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9-2020

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Huai-Yi Xie
Institute of Nuclear Energy Research

P. T. Leung
Portland State University, hopl@pdx.edu

Journal of Physics Communications

Electromagnetic reciprocity in the presence of topological insulators

Huai-Yi Xie[1] ￼ and P T Leung[2] ￼

1   Division of Physics, Institute of Nuclear Energy Research, Taoyuan County 32546, Taiwan
2   Department of Physics, Portland State University, P O Box 751, Portland, Oregon 97207, United States of America

E-mail: damoxie@iner.gov.tw

Abstract

Electromagnetic reciprocity is studied in the presence of topological insulators (TI) with application of axion electrodynamics for harmonic electromagnetic fields. The corresponding generalized Lorentz and Feld-Tai type lemmas are derived in terms of the axion coupling parameter, and their correlation to the conditional symmetry in source-observer coordinates for the various Green dyadics is established subjected to different types of boundary conditions. Possible application of the results to the probing of the topological magneto-electric effects from TI is discussed.

Introduction

Reciprocity in wave propagation refers to the symmetry in the interchange of the source and the observer, and is an important concept in both classical and quantum physics concerning the transmission of acoustic, electromagnetic, and matter waves. While such symmetry is trivial when propagation is considered in infinite vacuum space, the situation can be rather nontrivial and intriguing in the presence of material medium or an external potential in the propagation of matter waves [1, 2]. In particular, the possible breakdown of this symmetry in both classical electromagnetic waves [3] and quantum matter waves [4] have recently received much attention among researchers in the field. Such breakdown can lead to applications in device design such as optical isolators and one-way quantum tunneling processes, for example.

Conventionally, optical reciprocity has been studied exclusively with the formulation of Maxwell's electrodynamics [1–3, 5], including approaches based on microscopic modeling of general dielectric media as collections of point dipoles [6]. Nevertheless, the study of the problem has also been extended to certain generalizations of Maxwell's equations such as the case with the existence of magnetic monopoles [7]. Moreover, a more significant extension of Maxwell's theory to include the axion fields has recently been realized with the discovery of the topological insulators (TI) [8, 9] with exciting development in the recent observation of the axionic charge-density waves in the Weyl semimetal [10]. It is of interest to recall that the illusive axion which was first introduced decades ago in a nonmetric theory of gravitation [11], and in an attempt to account for the 'strong CP problem' [12] turned out to lead to modified electrodynamics which can be realized in a wide class of topological materials. Thus we are motivated in the present work to extend the conventional study of reciprocity in wave propagation to situations with the presence of TI, via the application of axion electrodynamics (AED) to this system.

As is well-known, the mathematical formulation of reciprocity symmetry in electromagnetism can be achieved in the following three most effective ways: (1) the Lorentz and Feld-Tai lemmas; (2) the symmetry in the Green dyadic; and (3) the symmetry in the scattering matrix. While there have been some studies of (3) in recent literature which among to the asymmetry in left/right incidence in the case of 1D propagation through a topological insulator with the presence of certain surface waves [13], here in our present study we shall focus on the formulation based on (1) and (2) and the equivalence between them. Our goal is to provide rigorous mathematical derivation of the extended versions of each to axion electrodynamics as applied to TI, and to establish the correlation between them subjected to various general class of boundary conditions. Such correlation was established previously in the literature for conventional Maxwell electrodynamics in the presence of various linear media including anisotropic and nonlocal dielectric response [14].

Theoretical formulation: general results

We begin by recalling the following field equations in AED which was first shown to be applicable to TI as a low-frequency effective theory and can be expressed in Gaussian units as follows [15]:

$$\nabla \cdot \vec{D} = 4\pi\rho - 4\pi\kappa\nabla\theta \cdot \vec{B}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{1}{c}\frac{\partial\vec{B}}{\partial t} = 0$$

$$\nabla \times \vec{H} - \frac{1}{c}\frac{\partial\vec{D}}{\partial t} = \frac{4\pi}{c}\vec{J} + 4\pi\kappa\nabla\theta \times \vec{E}, \tag{1}$$

with notations as in our recent work [16] where $\kappa = \frac{\alpha}{4\pi^2}$ and α is the fine structure constant, θ is the axion coupling taken to be time-independent: $\theta(\vec{r}, t) = \theta(\vec{r})$. The corresponding equations in the frequency domain then take the following form [17]:

$$\nabla \cdot \vec{D} = 4\pi\rho - 4\pi\kappa\nabla\theta \cdot \vec{B}$$

$$\nabla \cdot \vec{B} = 0$$

$$\nabla \times \vec{E} + \frac{i\omega}{c}\vec{B} = 0$$

$$\nabla \times \vec{H} - \frac{i\omega}{c}\vec{D} = \frac{4\pi}{c}\vec{J} + 4\pi\kappa\nabla\theta \times \vec{E}. \tag{2}$$

Note that all the field quantities and material parameters in (2) are functions of frequency with the following linear constitutive relations which are linear and isotropic, but can be inhomogeneous: $\vec{D} = \varepsilon(\vec{r})\vec{E}$ and $\vec{B} = \mu(\vec{r})\vec{H}$.

(I) <u>Generalized Lorentz and Feld-Tai lemmas</u>

Consider two sources $\vec{J}_1$ and $\vec{J}_2$ at two different locations in the same medium with the axion field; and the corresponding electromagnetic fields associated with them as given by the modified Ampere's law in (2):

$$\nabla \times \vec{H}_1 - \frac{i\omega}{c}\vec{D}_1 = \frac{4\pi}{c}\vec{J}_1 + 4\pi\kappa\nabla\theta \times \vec{E}_1, \tag{3}$$

$$\nabla \times \vec{H}_2 - \frac{i\omega}{c}\vec{D}_2 = \frac{4\pi}{c}\vec{J}_2 + 4\pi\kappa\nabla\theta \times \vec{E}_2, \tag{4}$$

Taking the inner products of equation (3) with $\vec{E}_2$ and equation (4) with $\vec{E}_1$, one obtains with the use of the constitutive relation and a triple product identity the following results:

$$\begin{cases} (\nabla \times \vec{H}_1) \cdot \vec{E}_2 - \frac{i\omega\varepsilon}{c}\vec{E}_1 \cdot \vec{E}_2 = \frac{4\pi}{c}\vec{J}_1 \cdot \vec{E}_2 - 4\pi\kappa(\nabla\theta \times \vec{E}_2) \cdot \vec{E}_1 \\ (\nabla \times \vec{H}_2) \cdot \vec{E}_1 - \frac{i\omega\varepsilon}{c}\vec{E}_2 \cdot \vec{E}_1 = \frac{4\pi}{c}\vec{J}_2 \cdot \vec{E}_1 + 4\pi\kappa(\nabla\theta \times \vec{E}_2) \cdot \vec{E}_1. \end{cases} \tag{5}$$

Subtracting the two equations in (5) from one another and use the result for $\nabla \cdot (\vec{a} \times \vec{b})$, one obtains:

$$\nabla \cdot (\vec{H}_1 \times \vec{E}_2) + \vec{H}_1 \cdot (\nabla \times \vec{E}_2) - \nabla \cdot (\vec{H}_2 \times \vec{E}_1) - \vec{H}_2 \cdot (\nabla \times \vec{E}_1)$$

$$= \frac{4\pi}{c}(\vec{J}_1 \cdot \vec{E}_2 - \vec{J}_2 \cdot \vec{E}_1) - 8\pi\kappa(\nabla\theta \times \vec{E}_2) \cdot \vec{E}_1. \tag{6}$$

