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Haltiner, G.J.; Chase, E.M.

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Some further Results on Convective Currents

By G. J. HALTINER and E. M. CHASE, Lt, USN
U.S. Naval Postgraduate School

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Abstract

An earlier theory of steady state, saturated convective currents is modified to include the drag of the condensed liquid water on the rising air. The system of equations is numerically integrated for several cases in which the entrainment is allowed to assume negative values. Results appear to indicate improvement over the earlier model.

Introduction

In a recent paper (HALTINER, 1959) a theory was presented for a steady state, saturated convective current undergoing an exchange of heat and momentum with its environment by lateral diffusion, as well as dynamic entrainment of environment air. When mass continuity considerations would result in the ejection of mass (detrainment) assuming a constant cross-section area of the updraft, the detrainment was taken to be zero; and the cross-section area of the current was allowed to increase in order to satisfy the equation of continuity. This implies that when the current begins to decelerate, the entire mass slows down and spreads out laterally. While this process seems plausible, some observations of cumulus clouds indicate that detrainment does take place at certain times in such convective currents.

The present paper differs in two respects from the preceding theory. Firstly, a specific term is included in the momentum equation for the drag of the condensed water which is carried along by the updraft. Secondly, the computations include some examples in which detrainment is allowed to take place, assuming the mass lost to have zero vertical velocity.

List of Symbols

T	temperature
w	vertical velocity
q	specific humidity
p	pressure
ϱ	density of air
σ	cross-section area of current
l	specific liquid water content of air
$M =$	$\varrho w \sigma$ mass rate of flow
r	relative humidity
D	drag of the air on the liquid water
E	external forces per unit mass
h	cloud height
t	time variable
z	height variable
C_p	specific heat at constant pressure for air
c	specific heat of water
L	latent heat of condensation
R_d	gas constant for dry air
R_v	gas constant for water vapor
g	gravity force per unit mass
γ_e	lapse rate of environment
k_1	diffusion coefficient

Subscripts

e	refers to environment
v	refers to virtual
o	initial value

The Basic Equations

Most of the basic equations are similar to those of the preceding paper, hence the derivations will be omitted except where significant change has been made.

The continuity equation is unchanged and is expressible in the form

$$\frac{dN}{dz} = \frac{1}{w}\frac{dw}{dz} + \frac{1}{\varrho}\frac{d\sigma}{dz} - \frac{g}{R_d T_{ve}} - \frac{1}{T_v}\frac{dT_v}{dz} \quad (1)$$

The equation of motion may be written as

$$\frac{dw}{dt} + w\frac{dN}{dt} = E + w_e\frac{dN}{dt} \quad (2)$$

The external forces per unit mass E are the pressure and gravity forces, which may be combined in the form $g(T_v - T_{ve})/T_{ve}$, and the frictional forces. The last of these will include the loss of momentum by lateral diffusion and also a specific term representing the drag of the liquid water carried along by the ascending air. The equation of motion for liquid water rising with the air may be written as

$$l\frac{dw}{dt} = -lg + D \quad (3)$$

Here l is the mass of condensed water per unit mass of air, and D represents the drag of the air on the l grams of liquid water. Thus $-D$, which may be obtained from Eq. (3) is the retarding drag of the liquid water on the air. The loss of momentum by the convective current per unit mass per unit time through lateral diffusion to the environment is represented by a term of the form $-k_1(w - w_e)$, where k_1 has dimensions of $\sec^{-1}$. Thus the combined "frictional" forces per unit mass of air are

$$-k_1(w - w_e) - l\left(g + \frac{dw}{dt}\right) \quad (4)$$

It follows from Eqs. (2) and (4) that the equation of motion for the rising air is

$$\frac{dw}{dt} = g\frac{T_v - T_{ve}}{T_{ve}} - (w - w_e)\frac{dN}{dt} - k_1(w - w_e) - $$
$$- l\left(g + \frac{dw}{dt}\right)$$

