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Generalization of certain well-known inequalities for rational functions

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Abstract. Let P_m be a class of all polynomials of degree at most m and let $R_{m,n} = R_{m,n}(d_1, ..., d_n) = \{p(z)/w(z); p \in P_m, w(z) = \prod_{j=1}^{n}(z - d_j)$ where $|d_j| > 1, j = 1, ..., n$ and $m \leq n\}$ denote the class of rational functions. It is proved that if the rational function $r(z)$ having all its zeros in $|z| \leq 1$, then for $|z| = 1$

$$|r^{'}(z)| \geq \frac{1}{2}\{|B^{'}(z)| - (n - m)\}|r(z)|.$$

The main purpose of this paper is to improve the above inequality for rational functions $r(z)$ having all its zeros in $|z| \leq k \leq 1$ with t-fold zeros at the origin and some other related inequalities. The obtained results sharpen some well-known estimates for the derivative and polar derivative of polynomials.

Keywords: Rational functions, Polynomials, Polar derivative, Inequalities, Poles, Restricted Zeros.

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1 Introduction and statement of results

Let $p(z)$ be a polynomial of degree at most n. We denote by U_- and U_+ the regions inside and out side the set $U := \{z : |z| = 1\}$, respectiveily.

In 1930, Bernstein [2] revisited his inequality and established the following comparative result by assuming that $p(z)$ and $q(z)$ are polynomials such as $p(z)$ has at most of degree n and $q(z)$ has exactly n zeros in $U \cup U_-$ and for $z \in U$

$$|p(z)| \leq |q(z)|,$$

then for $z \in U$

$$|p'(z)| \leq |q'(z)|. \tag{1.1}$$

Let $D_\alpha p(z)$ denote the polar derivative of the polynomial $p(z)$ of degree n with respect to the point α; $\alpha \in \mathbb{C}$, then

$$D_\alpha p(z) = np(z) + (\alpha - z)p^{'}(z).$$

The polynomial $D_\alpha p(z)$ is of degree at most $n-1$ and it generalizes the ordinary derivative in the sense that

$$\lim_{\alpha\to\infty}[\frac{D_\alpha p(z)}{\alpha}] = p'(z). \tag{1.2}$$

In the past few years many papers were published concerning the polar derivative of polynomials (for example see ([3], [8])). Aziz and Rather[1] proved that if all zeros of $p(z)$ lie in $|z| \leq k \leq 1$, then for every real or complex number α with $|\alpha| \geq k$, we get

$$\max_{z\in U}|D_\alpha p(z)| \geq \frac{n}{1+k}(|\alpha| - k)\max_{z\in U}|p(z)|. \tag{1.3}$$

Let P_m be a class of all polynomials of degree at most m and $d_1, d_2, ..., d_n$ be n given points in U_+ . Consider the following space of rational functions with prescribed poles and with a finite limit at infinity:

$$R_{m,n} = R_{m,n}(d_1, ..., d_n) = \{\frac{p(z)}{w(z)} : p \in \mathrm{P}_m\},$$

where

$$w(z) = \Pi_{j=1}^{n}(z - d_j).$$

The inequalities of Bernstein and Erdös-Lax have been extended to the rational functions ([4], [7]) by replacing the polynomial p(z) with a rational functions r(z) and z^n with Blaschke product B(z) defined by

$$B(z) = \frac{w^*(z)}{w(z)} = \frac{z^n\overline{w(\frac{1}{\bar{z}})}}{w(z)} = \Pi_{j=1}^{n}(\frac{1-\bar{d}_j z}{z - d_j}).$$