Using Faraday's law with $\vec{B} = \mu\vec{H}$ in equation (2), one sees that the second and fourth terms on the LHS of (6) cancel, leading to the following generalized Lorentz lemma for AED as applied to TI:

$$\int_S \hat{n} \cdot (\vec{H}_1 \times \vec{E}_2 - \vec{H}_2 \times \vec{E}_1)\,da$$

$$= \frac{4\pi}{c}\int_V (\vec{J}_1 \cdot \vec{E}_2 - \vec{J}_2 \cdot \vec{E}_1)\,d^3\vec{r} - 8\pi\kappa\int_V \nabla\theta \times \vec{E}_2 \cdot \vec{E}_1 d^3\vec{r}. \tag{7}$$

Note that in going from (6) to (7), we have integrated the result over a volume V with boundary S, and have applied the divergence theorem to the LHS. It is obvious equation (7) reduces to the well-known Lorentz lemma for $\nabla\theta = 0$, reconfirming the well-known fact that an axion field which is constant and uniform will lead to no observable physical effect since the additional $(\theta\vec{E} \cdot \vec{B})$ term in the AED Lagrangian will simply reduce to a total differential in this case [15]. Hence for a homogeneous TI with a finite boundary, the modification to the Lorentz

lemma will emerge exclusively from the discontinuity of the axion coupling across the boundary on the surface of the TI, as in those studied intensively by researchers within the context of the so-called $\theta-$ electrodynamics [18, 19].

Next we derive the generalized Feld-Tai lemma for AED. To do this we consider the scalar products of (3) and (4) with the respective magnetic $\vec{H}$ fields instead, using both the electric and magnetic constitutive relations. By going through similar steps as above, we obtain the following result:

$$
\nabla \cdot (\vec{H}_1 \times \vec{H}_2) - \frac{i\omega}{c}\varepsilon\mu^{-1}(\vec{E}_1 \cdot \vec{B}_2 - \vec{E}_2 \cdot \vec{B}_1)
$$
$$
= \frac{4\pi}{c}(\vec{J}_1 \cdot \vec{H}_2 - \vec{J}_2 \cdot \vec{H}_1) + 4\pi\kappa\nabla\theta \cdot [\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1]. \tag{8}
$$

However, using Faraday's law will not lead to cancelation of the second term on the LHS in this case. Instead, with the application of various vector identities, it can be shown that equation (8) can be re-expressed in terms of a total differential on the LHS to take the following form:

$$
\nabla \cdot \left(\vec{H}_1 \times \vec{H}_2 - \frac{1}{c}\varepsilon\mu^{-1}\vec{E}_1 \times \vec{E}_2 \right)
$$
$$
= \frac{4\pi}{c}(\vec{J}_1 \cdot \vec{H}_2 - \vec{J}_2 \cdot \vec{H}_1) - \nabla(\varepsilon\mu^{-1}) \cdot (\vec{E}_1 \times \vec{E}_2) + 4\pi\kappa\nabla\theta \cdot [\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1]. \tag{9}
$$

Again, integrating over a volume V with boundary S, and with the application of the divergence theorem to the LHS, we obtain the following generalized Feld-Tai lemma for AED as applied to TI:

$$
\int_S \hat{n} \cdot \left(\vec{H}_1 \times \vec{H}_2 - \frac{1}{c}\varepsilon\mu^{-1}\vec{E}_1 \times \vec{E}_2 \right)da
$$
$$
= \frac{4\pi}{c}\int_V (\vec{J}_1 \cdot \vec{H}_2 - \vec{J}_2 \cdot \vec{H}_1)d^3\vec{r} - \int_V \nabla(\varepsilon\mu^{-1}) \cdot (\vec{E}_1 \times \vec{E}_2)d^3\vec{r} + 4\pi\kappa\int_V \nabla\theta \cdot [\vec{E}_1 \times \vec{H}_2 - \vec{E}_2 \times \vec{H}_1]d^3\vec{r}. \tag{10}
$$

In the limit for homogeneous media with both $\nabla(\varepsilon\mu^{-1}) = 0$ and $\nabla\theta = 0$, the result in (10) simply reduces to the conventional Feld-Tai lemma as appeared in the literature [20].

(II) <u>Green reciprocity</u>

It is well-known that one powerful way to formulate reciprocity for wave propagation is to refer to the symmetry of the corresponding Green function or Green dyadic for the wave equation [2, 20]. To achieve this for AED as applied to TI, we first introduce the electric and magnetic Green dyadic via the following definitions:

$$
\vec{E} = \frac{i\omega}{c}\int_V \overset{\leftrightarrow}{G}_e(\vec{r}, \vec{r}') \cdot \vec{J}(\vec{r}')d^3\vec{r}', \tag{11}
$$

$$
\vec{B} = \int_V \overset{\leftrightarrow}{G}_m(\vec{r}, \vec{r}') \cdot \vec{J}(\vec{r}')d^3\vec{r}'. \tag{12}
$$

Hence Faraday's law in equation (2) leads to the following simple relation between the two dyadics:

$$
\nabla \times \overset{\leftrightarrow}{G}_e(\vec{r}, \vec{r}') + \overset{\leftrightarrow}{G}_m(\vec{r}, \vec{r}') = 0. \tag{13}
$$

II (a) <u>The electric Green dyadic</u>

It is straightforward to derive from (2) the following wave equation for the electric dyadic $\overset{\leftrightarrow}{G}_e$ (see appendix A for a simple derivation) [17]:

$$
\nabla \times \mu^{-1}\nabla \times \overset{\leftrightarrow}{G}_e(\vec{r}, \vec{r}') - \frac{\omega^2}{c^2}\varepsilon\overset{\leftrightarrow}{G}_e(\vec{r}, \vec{r}') + \frac{4\pi\kappa i\omega}{c}\nabla\theta \times \overset{\leftrightarrow}{G}_e(\vec{r}, \vec{r}') = -\frac{4\pi}{c}\delta(\vec{r} - \vec{r}')\overset{\leftrightarrow}{I}. \tag{14}
$$

To study the symmetry property of $\overset{\leftrightarrow}{G}_e$, we employ the following generalized Green's theorem for dyadic [21]:

$$
\int_V \{[\nabla \times \lambda\nabla \times \overset{\leftrightarrow}{Q}]^T \cdot \overset{\leftrightarrow}{P} - [\overset{\leftrightarrow}{Q}]^T \cdot \nabla \times \lambda\nabla \times \overset{\leftrightarrow}{P}\}d^3\vec{r}
$$
$$
= \int_S \lambda\{[\hat{n} \times \overset{\leftrightarrow}{Q}]^T \cdot (\nabla \times \overset{\leftrightarrow}{P}) - [\nabla \times \overset{\leftrightarrow}{Q}]^T \cdot (\hat{n} \times \overset{\leftrightarrow}{P})\}da, \tag{15}
$$

where [ ]^T stands for transpose. With $\vec{\vec{P}} = \vec{\vec{G}}_e(\vec{r}, \vec{r}')$, $\vec{\vec{Q}} = \vec{\vec{G}}_e(\vec{r}, \vec{r}'')$ and $\lambda = \mu^{-1}$ into (15), we obtain

$$\int_V \{[\nabla \times \mu^{-1}\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_e(\vec{r}, \vec{r}') - [\vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot \nabla \times \mu^{-1}\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}')\} d^3\vec{r}$$

$$= \int_S \mu^{-1}\{[\hat{n} \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot (\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}')) - [\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot (\hat{n} \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'))\} da$$

$$= -\int_S \mu^{-1}\{[\vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot (\hat{n} \times \nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}')) - [\hat{n} \times \nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot (\vec{\vec{G}}_e(\vec{r}, \vec{r}'))\} da, \quad (16)$$

where we have used the dyadic triple product rule $[\vec{\vec{c}}]^T \cdot (\vec{a} \times \vec{b}) = -[\vec{a} \times \vec{\vec{c}}]^T \cdot \vec{b}$ to get to the final result. Hence from either the dyadic Dirichlet condition:

$$\hat{n} \times \vec{\vec{G}}_e(\vec{r}, \vec{r}')|_{\vec{r} \in S} = 0, \quad (17)$$

or the dyadic Neumann condition:

$$\hat{n} \times \nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}')|_{\vec{r} \in S} = 0, \quad (18)$$

