Suppose that $r_1 \leqq r_2 \leqq \ldots\ldots \leqq r_n \leqq \ldots\ldots$, where $r_n=|a_n|$, and consider the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{r_n^{\alpha}}$ where α is a certain positive number. If $\sum\limits_{n=1}^{\infty}\dfrac{1}{r_n^{\alpha}}$ be divergent for any positive value of α (however great), then we consider $\sum\limits_{n=1}^{\infty}\dfrac{1}{e^{r_n^{\alpha}}}$, and so on.

Definition. Let e_p^x be an inverse function of $\log_p x$. If a positive integer $p'(\geqq 2)$ and a positive value (zero being included) ρ' be such that, for any prescribed positive value ε (however small), $\sum\limits_{n=1}^{\infty}\dfrac{1}{e_{p'-2}^{r_n^{\rho'-\varepsilon}}}$ [1] is divergent while $\sum\limits_{n=1}^{\infty}\dfrac{1}{e_{p'-2}^{r_n^{\rho'+\varepsilon}}}$ is convergent, then the *exponent of con-*

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[1] For $\rho'=0$, we take $\sum\limits_{n=1}^{\infty}\dfrac{1}{e_{p'-3}^{r_n^{G}}}$ instead of $\sum\limits_{n=1}^{\infty}\dfrac{1}{e_{p'-2}^{r_n^{\rho'-\varepsilon}}}$ where G is any positive value.

vergency of the moduli of the zero points of $F(z)$ is said to be $\omega^{p'-2}\cdot\rho'$. This definition is an extension of that for $p'=2$.[1]

Lemma. If a series $\sum_{n=1}^{\infty} u_n$ *whose terms are positive and decreasing be convergent, then* $\lim_{n=\infty} nu_n = 0$.[2]

Theorem 1. Let the zero points of a transcendental integral function $F(z)$ *be* $a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$ and $0<r_n \leqq r_{n+1}$ $(n=1, 2, \ldots\ldots)$ *where* $r_n = |a_n|$. *If the exponent of convergency of* $r_1, r_2, \ldots\ldots r_n, \ldots\ldots$ *be* $\omega^{p'-2}\cdot\rho'$, *the upper limit of* $\dfrac{\log_{p'-1} n}{\log r_n}$ *when* n *is infinite will be equal to* ρ'.

If $\overline{\lim_{n=\infty}} \dfrac{\log n}{\log r_n} = \infty$, we consider $\overline{\lim_{n=\infty}} \dfrac{\log_2 n}{\log r_n}$ and so on. Suppose that $\overline{\lim_{n=\infty}} \dfrac{\log_{p_1-1} n}{\log r_n} = \rho_1$ is finite while $\overline{\lim_{n=\infty}} \dfrac{\log_{p_1-2} n}{\log r_n} = \infty$. Then for any prescribed positive value ε, there is a corresponding positive number N such that

$$\frac{\log_{p_1-1} n}{\log r_n} \leqq \rho_1 + \frac{\varepsilon}{2} \qquad \text{for} \qquad n \geqq N.$$

Accordingly there is a positive value δ such that

$$\frac{\log_{p_1-1}(n^{1+\delta})}{\log r_n} \leqq \rho_1 + \varepsilon \qquad \text{for} \qquad n \geqq N,$$

or

$$n^{1+\delta} \leqq e_{p_1-2}^{r_n^{\rho_1+\varepsilon}} \qquad \text{for} \qquad n \geqq N,$$

so that

$$\sum_{n=N}^{\infty} \frac{1}{n^{1+\delta}} \geqq \sum_{n=N}^{\infty} \frac{1}{e_{p_1-2}^{r_n^{\rho_1+\varepsilon}}}.$$

But $\sum_{n=1}^{\infty} \dfrac{1}{n^{1+\delta}}$ is convergent and accordingly so also for $\sum_{n=1}^{\infty} \dfrac{1}{e_{p_1-2}^{r_n^{\rho_1+\varepsilon}}}$.

Therefore, we have

$$\omega^{p_1-2}(\rho_1+\varepsilon) \geqq \omega^{p'-2}\cdot\rho',$$

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[1] Borel, *loc. cit.*, p. 18.

[2] Borel, *loc. cit.*, p. 17.

which is valid for any positive value of ε, so that

$$(1) \qquad \omega^{p_1-2} \cdot \rho_1 \geqq \omega^{p'-2} \cdot \rho'.$$

Next, we consider the convergent series $\sum_{n=1}^{\infty} \frac{1}{e_{p'-2}^{r_n^{\rho'+\varepsilon}}}$. By the foregoing lemma, we have $\lim_{n\to\infty} \frac{n}{e_{p'-2}^{r_n^{\rho'+\varepsilon}}} = 0$, from which it follows that

$$n < e_{p'-2}^{r_n^{\rho'+\varepsilon}} \qquad \text{for} \qquad n \geqq N(\varepsilon),$$

where $N(\varepsilon)$ is a certain positive value which depends on ε. Therefore

$$(2) \qquad \overline{\lim_{n=\infty}} \frac{\log_{p'-1} n}{\log r_n} < \rho' + \varepsilon.$$

We now suppose that $p_1 > p'$. As

$$\overline{\lim_{n=\infty}} \frac{\log_{p_1-2} n}{\log r_n} = \overline{\lim_{n=\infty}} \left(\frac{\log_{p_1-2} n}{\log_{p_1-3} n} \cdot \frac{\log_{p_1-3} n}{\log_{p_1-4} n} \cdots \frac{\log_{p'} n}{\log_{p'-1} n} \cdot \frac{\log_{p'-1} n}{\log r_n} \right)$$

and

$$\overline{\lim_{n=\infty}} \frac{\log_{p_1-2} n}{\log_{p_1-3} n} = 0, \ldots, \overline{\lim_{n=\infty}} \frac{\log_{p'} n}{\log_{p'-1} n} = 0, \ \overline{\lim_{n=\infty}} \frac{\log_{p'-1} n}{\log r_n} < \rho' + \varepsilon \text{ by (2)},$$

we have

$$\overline{\lim_{n=\infty}} \frac{\log_{p_1-2} n}{\log r_n} < \rho' + \varepsilon,$$

which is contrary to the assumption that

$$\overline{\lim_{n=\infty}} \frac{\log_{p_1-2} n}{\log r_n} = \infty.$$

Hence

$$(3) \qquad p_1 = p',$$

so that we have by (1)

$$(1)' \qquad \rho_1 \geqq \rho'.$$

As, by (2) and (3),

$$\rho_1=\overline{\lim_{n=\infty}}\frac{\log_{p_1-1}n}{\log r_n}=\overline{\lim_{n=\infty}}\frac{\log_{p'-1}n}{\log r_n}<\rho'+\varepsilon,$$

we have $\rho_1<\rho'+\varepsilon$, which holds for any positive value of ε; so that

(4) $$\rho_1\leqq\rho'.$$

By (1)′ and (4), we have $\rho_1=\rho$, proving the proposition.

Similarly, supposing that $\underline{\lim}_{n=\infty}\frac{\log_{p_2-2}n}{\log r_n}=\infty$, while $\underline{\lim}_{n=\infty}\frac{\log_{p_2-1}n}{\log r_n}=\rho_2$, we have $\overset{p_2-2}{\omega}.\rho_2\leqq\overset{p'-2}{\omega}.\rho'$.

Definition. Let the zero points of a transcendental integral function $F(z)$ be $a_1, a_2, \ldots, a_n, \ldots$, and $0<r_n\leqq r_{n+1}$ $(n=1, 2, \ldots\ldots)$ where $r_n=|a_n|$. Supposing that the exponent of convergency of $r_1, r_2, \ldots, r_n, \ldots$ be $\overset{p'-2}{\omega}.\rho'$, if $\overline{\lim_{n=\infty}}\frac{\log_{p'-1}n}{\log r_n}=\underline{\lim}_{n=\infty}\frac{\log_{p'-1}n}{\log r_n}=\rho'$, we say that *the order of infinitude of* r_n $(n=1, 2, \ldots\ldots)$ *is determinate* and is equal to $\frac{1}{\rho'}\cdot\frac{1}{\omega^{p'-2}}$, and in the other case, *indeterminate*.

2. Let the greatest integral value of n which satisfies at least one of $r_n\leqq 1$ and $n\leqq e^1_{p'-2}$ be n_1, and we determine the integers p_n's as follows:

$$(5)\qquad\begin{cases}p_n=0, & \text{for } n\leqq n_1\\ p_n<\dfrac{\log n\log_2 n\ldots\ldots\log_{p'-1}n}{\log r_n}\leqq p_n+1, & \text{for } n>n_1.\end{cases}$$

Let

$$f(z)\equiv z^\lambda\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}.$$

i) Supposing that $p'=2$, we have $\overline{\lim_{n=\infty}}\frac{\log n}{\log r_n}=\rho'$. If ρ' be not an integer, we take $p_n<\rho'<p_n+1$ for $n=1, 2, \ldots\ldots$. Then p_n's thus determined will satisfy the lower relation in (5) for sufficiently great values of n, provided that there is no integer which is less than $\overline{\lim_{n=\infty}}\frac{\log n}{\log r_n}=\rho'$ and is not less than $\underline{\lim}_{n=\infty}\frac{\log n}{\log r_n}$. If ρ' be an integer, we take $p_n=\rho'$ or $\rho'-1$, $(n=1, 2, \ldots\ldots)$, according as $\sum_{n=\infty}^{\infty}\frac{1}{r_n^{\rho'}}$ is

divergent or convergent. Thus, $\sum_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$ is convergent uniformly and unconditionarily for all finite values of z,[1] so that

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

is an entire function whose zero points are the origin (of multiplicity λ) and $z=a_1,\ a_2,\ \cdots\cdots,\ a_n,\ \cdots\cdots$.

ii) Next, we suppose that $p'\geqq 3$. For any arbitrarily assigned value of z, whose absolute value is r, determine an integer n_2 such that

$$(6)\qquad r_{n_2}^{1-\frac{1}{\log_{p'-1}n_2}}\leqq r<r_{n_2+1}^{1-\frac{1}{\log_{p'-1}(n_2+1)}}.$$

Let the greater of n_1 and n_2 be N, and put

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

$$\equiv z^{\lambda}\prod_{n=1}^{N}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}\prod_{N+1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}.$$

in which $z^{\lambda}\prod_{n=1}^{N}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^3+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{p_n}\right)^{p_n}}$ is an entire function.

As $r<r_{n_2+1}^{1-\frac{1}{\log_{p'-1}(n_2+1)}}$ by (6), there is a positive value ε which satisfies

$$(7)\qquad r=r_{n_2+1}^{1-\frac{1+\varepsilon}{\log_{p'-1}(n_2+1)}}.$$

Moreover as $r_1\leqq r_2\leqq\cdots\cdots\leqq r_n\leqq\cdots\cdots$, we have

$$(8)\qquad r_n^{1-\frac{1+\varepsilon}{\log_{p'-1}n}}<r_{n+1}^{1-\frac{1+\varepsilon}{\log_{p'-1}(n+1)}}\qquad\text{for}\quad n>N,$$

and accordingly by (7) and (8)

$$(9)\qquad \log r_n-\log r\geqq(1+\varepsilon)\frac{\log r_n}{\log_{p'-1}n}\qquad\text{for}\quad n>N.$$

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1 Forsyth, *Theory of Functions*, 3rd edition, p. 94.