Utilizing the assumption of steady state leads to the form

$$(1 + l)\frac{dw}{dz} = \frac{g}{w}\left(\frac{T_v - T_{ve}}{T_{ve}} - l\right) - (w - w_e)\frac{dN}{dz} - k_1 \quad (5)$$

The thermodynamic equation is essentially unchanged from the previous theory, namely

$$-L\frac{dq}{dz} - cl\frac{dT}{dz} - [c_p(T - T_e) + L(q - q_e)]\frac{dN}{dz} -$$
$$- \frac{k_1}{w}\left[c_p(T - T_e) + L(q - q_e)\right] =$$
$$= c_p\frac{dT}{dz} + q\frac{T_v}{T_{ve}} \quad (6)$$

The second term on the left, which represents the heat energy given up by the liquid water as it ascends with the air, is only a few percent of the first term on the right, namely, the enthalpy change of the air. The former is included here mainly for consistency since the effect of the drag of this liquid on the air is included in the momentum equation. However since it was considered negligible and omitted in the previous results, it will also be omitted in the computations for the present model in order to afford a more accurate measure of the more important effects of the drag term and detrainment. It might also be pointed out here that in nature the water droplets acquire a downward velocity relative to the air and Eq. (3) would not apply in general.

The Clapeyron equation for the saturation vapor pressure together with the definition of specific humidity gives

$$\frac{1}{q}\frac{dq}{dz} = \frac{L}{R_v T^2}\frac{dT}{dz} + \frac{g}{R_d T_{ve}} \quad (7)$$

Since detrainment is being considered, the law for the liquid water content l must be piecewise defined as follows:

(i) during entrainment,

$$\frac{dl}{dz} = -\frac{dq}{dz} - (l + q - q_e)\frac{dN}{dz} - \frac{k_1}{w}(q - q_e) \quad (8)$$

(ii) during detrainment,

$$\frac{dl}{dz} = -\frac{dq}{dz} - \frac{k_1}{w}(q - q_e) \quad (9)$$

Using the definition of virtual temperature

$$T_v = T(1 + .61\, q) \tag{10}$$

it may be shown that

$$\frac{dT_v}{dz} \doteq (1 + 12.5\, q)\frac{dT}{dz} \tag{11}$$

In this study the cross-section area of the current will be assumed constant. Thus with

$$d\sigma = 0 \tag{12}$$

and some minor approximations, the preceding equations may be expressed as the following system which is in a convenient form for the numerical integration.

$$\frac{dT}{dz} = \left\{ \frac{1}{2}\left[q - q_e + \frac{c_p}{L}(T - T_e)\right] \times \left[\frac{g}{w^2}\left(\frac{T_{ve} - T_v}{T_{ve}} - l\right) - \frac{k_1}{w} - \frac{g}{R_d T_{ve}}\right] + \frac{k_1}{w}\left[q - q_e + \frac{c_p}{L}(T - T_e)\right] + \frac{g T_v}{L T_{ve}} + \frac{gq}{R_d T_{ve}} \right\} \left\{ -\frac{Lq}{R_v T^2} - \frac{c_p}{L} - \frac{cl}{L} + \frac{(1 + 1.25\, q)}{2 T_v} \times \left[q - q_e + \frac{c_p}{L}(T - T_e)\right] \right\}^{-1} \tag{13}$$

$$\frac{dN}{dz} = \frac{1}{2}\left[\frac{g}{w^2}\left(\frac{T_v - T_{ve}}{T_{ve}} - l\right) - \frac{k_1}{w} - \frac{g}{R_d T_{ve}} - \frac{1}{T_v}\frac{dT_v}{dz}\right] \tag{14}$$

$$\frac{dw}{dz} = \frac{1}{(1 + l)}\left[-(w - w_e)\frac{dN}{dz} - k_1 + \frac{g}{w} \times \left(\frac{T_v - T_{ve}}{T_{ve}} - l\right)\right] \tag{15}$$

$$\frac{dq}{dz} = \frac{Lq}{R_v T^2}\frac{dT}{dz} + \frac{gq}{R_d T_{ve}} \tag{16}$$