Equation (16) leads to the following conclusion:

$$\int_V \{[\nabla \times \mu^{-1}\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_e(\vec{r}, \vec{r}') - [\vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot \nabla \times \mu^{-1}\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}')\} d^3\vec{r} = 0. \quad (19)$$

Using the wave equation in equation (14), the result in (19) finally leads to:

$$-\frac{4\pi\kappa i\omega}{c}\int_V [\nabla\theta \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_e(\vec{r}, \vec{r}') d^3\vec{r} - \frac{4\pi}{c}\vec{\vec{G}}_e(\vec{r}'', \vec{r}')$$

$$+\frac{4\pi\kappa i\omega}{c}\int_V [\vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \vec{\vec{G}}_e(\vec{r}, \vec{r}') d^3\vec{r} + \frac{4\pi}{c}[\vec{\vec{G}}_e(\vec{r}', \vec{r}'')]^T = 0. \quad (20)$$

Furthermore, with the application of the dyadic triple product rule as follows:

$$[\nabla\theta \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_e(\vec{r}, \vec{r}') = -[\vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'), \quad (21)$$

equation (20) leads to the final result showing the general possible correction to the symmetry of the electric Green dyadic due to the axion coupling term:

$$\vec{\vec{G}}_e(\vec{r}'', \vec{r}') = [\vec{\vec{G}}_e(\vec{r}', \vec{r}'')]^T + 2\kappa i\omega \int_V [\vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \vec{\vec{G}}_e(\vec{r}, \vec{r}') d^3\vec{r}. \quad (22)$$

It is possible to show explicitly the equivalence between the generalized Lorentz lemma in equation (7) and the result in (22) (see appendix B). Once again, (22) will guarantee the symmetry of $\vec{\vec{G}}_e$ when $\nabla\theta = 0$ as in the conventional Maxwell electrodynamics with linear dielectric media [14]. Note also that although the axion term in (22) seems to imply the breaking of Green reciprocity in general in AED, there are important cases when such symmetry still prevails when applied to certain TI's as illustrated in the section below.

II (b) The magnetic Green dyadic
Next we study the reciprocal symmetry of the magnetic Green dyadic in AED. To achieve this, we shall introduce *two* types of electric dyadics $\vec{\vec{G}}_{e1}$ and $\vec{\vec{G}}_{e2}$ with one satisfying the conventional Dirichlet condition:

$$\hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')|_{\vec{r} \in S} = 0, \quad (23)$$

and the other satisfying the Neumann condition as follows:

$$\hat{n} \times \nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')|_{\vec{r} \in S} = 0. \quad (24)$$

We shall show that the two magnetic dyadics, $\vec{\vec{G}}_{m1}$ and $\vec{\vec{G}}_{m2}$, corresponding to the two electric dyadics defined in (22) and (24) via equation (13), will reveal similar relation to the generalized Feld-Tai lemma in (10) as that between (7) and (22) for the electric case. In addition, these two magnetic dyadics will also exhibit reciprocal symmetry similar to that revealed from (22) for the electric dyadics.
We start with equation (15) and let $\vec{\vec{P}} = \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')$, $\vec{\vec{Q}} = \mu^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')$ and $\lambda = \mu^{-1}$, together with the boundary conditions in (23) and (24), we obtain:

$$\int_V [\nabla \times \mu^{-1}\nabla \times \mu^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}') d^3\vec{r}$$

$$-\int_V [\mu^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla \times \mu^{-1}\nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}') d^3\vec{r} = 0. \quad (25)$$

To explore the consequence of the result in (25), we employ the wave equation in (14) to expand each of the two integrals in (25) into multiple terms and obtain:

$$\int_V [\nabla \times \mu^{-1}\nabla \times \mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= \frac{\omega^2}{c^2}\int_V [\nabla \times \mu^{-1}\varepsilon\ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$- \frac{4\pi\kappa i\omega}{c}\int_V [\nabla \times \mu^{-1}\nabla\theta \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$- \frac{4\pi}{c}\int_V [\nabla \times \mu^{-1}\ddot{I}\delta(\vec{r} - \vec{r}'')]^T \cdot \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}, \tag{26}$$

and

$$\int_V [\mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla \times \mu^{-1}\ddot{\nabla} \times \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= \frac{\omega^2}{c^2}\int_V [\mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \varepsilon\ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$- \frac{4\pi\kappa i\omega}{c}\int_V [\mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$- \frac{4\pi}{c}\int_V [\mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \ddot{I}\delta(\vec{r} - \vec{r}')d^3\vec{r}. \tag{27}$$

While the $\delta-$ function integral in (27) is trivial and gives:

$$\int_V [\mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \ddot{I}\delta(\vec{r} - \vec{r}')d^3\vec{r} = \mu^{-1}(\vec{r}')[\nabla' \times \ddot{G}_{e2}(\vec{r}', \vec{r}'')]^T, \tag{28}$$

that in (26) can also be simplified by using the following dyadic-dyadic divergence theorem (see appendix C for a proof):

$$\int_V \{[\ddot{B}]^T \cdot \nabla \times \ddot{A} - [\nabla \times \ddot{B}]^T \cdot \ddot{A}\}d^3\vec{r} = \int_S [\ddot{B}]^T \cdot \hat{n} \times \ddot{A}da. \tag{29}$$

Setting $\ddot{A} = \ddot{G}_{e1}(\vec{r}, \vec{r}')$ and $\ddot{B} = \mu^{-1}\ddot{I}\delta(\vec{r} - \vec{r}'')$ leads to:

$$\int_V \{[\mu^{-1}\ddot{I}\delta(\vec{r} - \vec{r}'')]^T \cdot \nabla \times \ddot{G}_{e1}(\vec{r}, \vec{r}') - [\nabla \times \mu^{-1}\ddot{I}\delta(\vec{r} - \vec{r}'')]^T \cdot \ddot{G}_{e1}(\vec{r}, \vec{r}')\}d^3\vec{r}$$

$$= \int_S [\mu^{-1}\ddot{I}\delta(\vec{r} - \vec{r}'')]^T \cdot \hat{n} \times \ddot{G}_{e1}(\vec{r}, \vec{r}')da = 0, \tag{30}$$

where the condition in (23) has been used. Hence we obtain:

$$\int_V [\nabla \times \mu^{-1}\ddot{I}\delta(\vec{r} - \vec{r}'')]^T \cdot \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r} = \mu^{-1}(\vec{r}'')\nabla'' \times \ddot{G}_{e1}(\vec{r}'', \vec{r}'). \tag{31}$$