We have however by (5)

$$p_n + 1 \geqq \frac{\log n \log_2 n \ldots\ldots \log_{p'-1} n}{\log r_n} \qquad \text{for } n > N\ (\geqq n_1),$$

so that by (9)

$$(p_n + 1)(\log r_n - \log r) \geqq \frac{\log n \log_2 n \ldots\ldots \log_{p'-1} n}{\log r_n}(1+\varepsilon)\frac{\log r_n}{\log_{p'-1} n}$$

$$\geqq (1+\varepsilon)\log n \qquad\qquad \text{for} \quad n > N,$$

or

$$\left(\frac{r_n}{r}\right)^{p_n+1} \geqq n^{1+\varepsilon} \qquad \text{for} \quad n > N.$$

Accordingly

$$\sum_{n=N+1}^{\infty}\left(\frac{r}{r_n}\right)^{p_n+1} \leqq \sum_{n=N+1}^{\infty}\frac{1}{n^{1+\varepsilon}} < \sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon}} \qquad \text{(convergent)},$$

and we may put

$$(10) \qquad \sum_{n=N+1}^{\infty}\left(\frac{r}{r_n}\right)^{p_n+1} = A.$$

As $\overline{\lim_{n=\infty}}\ \dfrac{\log_{p'-1} n}{\log r_n} = \rho'$, there is a finite positive value B such that $\dfrac{\log_{p'-1} n}{\log r_n} < B$ for $n > N$, and we have by (6)

$$r < r_n^{1-\frac{1}{\log_{p'-1} n}} = r_n e^{-\frac{\log r_n}{\log_{p'-1} n}} < r_n e^{-\frac{1}{B}} \quad \text{for} \quad n > N, \text{ so that}$$

$$(11) \qquad 1 - \frac{r}{r_n} > 1 - e^{-\frac{1}{B}} \qquad \text{for} \quad n > N.$$

Now

$$\left|\prod_{n=N+1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}\right|$$

$$\leqq \prod_{n=N+1}^{\infty} e^{\frac{1}{p_n+1}\left(\frac{r}{r_n}\right)^{p_n+1}+\frac{1}{p_n+2}\left(\frac{r}{r_n}\right)^{p_n+2}+\ldots\ldots}$$

$$< \prod_{n=N+1}^{\infty} e^{\left(\frac{r}{r_n}\right)^{p_n+1}+\left(\frac{r}{r_n}\right)^{p_n+2}+\ldots\ldots\ldots\ldots}$$

$$= \prod_{n=N+1}^{\infty} e^{\frac{\left(\frac{r}{r_n}\right)^{p_n+1}}{1-\frac{r}{r_n}}}$$

$$< e^{\frac{A}{1-e^{-B}}}, \qquad \text{by (10) and (11).}$$

Consequently $\prod_{n+1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_a}\right)^{p_n}}$ is convergent uniformly and unconditionally for all finite values of z, so that

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

is an entire function whose zero points are the origin (of multiplicity λ) and $z=a_1, a_2, \ldots\ldots, a_n \ldots\ldots$.

In all cases $f(z)$ is an entire function which has the same zero points of the same orders as $F(z)$. Hence $\frac{F(z)}{f(z)}$ is an entire function which has no zero points, and we may put $\frac{F(z)}{f(z)}\equiv e^{Q(z)}$, where $Q(z)$ is an entire function. Thus $F(z)$ *may be written in the form*

$$F(z)\equiv z^{\lambda}e^{Q(z)}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{2}\left(\frac{z}{p_n}\right)^{p_n}},$$

where $Q(z)$ *is an entire function and* p_n*'s are integers determined as in* (5).

3. *Theorem* 2. *Let*

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

be an entire function whose exponent of convergency of the moduli of the zero points is $w^{p'-2}\rho'$, *where* p_n*'s are determined as to satisfy* (5). *Then for any prescribed positive value* ε, *there corresponds a positive value* R *such that*

$$|f(z)| \leqq e_{p'-1}^{r^{\rho'+\varepsilon}} \qquad \textit{for all} \quad |z|=r\geqq R,$$

that is, the order of $f(z)$ *can not exceed the exponent of convergency.*

The proof of this theorem for $p'=2$ is given in Borel, *loc. cit.*, p. 61, and the following proof is for $p'\geqq 3$. Determine n_1 and n_2 as in §2. Then for sufficiently great values of r, we have $n_2 \geqq n_1$, and

$$f(z) \equiv z^{\lambda} \prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right) \prod_{n_1+1}^{n_2}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\dots\dots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

$$\times \prod_{n_2+1}^{\infty}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\dots\dots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}.$$

$z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)$ being a polynominal of the $(n_1+\lambda)^{th}$ degree, we have

$$\left|z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)\right| \leqq e_{p'-1}^{r^{\rho'}}$$

for all values of r greater than R which is a suitably chosen positive value. Secondly, we consider

$$\prod_{n_1+1}^{n_2}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\dots\dots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}.$$

$$\left|\prod_{n_1+1}^{n_2}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\dots\dots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}\right|$$

$$\leqq \prod_{n_1+1}^{n_2}\left(1+\frac{r}{r_n}\right) e^{\frac{r}{r_n}+\frac{1}{2}\left(\frac{r}{r_n}\right)^2+\dots\dots+\frac{1}{p_n}\left(\frac{r}{r_n}\right)^{p_n}}$$

$$< \prod_{n_1+1}^{n_2} e^{2\frac{r}{r_n}+\frac{1}{2}\left(\frac{r}{r_n}\right)^2+\dots\dots+\frac{1}{p_n}\left(\frac{r}{r_n}\right)^{p_n}}$$

$$< \prod_{n_1+1}^{n_2} e^{(p_n+1)r^{p_n}}$$

$$< e^{n_2(p_k+1)r^{p_k}},$$

where p_k is the greatest of $p_{n_1+1}, p_{n_1+2}, \dots\dots, p_{n_2}$. We have, however, by (5)

$$p_n < \frac{\log n \log_2 n \dots\dots \log_{p'-1} n}{\log r_n} \quad \text{for} \quad n > n_1,$$

so that $p_n < \log n \log_2 n \dots\dots \log_{p'-1} n = (\log n)^{1+\varepsilon'}$ where ε' is a positive value which can be made as small as we please by taking n sufficiently great. Accordingly

$$\left|\prod_{n_1+1}^{n_2}\right| < e^{n_2\{(\log k)^{1+\varepsilon'}+1\} r^{(\log k)^{1+\varepsilon'}}}$$

$$\leqq e^{n_2\{(\log n_2)^{1+\varepsilon'}+1\} r^{(\log n_2)^{1+\varepsilon'}}}.$$

As $\overline{\lim_{n=\infty}} \dfrac{\log_{p'-1} n}{\log r_n} = \rho'$ and $r_{n_2}^{1-\frac{1}{\log_{p'-1} n}} \leqq r$ by (6), we have $n_2 \leqq e_{p'-2}^{r^{\rho'+\varepsilon''}}$ where ε'' is a certain positive value which can be made as small as we please by taking n_2 sufficiently great; so that

$$\left|\prod_{n_1+1}^{n_2}\right| \leqq e^{e_{p'-2}^{r^{\rho'+\varepsilon''}} \cdot \left\{\left(e_{p'-3}^{r^{\rho'+\varepsilon''}}\right)^{1+\varepsilon'}+1\right\} e^{\left(e_{p'-3}^{r^{\rho'+\varepsilon''}}\right)^{1+\varepsilon'} \log r}}$$

$$\leqq e_{p'-1}^{r^{\rho'+\varepsilon_1}}.$$

where ε_1 is a certain positive value of the same property as ε''. As n_2 increases with r, we have

$$\left|\prod_{n_1+1}^{n_2}\right| \leqq e_{p'-1}^{r^{\rho'+\varepsilon_1}}$$

for $r \geqq R_2$, where R_2 is a certain positive value which corresponds to ε_1.

Lastly, for $\prod_{n_2+1}^{\infty}\left(1-\dfrac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$, we have only to repeat the process carried out in §2 and the same result must follow

$$\left|\prod_{n_2+1}^{\infty}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}\right| < e^{\frac{A}{1-e^{-\frac{1}{B}}}},$$

or

$$\left|\prod_{n_2+1}^{\infty}\right| \leqq e_{p'-1}^{r^{\rho'}}$$

for $r \geqq R_3$, where R_3 is a suitably chosen positive value. We have accordingly

$$|f(z)| \equiv \left|z^{\lambda}\prod_{n=1}^{n_1}\right| \cdot \left|\prod_{n_1+1}^{n_2}\right| \cdot \left|\prod_{n_2+1}^{\infty}\right|$$

$$\leqq e_{p'-1}^{r^{\rho'}} \cdot e_{p'-1}^{r^{\rho'+\varepsilon_1}} \cdot e_{p'-1}^{r^{\rho'}}$$

for $r > R_1, R_2, R_3$, or

$$|f(z)| \leqq e_{p'-1}^{r^{\rho'+\varepsilon}}$$

for all value of $r > R$, where R is a certain positive value corresponding to ε.

4. *Lemma. Let the zero points of a transcendental integral function* $\varphi(z)$ *be* $z = a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$ *and* $0 < r_n \leqq r_{n+1}$ $(n = 1, 2, \ldots\ldots)$ *where* $r_n = |a_n|$. *For an arbitrarily assigned value of* $|z| \equiv r$, *find* r_n *as to satisfy*

$$sr_n \leqq r < sr_{n+1},$$

where s *is a certain integer greater than* 2. *Then*

$$n \log(s-1) < \log M_1(r),$$

where $M_1(r)$ *is the maximum value of* $|\varphi(z)|$ *for* $|z| = r$.

The proof of this lemma is given in Borel, *loc. cit.*, p. 73.

Theorem 3. *Extension of Hadamard's first theorem*[1] *to entire functions of the transfinite orders.*

Let $F(z)$ *be an entire function of the transfinite order* $\omega^{p\dot{-}2}\rho$ *and the exponent of convergency of the moduli of the zero points of* $F(z)$ *be* $\omega^{p'\dot{-}2}\rho'$. *Then* $\omega^{p'\dot{-}2}\rho' \leqq \omega^{p\dot{-}2}\rho$.

Suppose that the zero points of $F(z)$, which are different from the origin, be $z = a_1, a_2, \ldots\ldots, a_n, \ldots\ldots$, and that $0 < r_n \leqq r_{n+1}$ $(n = 1, 2, \ldots\ldots)$ where $r_n = |a_n|$. Then the exponent convergency of $r_1, r_2, \ldots\ldots$ is $\omega^{p'\dot{-}2}\rho'$. As $F(z)$ is of the order $\omega^{p\dot{-}2}\rho$, for any prescribed positive value ε, there corresponds a positive value R such that

$$|F(z)| \leqq e_{p-1}^{r^{\rho+\varepsilon}} \qquad \text{for} \qquad |z| = r \geqq R.$$

We have, however, by the lemma

$$\begin{aligned} n \log(s-1) &< \log M(r) \\ &\leqq \log e_{p-1}^{r^{\rho+\varepsilon}} \\ &= e_{p-2}^{r^{\rho+\varepsilon}} \end{aligned}$$

for $r \geqq R$, where $sr_n \leqq r < sr_{n+i}$. Accordingly,

$$n+1 < \frac{n+1}{n} \frac{1}{\log(s-1)} e_{p-2}^{r^{\rho+\varepsilon}} \leqq \frac{2}{\log(s-1)} e_{p-2}^{r^{\rho+\varepsilon}} < \frac{2}{\log(s-1)} e_{p-2}^{(sr_{n+1})^{\rho+\varepsilon}},$$

from which it follows that

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[1] L'exposant de convergence ρ de la suite des r_n est au plus égal à ρ'. (Borel, *loc. cit.*, p. 74.)

$$(n+1)^{1+\varepsilon'} \leqq \left\{\frac{2}{\log(s-1)}\right\}^{1+\varepsilon'} \cdot e_{p-2}^{r_{n+1}^{\rho+\varepsilon''}} \qquad (\varepsilon'>0)$$

where $\varepsilon''(>\varepsilon)$ is a certain positive value which can be made as small as we please by taking R sufficiently great and ε', sufficiently small. But $\sum_{n=1}^{\infty}\frac{1}{n^{1+\varepsilon'}}$ is convergent and so also for $\sum_{n=1}^{\infty}\frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon''}}}$, so that

$$\omega^{p'\dot{-}2}\rho' \leqq \omega^{p-2}(\rho+\varepsilon''),$$

which holds for any positive value ε''. Hence we have

$$\omega^{p'\dot{-}2}\rho' \leqq \omega^{p\dot{-}2}\rho,$$

proving the proposition.

5. *Theorem 4. Extension of Hadamard's second theorem*[1] *to entire functions of the transfinite orders.*

Let

$$f(z) \equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

be an entire function of the order $\omega^{p\dot{-}2}\rho$. Then, for any positive values ε (however small) and G (however great) there is a circle whose radius is greater than G and on which

$$|f(z)| \geqq e^{-e_{p-2}^{r^{\rho+\varepsilon}}}.$$

Let all the annular domains which are expressed by $r_n - \frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon'}}} < x < r_n + \frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon'}}}$ be excluded from the total z-plane, where ε' is a certain positive number. Determine n_1 as in §2, and for an arbitrarily assigned value of z $(r > r_{n_1})$ in the remaining domain, determine n_2 as in §2, which is necessarily greater than or equal to n_1. Then

$$f(z) \equiv z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)\prod_{n_1+1}^{n_2}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$
$$\times\prod_{n_2+1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

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[1] Étant donnés un produit canonique $G(z)$ de facteurs primaires d'ordre ρ et un nombre positif arbitraire ε, on peut touver une infinité de rayons indefiniment croissants sur chacun desquels on a l'inégalité $|G(z)| > e^{-r^{\rho+\varepsilon}}$. (Borel. *loc. cit.*, p. 76).