$$\frac{dl}{dz} = -\frac{dq}{dz} - (l + q - q_e)\frac{dN}{dz} - \frac{k_1}{w}(q - q_e) \tag{17a}$$

or

$$\frac{dl}{dz} = -\frac{dq}{dz} - \frac{k_1}{w}(q - q_e) \tag{17b}$$

With appropriate initial and environmental conditions this system of equations may be integrated by numerical methods to yield values of T, N, w, q and l. The factor $(1 + l)^{-1}$ in Eq. (15) may be replaced by 1 with a high degree of accuracy. With this minor approximation, the above system differs from the equivalent system in the previous model H [2] in two respects. Firstly, because of the drag of liquid water, the term $[(T_v - T_{ve})/T_{ve}] - l$ appears in several equations above replacing $(T_v - T_{ve})/T_{ve}$ in the previous theory. Secondly, because some of the computations to follow include detrainment, Eq. (17 b) must be used to determine the liquid water content after the detrainment begins. In the computations to follow, it will also be assumed that $w_e = 0$, as in the previous computations.

Results and Conclusions

The system of differential equations 13—17 was integrated by the Runge-Kutta-Gil method (GILL, 1951) on one electronic digital computer. Four sets of initial and environmental conditions were used, each identical to a set of conditions already computed for the previous model.

For purposes of comparison, cases (a) and (b) have the same initial and environmental conditions; however the term representing the drag of the liquid water $- D$ was omitted in case (a), while retaining the detrainment feature of the present model. Cases (a), (c), and (d) include both water drag and detrainment. The conditions corresponding to the various cases are as follows:

Case (a)
$T_0 = 20°$ C, $T_{e0} = 19°$ C
$w_0 = 1$ m/sec, $w_e = 0$
$\gamma_e = .7°$ C/100 m, $r = 80$ %
$k_1 = .001$ sec^{-1}, $D = 0$

Case (b)
same as case (a) except $D \pm 0$

Case (c)
same as case (b) except that $\gamma_e = .6°$ C/100 m

Case (d)
same as case (b) except that $r = 90$ %

Figures 1, 2, 3, and 4 present a comparsion of, respectively, temperature excess over environ-

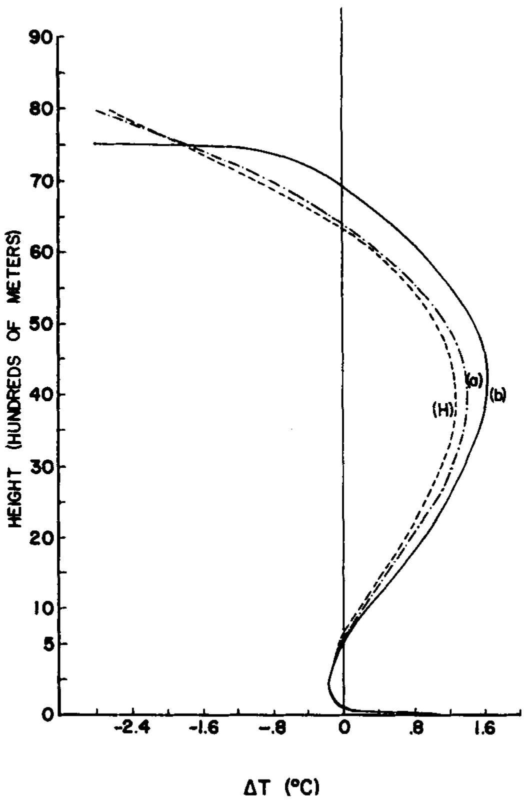

Figure 1. Temperature excess over environment as a function of height for Case (a), Case (b) and after HALTINER (1959).