Substituting (26), (27) into (25); and with the $\delta-$ function integral in each be replaced by (28) and (31), respectively, we finally obtain the following result:

$$\frac{\omega^2}{c^2}\int_V [\nabla \times \mu^{-1}\varepsilon\ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$- \frac{4\pi\kappa i\omega}{c}\int_V [\nabla \times \mu^{-1}\nabla\theta \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r} - \frac{4\pi}{c}\mu^{-1}(\vec{r}'')\nabla'' \times \ddot{G}_{e1}(\vec{r}'', \vec{r}')$$

$$- \frac{\omega^2}{c^2}\int_V [\mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \varepsilon\ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$+ \frac{4\pi\kappa i\omega}{c}\int_V [\mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r} + \frac{4\pi}{c}\mu^{-1}(\vec{r}')[\nabla' \times \ddot{G}_{e2}(\vec{r}', \vec{r}'')]^T = 0. \tag{32}$$

The two integrals with the coefficient $\frac{\omega^2}{c^2}$ can be combined using the following identity:

$$\nabla \times \mu^{-1}\varepsilon\ddot{G}_{e2}(\vec{r}, \vec{r}'') = \mu^{-1}\varepsilon\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'') + \nabla(\mu^{-1}\varepsilon) \times \ddot{G}_{e2}(\vec{r}, \vec{r}''), \tag{33}$$
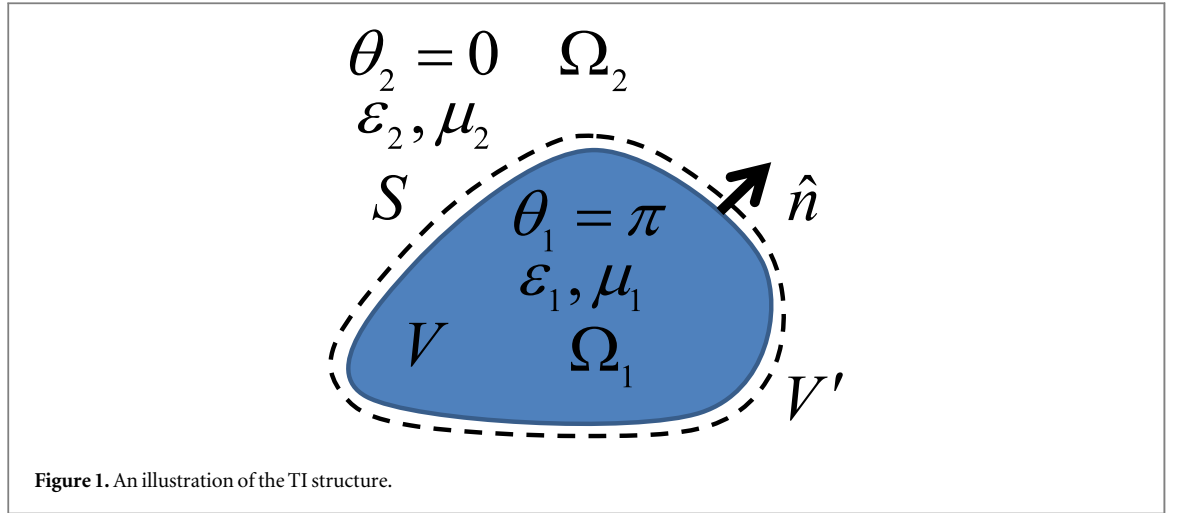
Figure 1. An illustration of the TI structure.

Equation (32) can then be re-written in the following form:

$$\frac{4\pi}{c}\mu^{-1}(\vec{r}'')\nabla'' \times \vec{\vec{G}}_{e1}(\vec{r}'', \vec{r}')$$

$$= \frac{4\pi}{c}\mu^{-1}(\vec{r}')[\nabla' \times \vec{\vec{G}}_{e2}(\vec{r}', \vec{r}'')]^T + \frac{\omega^2}{c^2}\int_V [\nabla(\mu^{-1}\varepsilon) \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$- \frac{4\pi\kappa i\omega}{c}\int_V [\nabla \times \mu^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$+ \frac{4\pi\kappa i\omega}{c}\int_V [\mu^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}. \tag{34}$$

(34) then leads to the following reciprocal relation between the two magnetic dyadics:

$$\mu^{-1}(\vec{r}'')\vec{\vec{G}}_{m1}(\vec{r}'', \vec{r}') = \mu^{-1}(\vec{r}')[\vec{\vec{G}}_{m2}(\vec{r}', \vec{r}'')]^T - \frac{\omega^2}{4\pi c}\int_V [\nabla(\mu^{-1}\varepsilon) \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$+ \kappa i\omega\int_V [\nabla \times \mu^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$- \kappa i\omega\int_V [\mu^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r} \tag{35}$$

where equation (13) has been used.

Hence we obtain the generalized reciprocal relation as in (35) for the magnetic dyadics in analogy to (22) for the electric dyadic. Again, the equivalence between (35) and the Feld-Tai lemma can be shown explicitly (appendix D). Note that for uniform and non-topological media with $\nabla\theta = 0$ and $\nabla(\mu^{-1}\varepsilon) = 0$, equation (35) reduces to the symmetry relation between the two dyadics $\vec{\vec{G}}_{m1}$ and $\vec{\vec{G}}_{m2}$.

Application to TI with constant θ and finite boundary

Here we shall apply our above results to an important class of TI with constant θ and finite boundary which has been widely studied in the literature [18, 19]. In this case, the effect of the axion coupling term will emerge only as a surface effect when θ experiences a jump crossing the boundary. To be specific, we shall consider the source and observer both located inside the TI and the outside is an ordinary non-topological dielectric medium. Other situations can be studied in a similar way by specifying the permittivity and permeability appropriately according to the positions of these source and observer. We shall demonstrate reciprocal symmetry using the results in (22) and (35) with more restricted boundary conditions applied to the respective dyadic.

(I) <u>Symmetry for the electric Green dyadic</u>

Let us consider a time-reversal symmetric TI in region Ω_1 with the outside Ω_2 an ordinary dielectric with $\theta = \pi$ and $\theta = 0$, respectively, as shown in figure 1. Let the volume of the region Ω_1 be V and the corresponding boundary be S with $\hat{n}$ an outward normal unit vector as shown. Hence $\nabla\theta$ can be expressed as follows [18, 19]:

$$\nabla\theta = (\theta_2 - \theta_1)\delta(\vec{r} - \vec{r}_s)\hat{n} = -\pi\delta(\vec{r} - \vec{r}_s)\hat{n}, \tag{36}$$

where $\vec{r}_s$ is the position vector on the boundary S. Without first imposing the boundary conditions in (17) and (18), the result in (20) will be modified to contain an additional surface integral as given by (16) to yield the following result:

$$-\frac{4\pi}{c}\ddot{G}_e(\vec{r}'', \vec{r}') + \frac{8\pi\kappa i\omega}{c}\int_V [\ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \ddot{G}_e(\vec{r}, \vec{r}')d^3\vec{r} + \frac{4\pi}{c}[\ddot{G}_e(\vec{r}', \vec{r}'')]^T$$

$$=\int_S \mu_1^{-1}\{[\hat{n} \times \ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot (\nabla \times \ddot{G}_e(\vec{r}, \vec{r}')) - [\nabla \times \ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot (\hat{n} \times \ddot{G}_e(\vec{r}, \vec{r}'))\}da. \quad (37)$$

Now consider an infinitesimal extension of the volume V to V' (see figure 1), we can use the result in (36) to evaluate the integral on the LHS of (37) as follows:

$$\int_V [\ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \ddot{G}_e(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= \lim_{V' \to V} \int_{V'} [\ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \ddot{G}_e(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= -\pi \lim_{V' \to V} \int_{V'} \delta(\vec{r} - \vec{r}_s)[\ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot \hat{n} \times \ddot{G}_e(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= -\pi \int_S [\ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot \hat{n} \times \ddot{G}_e(\vec{r}, \vec{r}')da. \quad (38)$$

