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on the order of $p = .01$ was reasonable. For this p, $1 - \Phi(z)$ is 7.9×10^{-27}, so one could hardly be more pessimistic.

Nevertheless, the candidate and his advisors decided to proceed with the recount. The recount was never completed. After some fraction of the votes had been recounted, the plurality for Candidate 2 had increased and Candidate 1 called a halt to further recounting.

5. CONCLUSION

As noted in the introduction, the model employed here is not appropriate when voting machines or automatic tabulating equipment are employed. The errors that might be encountered in such cases include transposition of digits or interchanging the entire vote of two candidates. In such a case, errors will be clustered and will not be the cumulative effect of misclassifying a number of single ballots.

In other controversial electoral outcomes, the recount results have depended on matters such as identifying illegally cast ballots or judicial decisions concerning possible disqualification of improperly marked paper ballots. When electoral outcomes have depended on contesting "very few" ballots, such issues have played a significant note in deciding the outcome. The available literature on statistical models for election recounts is quite small, but here is a brief list of references.

Stinnett and Blackstrom (1964) gave a historical account of the very complex recount of the 1962 Minnesota gubernatorial election. Judicial review of contested ballots played a major role in this election.

Israels (1975) studied the 1962 Massachusetts gubernatorial election. He compared the votes before and after the recount to get an estimate of the variability. It is interesting that an empirical analysis in that paper agrees substantially with the probabilistic result given here.

Finkelstein and Robbins (1973) suggested randomly removing a number of votes equal to the number of illegal votes from the tally, when there is a known number of illegal votes cast. This model was discussed and generalized by Gilliland and Meier (1986), who employed six elections as illustrations.

Downs, Gilliland, and Katz (1978), Gilliland and Meier (1986), and Gilliland (1984) discussed the 1975 mayoral election in Flint, Michigan. A "block" of ballots to be tabulated by machine were improperly "coded," so all votes in the block that should have gone to Candidate 1 went to Candidate 2 and conversely. They gave a procedure for estimating the actual vote received by each candidate.

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Integral Identities for Random Variables

EDWARD B. ROCKOWER*

Using a general method for deriving identities for random variables, we find a number of new results involving characteristic functions and generating functions. The method is simply to promote a parameter in an integral relation to the status of a random variable and then take expected values of both sides of the equation. Results include formulas for calculating the characteristic functions for x^2, $\sqrt{x}$, $1/x$, $x^2 + x$, $R^2 = x^2 + y^2$, and so forth in terms of integral transforms of the characteristic functions for x and (x, y), and so forth. Generalizations to higher dimensions can be obtained using the same method. Expressions for inverse/fractional moments, $E\{n!\}$, and so forth are also presented, demonstrating the method.

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1. INTRODUCTION

It is often easier to study a random process using transforms of the relevant probability distributions. Such transforms as the characteristic function and the probability generating function simplify manipulations involving convolutions of probability distributions and allow us to apply powerful methods from complex analysis and integral transform theory to the solution of differential-difference equations arising in the study of probability and stochastic processes. The value of techniques for manipulating such transforms and of "methods for constructing new characteristic functions out of given ones" is well known (see Feller 1965, p. 477). In fact, the theory of probability "depends to a large extent on the method of characteristic functions" (Feller 1960, p. 248). The characteristic function, $C(\omega)$, and the moment generating function, $M(\theta)$, are used for general random variables (rv's), whereas the probability generating function, $G(z)$, is defined for integer-valued rv's

a. $\sqrt{i/4\pi\gamma}\int_{-\infty}^{\infty} \exp[-i\xi^2/(4\gamma)]\, C_x(\xi)\, d\xi = C_{x^2}(\gamma).$

b. $i/(4\pi\sqrt{\gamma\delta})\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left[\frac{(-i\xi^2)}{4\gamma} - \frac{i\varepsilon^2}{4\delta}\right] C_{x,y}(\xi,\epsilon)\, d\xi d\epsilon = C_{x^2 y^2}(\gamma, \delta).$