We now consider $z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)$.

$$\left|z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)\right| \geqq r^{\lambda}\left(\frac{r}{r_1}-1\right)\left(\frac{r}{r_2}-1\right)\cdots\cdots\left(\frac{r}{r_{n_1}}-1\right)$$

$$\geqq r^{\lambda}\left(\frac{r}{r_{n_1}}-1\right)^{n_1}$$

$$\geqq e^{-e_{p-2}^{r^{\rho+\varepsilon_1}}}$$

for $r\geqq R_1$, where ε_1 is any assigned positive value and R_1 is the corresponding value to it. Secondly, we consider

$$\prod_{n_1+1}^{n_2}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}.$$

$$\left|\prod_{n_1+1}^{n_2}\right| \geqq \prod_{n_1+1}^{n_2}\left|1-\frac{r}{r_n}\right| e^{-\left\{\frac{r}{r_n}+\frac{1}{2}\left(\frac{r}{r_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{r}{r_n}\right)^{p_n}\right\}}$$

$$\geqq \prod_{n_1+1}^{n_2}\left|1-\frac{r_n\pm\frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon'}}}}{r_n}\right| e^{-\left\{\frac{r}{r_n}+\frac{1}{2}\left(\frac{r}{r_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{r}{r_n}\right)^{p_n}\right\}}$$

$$>\prod_{n_1+1}^{n_2}\frac{1}{r_n e_{p-2}^{r_n^{\rho+\varepsilon'}}}e^{-p_n r^{p_n}}$$

$$\geqq\left\{\frac{1}{r_{n_2}e_{p-2}^{r_{n_2}^{\rho+\varepsilon'}}}e^{-p_k r^{p_k}}\right\}^{n_2-n_1}$$

$$\geqq e^{-n_2\left\{\log r_{n_2}+e_{p-3}^{r_{n_2}^{\rho+\varepsilon'}}+p_k r^{p_k}\right\}},$$

where p_k is the greatest of $p_{n_1+1}, p_{n_1+2}, \ldots\ldots, p_{n_2}$. But we have, similarly as in §3,

$$p_k<(\log k)^{1+\varepsilon''}\leqq(\log n_2)^{1+\varepsilon''}$$

where ε'' is a positive value which can be made as small as we please, by taking n_2 sufficiently great. Hence, by similar reasoning as in §3, we have

$$\left|\prod_{n_1+1}^{n_2}\right|\geqq e^{-e_{p-2}^{r^{\rho+\varepsilon_2}}}$$

for $r\geqq R_2$, where ε_2 is any prescribed positive value and R_2 is the corresponding value to it. We consider lastly

$\prod_{n_2+1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$. By repeating the process carried out in §2, we have

$$\left|\prod_{n_2+1}^{\infty}\right|>\prod_{n_2+1}^{\infty}e^{-\frac{\left(\frac{r}{r_n}\right)^{p_n+1}}{1-\frac{r}{r_n}}}>e^{-\frac{A}{1-e^{-\frac{1}{B}}}},$$

and accordingly

$$\left|\prod_{n_2+1}^{\infty}\right|\geqq e^{-e_{p-2}^{r^{\rho+\varepsilon_3}}}$$

for $r\geqq R_3$, where ε_3 is any assigned positive value and R_3 is the corresponding value to it. Consequently we have

$$\left|f(z)\right|\equiv\left|z^{\lambda}\prod_{n=1}^{n_1}\cdot\prod_{n_1+1}^{n_2}\cdot\prod_{n_2+1}^{\infty}\right|$$

$$\geqq e^{-e_{p-2}^{r^{\rho+\varepsilon_1}}}\cdot e^{-e_{p-2}^{r^{\rho+\varepsilon_2}}}\cdot e^{-e_{p-2}^{r^{\rho+\varepsilon_3}}}$$

for $r\geqq R_1,\ R_2,\ R_3$, or

$$\geqq e^{-e_{p-2}^{r^{\rho+\varepsilon}}}$$

for $r\geqq R$, where ε is assigned positive value and R is the corresponding value to it.

We have excluded all the annular domains expressed by

$$r_n-\frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon'}}}<x<r_n+\frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon'}}},\qquad (n=1,\ 2,\ \ldots\ldots)$$

from the total z-plane and have considered the points in the remaining domain. But the sum of all the annular domains

$$\leqq\pi\sum_{n=1}^{\infty}\left\{\left(r_n+\frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon'}}}\right)^2-\left(r_n-\frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon'}}}\right)^2\right\}$$

$$=4\pi\sum_{n=1}^{\infty}\frac{r_n}{e_{p-2}^{r_n^{\rho+\varepsilon'}}}.$$

For any positive value $\sigma<\varepsilon'$ there corresponds a positive integer N such that

$$\frac{r_n}{e_{p-2}^{r_n^{\rho+\varepsilon'}}}\leqq\frac{1}{e_{p-2}^{r_n^{\rho+\sigma}}}\qquad\text{for}\qquad n>N.$$

As $\omega^{p'\dot{-}2}\rho' \leqq \omega^{p\dot{-}2}\rho$ (§4), $\sum\limits_{n=N}^{\infty}\dfrac{1}{e_{p-2}^{\,r_n^{\rho+\sigma}}}$ is convergent and so also for $\sum\limits_{n=1}^{\infty}\dfrac{r_n}{e_{p-2}^{\,r_n^{\rho+\varepsilon'}}}$; so that the sum of all the annular domains is finite and there exists a circle whose radius is greater than G and on which $|f(z)| \geqq e^{-e_{p-2}^{\,r^{\rho+\varepsilon}}}$.

Corollaly 1. *Let $f_1(z), f_2(z), \ldots\ldots f_k(z)$, be the canonical products of the orders $\omega^{p^{(1)}\dot{-}2}\rho_1$, $\omega^{p^{(2)}\dot{-}2}\rho_2$, ..., $\omega^{p^{(k)}\dot{-}2}\rho_k$ respectively. Then for any prescribed positive value ε, there is a circle whose radius is greater than any prescribed positive value G and on which*

$$|f_i(z)| \geqq e^{-e_{p^{(i)}-2}^{\,r^{\rho_i+\varepsilon}}}, \qquad (i=1, 2, \ldots\ldots, k).$$

Let the moduli of the zero points of $f_i(z)$ be $r_1^{(i)}, r_2^{(i)}, \ldots, r_n^{(i)}, \ldots\ldots$

Then, we have, by the theorem, $|f_i(z)| \geqq e^{-e_{p^{(i)}-2}^{\,r^{\rho_i+\varepsilon}}}$ for the values of r which are greater than G and satisfy

$$r_n^{(i)} + \frac{1}{e_{p^{(i)}-2}^{\,r_n^{(i)\,\rho_i+\varepsilon'}}} \leqq r \leqq r_{n+1}^{(i)} - \frac{1}{e_{p^{(i)}-2}^{\,r_{n+1}^{(i)\,\rho_i+\varepsilon'}}}$$

for a certain value of n. But $\sum\limits_{i=1}^{k}\sum\limits_{n=1}^{\infty}\dfrac{r_n^{(i)}}{e_{p^{(i)}-2}^{\,r_n^{(i)\,\rho_i+\varepsilon'}}}$ being finite, the sum of all the annular domains is finite and there exists a circle whose radius is greater than G and on which

$$|f_i(z)| \geqq e^{-e_{p^{(i)}-2}^{\,r^{\rho_i+\varepsilon}}}, \qquad (i=1, 2, \ldots\ldots k).$$

Corollaly 2. *Let*

$$F(z) \equiv e^{F_1(z)} \cdot f_0(z)$$

$$F_1(z) \equiv e^{F_2(z)} \cdot f_1(z)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$F_{p-2}(z) \equiv e^{F_{p-1}(z)} \cdot f_{p-2}(z),$$

where $F_{p-1}(z)$ is a polynominal of the ρ^{th} degree and $f_i(z)$, $(i=0, 1, \ldots, p-2)$, is a canonical product of the order $\omega^{p-(2+i)}\rho_i^{(i)}$. If $\omega^{p\dot{-}2}\rho$ be greater *than* $\omega^{p^{(i)}-2}\cdot\rho_i$, $(i=0, 1, \ldots, p-2)$, *the same will be true of the order of* $F(z)$.

For any prescribed positive value ε, there exists, by corollary 1, a circle whose radius is greater than any prescribed positive value and on which

$$|f_i(z)| \geqq e^{-e_{p^{(i)}-(2+i)}^{r^{\rho_i+\varepsilon}}}, \qquad (i=0, 1, \ldots\ldots, p-2).$$

As $F_{p-1}(z)$ is a polynominal of the ρ^{th} degnee, $e^{F_{p-1}(z)}$ is a *regularly* increasing entire function of the ρ^{th} order, that is, the maximum value of $\left|e^{F_{p-1}(z)}\right|$ for $|z|=r$ lies between $e^{r^{\rho-\varepsilon}}$ and $e^{r^{\rho+\varepsilon}}$ for sufficiently great r. Hence, we have, on the circle

$$\max\left|e^{F_{p-1}(z)}\cdot f_{p-2}(z)\right| \geqq e^{r^{\rho-\varepsilon}}\cdot e^{-e_{p-p}^{r^{\rho+\varepsilon}_{p-2}(p-2)}} \geqq e^{r^{\rho-\varepsilon_1}},$$

since $\omega^{p-2}\cdot\rho > \omega^{p-2}\overset{(p-2)}{\cdot}\rho_{p-2}$. As $\left|e^{e^{F_{p-2}(z)}}\cdot f_{p-2}(z)\right|$ and $e^{\left|e^{F_{p-2}(z)}\cdot f_{p-2}(z)\right|}$ are of the same orders, we have, similarly,

$$\max\left|e^{e^{F_{p-1}(z)}\cdot f_{p-2}(z)}\cdot f_{p-3}(z)\right| \geqq e^{e^{r^{\rho-\varepsilon_1}}}\cdot e^{-e_{p-(p-1)}^{r^{\rho+\varepsilon}_{p-3}(p-3)}} \geqq e^{e^{r^{\rho-\varepsilon_2}}}$$

on that circle, since $\omega^{p-2}\cdot\rho > \omega^{p-2}\overset{(p-3)}{\cdot}\rho_{p-3}$. By repeating the same reasoning, we have

$$\max|F(z)| \equiv M(r) \geqq e_{p-1}^{r^{\rho-\varepsilon_{p-1}}}$$

on the assigned circle, where ε_{p-1} is a certain positive value which becomes as small as we please for sufficiently great r. Determine r so large that $\omega^{p-2}(\rho-\varepsilon_{p-1})$ becomes greater than the greatest of $\omega^{p-2}\overset{(i)}{\cdot}\rho_i$ $(i=0, 1, \ldots\ldots, p-2)$. Then the order of $F(z)$, being greater than or equal to $\omega^{p-2}(\rho-\varepsilon_{p-1})$, is greater than the greatest of $\omega^{p-2}\overset{(i)}{\cdot}\rho_i$, $(i=0, 1, \ldots\ldots, p-2)$, which is the proposition.