Using (38), the result in (37) can be rewritten as follows:

$$\frac{4\pi}{c}\ddot{G}_e(\vec{r}'', \vec{r}') = \frac{4\pi}{c}[\ddot{G}_e(\vec{r}', \vec{r}'')]^T - \frac{8\pi^2\kappa i\omega}{c}\int_S [\ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot \hat{n} \times \ddot{G}_e(\vec{r}, \vec{r}')da$$

$$- \int_S \mu_1^{-1}\{[\hat{n} \times \ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot (\nabla \times \ddot{G}_e(\vec{r}, \vec{r}')) - [\nabla \times \ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot (\hat{n} \times \ddot{G}_e(\vec{r}, \vec{r}'))\}da$$

$$= \frac{4\pi}{c}[\ddot{G}_e(\vec{r}', \vec{r}'')]^T - \int_S \{\mu_1^{-1}[\hat{n} \times \ddot{G}_e(\vec{r}, \vec{r}'')]^T \cdot (\nabla \times \ddot{G}_e(\vec{r}, \vec{r}'))\}da$$

$$+ \int_S \left\{ \left[\mu_1^{-1}\nabla \times \ddot{G}_e(\vec{r}, \vec{r}'') - \frac{8\pi^2\kappa i\omega}{c}\ddot{G}_e(\vec{r}, \vec{r}'')\right]^T \cdot (\hat{n} \times \ddot{G}_e(\vec{r}, \vec{r}'))\right\}da, \quad (39)$$

which leads to the following symmetric electric Green dyadic with the imposition of the Dirichlet boundary condition in (17):

$$\ddot{G}_e(\vec{r}'', \vec{r}') = [\ddot{G}_e(\vec{r}', \vec{r}'')]^T. \quad (40)$$

Hence we conclude that reciprocal symmetry is pertained to this case with the imposition of the conventional Dirichlet boundary condition, which is similar to the case with electro-magneto statics as established in [18, 19]. subjected to the class-I type boundary condition defined in those references. Moreover, the conventional Neumann boundary condition (equation (18)) will not ensure such reciprocal symmetry in general. Note that the above conclusion is not restricted to our specification of $\theta = \pi$ as for a time reversal symmetric TI. It remains unaffected even for a time reversal non-symmetrical TI with $\theta = (2n + 1)\pi$ [22], illustrating once again that reciprocity can still hold in such case just as in the case with the presence a linear dissipative medium [1–3].

(II) Symmetry for the magnetic dyadic
To illustrate the symmetry property of the magnetic dyadic, we shall consider the same TI system as in figure 1, and try to determine the appropriate boundary conditions of $\ddot{G}_{e1}$ and $\ddot{G}_{e2}$. To achieve this we shall first keep the surface terms in deriving equation (25) from equation (15) without applying the boundary conditions (23) and (24) to obtain:

$$\int_V [\nabla \times \mu^{-1}\nabla \times \mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$- \int_V [\mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla \times \mu^{-1}\nabla \times \ddot{G}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= \int_S \mu^{-1}\{[\mu^{-1}\hat{n} \times \nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot (\nabla \times \ddot{G}_{e1}(\vec{r}, \vec{r}'))\}da$$

$$- \int_S \mu^{-1}\{[\nabla \times \mu^{-1}\nabla \times \ddot{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot (\hat{n} \times \ddot{G}_{e1}(\vec{r}, \vec{r}'))\}da, \quad (41)$$

then the result in (34) will now take the form:

$$-\frac{4\pi}{c}\mu_1^{-1}\nabla'' \times \vec{\vec{G}}_{e1}(\vec{r}'', \vec{r}') + \frac{4\pi}{c}\mu_1^{-1}[\nabla' \times \vec{\vec{G}}_{e2}(\vec{r}', \vec{r}'')]^T$$

$$+\frac{\omega^2}{c^2}\int_V [\nabla(\mu^{-1}\varepsilon)] \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$-\frac{4\pi\kappa i\omega}{c}\int_V [\nabla \times \mu_1^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$+\frac{4\pi\kappa i\omega}{c}\int_V [\mu_1^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$=\int_S \mu_1^{-1}\{[\mu_1^{-1}\hat{n} \times \nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot (\nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}'))\}da$$

$$-\int_S \mu_1^{-1}\{[\nabla \times \mu_1^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot (\hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}'))\}da. \quad (42)$$

In a similar way as in equation (36), we have:

$$\nabla(\mu^{-1}\varepsilon) = (\mu_2^{-1}\varepsilon_2 - \mu_1^{-1}\varepsilon_1)\delta(\vec{r} - \vec{r}_s)\hat{n}, \quad (43)$$

and hence the integral in the second row of (42) can be evaluated as:

$$\int_V [\nabla(\mu^{-1}\varepsilon) \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= \lim_{V' \to V}\int_{V'} [\nabla(\mu^{-1}\varepsilon) \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= \lim_{V' \to V}\int_{V'} [(\mu_2^{-1}\varepsilon_2 - \mu_1^{-1}\varepsilon_1)\delta(\vec{r} - \vec{r}_s)\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= (\mu_2^{-1}\varepsilon_2 - \mu_1^{-1}\varepsilon_1)\int_S [\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')da. \quad (44)$$

Next, we again use (29) with $\vec{\vec{A}} = \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')$ and $\vec{\vec{B}} = \mu_1^{-1}\delta(\vec{r} - \vec{r}_s)\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')$ to obtain the following result:

$$\int_V [\nabla \times \mu_1^{-1}\delta(\vec{r} - \vec{r}_s)\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= \int_V [\mu_1^{-1}\delta(\vec{r} - \vec{r}_s)\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$- \int_S [\mu_1^{-1}\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')da. \quad (45)$$

With this, the integral in the third row of (42) can be evaluated as:

$$\int_V [\nabla \times \mu_1^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= \lim_{V' \to V}\int_{V'} [\nabla \times \mu_1^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= \lim_{V' \to V}(-\pi)\int_{V'} [\nabla \times \mu_1^{-1}\delta(\vec{r} - \vec{r}_s)\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$= -\pi\int_V [\mu_1^{-1}\delta(\vec{r} - \vec{r}_s)\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}$$

$$+ \pi\int_S [\mu_1^{-1}\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')da$$

$$= -\pi\int_S [\mu_1^{-1}\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')da$$

$$+ \pi\int_S [\mu_1^{-1}\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')da. \quad (46)$$

The fourth row in equation (42) simplifies as before to yield:

$$
\int_V [\mu_1^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}
$$

$$
= \lim_{V' \to V} \int_{V'} [\mu_1^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla\theta \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}
$$

$$
= \lim_{V' \to V} (-\pi) \int_{V'} \delta(\vec{r} - \vec{r}_s)[\mu_1^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')d^3\vec{r}
$$

$$
= -\pi \int_S [\mu_1^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')da. \tag{47}
$$