c. $i/(4\pi\gamma)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp\left\{\frac{[-i(\xi^2+\epsilon^2)]}{4\gamma}\right\} C_{x,y}(\xi,\epsilon)\, d\xi d\epsilon = C_{R^2}(\gamma).$

d. $\sqrt{i/(4\pi\gamma)}\int_{-\infty}^{\infty} \exp[-i\xi^2/(4\gamma)]\, C_x(\xi+\gamma)\, d\xi = C_{x^2+x}(\gamma).$

e. $s/\sqrt{\pi}\int_0^{\infty} \exp[-s^2\xi^2/4]\,\mathscr{L}_x(1/\xi^2)\, d\xi = \mathscr{L}_{\sqrt{x}}(s).$

f. $(-)\int_0^{\infty} \mathscr{L}_x'(\xi) J_0[2\sqrt{s\xi}]\, d\xi = \mathscr{L}_{1/x}(s).$

Figure 1. Transform Expressions.

and the Laplace transform of the pdf, $\mathscr{L}(s)$, is used for nonnegative random variables.

A number of interesting relations involving these transforms are found by promoting a parameter in an integral expression to the status of an rv and then taking expected values of both sides of the equation. In general, there is no guarantee that the resulting integrals can be evaluated in closed form for all distributions of interest, but the expression may be helpful in numerical work. In the probability context similar methods have long been used to solve problems by averaging conditional results over the conditioning variable. The methods presented here may further aid in the interpretation of complicated characteristic functions and facilitate the identification of independent processes that contribute to the result (see, e.g., Rockower and Abraham 1978). Apart from their usefulness in probabilistic applications, our results also provide another means of generating new integral identities from old ones.

Several identities are presented in Sections 2 and 3, demonstrating the method of derivation. Additional results are derived in the Appendix. Figures 1 and 2 summarize these identities for probability transforms and expected values, respectively. Some examples of calculations illustrating their use are carried out in Section 4.

2. IDENTITIES FOR TRANSFORMS OF DISTRIBUTIONS

Consider the well-known integral expressing the normalization of a Normal (Gaussian) distribution, in which x is an arbitrary constant,

$$1/\sqrt{2\pi\sigma^2}\int_{-\infty}^{\infty} \exp[-(\xi-x)^2/(2\sigma^2)]\, d\xi = 1.$$

If we change variables according to $\xi \to \xi/(2\gamma)$, $1/(2\sigma^2) \to i\gamma$, this becomes

$$\sqrt{i/(4\pi\gamma)}\int_{-\infty}^{\infty} \exp[-i\xi^2/(4\gamma) + i\xi x]\, d\xi = \exp[i\gamma x^2]. \tag{1}$$

We now promote x to be a real random variable and take expected values of both sides of the equation, assuming that the implicit interchange of orders of integration is justified (i.e., that $\int$ and E{ } commute). This yields the expression a of Figure 1 for the characteristic function of the square of an rv C_{x^2}, in terms of C_x. Generalizations of this result are derived in the Appendix and also appear in Figure 1.

Given the Laplace transform of the pdf of x, is it possible to calculate general expressions involving $\sqrt{x}$? To answer this, consider the definite integral (e.g., Gradshteyn and Ryzhik 1980, p. 341)

$$\int_0^{\infty} \exp[-a/\xi^2 - b\xi^2]\, d\xi = \sqrt{\pi/(4b)}\, \exp[-2\sqrt{ab}]. \tag{2}$$

Let $a \to x$, $b \to s^2/4$ to obtain the identity

$$\int_0^{\infty} \exp[-x/\xi^2 - s^2\xi^2/4]\, d\xi = \sqrt{\pi}/s\, \exp[-s\sqrt{x}]. \tag{3}$$

Now, promote x to be a nonnegative rv and average over x to obtain the Laplace transform of the pdf of $\sqrt{X}$, $\mathscr{L}_{\sqrt{x}}(s)$. This is presented as expression e in Figure 1. Alternatively, a similar integral from Gradshteyn and Ryzhik (1980, p. 399) allows one to express $\mathscr{L}_{\sqrt{x}}(s)$ in terms of the characteristic function C_x. Using another definite integral as the starting point, the Laplace transform for $1/x$ is also evaluated in the Appendix.