Li et al.([6], [7]) obtained Bernstein-type inequalities for rational function $r(z)$. They proved that if $r(z) \in R_{m,n}$ and all the zeros of $r(z)$ in $U \cup U_-$, then for $z \in U$

$$|r^{'}(z)| \geq \frac{1}{2}\{|B^{'}(z)| - (n-m)\}|r(z)|. \tag{1.4}$$

Xin Li [6] extended the inequality (1.1) for rational functions by showing that, if $r(z)$, $s(z) \in R_{n,n}$ such that $s(z)$ has all its n zeros in $U \cup U_-$ and for $z \in U$

$$|r(z)| \leq |s(z)|,$$

then for $z \in U$

$$|r'(z)| \leq |s'(z)|. \tag{1.5}$$

Recently, Hans and Tripathi [5] proved that, if $r(z)$, $s(z) \in R_{n,n}$ such that $s(z)$ has all its n zeros in $U \cup U_-$ and $|r(z)| \leq |s(z)|$ for $z \in U$, then for every real or complex number β with $|\beta| \leq 1$ and $z \in U$

$$|\ zr'(z) + \frac{\beta}{2}|B'(z)|r(z)\ | \leq |\ zs'(z) + \frac{\beta}{2}|B'(z)|s(z)\ |. \tag{1.6}$$

Also, they obtained that if $r(z) \in R_{n,n}$, then for every real or complex number β with $|\beta| \leq 1$ and $z \in U$

$$|\ zr'(z) + \frac{\beta}{2}|B'(z)|r(z)\ | \leq |1 + \frac{\beta}{2}||B'(z)| \max_{z \in U} |r(z)|. \tag{1.7}$$

In this paper, we first prove the following theorem which not only leads to several conclutions about inequality for rational function, but also generalize inequality (1.4).

Theorem 1.1. If $r(z) \in R_{m,n}$ has a zero of order μ at z_0 with $|z_0| > k, k \leq 1$, and the remaining $m - \mu$ zeros are in $|z| \leq k$, then for $z \in U$

$$\begin{aligned} \max_{z \in U} |r'(z)| \geq \frac{1}{2}\{ \left(\frac{1 - |z_0|}{1 + |z_0|} \right)^{\mu} \left(|B'(z)| + \frac{2(m - \mu)}{1 + k} - n \right) \\ - \frac{2\mu}{1 + |z_0|} \} \max_{z \in U} |r(z)|. \end{aligned} \tag{1.8}$$

For $\mu = 0$ in Theorem 1.1, we have the following generalization of the inequality (1.4).

Corollary 1.1. If $r(z) \in R_{m,n}$ has all its zeros in $|z| \leq k \leq 1$, then for $z \in U$

$$\max_{z \in U} |r'(z)| \geq \frac{1}{2} \left\{ |B'(z)| + \frac{2m}{1 + k} - n \right\} \max_{z \in U} |r(z)|. \tag{1.9}$$

Furthermore, if we take $k = 1$ and $m = n$ in inequality (1.8), then we have the following result.

Corollary 1.2. If $r(z) \in R_{n,n}$ has a zero of order μ at z_0 with $|z_0| > 1$, and the remaining $n - \mu$ zeros are in $U \cup U_-$, then for $z \in U$

$$\max_{z \in U} |r'(z)| \geq \frac{1}{2} \left\{ \left(\frac{1 - |z_0|}{1 + |z_0|} \right)^{\mu} \left(|B'(z)| - \mu \right) - \frac{2\mu}{1 + |z_0|} \right\} \max_{z \in U} |r(z)|.$$

Remark 1.1. If we consider $p(z)$ as a polynomial of degree m, then for rational function $r(z) = \dfrac{p(z)}{(z-\alpha)^n}$, we have

$$r'(z) = \left(\frac{p(z)}{(z-\alpha)^n}\right)' = -\left[\frac{(n-m)p(z) + D_\alpha p(z)}{(z-\alpha)^{n+1}}\right].$$

Also for $B(z) = \dfrac{w^*(z)}{w(z)}$, we have $B'(z) = \dfrac{n(|\alpha|^2-1)}{(z-\alpha)^2}\left(\dfrac{1-\overline{\alpha}z}{z-\alpha}\right)^{n-1}$, hence for $z \in U$, $|B'(z)| = \dfrac{n(|\alpha|^2-1)}{|z-\alpha|^2}$. Now by taking $m = n$ and $d_j = \alpha$; $j = 1, 2, ..., n$ in Theorem 1.1 for $z \in U$, we get