Substituting equations (44), (46) and (47) into (42), we obtain:

$$
-\frac{4\pi}{c}\mu_1^{-1}\nabla'' \times \vec{\vec{G}}_{e1}(\vec{r}'', \vec{r}') + \mu_1^{-1}\frac{4\pi}{c}[\nabla' \times \vec{\vec{G}}_{e2}(\vec{r}', \vec{r}'')]^T
$$

$$
+\frac{\omega^2}{c^2}(\mu_2^{-1}\varepsilon_2 - \mu_1^{-1}\varepsilon_1) \int_S [\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')da
$$

$$
+\frac{4\pi^2\kappa i\omega}{c} \int_S [\mu_1^{-1}\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')da
$$

$$
-\frac{4\pi^2\kappa i\omega}{c} \int_S [\mu_1^{-1}\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')da
$$

$$
-\frac{4\pi^2\kappa i\omega}{c} \int_S [\mu_1^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')da
$$

$$
= \int_S \mu_1^{-1}\{[\mu_1^{-1}\hat{n} \times \nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot (\nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}'))\}da
$$

$$
- \int_S \mu_1^{-1}\{[\nabla \times \mu_1^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot (\hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}'))\}da, \tag{48}
$$

which, with equation (13), can be rewritten in the following form in terms of only surface integrals:

$$
\frac{4\pi}{c}\mu_1^{-1}\vec{\vec{G}}_{m1}(\vec{r}'', \vec{r}') - \frac{4\pi}{c}\mu_1^{-1}[\vec{\vec{G}}_{m2}(\vec{r}', \vec{r}'')]^T
$$

$$
= \int_S \left\{ \left[\mu_1^{-2}\hat{n} \times \nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'') - \frac{4\pi^2\kappa i\omega}{c}\mu_1^{-1}\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'') \right]^T \cdot (\nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')) \right\}da
$$

$$
+ \int_S \left\{ \left[\frac{\omega^2}{c^2}(\mu_2^{-1}\varepsilon_2 - \mu_1^{-1}\varepsilon_1) \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'') + \frac{4\pi^2\kappa i\omega}{c}\mu_1^{-1}\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'') \right. \right.
$$

$$
\left. \left. + \frac{4\pi^2\kappa i\omega}{c}\mu_1^{-1}\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'') - \mu_1^{-2}\nabla \times \nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'') \right]^T \cdot (\hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')) \right\}da. \tag{49}
$$

Now if we impose the conventional Dirichlet condition equation (23) for $\vec{\vec{G}}_{e1}$ but the following Robin (mixed) condition [23] for $\vec{\vec{G}}_{e2}$:

$$
\mu_1^{-1}\hat{n} \times \nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'') + \frac{4\pi^2\kappa i\omega}{c}\hat{n} \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'') \bigg|_{\vec{r} \in S} = 0, \tag{50}
$$

in place of the conventional Neumann condition, we obtain from (49) the following symmetric relation of magnetic dyadic Green's function for $\vec{\vec{G}}_{m1}$ and $\vec{\vec{G}}_{m2}$:

$$
\vec{\vec{G}}_{m1}(\vec{r}'', \vec{r}') = [\vec{\vec{G}}_{m2}(\vec{r}', \vec{r}'')]^T. \tag{51}
$$

Discussion and conclusion

In this work, we have extended the usual mathematical formulation for electromagnetic reciprocity to the case when topological insulators are present. With application of axion electrodynamics, we have derived generalized

Lorentz lemma, generalized Feld-Tai lemma, and the modified Green reciprocity for this situation. In particular, we have demonstrated the validity of reciprocal symmetry in the presence of a TI with constant axion coupling and finite extent subjected to the appropriate boundary conditions of the electric and magnetic dyadic Green's functions, consistent with previous literature which has considered such TI's under electro-magneto-statics condition [18, 19].

As is well-known, one challenge for studying the electromagnetic properties of TI's is the probing of the topological magneto-electric (TME) effect due to the axion coupling term. Several recent experiments have successfully observed such effects via the study of Faraday/Kerr rotation of the polarization of incident THz EM waves on such systems [24–26], and recently we have also explored possible observation of TME from red-shifted resonances of plasmonic nanoshells [16]. Our results obtained in this study may provide an alternative approach to the study of TME via the probing of possible violation of the conventional Lorentz lemma (equation (7)), in which case implication for the Green reciprocity will be that the wave equation (equation (14)) will not admit physical solutions satisfying the Dirichlet condition for the case of TI with a constant axion coupling and finite extent. It is of interest to note that recent studies of surface waves at a TI boundary interfaced with an anisotropic medium can lead to nonreciprocal left/right propagation of optical signals [13, 27]. Since one of the most significant applications of optical nonreciprocity is in the design of various optical isolators [1, 28], it will be of interest to generalize our present formulation to the anisotropic case to study how the axion parameter and various boundary conditions may be manifested in the various generalized lemmas and Green reciprocity constraints.

Acknowledgments

Appendix A. Derivation of equation (14)

Taking the curl of the third equation (Faraday's law) in equation (2), we obtain:

$$\nabla \times \mu^{-1}\nabla \times \vec{E} + \frac{i\omega}{c}\nabla \times \vec{H} = 0. \tag{A1}$$

Combining with the fourth equation in (2) leads to:

$$\nabla \times \mu^{-1}\nabla \times \vec{E} + \frac{i\omega}{c}\left(\frac{i\omega}{c}\vec{D} + \frac{4\pi}{c}\vec{J} + 4\pi\kappa\nabla\theta \times \vec{E}\right) = 0. \tag{A2}$$

Hence we have:

$$\nabla \times \mu^{-1}\nabla \times \vec{E} - \frac{\omega^2}{c^2}\varepsilon\vec{E} + \frac{4\pi\kappa i\omega}{c}\nabla\theta \times \vec{E} = -\frac{4\pi i\omega}{c^2}\vec{J}. \tag{A3}$$

The wave equation for the electric Green dyadic follows with the application of equations (11) to (A3) to yield:

$$\nabla \times \mu^{-1}\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}') - \frac{\omega^2}{c^2}\varepsilon\vec{\vec{G}}_e(\vec{r}, \vec{r}') + \frac{4\pi\kappa i\omega}{c}\nabla\theta \times \vec{\vec{G}}_e(\vec{r}, \vec{r}') = -\frac{4\pi}{c}\delta(\vec{r} - \vec{r}')\vec{\vec{I}}, \tag{A4}$$

which is the result in equation (14).

Appendix B. Equivalence between the Lorentz lemma and Green reciprocity

To prove the equivalence between equations (7) and (22), we consider the currents of two harmonic point dipoles as follows:

$$\vec{J}_1 = -i\omega p \delta(\vec{r} - \vec{r}'')\hat{e}_i$$
$$\vec{J}_2 = -i\omega p \delta(\vec{r} - \vec{r}')\hat{e}_j, \tag{B1}$$

and the corresponding electric fields as obtained from equation (11):

$$\vec{E}_1 = \frac{\omega^2 p}{c}\int_V \vec{\vec{G}}_e(\vec{r}, \vec{r}') \cdot \delta(\vec{r}' - \vec{r}'')\hat{e}_i d^3\vec{r}' = \frac{\omega^2 p}{c}\vec{\vec{G}}_e(\vec{r}, \vec{r}') \cdot \hat{e}_i \equiv \frac{\omega^2 p}{c}\vec{G}_{ei}(\vec{r}, \vec{r}'')$$

$$\vec{E}_2 = \frac{\omega^2 p}{c}\int_V \vec{\vec{G}}_e(\vec{r}, \vec{r}'') \cdot \delta(\vec{r}'' - \vec{r}')\hat{e}_j d^3\vec{r}'' = \frac{\omega^2 p}{c}\vec{\vec{G}}_e(\vec{r}, \vec{r}') \cdot \hat{e}_j \equiv \frac{\omega^2 p}{c}\vec{G}_{ej}(\vec{r}, \vec{r}'), \tag{B2}$$

with the vectors $\vec{G}_{ei}$ and $\vec{G}_{ej}$ being the column components of the electric dyadic $\vec{\vec{G}}_e$.