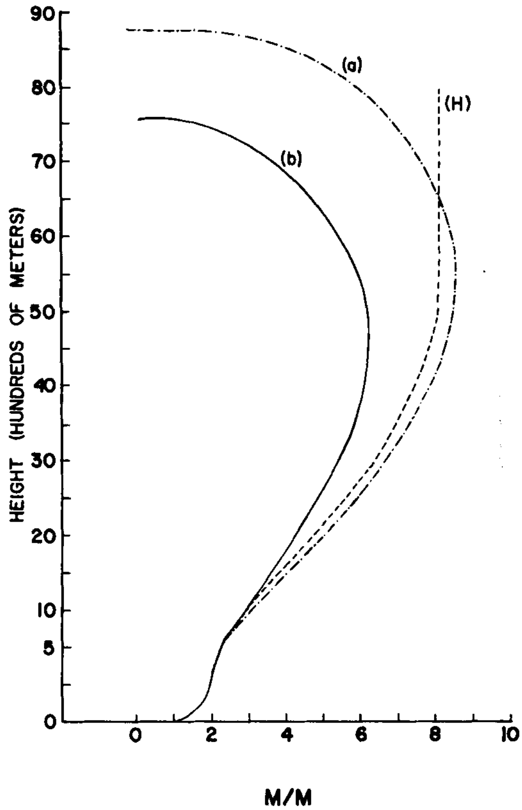

Figure 2. Fractional mass change M/M_0 a as function of height for Case (a), Case (b) and after HALTINER (1959).

ment $\Delta T = T - T_e$, fractional increase of rate of mass flow M/M_0, vertical velocity w, liquid water content and also the corresponding results of the previous theory (H). Cases (a) and (H) should be the same up to the point where detrainment begins, however a small systematic difference may be noted. This minor difference is due to slightly different computational procedures, primarily in the determination of the environmental parameters.

Figure 1 shows that the differences in temperature among the three models generally are only of the order of a few tenths of a degree, which would make it rather difficult to indicate the preferable model on the basis of temperature observations in clouds. Figures 2 and 3 show that permitting detrainment, while omitting the drag of the liquid water (case (a)), gives the greatest cloud height and vertical velocity. This is to be expected since the ejection of mass (detrainment) tends to accelerate the remaining mass of the ascending current. On the other hand, the inclusion of the additional frictional drag of the liquid water tends to reduce the cloud height, vertical velocity and, in turn, the total cloud mass, the latter being represented by the term M/M_0.

It is interesting to note that the apposing effects of detrainment and the retarding drag of the liquid water in case (b) result in nearly the same cloud height as (H) where both are omitted. However, the maximum vertical velocity attained in case (b) is significantly less than that of the previous theory (H), or case (a) which does not include the drag of the liquid water. This is to be expected since retarding drag of the liquid water begins immediately, whereas the accelerating effect of

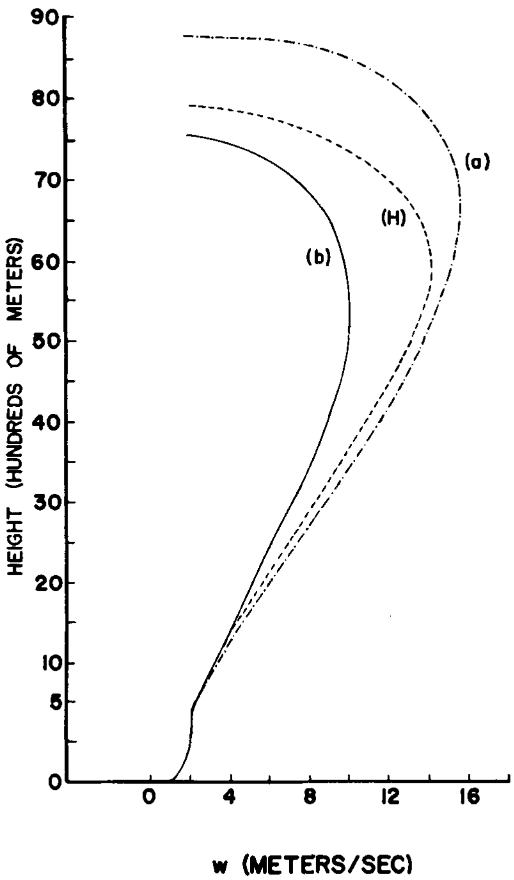

Figure 3. Vertical velocity as a function of height for Case (a), Case (b) and after HALTINER (1959).