$$\max_{z\in U} \frac{|D_\alpha p(z)|}{|z-\alpha|^{n+1}} \geq \frac{1}{2}\{\left(\frac{1-|z_0|}{1+|z_0|}\right)^\mu \left(\frac{n(|\alpha|^2-1)}{|z-\alpha|^2} - \mu\right) - \frac{2\mu}{1+|z_0|}\} \max_{z\in U} \frac{|p(z)|}{|z-\alpha|^n},$$

that is

$$\max_{z\in U} |D_\alpha p(z)| \geq \frac{1}{2}\{\left(\frac{1-|z_0|}{1+|z_0|}\right)^\mu \left(\frac{n(|\alpha|^2-1)}{|z-\alpha|} - \mu|z-\alpha|\right) - \frac{2\mu|z-\alpha|}{1+|z_0|}\} \max_{z\in U} |p(z)|,$$

which implies

$$\max_{z\in U} |D_\alpha p(z)| \geq \frac{1}{2}\{\left(\frac{1-|z_0|}{1+|z_0|}\right)^\mu \left(\frac{n(|\alpha|^2-1)}{1+|\alpha|} - \mu(1+|\alpha|)\right) - \frac{2\mu}{1+|z_0|}(1+|\alpha|)\} \max_{z\in U} |p(z)|.$$

Therefore, we obtain the following result on the polar derivatives of a polynomial which is an improvement and generalization of the inequality (1.3).

Corollary 1.3. If $p(z) \in P_n$ has a zero of order μ at z_0 with $|z_0| > 1$, and the remaining $n - \mu$ zeros are in $U \cup U_-$, then for every real or complex number α with $|\alpha| \geq 1$ and $z \in U$

$$\max_{z\in U} |D_\alpha p(z)| \geq \frac{1}{2}\{n\left(\frac{1-|z_0|}{1+|z_0|}\right)^\mu (|\alpha|-1) - \left[\mu\left(\frac{1-|z_0|}{1+|z_0|}\right)^\mu + \frac{2\mu}{1+|z_0|}\right](|\alpha|+1)\} \max_{z\in U} |p(z)|. \tag{1.10}$$

Dividing two sides of inequality (1.10) by $|\alpha|$ and letting $|\alpha| \longrightarrow \infty$, we have the following extension of a result which is proved by Turán[10].

Corollary 1.4. If $p(z) \in P_n$ has a zero of order μ at z_0 with $|z_0| > 1$, and the remaining $n-\mu$ zeros are in $U \cup U_-$, then for $z \in U$

$$\max_{z\in U}|p'(z)| \geq \frac{1}{2}\left\{(n-\mu)\left(\frac{1-|z_0|}{1+|z_0|}\right)^{\mu} - \frac{2\mu}{1+|z_0|}\right\}\max_{z\in U}|p(z)|.$$

Next, we obtain the following generalization of inequality (1.6) as follows:

Theorem 1.2. Let $r(z), s(z) \in R_{m,n}$ and assume $s(z)$ has all its zeros in $|z| \leq k \leq 1$. If $r(z)$ and $s(z)$ have zeros of order t at origin and for $z \in U$

$$|r(z)| \leq |s(z)|,$$

then for every real or complex number ρ with $|\rho| \leq \dfrac{1}{2}$

$$\left|zr'(z) + \rho\left(|B'(z)| + \frac{k(2t-n)+(2m-n)}{1+k}\right)r(z)\right|$$

$$\leq \left|zs'(z) + \rho\left(|B'(z)| + \frac{k(2t-n)+(2m-n)}{1+k}\right)s(z)\right|. \tag{1.11}$$

If we take $t=0$, $k=1$ and $s(z) = B(z)\max_{z\in U}|r(z)|$ in inequality (1.11), then we have the following generalization of inequality (1.7).

Corollary 1.5. If $r(z) \in R_{m,n}$, then for every real or complex ρ with $|\rho| \leq \dfrac{1}{2}$ and for $z \in U$

$$\left|zr'(z) + \rho\left\{|B'(z)| - (n-m)r(z)\right\}\right| \leq$$

$$\{|1+\rho||B'(z)| + (n-m)|\rho|\}\max_{z\in U}|r(z)|.$$

Finally, by involving the coefficients c_0 and c_{m-t} of $p(z)$, we give a refinement of Corollary 1.1 by proving the following theorem.