Next we substitute (B1) and (B2) into equation (7) and with the help of Faraday's law to obtain:

$$\frac{\omega p}{i}\int_S \frac{1}{\mu}\{(\nabla \times \vec{G}_{ei}(\vec{r}, \vec{r}'')) \cdot (\hat{n} \times \vec{G}_{ej}(\vec{r}, \vec{r}')) - (\nabla \times \vec{G}_{ej}(\vec{r}, \vec{r}')) \cdot (\hat{n} \times \vec{G}_{ei}(\vec{r}, \vec{r}''))\}da$$

$$= -\frac{\omega p}{i}\int_S \frac{1}{\mu}\{\vec{G}_{ej}(\vec{r}, \vec{r}') \cdot [\hat{n} \times (\nabla \times \vec{G}_{ei}(\vec{r}, \vec{r}''))] - \vec{G}_{ei}(\vec{r}, \vec{r}'') \cdot [\hat{n} \times (\nabla \times \vec{G}_{ej}(\vec{r}, \vec{r}'))]\}da$$

$$= -\frac{4\pi i\omega p}{c}\{[\vec{\vec{G}}_e(\vec{r}'', \vec{r}')]_{ij} - [\vec{\vec{G}}_e(\vec{r}', \vec{r}'')]_{ji}\} - \frac{8\pi\kappa\omega^2 p}{c}\int_V (\nabla\theta \times \vec{G}_{ej}(\vec{r}, \vec{r}')) \cdot \vec{G}_{ei}(\vec{r}, \vec{r}'')d^3\vec{r}, \tag{B3}$$

where the vector triple product has been used in going from the first to the second row. (B3) can then be rewritten in dyadic form as follows:

$$\frac{\omega p}{i}\int_S \frac{1}{\mu}\{[\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot (\hat{n} \times \vec{\vec{G}}_e(\vec{r}, \vec{r}')) - [\hat{n} \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot (\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'))\}da$$

$$= -\frac{\omega p}{i}\int_S \frac{1}{\mu}\{[\hat{n} \times (\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}''))]^T \cdot \vec{\vec{G}}_e(\vec{r}, \vec{r}') - [\vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot [\hat{n} \times (\nabla \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'))]\}da$$

$$= -\frac{4\pi i\omega p}{c}\{[\vec{\vec{G}}_e(\vec{r}'', \vec{r}')]_{ij}\hat{e}_i\hat{e}_j - [\vec{\vec{G}}_e(\vec{r}', \vec{r}'')]_{ji}\hat{e}_i\hat{e}_j\} - \frac{8\pi\kappa\omega^2 p}{c}\int_V [\vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot (\nabla\theta \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'))d^3\vec{r}. \tag{B4}$$

Hence from either the dyadic Dirichlet condition (equation (17)) or the dyadic Neumann condition (equation (18)), the surface integrals will vanish and equation (B4) reduces to the following form:

$$\vec{\vec{G}}_e(\vec{r}'', \vec{r}') = [\vec{\vec{G}}_e(\vec{r}', \vec{r}'')]^T + 2\omega\kappa i\int_V [\vec{\vec{G}}_e(\vec{r}, \vec{r}'')]^T \cdot (\nabla\theta \times \vec{\vec{G}}_e(\vec{r}, \vec{r}'))d^3\vec{r}, \tag{B5}$$

which is the result in equation (22).

Appendix C. Derivation of the dyadic-dyadic divergence theorem

To derive equation (29), we consider the vector-dyadic divergence theorem as quoted from [20, 21]:

$$\int_V \{[\vec{\vec{B}}]^T \cdot \nabla \times \vec{A} - [\nabla \times \vec{\vec{B}}]^T \cdot \vec{A}\}d^3\vec{r} = \int_S \hat{n} \cdot (\vec{A} \times \vec{\vec{B}})da. \tag{C1}$$

By rewriting the RHS of (C1) as

$$\hat{n} \cdot (\vec{A} \times \vec{\vec{B}}) = (\hat{n} \times \vec{A}) \cdot \vec{\vec{B}} = [\vec{\vec{B}}]^T \cdot (\hat{n} \times \vec{A}), \tag{C2}$$

Equation (C1) leads to:

$$\int_V \{[\vec{\vec{B}}]^T \cdot \nabla \times \vec{A} - [\nabla \times \vec{\vec{B}}]^T \cdot \vec{A}\}d^3\vec{r} = \int_S [\vec{\vec{B}}]^T \cdot (\hat{n} \times \vec{A})da. \tag{C3}$$

It is straightforward to extend the result in (C3) with the vector $\vec{A}$ be replaced by a higher rank tensor, leading to the result in equation (29).

Appendix D. Equivalence between the Feld-Tai lemma and Green reciprocity

To demonstrate the equivalence between equations (10) and (35), we consider the electric fields due to the same two sources in (B1), but being 'propagated' by the two dyadics as defined by the conditions in equations (23) and (24):

$$\vec{E}_1 = \frac{\omega^2 p}{c} \vec{G}_{e2i}(\vec{r}, \vec{r}'')$$

$$\vec{E}_2 = \frac{\omega^2 p}{c} \vec{G}_{e1j}(\vec{r}, \vec{r}'), \tag{D1}$$

where $\vec{G}_{e2i}$ and $\vec{G}_{e1j}$ are the column components of the corresponding dyadic functions. Substituting (B1) and (D1) into equation (10) leads to the following:

$$\frac{\omega^2 p}{c} \int_S \hat{n} \cdot \left[\frac{c}{i\omega\mu} \frac{c}{i\omega\mu} (\nabla \times \vec{G}_{e2i}(\vec{r}, \vec{r}'')) \times (\nabla \times \vec{G}_{e1j}(\vec{r}, \vec{r}')) - \frac{1}{c} \varepsilon\mu^{-1} \vec{G}_{e2i}(\vec{r}, \vec{r}'') \vec{G}_{e1j}(\vec{r}, \vec{r}') \right] da$$

$$= 4\pi p [\mu^{-1}(\vec{r}'') \nabla \times \vec{G}_{e1j}(\vec{r}'', \vec{r}') \cdot \hat{e}_i - \mu^{-1}(\vec{r}') \nabla \times \vec{G}_{e2i}(\vec{r}', \vec{r}'') \cdot \hat{e}_j]$$

$$- \frac{\omega^2 p}{c} \int_V \nabla(\varepsilon\mu^{-1}) \cdot (\vec{G}_{e2i}(\vec{r}, \vec{r}'') \times \vec{G}_{e1j}(\vec{r}, \vec{r}')) d^3\vec{r}$$

$$- \frac{4\pi\kappa\omega p}{i} \int_V \mu^{-1} \nabla\theta \cdot [\vec{G}_{e2i}(\vec{r}, \vec{r}'') \times (\nabla \times \vec{G}_{e1j}(\vec{r}, \vec{r}')) - \vec{G}_{e1j}(\vec{r}, \vec{r}') \times (\nabla \times \vec{G}_{e2i}(\vec{r}, \vec{r}''))] d^3\vec{r}. \tag{D2}$$