Corollary 3. *If*

$$F(z) \equiv e^{Q(z)} z^{\lambda} \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

be of the $\omega^{p-2}\cdot\rho^{th}$ *order*, $e^{Q(z)}$ *will be, at most, of the same order.*

Suppose that the order $\omega^{p'-2}\cdot\rho'$ of $e^{Q(z)}$ be higher than $\omega^{p-2}\cdot\rho$, and determine two positive value ε and ε' such that

$$\omega^{p'-2}\cdot(\rho'-\varepsilon') > \omega^{p-2}(\rho+\varepsilon).$$

Then there exists a positive value R_1 corresponding to ε, such that

$$|F(z)| \leqq e_{p-1}^{r^{\rho+\varepsilon}} \qquad \text{for} \qquad r \geqq R_1.$$

By theorems 2, 3 and 4, we have

$$\left| z^{\lambda} \prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right) e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\cdots\cdots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}} \right| > e^{-e_{p-2}^{r^{\rho+\varepsilon}}}$$

for any z $(r \geqq R_2)$ in the exterior of the annular domains

$$r_n - \frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon}}} < x < r_n + \frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon}}}, \qquad (n=1, 2, \ldots\ldots),$$

and, accordingly, we have $\left| e^{Q(z)} \right| < e^{2e_{p-2}^{r^{\rho+\varepsilon}}}$ for z $(r \geqq R_1, R_2)$ in the same region. The order of $e^{Q(z)}$ however being $\omega^{p'-2}\rho'$,

$$e_{p'-1}^{r^{\rho'-\varepsilon'}} \leqq \left\{ \text{the maximum value of } \left| e^{Q(z)} \right| \text{ for } |z|=r \right\} \leqq e_{p'-1}^{r^{\rho'+\varepsilon'}}$$

must be satisfied for infinitely many values of z which diverges to infinity, and from what has been proved, they must be in the interior of the annular domains. The maximum value of $|e^{Q(z)}|$ for $|z|=r$ increasing with r, we have

$$\begin{aligned} e_{p'-1}^{\left(r_n-\frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon}}}\right)^{\rho'-\varepsilon'}} &\leqq e_{p'-1}^{r^{\rho'-\varepsilon'}} \\ &\leqq \text{the maximum value of } |e^{Q(z)}| \text{ for } |z|=r \\ &\leqq \text{the maximum value of } |e^{Q(z)}| \text{ for } |z|=r_n+\frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon}}} \\ &\leqq e^{2e_{p-2}^{\left(r_n+\frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon}}}\right)^{\rho+\varepsilon}}}, \end{aligned}$$

where
$$r_n - \frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon}}} < r < r_n + \frac{1}{e_{p-2}^{r_n^{\rho+\varepsilon}}}.$$

This is however impossible for sufficiently great n, since $\omega^{p'-2}(\rho'-\varepsilon') > \omega^{p-2}(\rho+\varepsilon)$ and $\lim\limits_{n=\infty} \dfrac{1}{e_{p-2}^{r_n^{\rho+\varepsilon}}} = 0$. Hence, the proposition is true.

Corollary 4. *Let*

$$F(z) \equiv e^{F_1(z)} \cdot f_0(z)$$
$$F_1(z) \equiv e^{F_2(z)} \cdot f_1(z)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$F_{p-2}(z) \equiv e^{F_{p-1}(z)} \cdot f_{p-2}(z),$$

where $F_{p-1}(z)$ *is a polynominal and* $f_i(z)$, $(i=0, 1, \ldots, p-2)$, *is a canonical product of the order* $\omega^{p-(2+i)}\overset{(i)}{\rho_i}$. *Then* $\omega^{p-2}\overset{(i)}{\rho_i}$, $(i=0, 1, \ldots, p-2)$, *is at most equal to the order of* $F(z)$.

- Let the order of $F(z)$ be $\omega^{p\dot{-}2}\rho$. Then by theorems 2 (§3) and 3 (§4), the order $\omega^{p\dot{-}2}\overset{(0)}{\rho_0}$ of $f_0(z)$ is, at most, equal to $\omega^{p\dot{-}2}\rho$. As $e^{F_1(z)}$ is, by corollary 3, at most of the order $\omega^{p\dot{-}2}\rho$, $F_1(z)\equiv e^{F_2(z)}\cdot f_1(z)$ is at most of the $\omega^{p\dot{-}(2+1)}\rho^{th}$ order. Hence, again by theorems 2 and 3, the order $\omega^{p\dot{-}(2+1)}\overset{(1)}{\rho_1}$ of $f_1(z)$ is not greater than $\omega^{p\dot{-}(2+1)}\rho$, or $\omega^{p\dot{-}2}\overset{(1)}{\rho_1}\equiv\omega^{p\dot{-}2}\rho$. We have similarly by the alternate applications of theorems 2 and 3 and of corollary 3, $\qquad \omega^{p\dot{-}2}\overset{(i)}{\rho_i}\leqq\omega^{p\dot{-}2}\rho \qquad (i=0, 1, \ldots\ldots, p-2).$

6. *Theorem 5. If the order of infinitude of* $r_n=|a_n|$, $(n=1, 2,\ldots\ldots)$, *be determinate, the function.*

$$F(z)\equiv e^{Q(z)}\cdot z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$

will be increasing regularly.

We assume that $F(z)$ is increasing irregularly, and that the exponent of convergency of r_n, $(n=1, 2, \ldots\ldots)$, is equal to the order of $F(z)$[1]. Supposing that ε is any prescribed positive value, and that R is the corresponding value, we have by theorem 2 (§3),

$$|F(z)|\leqq e_{p-1}^{r^{\rho+\varepsilon}} \qquad \text{for all} \qquad r\geqq R.$$

As $F(z)$ is, by assumption, increasing irregularly, there is a number $\omega^{p'\dot{-}2}\sigma$, finite or transfinite, such that $\omega^{p\dot{-}2}\rho>\omega^{p'\dot{-}2}\sigma$ and $|F(z)|\leqq e_{p'-}^{r^{\sigma}}$ for infinitely many values of r which increase without limit. Supposing that $|z|=r$ be one of such values, determine a positive integer n as to satisfy

$$sr_n\leqq r< sr_{n+1},$$

where s is a prescribed positive integer greater than 2. Then we have, by the lemma in §4.

$$n\log(s-1)<\log M(r)< e_{p'-2}^{r^{\sigma}}< e_{p'-2}^{(sr_n+1)^{\sigma}},$$

or

$$(12) \qquad r_{n+1}>\frac{\{\log_{p'-2}(n\log(s-1))\}^{\frac{1}{\sigma}}}{s}>\{\log_{p'-2}(n+1)\}^{\frac{1}{\sigma}-\varepsilon'},$$

where ε' may be taken as small as we please for sufficiently great n. The inequality (12) holds for infinitely many values of n. Supposing that those values be $n_1<n_2<\ldots\ldots<n_i<\ldots\ldots$, we have

$$\lim_{i=\infty}\frac{\log_{p'-1}n_i}{\log r_n}<\frac{\sigma}{1-\sigma\varepsilon'},$$

1 The assumption is legitimate by the generalized theorem of Picard in §7, Chap. II., the proof of which is independent of the above theorem.

that is, the order of infinitude of r_n $(n=1, 2, \ldots\ldots)$, is indeterminate, proving the theorem.

Theorem 6. Let

$$f(z)\equiv z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{r_n}\right)^{p_n}}$$

be of the order $\omega^{p\dot{-}2}\rho$. *If the function* $f(z)$ *be increasing regularly, the order of infinitude of* $r_n=|a_n|$ *will be determinate.*

We assume that $p\geqq 3$[1]. Supposing that the order of infinitude of r_n is indeterminate, let

$$(13)\qquad \frac{\log_{p-1}n}{\log r_n}<\sigma\,{}^{2}$$

for infinitely many integral values of n, where σ is subjected to the condition that $\sigma<\rho$. Suppose that h $(>n_1)$ be an integer which satisfies (13), where n_1 is determined as in §2, and put

$$(14)\qquad (\log_{p-2}h)^{\frac{1}{s}}=r,$$

where s is subjected to the condition that

$$\sigma<s<\rho.$$

$$z^{\lambda}\prod_{n=1}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}$$
$$=z^{\lambda}\prod_{n=1}^{n_1}\cdot\prod_{n_1+1}^{h-1}\cdot\prod_{h}^{\infty}\cdot$$

$z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)$ being a polynominal of the $(n_1+\lambda)^{th}$ degree,

$\left|z^{\lambda}\prod_{n=1}^{n_1}\left(1-\frac{z}{a_n}\right)\right|<e_{p-1}^{r^s}$ for sufficiently great r. We have, similarly as in §3,

$$\left|\prod_{n_1+1}^{h-1}\right|<e^{h\{1+(\log h)^{1+\varepsilon'}\}}\,e^{(\log h)^{1\dot{+}\varepsilon'}\log r},$$

or in virtue of (14),

$$\left|\prod_{n_1+1}^{h-1}\right|\leqq e_{p-1}^{r^{(1+\varepsilon'')s}},$$

where ε'' is a certain positive value which may be taken as small as we please for sufficiently great h.

$$\left|\prod_{n=h}^{\infty}\left(1-\frac{z}{a_n}\right)e^{\frac{z}{a_n}+\frac{1}{2}\left(\frac{z}{a_n}\right)^2+\ldots\ldots+\frac{1}{p_n}\left(\frac{z}{a_n}\right)^{p_n}}\right|$$
$$\leqq\prod_{n=h}^{\infty}e^{\frac{1}{p_n+1}\left(\frac{r}{r_n}\right)^{p_n+1}+\frac{1}{p_n+2}\left(\frac{r}{r_n}\right)^{p_n+2}+\ldots\ldots\ldots}$$

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[1] For p=2, see Borel, *loc. cit.*, p. 110.

[2] By theorem 3 (§4), the exponent of convergency $\omega^{p'\dot{-}2}\rho'<\omega^{p\dot{-}2}\rho$.

We have, however, from (13) and (14)

$$r_h^{\sigma} > \log_{p-2} h = r^s.$$

As $\sigma < s$, we have $\frac{\sigma}{s} < 1 - \frac{1}{\log_{p-1} h}$ for sufficiently great h, and accordingly $r < r_h^{\frac{\sigma}{s}} < r_h^{1-\frac{1}{\log_{p-1} h}}$, which satisfies the inequality (6) in §2. Hence we have, as in §2, $\left|\prod_{n=h}^{\infty}\right| < e^{\frac{A}{1-e^{-\frac{1}{B}}}}$ where A and B are certain finite values such that $\sum_{n=h}^{\infty}\left(\frac{r}{r_n}\right)^{p_n+1} = A$ and $\frac{\log_{p-1} n}{\log r_n} < B$ for $n \geqq h$, so that $\left|\prod_{n=h}^{\infty}\right| < e_{p-1}^{r^s}$ for sufficiently great r. We have therefore

$$\left|\prod_{n=1}^{\infty}\right| < e_{p-1}^{r^s} \cdot e_{p-1}^{r^{(1+\varepsilon')s}} \cdot e_{p-1}^{r^s}$$

$$\leqq e_{p-1}^{r^{s+\varepsilon_1}} \qquad (15)$$

where ε_1 is a certain positive value which diminishes with $\frac{1}{r}$. As there are infinitely many values of h which satisfy (13), the relation (15) holds for infinitely many values of r which increase without limit, that is, $f(z)$ is increasing irregularly, which is the proposition.

CHAPTER II.

TRANSCENDENTAL ALGEBRAIC FUNCTIONS.

INTRODUCTION.

Supposing that the zero points of a transcendental integral function $P_i(z)$, $(i=1, 2, \ldots\ldots, k)$, be $z = a_{i1}, a_{i2}, \ldots\ldots, a_{ik}, \ldots\ldots, b_1, b_2, \ldots\ldots, b_k, \ldots\ldots$ where b_k's are the common zero ponts of $P_1(z)$, $P_2(z)$, ..., $P_k(z)$, put

$$f(z) \equiv \prod_{n=1}^{\infty}\left(1 - \frac{z}{b_n}\right) e^{\frac{z}{b_n} + \frac{1}{2}\left(\frac{z}{b_n}\right)^2 + \ldots\ldots + \frac{1}{p_n}\left(\frac{z}{b_n}\right)^{p_n}} \text{ [1]}$$

and

$$f_i(z) \equiv \prod_{n=1}^{\infty}\left(1 - \frac{z}{a_{in}}\right) e^{\frac{z}{a_{in}} + \frac{1}{2}\left(\frac{z}{a_{in}}\right)^2 + \ldots\ldots + \frac{1}{p_n^{(i)}}\left(\frac{z}{a_{in}}\right)^{p_n^{(i)}}}, (i = 1, 2, \ldots\ldots, k),$$

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[1] If the number of b_n's be finite, say N, $f(z) \equiv \prod_{n=1}^{N}\left(1 - \frac{z}{b_n}\right)$. Similarly for $f_i(z)$, $(i = 1, 2, \ldots\ldots, k)$.

in which p_n's and $p_n^{(i)}$'s are determined by the condition (5) in §2, chap. I. Put

$$P_i(z) \equiv e^{\overline{P}_i(z)} f(z) \cdot f_i(z), \qquad (i=1, 2, \ldots\ldots, k).$$

where $\overline{P}_i(z)$, $(i=1, 2, \ldots\ldots, k)$, is evidently an entire function. If the orders of $e^{\overline{P}_i(z)-\overline{P}_1(z)}$, $(i=2, \ldots\ldots, k)$, be all less than the greatest of all orders of $e^{\overline{P}_i(z)}$, $(i=1, 2, \ldots\ldots, k)$[1], put

$$K(z) \equiv e^{\overline{P}_1(z)} f(z)$$

and in the other case, put

$$K(z) \equiv f(z).$$

Definition. When $K(z)$ is a constant, we say that $P_1(z)$, $P_2(z)$, ..., $P_k(z)$ are *relative prime.*

Definition. A function $\omega(z)$ which is defined by

$$(1) \qquad P_0(z)\omega^n + P_1(z)\omega^{n-1} + \ldots\ldots + P_n(z) = 0,$$

where n is a positive integer and $P_0(z)$, $P_1(z)$, ..., $P_n(z)$ are transcendental integral function of z, is called a *transcendental algebraic function* of z.