Theorem 1.3. If $r(z) \in R_{m,n}$ has all its zeros in $U \cup U_-$ with t-fold zeros at the origin then for $z \in U$

$$Re\left\{\frac{zr'(z)}{r(z)}\right\} \geq \frac{1}{2}\left\{|B'(z)| - (n-m-t) + \frac{|c_{m-t}| - |c_0|}{|c_{m-t}| + |c_0|}\right\}.$$

We can immediately get from Theorem 1.3 the following result.

Corollary 1.6. If $r(z) \in R_{m,n}$ has all its zeros in $U \cup U_-$ with t-fold zeros at the origin, then for $z \in U$

$$|r'(z)| \geq \frac{1}{2}\left\{|B'(z)| - (n-m-t) + \frac{|c_{m-t}| - |c_0|}{|c_{m-t}| + |c_0|}\right\}|r(z)|. \tag{1.12}$$

Since all the zeros of $r(z)$ and therefore the zeros of $p(z) := \sum_{j=0}^{m-t} c_j z^j$ are in $U \cup U_-$, therefore $|c_{m-t}| \geq |c_0|$. Hence inequality (1.12) is an improvement of Corollary 1.1.

If we assume that $r(z)$ has a pole of order n at $z = \alpha$, $|\alpha| \geq 1$, then $r'(z) = -\dfrac{D_\alpha p(z)}{(z-\alpha)^{n+1}}$, where $D_\alpha p(z)$ is the polar derivative of $p(z)$.

Also for $z \in U$, $|B'(z)| = \dfrac{n(|\alpha|^2 - 1)}{|z-\alpha|^2}$.

Now by taking $m = n$ and $d_j = \alpha$; $j = 1, 2, ..., n$ in inequality (1.12) for $z \in U$, we get

$$|D_\alpha p(z)| \geq \frac{1}{2}\left\{\frac{n(|\alpha|^2-1)}{1+|\alpha|} + \left(t + \frac{|c_{n-t}| - |c_0|}{|c_{n-t}| + |c_0|}\right)(|\alpha| - 1)\right\}|p(z)|.$$

Therefore, we obtain the following result on $D_\alpha p(z)$, which is an improvement and generalization of the inequality (1.3) in particular case.

Corollary 1.7. If $p(z) \in P_n$ has all its zeros in $U \cup U_-$, with t-fold zeros at the origin, then for every real or complex number α with $|\alpha| \geq 1$ and for $z \in U$

$$|D_\alpha p(z)| \geq \frac{|\alpha| - 1}{2}\left\{n + t + \frac{|c_{n-t}| - |c_0|}{|c_{n-t}| + |c_0|}\right\}|p(z)|. \tag{1.13}$$

Dividing two sides of inequality (1.13) by $|\alpha|$ and letting $|\alpha| \longrightarrow \infty$, we get the following generalization of the result due to Turán [10].

Corollary 1.8. If $p(z) \in P_n$ has all its zeros in $U \cup U_-$, with t-fold zeros at the origin, then for $z \in U$

$$|p'(z)| \geq \left\{\frac{n+t}{2} + \frac{1}{2}\frac{|c_{n-t}| - |c_0|}{|c_{n-t}| + |c_0|}\right\}|p(z)|.$$

2 Lemmas

For the proofs of these theorems, we need the following lemmas.

Lemma 2.1. If $z \in U$, then
(i) $\dfrac{zB'(z)}{B(z)} = |B'(z)|$.
(ii) $Re\left\{\dfrac{zw'(z)}{w(z)}\right\} = \dfrac{n - |B'(z)|}{2}$.