3. IDENTITIES FOR NONSTANDARD MOMENTS AND AVERAGES

The methods used in Section 2 can also be employed to obtain identities for expected values. Consider the elementary integral, where x is just a parameter

$$\int_0^{\infty} s^{n-1} \exp[-x\, s]\, ds = (n-1)!/x^n. \tag{4}$$

Now, promoting x to be a nonnegative rv, whose pdf falls off sufficiently rapidly as $x \to 0$ [e.g., an Erlang $(n+1)$], and taking expected values with respect to x yields

$$1/(n-1)!\int_0^{\infty} s^{n-1}\, \mathscr{L}_x(s)\, ds = \mathrm{E}\{1/x^n\}. \tag{5}$$

Letting $x \to (x + A)$ leads immediately to identity a in Figure 2. Analogous results for the moment generating function (MGF) were also derived by Chao and Strawderman

a. $[1/(n-1)!] \int_0^\infty s^{n-1}\exp(-sA)\mathcal{L}_x(s)ds = E(1/(x+A)^n)$.

b. $1/\sqrt{\pi} \int_0^\infty y^{1/2-1}\mathcal{L}_x(y)dy = E(1/\sqrt{x})$.

c. $2/\sqrt{\pi} \int_0^\infty M_x'(-t^2)dt = E\{\sqrt{x}\}$.

d. $\int_0^\infty \mathcal{L}_x(s)J_0(bs)ds = E(1/\sqrt{x^2+b^2})$.

e. $1/(2\pi) \int_{-\pi}^{\pi} \exp[-n\,i\,\theta]C_x(\sin\theta)d\theta = E\{J_n(x)\}$.

f. $\int_0^\infty G(z)\exp[-sz]dz = E(n!/s^{n+1})$.

g. $\int_0^1 G(z)dz = E[1/(n+1)]$.

h. $1/\sqrt{2\pi\sigma^2} \int_{-\infty}^\infty G(z^2)\exp[-z^2/(2\sigma^2)]dz = E\{(2n-1)!!\,\sigma^{2n}\}$.

i. $(2/\pi) \int_0^{\pi/2} G(\sin^2\theta)d\theta = E((2n-1)!!/(2n)!!)$.

j. $\int_0^1 G(1-y^2)dy = E((2n)!!/(2n+1)!!)$.

Figure 2. Expectation Values.

(1972) and Cressie, Davis, Folks, and Policello (1981) using methods similar to the above. Those authors also gave additional applications of this result.

To obtain fractional moments, first consider the integral

$$2\int_0^\infty \exp[-a\,t^2]dt = \sqrt{\pi/a}. \tag{6}$$

Let $a \to x$, a nonnegative rv, and take expected values. This yields an expression for $E\{1/\sqrt{x}\}$ in terms of the Laplace transform, $\mathcal{L}_{\sqrt{x}}(t^2)$. A slight variation yields a corresponding expression for $E\{\sqrt{x}\}$.

Making the change of variable $y = t^2$ results in expression b in Figure 2. This can be recognized as a fractional integration of order ½ of the Laplace transform (or MGF). Some of the other moments in this section can also be written as fractional integro-differentiations of MGF's or Laplace transforms. This fact, as well as other extensions (and related references) were discussed by Cressie and Borkent (1986) and Jones (1986, 1987).

For a more general fractional moment we might consider Lipschitz's integral (see Watson 1958) for the ordinary Bessel function of zero order, J_0,

$$\int_0^\infty \exp[-as]J_0(bs)ds = 1/\sqrt{a^2+b^2}. \tag{7}$$

Promoting $a \to x$, a nonnegative rv, and taking expected values of both sides we obtain expression d in Figure 2 for $E\{1/(x^2+b^2)^{1/2}\}$. Successively differentiating this identity with respect to the parameter b produces a family of similar identities.