Definition. When the left hand side of (1) be not decomposable into factors of the same form, and $P_0(z)$, $P_1'(z)$, ..., $P_n(z)$ be relative prime, (1) is called an *irreducible equation.*

Definition. In an irreducible equation (1), if $\omega^{p-2}_{\nu}\rho$ be the highest of the orders of $P_0(z)$, $P_1(z)$, ..., $P_n(z)$, (1) is called a *transcendental algebraic equation of the $\omega^{p-2}_{\nu}\rho^{th}$ order and the n^{th} degree* and the function defined by (1) is called a *transcendental algebraic function of the $\omega^{p-2}_{\nu}\rho^{th}$ order and the n^{th} degree.*

1. *Theorem 1. A transcendental algebraic function behaves algebraically in any finite part of the plane, the number of the branches being constant; and conversely.*

We may assume, without loss of generality, that (1) is an irreducible equation. Then the function $\omega(z)$ defined by (1) is evidently n-valued. As $P_0(z)$, $P_1(z)$, ..., $P_n(z)$ behave regularly in any finite domain D, $\omega(z)$ behaves algebraically in the neighbourhood of z in D, at which $P_0(z)$ and the discriminant $\mathfrak{D}(z)$ of (1) do not vanish. The zero points of $P_0(z)$ in D are finite in number, and at those points, at least one of the branches of $\omega(z)$ becomes infinity, the infinities, how-

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[1] This occurs only when all $e^{\overline{P}_i(z)}$, $(i=1, 2, \ldots\ldots, k)$, are of the same orders.

ever, being of finite orders, since $P_0(z)$, $P_1(z)$, ..., $P_n(z)$ behave like polynominals in D. The equation (1) being irreducible, $\mathfrak{D}(z)$ does not vanish identically in D, and it is an entire function. Accordingly, the zero points of $\mathfrak{D}(z)$ in D are finite in number and so also for the branch points of $\omega(z)$. Moreover, as $P_0(z)$, $P_1(z)$, ..., $P_n(z)$ behave regularly in D, $\omega(z)$ behaves algebraically in the vicinities of the branch points. In short, $\omega(z)$ has, at most, only a finite number of singular points in D, and the singularities are poles, branch points, or the combinations of them, that is, $\omega(z)$ behaves algebraically in any finite domain. Conversely, suppose that $\omega(z)$ is n-valued and behaves algebraically in any finite domain. Then any symmetric polynominal of the branches ω_1, ω_2, ..., ω_n of $\omega(z)$, being one-valued, has, as singularities, only a finite number of poles in any finite domain, so that it is a meromorphic function of z. Accorningly, $\omega(z)$ is the solution of

$$\prod_{i=1}^{n}(\omega-\omega_i)\equiv\omega^n+R_1(z)\omega^{n-1}+\ldots\ldots+R_n(z)=0,$$

where $R_1(z)$, $R_2(z)$, ..., $R_n(z)$ are meromorphic functions of z. Hence, putting $R_1(z)\equiv\dfrac{P_1(z)}{P_0(z)}$, ..., $R_n(z)\equiv\dfrac{P_n(z)}{P_0(z)}$ where $P_0(z)$, $P_1(z)$, ..., $P_n(z)$ are entire functions of z, we have the proposition.

2. *Theorem 2. Let $\omega(z)$ be a transcendental algebraic function of the $\omega^{p-2}\rho^{th}$ order defined by*

$$(1)\qquad P_0(z)\omega^n+P_1(z)\omega^{n-1}+\ldots\ldots+P_n(z)=0.$$

Then for any prescribed positive value ε, there corresponds a finite value R such that

$$|P_0(z)\omega|\leqq e_{p-1}^{r^{\rho+\varepsilon}}\qquad \textit{for all}\quad r\equiv|z|\geqq R.$$

Suppose, if possible, there were a sequence of values z_1, z_2, ..., z_i, ... which satisfy $\lim_{i=\infty} z_i=\infty$ and $|P_0(z)\omega|>e_{p-1}^{r^{\rho+\varepsilon}}$. Then by (1)

$$|P_0\omega|\leqq|P_1|+|P_2P_0|\cdot|P_0\omega|^{-1}+\ldots\ldots+|P_nP_0^{n-1}|\cdot|P_0\omega|^{-(n-1)}$$
$$\leqq|P|\frac{1-|P_0\omega|^{-n}}{1-|P_0\omega|^{-1}},$$

where $|P|$ is the greatest of $|P_1|$, $|P_2P_0|$, ..., $|P_nP_0^{n-1}|$. As $|P_0(z_i)\omega(z_i)|>e_{p-1}^{r_i^{\rho+\varepsilon}}$ by assumption, where $r_i=|z_i|$, $(i=1,\ 2,\ldots\ldots)$, we have

$$\lim_{i=\infty}\frac{1-|P_0(z_i)\omega(z_i)|^{-n}}{1-|P_0(z_i)\omega(z_i)|^{-1}}=1,$$

so that

$$\frac{1-|P_0(z_i)\omega(z_i)|^{-n}}{1-|P_0(z_i)\omega(z_i)|^{-1}} \leqq 2 \qquad \text{for} \quad r_i \geqq R_1.$$

$P_1(z)$, $P_2(z)P_0(z)$, ..., $P_n(z)P_0(z)^{n-1}$ being entire functions at most of the $\omega^{p-2}\rho$th order, we have $|P| \leqq \frac{1}{2} e_{p-1}^{r^{\rho+\varepsilon}}$ for $r \geqq R_2$. Then for all z_i such as $r_i \geqq R_1$, R_2, we have

$$e_{p-1}^{r_i^{\rho+\varepsilon}} < |P_0(z_i)\omega(z_i)| \leqq \frac{1}{2} e_{p-1}^{r_i^{\rho+\varepsilon}} \cdot 2 = e_{p-1}^{r_i^{\rho+\varepsilon}}$$

which is a contradiction. Hence we have the proposition.

Corollary 1. *For any positive value* ε, *there corresponds a finite value* R *such that* $\left|\frac{P_n(z)}{\omega(z)}\right| \leqq e_{p-1}^{r^{\rho+\varepsilon}}$ *for all* $r \equiv |z| \geqq R$.

Corollary 2. *Extension of Hadamard's second theorem*[1] *to transcendental algebraic functions.*

Let $\omega(z)$ *be a transcendental algebraic function of the order* $\omega^{p-2}\rho$, *defined by* (1). *Then for any prescribed values* ε (*however small*) *and* G (*however great*), *there will be a circle whose radius is greater than* G *and on which* $|\omega(z)| \leqq e_{p-1}^{|z|^{\rho+\varepsilon}}$. *Similarly, there will be another circle of the same property, on which* $|\omega(z)| \geqq e^{-e_{p-2}^{|z|^{\rho+\varepsilon}}}$.

Supposing that $P_0(z) \equiv e^{P(z)} \cdot f(z)$ where $f(z)$ is the canonical product of the order $\omega^{p'-2}\rho'$ ($\leqq \omega^{p-2}\rho$) of all the zero points of $P_0(z)$, divide both members of (1) by $e^{P(z)}$, by which the order of (1) is invariable. Then we have, by the theorem,

$$|f(z)\omega(z)| \leqq e_{p-1}^{|z|^{\rho+\varepsilon}} \qquad \text{for} \quad |z| \geqq R.$$

By Hadamard's 2^{nd} theorem[2] (for $p'=2$) or by its extension[3] (for $p' \geqq 3$), there is a circle whose radius is greater than any prescribed positive value G, and on which

$$|f(z)| > e^{-e_{p'-2}^{|z|^{\rho'+\varepsilon}}} \geqq e^{-e_{p-2}^{|z|^{\rho+\varepsilon}}}$$

so that

$$|\omega(z)| = |f(z)\omega(z)| \cdot |f(z)|^{-1} \leqq e^{2e_{p-2}^{|z|^{\rho+\varepsilon}}} < e_{p-1}^{|z|^{\rho+\varepsilon'}}$$

where ε' may be taken as small as we please by taking $|z|$ sufficiently great. Thus the first part of the corollary is proved. In precisely the same way, the second part of the corollary may be proved by aid of corollary 1 instead of theorem 2.

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[1] See §5, Chap. 1.

[2] See foot-note in §5, Chap. 1.

[3] See §5, Chap. 1.

3. *Theorem 3. Let*

$$(2)\qquad I_1(z, w) \equiv L_0(z)\omega^{m+n} + L_1(z)\omega^{m+n-1} + \ldots\ldots + L_{m+n}(z),$$
$$(3)\qquad I_2(z, \omega) \equiv M_0(z)\omega^{m} + M_1(z)\omega^{m-1} + \ldots\ldots + M_m(z),$$
$$(4)\qquad I_3(z, \omega) \equiv N_0(z)\omega^{n} + N_1(z)\omega^{n-1} + \ldots\ldots + N_n(z),$$

where $L_i(z)$, $(i=0, 1, \ldots\ldots, m+n)$, *are all entire functions and* $M_i(z)$, $(i=0, 1, \ldots\ldots, m)$, *and* $N_i(z)$, $(i=0, 1, \ldots\ldots, n)$, *are generally meromorphic functions. Supposing that*

$$(5)\qquad I_1(z, \omega) \equiv I_2(z, \omega) \cdot I_3(z, \omega),$$

we may determine $\overline{I_2}(z, \omega)$ *and* $\overline{I_3}(z, \omega)$ *such that their degrees in* ω *are equal to those of* $I_2(z, w)$ *and* $I_3(z, \omega)$ *respectively, while their coefficients are entire functions and the functional relation* (5) *still holds.*

Let $M_i(z) \equiv \dfrac{\overline{M_i}(z)}{M(z)}$, $(i=0, 1, \ldots\ldots, m)$, where $M(z)$, $\overline{M_0}(z), \ldots\ldots, \overline{M_m}(z)$ are all entire functions and $M(z)$ is the canonical product of the primary factors of the infinity points, at least, of one of $M_i(z)$ (that is to say, in the case that all $M_i(z)$ are rational, $M(z)$ is the least common multiple of their denominators). Similarly, we determine $N(z)$, $\overline{N_0}(z)$, $\overline{N_1}(z), \ldots\ldots, \overline{N_n}(z)$. Supposing that a be an arbitrary zero point of $M(z)$, there exists, at least one $\overline{M_i}(z)$ which is indivisible by $(z-a)$. Let $\overline{M_h}(z)$ be the one whose suffix is the least among them. Similarly, if there are $\overline{N}$ which are not divisible by $z-a$, let $\overline{N_k}(z)$ be the first one. Comparing the coefficients of $\omega^{m+n-(h+k)}$ in (5), we have

$$M(z)N(z)L_{h+k}(z) = \overline{M_0}(z)\overline{N_{h+k}}(z) + \overline{M_1}(z)\overline{N_{h+k-1}}(z) + \ldots\ldots + \overline{M_{h-1}}(z)\overline{N_{k+1}}(z)$$
$$+ \overline{M_h}(z)\overline{N_k}(z) + \overline{M_{h+1}}(z)\overline{N_{k-1}}(z) + \ldots\ldots + \overline{M_{h+k}}(z)\overline{N_0}(z),$$

in which all $\overline{M_i}(z)$'s whose suffixes are greater than m, and all $\overline{N_i}(z)$'s whose suffixes are greater than n are zero. Here, on both sides, all terms except $\overline{M_h}(z)\overline{N_k}(z)$ of the right hand are divisible by $(z-a)$, which is impossible. It follows, therefore, that all $\overline{N_i}(z)$'s must be divisible by $(z-a)$. Consequently, a being an arbitrary zero point of $M(z)$, all $\overline{N_i}(z)$'s are divisible by $M(z)$. Similarly all $\overline{M_i}(z)$'s are divisible by $N(z)$. Put

$$\overline{\overline{M_i}}(z) \equiv \frac{\overline{M_i}(z)}{N(z)}, \qquad (i=0, 1, \ldots\ldots, m),$$
$$\overline{\overline{N_i}}(z) \equiv \frac{\overline{N_i}(z)}{M(z)}, \qquad (i=0, 1, \ldots\ldots, n).$$

Then $\overline{\overline{M}}(z)$, $(i=0, 1, \ldots\ldots, m)$, and $\overline{\overline{N_i}}(z)$, $(i=0, 1, \ldots\ldots, n)$, are all entire functions and

$$\overline{I}_2(z,\omega)\equiv\overline{\overline{M}}_0(z)\omega^m+\ldots\ldots+\overline{\overline{M}}_m(z),$$
$$\overline{I}_3(z,\omega)\equiv\overline{\overline{N}}_0(z)\omega^n+\ldots\ldots=\overline{\overline{N}}_n(z)$$

are desired functions.