Proof.
(i). It is proved by Li [7].
(ii). Since $B(z) = \dfrac{w^*(z)}{w(z)}$, then $\dfrac{zB'(z)}{B(z)} = \dfrac{z(w^*(z))'}{w^*(z)} - \dfrac{zw'(z)}{w(z)}$.
Hence by (i) for $z \in U$

$$|B'(z)| = \frac{z(w^*(z))'}{w^*(z)} - \frac{zw'(z)}{w(z)}$$

which gives

$$Re\{\frac{z(w^*(z))'}{w^*(z)}\} - Re\{\frac{zw'(z)}{w(z)}\} = |B'(z)|. \tag{2.1}$$

For $w^*(z) = z^n \overline{w(\frac{1}{\bar{z}})}$, we have

$$z(w^*(z))' = nz^n\overline{w(\frac{1}{\bar{z}})} - z^{n-1}\overline{w'(\frac{1}{\bar{z}})},$$

and one can easily verify that for $z \in U$

$$\frac{z(w^*(z))'}{w^*(z)} = n - \overline{(\frac{zw'(z)}{w(z)})},$$

therefore

$$Re\{\frac{z(w^*(z))'}{w^*(z)}\} + Re\{\frac{zw'(z)}{w(z)}\} = n. \tag{2.2}$$

Using (2.1) in (2.2), we get for $z \in U$

$$Re\{\frac{zw'(z)}{w(z)}\} = \frac{n - |B'(z)|}{2},$$

which is the required result.

Lemma 2.2. Let $r(z) \in R_{m,n}$ has all its zeros in $|z| \leq k \leq 1$, with t-fold zeros at the origin and $m \leq n$, then for $z \in U$

$$Re\left\{\frac{zr'(z)}{r(z)}\right\} \geq \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n) + (2m-n)}{1+k}\right\}.$$

Proof. By the hypothesis of Lemma 2.2

$$r(z)=\frac{z^tp(z)}{w(z)}=\frac{z^t\prod_{i=1}^{m-t}(z-b_i)}{\prod_{j=1}^{n}(z-d_j)},$$

where b_i, $|b_i|\leq k\leq 1$, $i=1,...m-t$, are the zeros of $r(z)$. Hence

$$\frac{zr'(z)}{r(z)}=t+\frac{zp'(z)}{p(z)}-\frac{zw'(z)}{w(z)}=t+(\sum_{i=1}^{m-t}\frac{z}{z-b_i})-\frac{zw'(z)}{w(z)}.$$

Now by (ii) of Lemma 2.1, for $z\in U$

$$Re\left\{\frac{zr'(z)}{r(z)}\right\}=t+Re(\sum_{i=1}^{m-t}\frac{z}{z-b_i})-\frac{n-|B'(z)|}{2}\geq$$

$$t+\frac{m-t}{1+k}-\frac{n-|B'(z)|}{2}=\frac{1}{2}\left\{|B'(z)|+\frac{k(2t-n)+(2m-n)}{1+k}\right\},$$

which proves lemma 2.2 completely.
We need the following lemmas due to Li [6] and Osserman [9] respectively.

Lemma 2.3. Let A and B be any two complex numbers.Then
(i) If $|A|\geq|B|$ and $B\neq 0$, then $A\neq\delta B$ for all complex numbers δ satisfying $|\delta|<1$.
(ii) Conversely, if $A\neq\delta B$ for all complex numbers δ satisfying $|\delta|<1$, then $|A|\geq|B|$.

Lemma 2.4. Let $f:D\to D$ be holomorphic. Assume that $f(0)=0$. Further assume that there is $b\in\partial D$, so that f extends continuously to b, $|f(b)|=1$ and $f'(b)$ exists, then

$$|f'(b)|\geq\frac{2}{1+|f'(0)|}.$$

3 Proof of theorems

Proof of Theorem 1.1. Let $r(z)=(z-z_0)^\mu s(z)\in R_{m,n}$, where $s(z)\in R_{m-\mu,n}$ having all its zeros in $|z|\leq k\leq 1$. Then

$$r'(z)=(z-z_0)^\mu s'(z)+\mu(z-z_0)^{\mu-1}s(z)$$

or

$$|r'(z)|=|(z-z_0)^\mu s'(z)+\mu(z-z_0)^{\mu-1}s(z)|$$

$$\geq |(z-z_0)^{\mu}s'(z)| - \mu|(z-z_0)^{\mu-1}s(z)|,$$

which implies

$$\max_{z\in U}|r'(z)| \geq \max_{z\in U}|(z-z_0)^{\mu}s'(z)| - \mu\max_{z\in U}|(z-z_0)^{\mu-1}s(z)|.$$

or

$$\max_{z\in U}|r'(z)| \geq |1-|z_0||^{\mu}\max_{z\in U}|s'(z)| - \mu(1+|z_0|)^{\mu-1}\max_{z\in U}|s(z)|. \tag{3.1}$$