An identity for an integer-valued rv is obtained by again considering the well-known integral defining the gamma function $\Gamma(n+1)$,

$$\int_0^\infty z^n \exp[-s\,z]dz = n!/s^{n+1}. \tag{8}$$

This time let n be the rv and average over it, yielding expression f in Figure 2. In particular, when $s = 1$, this yields $E\{n!\}$ when it exists; that is, the Laplace transform of the probability generating function, evaluated at $s = 1$, is just $E\{n!\}$ This is in contrast to the factorial moments $E\{n(n-1)\cdots(n-k+1)\}$ obtained by differentiating $G(z)$. For noninteger rv's we obtain a corresponding expression for $E\{\Gamma(x)\}$.

Additional expressions for integer-valued random variables in Figure 2 are derived in the Appendix. Note the double factorial symbol !!, which means, for example, 5!! = 5·3·1.

4. SOME APPLICATIONS OF THE IDENTITIES

If x has a Normal distribution with zero mean, then $C_x(\xi) = \exp\{-\xi^2\sigma^2/2\}$. Using this in expression a of Figure 1 and performing the integration, we have

$$\begin{aligned} C_{x^2}(\gamma) &= \sqrt{i/(4\pi\gamma)}\int_{-\infty}^\infty \exp[-i\xi^2/(4\gamma) - \xi^2\sigma^2/2]d\xi \\ &= 1/\sqrt{1-2i\sigma^2\gamma}. \end{aligned} \tag{9}$$

This is recognized as the characteristic function for a χ^2 distribution with 1 df. (Similarly, if x, y have independent normal distributions with the same value of the variance, then $R^2 = X^2 + Y^2$ has a negative exponential distribution, which follows trivially from expression c in Fig. 1.)

Now, let x have a Normal distribution with nonzero mean, μ; then $C_x(\xi) = \exp\{i\mu\xi - \xi^2\sigma^2/2\}$. Substituting this in expression a of Figure 1 yields

$$\begin{aligned} C_{x^2}(\gamma) &= \sqrt{i/(4\pi\gamma)}\int_{-\infty}^\infty \exp[i\mu\xi - i\xi^2/(4\gamma) - \xi^2\sigma^2/2]d\xi \\ &= 1/\sqrt{1-2i\sigma^2\gamma}\exp\{i\mu^2\gamma/(1-2i\sigma^2\gamma)\}, \end{aligned} \tag{10}$$

which is the characteristic function of an offset χ^2 distribution.

To calculate $E\{1/\sqrt{x}\}$, where x is an exponential rv, use $\mathcal{L}(s) = \lambda/(\lambda + s)$. Substituting this in expression b of Figure 2 and using a standard integral, we obtain

$$E\{1/\sqrt{x}\} = \pi\sqrt{\lambda}. \quad (11)$$

This is easily verified by a direct calculation. $E\{\sqrt{x}\}$ is also easily verified to be the result produced by expression c of Figure 2.

A less common application is obtained by inserting the Laplace transform of the pdf for an exponential distribution into expression d of Figure 2:

$$E(1/\sqrt{x^2 + b^2}) = \int_0^\infty [\lambda/(\lambda + s)]J_0(bs)ds. \quad (12)$$

This integral was tabulated by Gradshteyn and Ryzhik (1980, p. 685), resulting in

$$E(1/\sqrt{x^2 + b^2}) = \lambda(\pi/2)\,[H_0(b\lambda) - N_0(b\lambda)], \quad (13)$$

where H_0 and N_0 are Struve and Neumann functions, respectively, of zero order (N_0 can be replaced with Y_0, the Bessel function of the second kind). For example, taking $b = 4$ and $\lambda = 1$ [$H_0(4) = .13501$ and $Y_0(4) = -.01694$; see Abramowitz and Stegun 1965], we find for the exponential distribution,

$$E(1/\sqrt{x^2 + b^2}) = .2387, \quad (14)$$

which we also confirmed by direct Gauss–Laguerre integration of the left side.