4. *Theorem 4. Let* $L_i(z)$, $M_i(z)$ *and* $N_i(z)$ *in* (5) *be entire functions* and $M_i(z)$, $(i=0, 1, \ldots\ldots, m)$, *be relative prime.*[1] *If* $\omega^{p_1\dot{-}2}\rho_1$, $\omega^{p_2\dot{-}2}\rho_2$ and $\omega^{p_3\dot{-}2}\rho_3$ *be the highest orders among those of* $L_i(z)$, $(i=0, 1, \ldots\ldots, m+n)$, *of* $M_i(z)$, $(i=0, 1, \ldots\ldots, m)$, *and of* $N_i(z)$, $(i=0, 1, \ldots\ldots, n)$, *respectively,* $\omega^{p_1\dot{-}2}\rho_1$ *will be equal to the greater of* $\omega^{p_2\dot{-}2}\rho_2$ *and* $\omega^{p_3\dot{-}2}\rho_3$.

$$(6)\begin{cases} L_0=M_0N_0 \\ L_1=M_0N_1+M_1N_0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_i=M_0N_i+M_1N_{i-1}+\ldots\ldots+M_iN_0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ L_{m+n}=M_mN_n. \end{cases}$$

As $L_0=M_0N_0$ by (6), the canonical products in M_0 and N_0 can not be of higher order than L_0, that is, at most of the $\omega^{p_1\dot{-}2}\rho_1{}^{th}$ order, so that we may put $M_0\equiv\overline{M}_0e^k$ and $N_0\equiv\overline{N}_0\cdot e^{-k}$ respectively where $\overline{M}_0$ and $\overline{N}_0$ are entire functions at most of the $\omega^{p_1\dot{-}2}\rho_1{}^{th}$ order and k is identically zero or e^k is an entire function of order higher than $\omega^{p_1\dot{-}2}\rho_1$ and has no zero point. Putting $M_i\equiv\overline{M}_i\cdot e^k$ $(i=1, 2, \ldots\ldots, m)$, and $N_i\equiv\overline{N}_i\cdot e^{-k}$, $(i=1, 2, \ldots\ldots, n)$, and solving $\overline{M}_i$ (or $\overline{N}_i$) from $m+n$ equations except the first in (6), $\overline{M}_i$ (or $\overline{N}_i$) may be determined as an algebraic function of L_1, L_2, ..., L_{m+n}, $\overline{M}_0$, and $\overline{N}_0$.

$$A_0(L_1, \ldots\ldots, L_{m+n}, \overline{M}_0, \overline{N}_0)\overline{M}_i^s+\ldots\ldots+A_s(L_1, \ldots\ldots, L_{m+n}, \overline{M}_0, \overline{N}_0)=0,$$

where s is a positive integer, and all A's are polynominals of L_1, L_2, ..., L_{m+n}, $\overline{M}_0$, $\overline{N}_0$, that is, entire functions at most of the $\omega^{p_1\dot{-}2}\rho_1{}^{th}$ order. Accordingly we have, by theorem 2 in §2, $|A_0\overline{M}_i|\angle e_{p_1-1}^{r^{\rho_1+\varepsilon}}$ for $r\geqq R$, that is, $A_0\overline{M}_i$ is an entire function at most of the $\omega^{p_1\dot{-}2}\rho_1{}^{th}$ order. Put $A_0\equiv e^{Q_0}\cdot f_0$ and $\overline{M}_i\equiv e^{Q_1}\cdot f_1$ where Q_0, Q_1 are entire functions and f_0, f_1 are the canonical products of the zero points of A_0 and $\overline{M}_i$ respectively. As $A_0\overline{M}_i\equiv e^{Q_0+Q_1}f_0\cdot f_1$ is at most of the order $\omega^{p_1\dot{-}2}\rho_1$, the same is true, by §4, Chap. 1, of $f_0\cdot f_1$ and accordingly of f_1. Similarly, as A_0 and $A_0\overline{M}_i$ are at most of the order $\omega^{p_1\dot{-}2}\rho_1$, the same is true by cor. 3 in §5, Chap. 1, of e^{Q_0} and $e^{Q_0+Q_1}$, so also that of e^{Q_1}. Consequently, $e^{Q_1}\cdot f_1\equiv\overline{M}_i$ is at most of the order $\omega^{p_1\dot{-}2}\rho_1$. But $M_i\equiv e^k\overline{M}_i$, $(i=0, 1,$

1 See Introduction.

......, m), being relative prime by assumption, we have $k \equiv 0$, so that all M's are at most of the $\omega^{p_1}\dot{-}^{2}\rho_1{}^{th}$ order, and so also for all N's. Hence we have $\omega^{p_1}\dot{-}^{2}\rho_1 \geqq \omega^{p_2}\dot{-}^{2}\rho_2$ and $\omega^{p_1}\dot{-}^{2}\rho_1 \geqq \omega^{p_3}\dot{-}^{2}\rho_3$. Now, suppose that $\omega^{p_1}\dot{-}^{2}\rho_1 > \omega^{p_2}\dot{-}^{2}\rho_2$, $\omega^{p_3}\dot{-}^{2}\rho_3$. Then in virtue of $m+n+1$ equations in (6), the orders of $L_0, L_1, \ldots\ldots, L_{m+n}$ are not greater than the greater of $\omega^{p_2}\dot{-}^{2}\rho_2$ and $\omega^{p_3}\dot{-}^{2}\rho_3$ and accordingly are less than $\omega^{p_1}\dot{-}^{2}\rho_1$, which is a contradiction.

5. *Theorem 5. No transcendental algebraic equation is satisfied by a transcendental algebraic function of the higher order.*

Let $\omega(z)$ be a transcendental algebraic function of the $\omega^{p_1}\dot{-}^{2}\rho_1{}^{th}$ order defined by

$$(1) \quad P_0(z)\omega^n + P_1(z)\omega^{n-1} + \ldots\ldots + P_n(z) = 0$$

and

$$(7) \quad Q_0(z)\omega^m + Q_1(z)\omega^{m-1} + \ldots\ldots + Q_m(z) = 0$$

be a transcendental algebraic equation of the $\omega^{p_2}\dot{-}^{2}\rho_2{}^{th}$ order where $\omega^{p_2}\dot{-}^{2}\rho_2 < \omega^{p_1}\dot{-}^{2}\rho_1$. As (1) is irreducible, in order that (7) be satisfied by $\omega(z)$ defined by (1), it must be decomposable, that is

$$Q_0(z)\omega^m + Q_1(z)\omega^{m-1} + \ldots\ldots + Q_m(z) \equiv \Big(P_0(z)\omega^n + \ldots\ldots + P_n(z)\Big) \times \Big(R_0(z)\omega^{m-n} + R_1(z)\omega^{m-n-1} + \ldots\ldots + R_{m-n}(z)\Big).$$

$Q_0(z), Q_1(z), \ldots\ldots, Q_m(z)$ being relative prime, $R_0(z), \ldots\ldots, R_{m-n}(z)$ are, by §3, entire functions and accordingly by §4, $P_0(z), P_1(z), \ldots\ldots, P_n(z)$ are at most of the order $\omega^{p_2}\dot{-}^{2}\rho_2$, that is, of orders lower than $\omega^{p_1}\dot{-}^{2}\rho$, which is impossible.

Corollary. No transcendental algebraic function satisfies an irreducible equation of the higher order.

This may be proved in the same way as the above theorem.

6. *Theorem 6. Let $R_i(z)$, $(i=1, 2. \ldots\ldots, m)$, being different from zero, be a rational or a meromorphic function of a lower order than $e^{K_i(z)}$, where $K_i(z)$ is a polynominal or an entire function of order lower than Ω; let $R_0(z)$ be a rational function (zero being included) or a meromorphic function of a lower order than all $e^{K_i(z)}$, $(i=1. 2, \ldots\ldots, m)$. Then, in order that*

$$(8) \qquad R_0(z) + R_1(z)e^{K_1(z)} + \ldots\ldots + R_m(z)e^{K_m(z)} \equiv 0,$$

it is necessary that, for each $K_i(z)$, there exists, at least one $K_j(z)$ $(i \neq j)$ such that both $e^{K_i(z)}$ and *$e^{K_j(z)}$ are of higher orders than $e^{K_i(z)-K_j(z)}$, and all the sums of those terms relating to one another in such a manner, and accordingly R_0 also, vanish identically.*

Supposing that the theorem is untrue, we may assume, then that in (8), no $K_i(z)$ has the property above described; for otherwire we combine all the terms relating to one another in such a manner into one term, by which the coefficients of the combined terms, by assumption, do not all vanish identically. First, we assume that $R_0(z)\equiv 0$. Then we may assume, without loss of generality, that $e^{K_1(z)}$ is one of the lowest orders in $e^{K_i(z)}$, $(i=1, 2, \ldots\ldots, m)$. Dividing both members of (8) by $R_1(z)e^{K_1(z)}$, and differentiating with respect to z, we have

$$(8)' \qquad \sum_{i=2}^{m}\left\{\frac{d\left(\frac{R_i}{R_1}\right)}{dz}+\frac{R_i}{R_1}\frac{d(K_i-K_1)}{dz}\right\}e^{K_i-K_1}\equiv 0$$

of which any coefficient $\dfrac{d\left(\frac{R_i}{R_1}\right)}{dz}+\dfrac{R_i}{R_1}\dfrac{d(K_i-K_1)}{dz}$ does not vanish identically. For if $\dfrac{d\left(\frac{R_i}{R_1}\right)}{dz}+\dfrac{R_i}{R_1}\dfrac{d(K_i-K_1)}{dz}\equiv 0$, we have, by integration, $\dfrac{R_i}{R_1}\equiv Ce^{-(K_i-K_1)}$ where C is a constant different from zero. By the assumption that at least one of e^{K_1} and e^{K_i} is not of a higher order than $e^{K_i-K_1}$ and that e^{K_1} is one of the lowest orders, $e^{K_i-K_1}$ is not of a lower order than e^{K_1}. Suppose that $e^{K_i-K_1}$ be of an order lower than e^{K_i}. As the order of the product of the finite number of entire functions is not greater than all of their orders, the order of $e^{K_i}\equiv e^{K_i-K_1}\cdot e^{K_1}$ is not greater than that of $e^{K_i-K_1}$, and accordingly by assumption, is less than itself, which is a contradiction. Similarly $e^{K_i-K_1}$ is not of a higher order than e^{K_i}. Hence e^{K_i} and $e^{K_i-K_1}$ are of the same orders. By the assumption that R_1 and R_i are of lower orders than e^{K_1} and e^{K_i} respectively, and that e^{K_1} is one of the lowest orders in e^{K_i}, $(i=1, 2, \ldots\ldots, m)$, the order of $\dfrac{R_i}{R_1}$[1] is less than that of e^{K_i}, that is, is less than that of $e^{K_i-K_1}$, and the above identity does not hold. The order of an entire function does not increase by differentiation,[2] and this theorem may easily be extended to mermorphic functions, so that $\dfrac{d}{dz}\left(\dfrac{R_i}{R_1}\right)$ is not of a higher order than $\dfrac{R_i}{R_1}$. Similarly, $\dfrac{d(K_i-K_1)}{dz}$ is not of an order higher than K_i-K_1, which is of a lower order than $e^{K_i-K_1}$. Hence, it follows that $e^{K_i-K_1}$ is of an order higher than

1 All rational functions are of the order *zero.*

2 See Borel, *Leçons sur les Fonctions méromorphes,* p. 60.

$\frac{d}{dz}\left(\frac{R_i}{R_1}\right)+\frac{R_i}{R_1}\frac{d(K_i-K_1)}{dz}$. Consequently, we have

$$R_2^{(1)}e^{K_2-K_1}+R_3^{(1)}e^{K_3-K_1}+\dots\dots+R_m^{(1)}e^{K_m-K_1}\equiv 0,$$

where $R_i^{(1)}$, being different from zero, is a rational or *a* meromorphic function of a lower order than $e^{K_i-K_1}$. By repeating the same reasoning, we would have

$$R_m^{(m-1)}e^{K_m-K_{m-1}}\equiv 0,$$

which is impossible, since $R_m^{(m-1)}$ is different from zero.