By lemma 2.2, for $z \in U$

$$\max_{z\in U}|s'(z)| \geq \frac{1}{2}\left\{|B'(z)| + \frac{2(m-\mu)}{1+k} - n\right\}\max_{z\in U}|s(z)|.$$

By applying this inequality in (3.1), we get

$$\max_{z\in U}|r'(z)| \geq \frac{1}{2}\{|1-|z_0||^{\mu}\left(|B'(z)| + \frac{2(m-\mu)}{1+k} - n\right) \tag{3.2}$$

$$-2\mu(|1+|z_0|)^{\mu-1}\}\max_{z\in U}|s(z)|.$$

For $z \in U$, we obtain

$$|s(z)| = \frac{1}{|z-z_0|^{\mu}}|r(z)| \geq \frac{1}{(|1+|z_0|)^{\mu}}|r(z)|$$

or

$$\max_{z\in U}|s(z)| \geq \frac{1}{(|1+|z_0|)^{\mu}}\max_{z\in U}|r(z)|. \tag{3.3}$$

Using (3.3) in (3.2), we get (1.8). This completes the proof of Theorem 1.1.

Proof of Theorem 1.2. First supposing that $s(z) \neq 0$ for $z \in U$, then for every complex number α with $|\alpha| < 1$, it follows by Rouche's Theorem that $\alpha r(z) + s(z)$ has all zeros in $|z| \leq k < 1$ with t-fold zeros in origin. Now $\alpha r(z) + s(z) \neq 0$ in $U \cup U_+$, hence by Lemma 2.2 for $z \in U$, we get

$$\left|\frac{z[\alpha r(z)+s(z)]'}{\alpha r(z)+s(z)}\right| \geq \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n)+(2m-n)}{1+k}\right\},$$

or

$$|z(\alpha r'(z)+s'(z))|$$
$$\geq \frac{1}{2}\left\{|B'(z)| + \frac{k(2t-n)+(2m-n)}{1+k}\right\}|\alpha r(z)+s(z)|.$$

Since $|B'(z)| \neq 0$ (see formula 14 in [7]), it follows by using (i) of Lemma 2.3 for every real or complex β with $|\beta| < 1$,

$$z\{\alpha r'(z) + s'(z)\} + \frac{\beta}{2}\left\{|B'(z)| + \frac{k(2t-n)+(2m-n)}{1+k}\right\}\{\alpha r(z) + s(z)\} \neq 0$$

in $U \cup U_+$. This implies that

$$\alpha\left\{zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{k(2t-n)+(2m-n)}{1+k}\right) r(z)\right\}$$
$$\neq -\left\{zs'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{k(2t-n)+(2m-n)}{1+k}\right) s(z)\right\}.$$

Now using (ii) of Lemma 2.3, we get for α with $|\alpha| < 1$ and $z \in U$,

$$\left|zs'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{k(2t-n)+(2m-n)}{1+k}\right) s(z)\right|$$
$$\geq \left|zr'(z) + \frac{\beta}{2}\left(|B'(z)| + \frac{k(2t-n)+(2m-n)}{1+k}\right) r(z)\right|.$$

Taking $\rho := \frac{\beta}{2}$ gives us the desired inequality when $|\rho| < \frac{1}{2}$.
Finally, by continuity, the same must be hold for those zeros of $s(z)$ lie on U and for $|\rho| \leq \frac{1}{2}$. This completes the proof of Theorem 1.2.