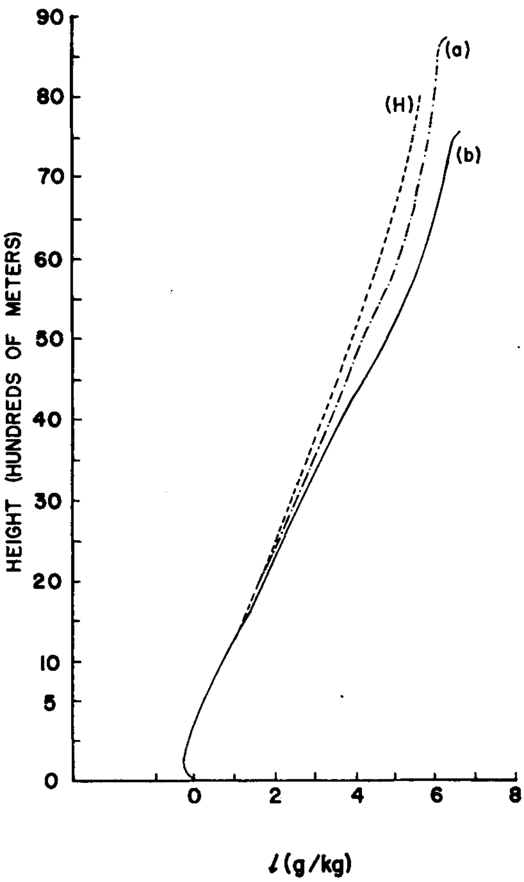

Figure 4. Liquid water content as a function of height for Case (a), Case (b) and after HALTINER (1959).

the detrainment does not begin until after the maximum vertical velocity has been attained.

Differences in the liquid water content of the cloud are rather small, averaging about 10 %. Table 1 summarizes the results of the computations.

In the previous paper computed values were compared to some observational data from an actual cloud. In general the agreement between observed and computed parameters was fairly good; however the computed vertical velocity was somewhat in excess of observed values. To the extent that the modifications reduce vertical velocity, while comparing equally favorably in other respects, the modified model appears to be an improvement over the previous theory.

For a more complete understanding of convective phenomena further work is necessary in regard to the cross-section area of the convective currents, the environment vertical velocity and the horizontal distribution of convective cells. These questions will require the introduction of horizontal variations and in essence a three-dimensional analysis of convection. Moreover, more knowledge must be gained about the initiation of the downdraft in a large convective cloud, as well as about the initial development of the convective current itself. In connection with the latter, some computations by MALKUS and WITT (1959) show a bubble-like structure during the development of a convective element.

Table 1. Comparison of maximum values of ΔT, M/M, l, w and cloud height, h, with maximum values after Haltiner (2).

Case Parameter	(H)	Case (a)	Difference
ΔT (° C)	1.26	1.39	.13
M/M_0	8.13	8.56	.43
w (m/s)	14.12	15.52	1.40
l (g/kg)	6.0	6.4	.4
h (m)	7900	8750	850
	(H)	Case (b)	
ΔT	1.26	1.60	.34
M/M_0	8.13	6.22	1.81
w	14.12	10.13	3.99
l	6.0	6.83	.83
h	7900	7550	350
	Case (a)	Case (b)	
ΔT	1.39	1.60	.21
M/M_0	8.56	6.22	2.34
w	15.52	10.13	5.39
l	6.4	6.83	.43
h	8750	7550	1200
	(H)	Case (c).	
ΔT	— .01	.97	.98
M/M_0	2.88	1.92	.96
w	4.07	2.12	1.95
l	5.9	4.2	1.7
h	4750	4500	250
	(H)	Case (d)	
ΔT	2.47	2.76	.29
M/M_0	11.80	9.75	2.05
w	22.28	17.78	4.50
l	7.1	7.7	.6
h	9500	9250	250

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