Again, Faraday's law has been utilized to express the magnetic in terms of the electric fields in deriving (D2). Using the vector triple product, the first row in equation (D2) becomes (excluding the $\frac{\omega^2 p}{c}$ factor):

$$\int_S \hat{n} \cdot \left[\left(\frac{c}{i\omega\mu} \right)^2 (\nabla \times \vec{G}_{e2i}(\vec{r}, \vec{r}'')) \times (\nabla \times \vec{G}_{e1j}(\vec{r}, \vec{r}')) - \frac{1}{c} \varepsilon\mu^{-1} \vec{G}_{e2i}(\vec{r}, \vec{r}'') \times \vec{G}_{e1j}(\vec{r}, \vec{r}') \right] da$$

$$= -\hat{n} \cdot \int_S \left\{ \left(\frac{c}{i\omega\mu} \right)^2 [(\nabla \times \vec{G}_{e1j}(\vec{r}, \vec{r}')) \times (\nabla \times \vec{G}_{e2i}(\vec{r}, \vec{r}''))] + \frac{1}{c} \varepsilon\mu^{-1} \vec{G}_{e2i}(\vec{r}, \vec{r}'') \vec{G}_{e1j}(\vec{r}, \vec{r}') \right\} da$$

$$= -\int_S \left(\frac{c}{i\omega\mu} \right)^2 (\nabla \times \vec{G}_{e1j}(\vec{r}, \vec{r}')) \cdot [\hat{n} \times (\nabla \times \vec{G}_{e2i}(\vec{r}, \vec{r}''))] da - \int_S \frac{1}{c} \varepsilon\mu^{-1} \vec{G}_{e2i}(\vec{r}, \vec{r}'') \cdot [\hat{n} \times \vec{G}_{e1j}(\vec{r}, \vec{r}')] da. \tag{D3}$$

Substitute equations (D3) into (D2), we obtain:

$$-\frac{\omega^2 p}{c} \int_S \left(\frac{c}{i\omega\mu} \right)^2 [\hat{n} \times \nabla \times \overset{\leftrightarrow}{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \nabla \times \overset{\leftrightarrow}{G}_{e1}(\vec{r}, \vec{r}') da$$

$$-\frac{\omega^2 p}{c} \int_S \frac{1}{c} \varepsilon\mu^{-1} [\overset{\leftrightarrow}{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot (\hat{n} \times \overset{\leftrightarrow}{G}_{e1}(\vec{r}, \vec{r}')) da$$

$$= 4\pi p \{ \mu^{-1}(\vec{r}'') \nabla'' \times \overset{\leftrightarrow}{G}_{e1}(\vec{r}'', \vec{r}') - \mu^{-1}(\vec{r}') [\nabla' \times \overset{\leftrightarrow}{G}_{e2}(\vec{r}', \vec{r}'')]^T \}$$

$$-\frac{\omega^2 p}{c} \int_V [\nabla(\varepsilon\mu^{-1}) \times \overset{\leftrightarrow}{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \overset{\leftrightarrow}{G}_{e1}(\vec{r}, \vec{r}') d^3\vec{r}$$

$$+\frac{4\pi\kappa\omega p}{i} \int_V [\nabla \times \overset{\leftrightarrow}{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot [\mu^{-1} \nabla\theta \times \overset{\leftrightarrow}{G}_{e1}(\vec{r}, \vec{r}')] d^3\vec{r}$$

$$+\frac{4\pi\kappa\omega p}{i} \int_V [\overset{\leftrightarrow}{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot [\mu^{-1} \nabla\theta \times (\nabla \times \overset{\leftrightarrow}{G}_{e1}(\vec{r}, \vec{r}'))] d^3\vec{r}, \tag{D4}$$

and upon using the boundary conditions in equations (23) and (24) leads to:

$$4\pi p \{ \mu^{-1}(\vec{r}'') [\nabla'' \times \overset{\leftrightarrow}{G}_{e1}(\vec{r}'', \vec{r}')] - \mu^{-1}(\vec{r}') [\nabla' \times \overset{\leftrightarrow}{G}_{e2}(\vec{r}', \vec{r}'')]^T \}$$

$$-\frac{\omega^2 p}{c} \int_V [\nabla(\varepsilon\mu^{-1}) \times \overset{\leftrightarrow}{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \overset{\leftrightarrow}{G}_{e1}(\vec{r}, \vec{r}') d^3\vec{r}$$

$$+\frac{4\pi\kappa\omega p}{i} \int_V [\nabla \times \overset{\leftrightarrow}{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot [\mu^{-1} \nabla\theta \times \overset{\leftrightarrow}{G}_{e1}(\vec{r}, \vec{r}')] d^3\vec{r}$$

$$+\frac{4\pi\kappa\omega p}{i} \int_V [\overset{\leftrightarrow}{G}_{e2}(\vec{r}, \vec{r}'')]^T \cdot [\mu^{-1} \nabla\theta \times (\nabla \times \overset{\leftrightarrow}{G}_{e1}(\vec{r}, \vec{r}'))] d^3\vec{r} = 0. \tag{D5}$$

Note that the last term on the LHS of (D5) can be rewritten as follows using the dyadic triple product rule:

$$\int_V [\vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot [\mu^{-1}\nabla\theta \times (\nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}'))] d^3\vec{r}$$

$$= -\int_V [\mu^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot (\nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')) \times d^3\vec{r}, \tag{D6}$$

hence the dyadic-dyadic divergence theorem in (29) leads to:

$$\int_V [\mu^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot (\nabla \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')) d^3\vec{r}$$

$$- \int_V [\nabla \times (\mu^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}''))]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}') d^3\vec{r}$$

$$= \int_S [\mu^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot [\hat{n} \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')] da = 0, \tag{D7}$$

where we use the boundary condition in equation (23). Hence equation (D5) becomes:

$$\{\mu^{-1}(\vec{r}'')\nabla'' \times \vec{\vec{G}}_{e1}(\vec{r}'', \vec{r}') - \mu^{-1}(\vec{r}')[\nabla' \times \vec{\vec{G}}_{e2}(\vec{r}',\vec{r}'')]^T\}$$

$$- \frac{\omega^2}{4\pi c}\int_V [\nabla(\varepsilon\mu^{-1}) \cdot \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}') d^3\vec{r}$$

$$- i\kappa\omega \int_V \mu^{-1}[\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot [\nabla\theta \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')] d^3\vec{r}$$

$$+ i\kappa\omega \int_V [\nabla \times (\mu^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}''))]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}') d^3\vec{r} = 0, \tag{D8}$$

which can be rewritten as (using equation (13)):

$$\mu^{-1}(\vec{r}'')\vec{\vec{G}}_{m1}(\vec{r}'', \vec{r}') = \mu^{-1}(\vec{r}')[\vec{\vec{G}}_{m2}(\vec{r}', \vec{r}'')]^T$$

$$- \frac{\omega^2}{4\pi c}\int_V [\nabla(\varepsilon\mu^{-1}) \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}') d^3\vec{r}$$

$$+ i\kappa\omega \int_V [\nabla \times (\mu^{-1}\nabla\theta \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}''))]^T \cdot \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}') d^3\vec{r}$$

$$- i\kappa\omega \int_V \mu^{-1}[\nabla \times \vec{\vec{G}}_{e2}(\vec{r}, \vec{r}'')]^T \cdot [\nabla\theta \times \vec{\vec{G}}_{e1}(\vec{r}, \vec{r}')] d^3\vec{r}. \tag{D9}$$

The result in (D9) is seen to be identical with that in equation (35) and hence the equivalence between the generalized Feld-Tai lemma and the magnetic Green reciprocity is established.

ORCID iDs

Huai-Yi Xie ⓘ https://orcid.org/0000-0001-6153-3300
P T Leung ⓘ https://orcid.org/0000-0003-4767-2237

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