Proof of Theorem 1.3. Let

$$r(z) = \frac{z^t p(z)}{w(z)} ,$$

where $p(z) = c_{m-t}\prod_{i=1}^{m-t}(z-b_i)$, $b_i \in U_-$, $i = 1, 2, ..., m-t$.
Therefore, we have

$$Re\left\{\frac{zr'(z)}{r(z)}\right\} = t + Re\left\{\frac{zp'(z)}{p(z)}\right\} - Re\left\{\frac{zw'(z)}{w(z)}\right\},$$

hence by (ii) of Lemma 2.1

$$Re\left\{\frac{zr'(z)}{r(z)}\right\} = t + Re\left\{\frac{zp'(z)}{p(z)}\right\} - \frac{n - |B'(z)|}{2}. \tag{3.4}$$

Now we calculate $Re\left\{\frac{zp'(z)}{p(z)}\right\}$.
Since $p(z)$ is a polynomial of degree $m-t$, which has all its zeros in U_-, therefore

the polynomial $p^*(z) = z^{m-t}\ \overline{p(\frac{1}{\overline{z}})} \neq 0$ in U_- .
Hence

$$H(z) = \frac{zp(z)}{p^*(z)} = z\ \frac{c_{m-t}}{\overline{c}_{m-t}} \prod_{i=1}^{m-t} \left(\frac{z - b_i}{1 - \overline{b_i} z} \right) \tag{3.5}$$

is analytic function in $U \cup U_-$ with $H(0) = 0$ and $|H(z)| = 1$ for $z \in U$.
Applying Lemma 2.4 to $H(z)$, we conclude for $z \in U$

$$|H'(z)| \geq \frac{2}{1 + |H'(0)|}\ . \tag{3.6}$$

Also,

$$z\frac{H'(z)}{H(z)} = 1 + \frac{zp'(z)}{p(z)} - \frac{zp^{*\prime}(z)}{p^*(z)}\ , \tag{3.7}$$

and for $z \in U$

$$p^{*\prime}(z) = (m-t)z^{m-t-1}\ \overline{p(\frac{1}{\overline{z}})} - z^{m-t-2}\ \overline{p'(\frac{1}{\overline{z}})}\ .$$

Therefore for $z \in U$

$$\frac{zp^{*\prime}(z)}{p^*(z)} = (m-t) - \overline{\left(\frac{zp'(z)}{p(z)} \right)}. \tag{3.8}$$

From (3.7) and (3.8), we get for $z \in U$

$$z\frac{H'(z)}{H(z)} = -(m-t-1) + 2Re\left\{ \frac{zp'(z)}{p(z)} \right\}.$$

Also,

$$z\frac{H'(z)}{H(z)} = \left| z\frac{H'(z)}{H(z)} \right| = |H'(z)|,$$

therefore

$$|H'(z)| = -(m-t-1) + 2Re\left\{ \frac{zp'(z)}{p(z)} \right\}. \tag{3.9}$$

Using (3.5), we obtain for $z \in U$

$$|H'(0)| = \prod_{i=1}^{m-t} |b_i| = \frac{|c_0|}{|c_{m-t}|}. \tag{3.10}$$

Since $p(z) \neq 0$ for $z \in U$, hence by (3.9), (3.10) and (3.6), we get

$$-(m-t-1) + 2Re\left\{ \frac{zp'(z)}{p(z)} \right\} \geq \frac{2|c_{m-t}|}{|c_0| + |c_{m-t}|},$$

or
$$Re\left\{\frac{zp'(z)}{p(z)}\right\} \geq \frac{m-t-1}{2} + \frac{|c_{m-t}|}{|c_0|+|c_{m-t}|}\ .$$
Using this inequality and (3.4), we get for $z \in U$
$$\begin{aligned} Re\left\{\frac{zr'(z)}{r(z)}\right\} &\geq t + \frac{m-t-1}{2} + \frac{|c_{m-t}|}{|c_0|+|c_{m-t}|} - \frac{n-|B'(z)|}{2} \\ &= \frac{1}{2}\left\{|B'(z)| - (n-m-t) + \frac{|c_{m-t}|-|c_0|}{|c_{m-t}|+|c_0|}\right\}, \end{aligned}$$
which is the required result.

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