Secondly, we assume that $R_0(z)\not\equiv 0$. In this case, we may prove the theorem, by dividing, at first, both members of (8) by R_0 whose order is less than those of e^{K_i}, $(i=1, 2, \dots\dots, m)$, and then by proceeding in the same way as before.

7. We are now to prove the generalized theorem of Picard. Let

$$(1)\qquad P_0(z)\omega^n+P_1(z)\omega^{n-1}+\dots\dots P_n(z)=0,$$

and

$$(9)\qquad Q_0^{(i)}(z)\omega_m+Q_1^{(i)}(z)\omega^{m-1}+\dots\dots+Q_m^{(i)}(z)=0,\ (1,\ 2,\ \dots\dots),$$

be irreducible equations of the $\omega^{p}\dot{-}^{2}\rho^{th}$ order and the n^{th} degree, and of the $\omega^{p_i}\dot{-}^{2}\rho_i^{th}$ order and the m^{th} degree respectively. In order that $\omega_i=\omega$ at a certain point, it is necessary and sufficient that

$$\begin{vmatrix} P_0, & P_1, & \dots\dots\dots, & P_n, & 0, & 0, & \dots\dots\dots\dots, & 0 \\ 0, & P_0, & \dots\dots\dots\dots, & P_n, & 0, & \dots\dots\dots\dots, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & \dots\dots, & 0, & P_0, & \dots\dots\dots\dots\dots\dots\dots\dots, & P_n \\ Q_0^{(i)}, & Q_1^{(i)}, & \dots\dots\dots, & Q_m^{(i)}, & 0, & 0, & \dots\dots\dots\dots, & 0 \\ 0, & Q_0^{(i)}, & \dots\dots\dots\dots\dots, & Q_m^{(i)}, & 0, & \dots\dots\dots, & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0, & \dots\dots\dots\dots\dots, & 0, & Q_0^{(i)}, & \dots\dots\dots\dots, & Q_m^{(i)} \end{vmatrix}\equiv\psi_{i1}\varphi_1+\psi_{i2}\varphi_2+\dots\dots+\psi_{iM}\varphi_M$$

vanishes at that point, where $M={}_{m+n}C_m=\frac{(m+n)!}{m!\,n!}$ and ψ and φ are homogeneous polynominals of the n^{th} degree in $Q^{(i)}$ and of the m^{th} degree in P respectively.

Theorem 7. Picard's theorem generalized.[1]

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[1] Le theorème de M. Picard:—Une fonction entière $F(z)$ telle que les équations $F(z)=a$, $F(z)=b$, $a\neq b$, naient pas de racines, se réduit nécessairement à une constante. (Borel: *Leçons sur les fonctions entières*, p. 88).

Extention aux fonctions méromorphes:—Etant donnée une fonction méromorphe $f(z)$ d'ordre ρ et une autre fonction méromorphe quelconque $\varphi(z)$ d'ordre inférieur, parmi les équations $f(z)=\varphi(z)$ il n'y en a pas en général déxceptionnelles, et s'il en a, il y en a deux an plus. (Borel:—*Leçons sur les fonctions méromorphes*, p. 66).

Let $\omega(z)$ be a transcendental algebraic function of the $\omega^{p\dot{-}2}\rho^{th}$ order and the n^{th} degree and $\omega_i(z)$, $(i=1, 2, \ldots\ldots, r)$, be any transcendental algebraic functions of the $\omega^{p_i\dot{-}2}\rho_i^{th}$ order $(\omega^{p_i\dot{-}2}\rho_i<\omega^{p\dot{-}2}\rho)$ and the m^{th} degree respectively, under the condition that none of the determinants

$$(10)\qquad \begin{vmatrix} \psi_{r_1 1} & \psi_{r_2 1} & \cdots\cdots\cdots & \psi_{r_M 1} \\ \psi_{r_1 2} & \psi_{r_2 2} & \cdots\cdots\cdots & \psi_{r_M 2} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \psi_{r_1 M} & \psi_{r_2 M} & \cdots\cdots & \psi_{r_M M} \end{vmatrix} \quad \begin{pmatrix} r_1, r_2, \ldots\ldots r_M = 1, 2, \ldots\ldots, r \\ r_1 < r_2 < \ldots\ldots\ldots\ldots < r_M \end{pmatrix},$$

vanishes identically.[1] *Then all the orders of the canonical products of the primary factors corresponding to the points which satisfy $\omega(z)=\omega_i(z)$, $(i=1, 2, \ldots\ldots, r)$, can not be inferior to $\omega^{p\dot{-}2}\rho$, unless $r\leqq N=2(M-1)$.*

Let (1) and (9) be the equations which define $\omega(z)$ and $\omega_i(z)$ respectivly, and $\omega^{q_i\dot{-}2}\sigma_i$ be the order of the canonical product of the primary factors corresponding to the points which satisfy $\omega(z)=\omega_i(z)$. Let $\omega^{p'\dot{-}2}\rho'$ be the greatest of $\omega^{p_i\dot{-}2}\rho_i$ $(i=1, 2, \ldots\ldots, r)$ and $\omega^{q_i\dot{-}2}\sigma_i$ $(i=1, 2, \ldots\ldots, r)$. Then $\omega^{p'\dot{-}2}\rho<\omega^{p\dot{-}2}\rho$. Now we have

$$(11)\qquad \psi_{i1}\varphi_1+\psi_{i2}\varphi_2+\ldots\ldots+\psi_{iM}\varphi_M\equiv R_i e^{K_i},^{2}\quad (i=1, 2, \ldots\ldots, r).$$

where R_i is the canonical product of the primary factors corresponding to the points which satisfy $\omega(z)=\omega_i(z)$, so that it is an entire function of the $\omega^{q_i\dot{-}2}\sigma_i^{th}$ order. Accordingly, e^{K_i} is an entire function at most of the $\omega^{p\dot{-}2}\rho^{th}$ order. Suppose that $r>M$[3] and eliminating $\varphi_1, \varphi_2, \ldots\ldots, \varphi_M$ from any $M+1$ equations, say equations corresponding to $i=1, 2, \ldots\ldots, M+1$, we have

$$(12)\qquad \begin{vmatrix} \psi_{11}, & \psi_{12}, & \ldots\ldots\ldots\ldots, & \psi_{1M}, & R_1 e^{K_1} \\ \psi_{21}, & \psi_{22}, & \ldots\ldots\ldots\ldots, & \psi_{2M}, & R_2 e^{K_2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_{M1}, & \psi_{M2}, & \ldots\ldots\ldots\ldots, & \psi_{MM}, & R_M e^{K_M} \\ \psi_{M+11}, & \psi_{M+12}, & \ldots, & \psi_{M+1M}, & R_{M+1}e^{K_{M+1}} \end{vmatrix} \equiv 0,$$

in which the coefficient $\pm R_i(\psi_{11}\psi_{22}\ldots\ldots\psi_{i-1\,i-1}\psi_{i+1i}\ldots\ldots\psi_{M+1M})$ of e^{K_i}, by assumption, does not vanish identically and is at most of the $\omega^{p'\dot{-}2}\rho'^{th}$ order. For each e^{K_i} of the $\omega^{p\dot{-}2}\rho^{th}$ order, there exists at least one e^{K} $(j\neq i)$, in e^{K_1}, e^{K_2}, ..., $e^{K_{M+1}}$ such that $e^{K_i-K_j}$ is of an order lower

[1] When $m=1$, the condition will be satisfied if $\omega_i(i=1, 2, \ldots\ldots, r)$, are different from one another.

[2] By theorem 5 in §5, $R_i\neq 0$.

[3] If $r\leqq M$, then M being $\geqq 2$, we have $r\leqq M\leqq 2(M-1)=N$.

than those of e^{K_i} and e^{K_j}, or is of an order lower than $\omega^{p-2}\rho$, and accordingly e^{K_j} is of the $\omega^{p-2}\rho^{th}$ order (§6). Now we divide e^{K_1}, e^{K_2}, ..., e^{K_r} into groups such that all e^{K_i}'s which are of order lower than $\omega^{p-2}\rho$ belong to one group, and for other e^{K_i}'s of the order $\omega^{p-2}\rho$, if any two of them have the relation above mentioned, they belong to one and the same group; otherwise, to different groups. Let those groups be G_1, G_2, ..., G_t and s_j be the number of e^{K_i}'s which belong to G_j, $(j=1, 2, ..., t)$. Then we hove evidently

$$(13) \qquad s_1+s_2+......+s_t=r.$$

Now all s_j's are less than M. For otherwise, suppose that e^{K_1}, e^{K_2}, ..., e^{K_M} belong to one and the same group. Then as the determinant $(\psi_{11}, \psi_{12}, ..., \psi_{MM})$ by assumption, does not vanish identically, we may express, by solving M equations in (11) corresponding to $i=1, 2, ..., M$, $\varphi_1, \varphi_2, ..., \varphi_M$ as function of the form

$$\varphi_i \equiv R_i e^{K_1}, \quad (i=1, 2, ..., M),$$

where R_i, $(1, 2, ..., M)$, are entire functions of orders lower than $\omega^{p-2}\rho$, and e^{K_1} is an entire function, at most, of the $\omega^{p-2}\rho^{th}$ order. But among $\varphi_1, \varphi_2, ..., \varphi_M$, there exist $P_0^{m-1}P_j$ $(j=0, 1, ..., m)$, and we have

$$P_i \equiv S_i e^{\frac{K_1}{m}}, \qquad (i=0, 1, ..., m),$$

where S_i, $(i=0, 1, ..., m)$, are entire functions of orders lower than $\omega^{p-2}\rho$. Substituting these values in (1), we have

$$S_0(z)\omega^n + S_1(z)\omega^{n-1} + ... + S_n(z) = 0,$$

that is, $\omega(z)$ is of order lower than $\omega^{p-2}\rho$, which is a contradiction. Hence $s_j \leqq M-1$, $(j=1, 2, ..., t)$, and consequently we have, from (13),

$$(14) \qquad t \geqq 2.$$

Next, the number of all e^{K_i}'s which belong to $G_2, G_3, ..., G_t$ is less than M. For otherwire, taking any M e^{K_i}'s, say $e^{K_1}, e^{K_2}, ..., e^{K_M}$, which belong to $G_2, G_3, ..., G_t$ and any e^{K_i}, say $e^{K_{M+1}}$, which belongs to G_1 and eliminating $\varphi_1, \varphi_2, ..., \varphi_M$ from $M+1$ equations in (11) which correspond to $i=1, 2, ..., M+1$, we have the identity (12). Hence there exists at least one e^{K_i} among $e^{K_1}, e^{K_2}, ..., e^{K_M}$, which belongs to the same group as $e^{K_{M+1}}$ (§6). This is however impossible, and accordingly we have

$$s_2+s_3+..........+s_t \leqq M-1.$$

Similarly,

$$s_1+s_3+\dots\dots\dots\dots+s_t\leqq M-1,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$s_1+s_2+\dots\dots\dots\dots+s_{t-1}\leqq M-1,$$

or, by addition,

$$(t-1)\ (s_1+s_2+\dots\dots+s_t)\leqq t(M-1),$$

by (13)

$$r\leqq\frac{t}{t-1}(M-1).$$

As $t\geqq 2$ by (14), we have $\frac{t}{t-1}\leqq 2$ and consequently

$$r\leqq 2(M-1).$$

Theorem 8. If a transcendental algebraic function $\omega(z)$, having only a finite number of branch points, be such that $\omega(z)=a$ and $\omega(z)=b$, $a\neq b$, have no root at finiteness, then it will be an ordinary algebraic function.