We now calculate the average of the nth-order Bessel function when x has an $N(0, \sigma)$ distribution with the use of expression e from Figure 2. After inserting the characteristic function for a normal distribution, using the trigonometric identity $\sin^2\theta = (1 - \cos 2\theta)/2$, and Bessel's integral identity, for $I_{n/2}$ we have

$$E\{J_n(x)\} = \exp[-\sigma^2/4]I_{n/2}(\sigma^2/4) \quad (15)$$

when n is even, and 0 when n is odd. This expression can be confirmed by evaluating the expected value directly with the help of an integral tabulated by Gradshteyn and Ryzhik (1980, p. 710).

Let $G(z)$ be the generating function for a Poisson distribution. $G(z) = \exp[\bar{n}(z-1)]$. Putting this in expression f of Figure 2 and integrating yields

$$E\{n!/s^{n+1}\} = 1/(s - \bar{n})\exp[-\bar{n}], \quad (16)$$

and, in particular, when $s = 1$,

$$E\{n!\} = 1/(1 - \bar{n})\exp[-\bar{n}]. \quad (17)$$

This is easily verified, and for a Poisson distribution $E\{n!\}$ is only finite for $\bar{n} < 1$.

If we substitute the generating function for a Poisson rv into expression h of Figure 2 and perform the integration, (letting $\sigma = 1$) we easily obtain

$$E\{(2n-1)!!\} = [1/(1 - 2\bar{n})^{1/2}]\exp[-\bar{n}]. \quad (18)$$

Note that this is only finite for $\bar{n} < 1/2$.

Again using the generating function for a Poisson rv, after using a trigonometric identity for $\sin^2\theta$ and Bessel's integral representation for J_0, expression i of Figure 2 yields

$$E\{(2n-1)!!/(2n)!!\} = \exp[-\bar{n}/2]J_0(i\,\bar{n}/2)$$
$$= \exp[-\bar{n}/2]I_0(\bar{n}/2), \quad (19)$$

where I_0 is the modified Bessel function of zero order. All of the applications shown here have been verified with direct numerical calculations.

5. CONCLUSION

Some of the identities presented may be derived or verified using other methods. For example, we originally obtained Equation (2) using the method of Rockower and Abraham (1978), which depends on the existence of all moments. Similarly, the expression for $E\{1/(n+1)\}$ follows easily from integrating, term by term, the infinite series definition of $G(z)$. In fact, expressions for fractional and/or inverse moments, including some of those derived in Section 3, have been expressed elsewhere (Cressie and Borkent 1986; Jones 1986, 1987) in a unified manner in terms of fractional integro-differentiations of the MGF, generalizing the usual formulas for moments and factorial moments.

But alternate derivations are not readily identified for all of our integral relations. By presenting our unified treatment (containing as a proper subset some of the previously mentioned formalisms) it becomes straightforward to obtain new integral identities for random variables by a judicious search of tables of integrals such as in Gradshteyn and Ryzhik (1980).

APPENDIX: ADDITIONAL RESULTS AND EXTENSIONS

Expression a (Fig. 1) can be extended to more than one dimension by multiplying Equation (1) by itself with $x \to y$, $\xi \to \varepsilon$, and $\gamma \to \delta$ to obtain

$$i/(4\pi\sqrt{\gamma\delta}) \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i\xi^2/(4\gamma) + i\xi x - i\varepsilon^2/(4\delta) + i\varepsilon y]d\xi d\varepsilon$$
$$= \exp[i\gamma x^2 + i\delta y^2]. \quad (A.1)$$

Again consider x, y to be rv's and take expected values of both sides, which gives the relation in expression b (Fig. 1) for $C_{x^2,y^2}(\gamma, \delta)$. If we now let $\delta = \gamma$ we obtain the corresponding result for C_{R^2} in expression c (Fig. 1), where $R^2 = X^2 + Y^2$ (see Rockower and Abraham 1978). This can be generalized further to three or more rv's in an analogous manner.

Multiply Equation (1) by $\exp(i\gamma x)$ and take expected values to obtain the characteristic function for $X^2 + X$, expression d (Fig. 1). Again, it is clear that this can be generalized further.

To obtain the Laplace transform of the pdf for the rv $1/x$ (i.e., $\mathcal{L}_{1/x}$, given $\mathcal{L}_x$), consider the integral

$$\int_0^\infty \exp[-a\xi]J_0(b\sqrt{\xi})d\xi = 1/a\exp[-b^2/(4a)] \quad (A.2)$$

(see Watson 1958). Multiply both sides by a and change the parameters $b \to 2\sqrt{s}$, $a \to x$ (the latter a nonnegative rv) to obtain

$$\int_0^\infty \mathrm{E}\{x \exp[-x\xi]\}J_0[2\sqrt{s\xi}]d\xi = \mathrm{E}\{\exp[-s/x]\}. \quad (\mathrm{A}.3)$$

In terms of the Laplace transform, this is expression f in Figure 1.

Multiply Equation (6) by a, let $a \to x$, and average to obtain

$$2/\sqrt{\pi}\int_0^\infty \mathrm{E}\{x \exp[-xt^2]\}dt = \mathrm{E}\{\sqrt{x}\}; \quad (\mathrm{A}.4)$$

switching to MGF's instead of Laplace transforms for this result (either could be used here) gives expression c in Figure 2. This can be generalized to obtain a formula for $\mathrm{E}\{x^{m+1/2}\}$, with m an integer, in a straightforward manner.

Consider one form of Bessel's integral for the nth-order ordinary Bessel function

$$1/(2\pi)\int_{-\pi}^{\pi} \exp[-n\,i\,\theta + i\,x\sin\theta]d\theta = J_n(x) \quad (\mathrm{A}.5)$$

(see Watson 1958), let x be an rv, and average over all x to obtain $\mathrm{E}\{J_n(x)\}$ in expression e (Fig. 2). Clearly, this result can be generalized in many ways and is somewhat reminiscent of the well-known formula

$$\mathrm{E}\{H(x)\} = 1/(2\pi)\int \tilde{H}(\omega)C_x(\omega)d\omega, \quad (\mathrm{A}.6)$$

where $\tilde{H}(\omega)$ is the Fourier transform of $H(x)$. The latter equation can, in the spirit of this article, be simply derived by taking expected values of x in the representation of $H(x)$ as the Fourier transform of $\tilde{H}(\omega)$.

Consider the integral

$$\int_0^1 z^n\,dz = 1/(n+1). \quad (\mathrm{A}.7)$$

Now, let n be a nonnegative integer-valued rv and average, obtaining identity g in Figure 2 for $\mathrm{E}\{1/(n+1)\}$, which also follows easily from the power series definition of $G(z)$ and is directly analogous to the usual result for $\mathrm{E}\{n\}$.

Consider the integral expressing the standard result for the even moments of the Normal distribution,

$$1/\sqrt{2\pi\sigma^2}\int_{-\infty}^{\infty} z^{2n}\exp[-z^2/(2\sigma^2)]dz = (2n-1)!!\,\sigma^{2n}, \quad (\mathrm{A}.8)$$

where the double factorial symbol means, for example, $5!! = 5\cdot 3\cdot 1$. Again take averages over n on both sides of the equality and obtain expression h (Fig. 2). When $\sigma = 1$ we have $\mathrm{E}\{(2n-1)!!\}$.

Consider the integrals

$$\int_0^{\pi/2} [\sin^2\theta]^n d\theta = \pi/2(2n-1)!!/(2n)!! \quad (\mathrm{A}.9)$$

and

$$\int_0^{\pi/2} \sin\theta\,[\sin^2\theta]^n d\theta = (2n)!!/(2n+1)!!, \quad (\mathrm{A}.10)$$

found in Gradshteyn and Ryzhik (1980, p. 369). Letting n be an rv and averaging over all values of n on each side of the (A.9) and (A.10), we obtain expression i in Figure 2, and with the change of variable $y = \cos^2\theta$ we obtain expression j.

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