Though this theorem is more restricted than the former one, it is very useful for later investigations. As $\omega(z)$ has only a finite number of branch points, any branch of $\omega(z)$ may be expressed as a regular function of $t=z^{\frac{1}{\lambda}}$ in the region $(G\leqq|t|<\infty)$ where λ is a certain positive integer and G, a certain positive value. As $\omega(z)=a$ and $\omega(z)=b$, $a\neq b$, have no root in the assigned region, the infinity point is, by Picard's theorem generalized,[1] a regular or a non-essential singular point of the function $\omega(t^\lambda)$ of t. Accordingly each of the coefficients $P_0(z)$, ..., $P_n(z)$ in

$$(1)\qquad P_0(z)\omega^n+P_1(z)\omega^{n-1}+\dots\dots+P_n(z)=0$$

has, as singularities, at most a pole at infinity, that is, $\omega(z)$ is an ordinary algebraic function of z.

8. We now proceed to consider the Riemann's surface for transcendental algebraic functions. They are entirely analogous to those for ordinary algebraic functions, except for the vicinity of the infinity point. As for the branch lines, we determine them as follows: If $\omega(z)$ be a transcendental algebraic function of the n^{th} degree, the Riemann's surface for it is n-sheeted. If the origin be a branch point, we take as the branch line, a half straight line in any direction, having the origin as its end point, and extending to infinity. For other branch points, $(r_i\theta_i)$,

[1] Voraussetzung: $f(x)$ sei für $0<|x-x_0|<\rho$ eindeutig-regulär, $\neq a$ und $\neq b$ (wo $a\neq b$ ist). Behauptung: x_0 ist keine wesentlich singuläre Stelle (sondern regulär oder ein Pol). (E. Landau, *Darstellung und Begründung einiger neuerer Ergebnisse der Funktionen-theorie* p. 96).

we arrange them in order of increasing moduli, and when there are branch points of equal moduli, we arrange them in order of increasing arguments, $(0 \leqq \theta_i < 2\pi)$. Then we have

$$(r_1\theta_1),\ (r_2\theta_2),\ \ldots\ldots,\ (r_i\theta_i),\ \ldots\ldots\ldots .$$

As the branch line corresponding to (r_1, θ_1), we take a half straight line, $R \geqq r_1$, $\theta = \theta_1$, where (R, θ) are the current coordinates, and if there exists another branch point (r_i, θ_i) such that $\theta_i = \theta_1$, we take a small semi-circle as a part of the branch line, whose center being (r_i, θ_i), (*see* Fig. 1). For $(r_2\theta_2)$, (r_3, θ_3), ..., similar process will be applicable. As the branch line corresponding to (r_i, θ_i), we take the line determined as above, but slightly deformed so as to have no common point with that corresponding to (r_1, θ_1), and so on. On the Riemann's surface determined as before, $\omega(z)$ is evidently a uniform function of position.

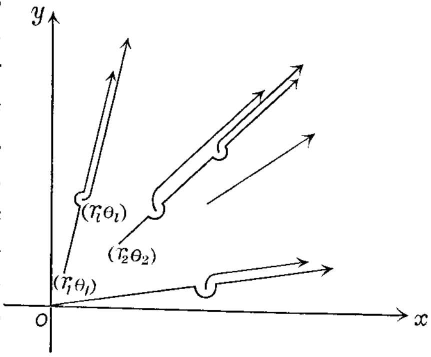

Fig. 1.

9. We shall prove some theorems concerning the Riemann's surface.

Theorem 9. Let S be the Riemann's surface for $\omega(z)$ defined by

$$(1) \qquad P_0(z)\omega^n + P_1(z)\omega^{n-1} + \ldots\ldots + P_n(z) = 0$$

and $f(z, \omega)$ be a uniform function of position on S and be branching out as S. If $f(z, \omega)$ be a holomorphic analytic function of position on the total surface S, the infinity point being included, it will be a constant

Let

$$(a_1, \omega_1),\ (a_2, \omega_2),\ \ldots\ldots,\ (a_n, \omega_n),\ (a_1 = a_2 = \ldots = a_n = a),$$

be n analytic points corresponding to $z = a$. In the vicinity of (a_i, ω_i), $(i = 1, 2, \ldots\ldots, n)$, we have $f(z, \omega) = f_i(z) = f_i(z\,|\,a)$. $S(f)$, being any symmetric polynominal of $f_1(z), f_2(z), \ldots\ldots, f_n(z)$, has the same value in whatever sheet z may lie and by whatever path z may have attained its position in that sheet. Hence it is a uniform function of z. Moreover, as $S(f)$ is holomorphic on the total surfaces S, the point at infinity being included, the same is true on the total z-plane, the point at infinity being included, and accordingly it is a constant. From this, it follows that, in

$$(f - f_1)(f - f_2)\ldots\ldots(f - f_n) \equiv f^n + S_1 f^{n-1} + \ldots\ldots S_n = 0,$$

all S_i, $(i = 1, 2, \ldots\ldots, n)$, are constants and hence $f(z, \omega)$ is also.

Theorem 10. Let $f(z, \omega)$ be a uniform function of position on S and be branching out as S. If all the singularities be poles, finite in number, in any finite part of S, $f(z, \omega)$ will be rational in ω and meromorphic in z; and conversely.

Let

$$(a_1\omega_1), (a_2\omega_2), \ldots\ldots, (a_n\omega_n), (a_1=a_2=\ldots\ldots a_n=a),$$

be n analytic points corresponding to $z=a$. In the vicinity of $(a_i\omega_i)$, we have $f(z, \omega)=f_i(z|a)$, $(i=1, 2, \ldots\ldots, n)$. Put

$$(15) \qquad \begin{cases} f_1+f_2+\ldots\ldots f_n=P_0 \\ f_1\omega_1+f_2\omega_2+\ldots\ldots+f_n\omega_n=P_1 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ f_1\omega_1^{n-1}+f_2\omega_2^{n-1}+\ldots\ldots+f_n\omega_n^{n-1}=P_{n-1}, \end{cases}$$

where P_i, $(i=0, 1, \ldots\ldots, n-1)$, are evidently one-valued functions of z and have only finite number of poles in any finite part of the z-plane, so that they are mormorphic functions of z. Next, we determine A_{n-1}, A_{n-2}, ……, A_1 so as to satisfy the following relations

$$(16) \qquad \omega_i^{n-1}+A_1\omega_i^{n-2}+\ldots\ldots+A_{n-1}=0, \quad (i=2, 3, \ldots\ldots, n).$$

Multiplying the equations in (15) by A_{n-1}, A_{n-2}, ……, A_1, 1 respectively and adding side by side, we have

$$\sum_{i=1}^{n} f_i(\omega_i^{n-1}+A_1\omega_i^{n-2}+\ldots\ldots+A_{n-1})=P_{n-1}+A_1P_{n-2}+\ldots\ldots A_{n-1}P_0,$$

which is reduced, in virtue of (16), to

$$(17) \quad f_1(\omega_1^{n-1}+A_1\omega_1^{n-2}+\ldots\ldots+A_{n-1})=P_{n-1}+A_1P_{n-2}+\ldots\ldots A_{n-1}P_0.$$

From (16), the roots of $\omega^{n-1}+A_1\omega^{n-2}+\ldots\ldots+A_{n-1}=0$ are ω_2, ω_3, ……, ω_n. Hence we have

$$\begin{aligned} \omega^n+\frac{Q_1}{Q_0}\omega^{n-2}+\ldots\ldots+\frac{Q_n}{Q_0} &\equiv(\omega-\omega_1)(\omega-\omega_2)\ldots\ldots(\omega-\omega_n) \\ &\equiv(\omega-\omega_1)(\omega^{n-1}+A_1\omega^{n-2}+\ldots\ldots+A_{n-1}) \\ &\equiv\omega^n+(A_1-\omega_1)\omega^{n-1}+\ldots\ldots-\omega_1A_{n-1}. \end{aligned}$$

Comparing the coefficients, we have

$$A_1=\frac{Q_0\omega_1+Q_1}{Q_0}$$

……………………

$$A_i = \frac{Q_0\omega_1^i + Q_1\omega_1^{i-1} + \ldots + Q_i}{Q_0}$$

....

i.e. A_i, $(i = 1, 2, \ldots\ldots, n-1)$, are polynominals of ω_1 whose coefficients are meromorphic functions of z. From (17), we have

$$f_1 = \frac{P_{n-1} + A_1P_{n-2} + \ldots\ldots + A_{n-1}P_0}{\omega_1^{n-1} + A_1\omega_1^{n-2} + \ldots\ldots + A_{n-1}}.$$

Proceeding just as above, we may obtain an analogous expression for each branch of ω, so that we have

$$f = \frac{P_{n-1} + A_1P_{n-2} + \ldots\ldots A_{n-1}P_0}{\omega^{n-1} + A_1\omega^{n-2} + \ldots\ldots + A_{n-1}},$$

where $A_i = \dfrac{Q_0\omega^i + Q_1\omega^{i-1} + \ldots\ldots + Q_i}{Q_0}$, $(i = 1, 2, \ldots\ldots, n-1)$, i.e. $f(z, \omega)$ is rational in ω and is mermorphic in z.

Conversely, let $f(z, \omega)$ be a function which is rational in ω and mermorphic in z. In virtue of the equation (1), it may be reduced to the form

$$f(z, w) \equiv \frac{R_0\omega^{n-1} + R_1\omega^{n-2} + \ldots\ldots + R_{n-1}}{S_0\omega^{n-1} + S_1\omega^{n-2} + \ldots\ldots + S_{n-1}},$$

where R_i and S_i $(i = 0, 1, \ldots\ldots, n-1)$, are entire functions of z. In the determinant

$$B \equiv \begin{vmatrix} Q_0, & Q_1, & \ldots\ldots\ldots\ldots, & Q_n, & 0, & 0, & \ldots\ldots\ldots\ldots, & 0 \\ 0, & Q_0, & \ldots\ldots\ldots\ldots\ldots\ldots, & Q_n, & 0, & \ldots\ldots\ldots\ldots, & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & \ldots\ldots\ldots, & 0, & Q_0, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, & & & Q_n \\ S_0, & S_1, & \ldots\ldots\ldots\ldots\ldots, & S_{n-1}, & 0, & 0, & \ldots\ldots\ldots, & 0 \\ 0, & S_0, & \ldots\ldots\ldots\ldots\ldots\ldots, & S_{n-1}, & 0, & \ldots\ldots\ldots, & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0, & \ldots\ldots\ldots, & 0, & S_0, & \ldots\ldots\ldots\ldots\ldots\ldots\ldots, & & & S_{n-1} \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n-1 \text{ rows} \\ \left.\vphantom{\begin{matrix}a\\a\\a\\a\end{matrix}}\right\} n \text{ rows} \end{matrix},$$

which is evidently an entire function of z, multiply the first $2(n-1)$ columns in order by ω^{2n-2}, ω^{2n-3}, ..., ω and add the products to the last column. Then we have

$$B \equiv \varphi(z, \omega)\,(S_0\omega^{n-1} + \ldots\ldots + S_{n-1}) + \psi(z, \omega)(Q_0\omega^n + Q_1\omega^{n-1} + \ldots\ldots + Q_n),$$

where φ and ψ are polynominals of ω whose coefficients are entire functions of z. But as $Q_0\omega^n + Q_1\omega^{n-1} + \ldots\ldots Q_n \equiv 0$ on S, we have

$$f \equiv \frac{R_0\omega^{n-1} + \ldots\ldots + R_{n-1}}{S_0\omega^{n-1} + \ldots\ldots + S_{n-1}} \cdot \frac{\varphi(z, \omega)}{\varphi(z, \omega)} \equiv \frac{T_0\omega^m + T_1\omega^{m-1} + \ldots\ldots + T_m}{B}$$

where m is a positive integer and T_i $(i=0, 1, \ldots\ldots, m)$, are entire functions of z. Accordingly, it follows from the identity, that $f(z, \omega)$ is one-valued on S and is branching out as S, and moreover it has, as singularities, only a finite number of poles in any finite part of S.

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