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Stochastic HJB Equations and Regular Singular Points

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Regular Singular Points

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Recall from Boyce and DiPrima, the ODE

$$0 = P(x)y'' + Q(x)y' + R(x)y$$

has a regular singular point at $x = 0$ if $\frac{xQ(x)}{P(x)}$ and $\frac{x^2R(x)}{P(x)}$ have finite limits as $x \to 0$.

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The usual way this happens is that $P(x), Q(x), R(x)$ are real analytic and

$$P(x) = x^2 \sum_{n=0}^{\infty} p_n x^n$$

$$Q(x) = x \sum_{n=0}^{\infty} q_n x^n$$

$$R(x) = \sum_{n=0}^{\infty} r_n x^n$$

Regular Singular Points

Euler assumed that for some ρ the solution had a series expansion of the form

$$y(x) = \sum_{m=0}^{\infty} a_m x^{\rho+m}$$

then

$$\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} (\rho+m) a_m x^{\rho+m-1} \\ y''(x) &= \sum_{m=0}^{\infty} (\rho+m)(\rho+m-1) a_m x^{\rho+m-2} \end{aligned}$$

Regular Singular Points

He plugged these expressions into the ODE and obtained

$$\begin{aligned} 0 &= \left(\sum_{n=0}^{\infty} p_n x^n\right)\left(\sum_{m=0}^{\infty} (\rho+m)(\rho+m-1)a_m x^{\rho+m}\right) \\ &\quad + \left(\sum_{n=0}^{\infty} q_n x^n\right)\left(\sum_{m=0}^{\infty} (\rho+m)a_m x^{\rho+m}\right) \\ &\quad + \left(\sum_{n=0}^{\infty} r_n x^n\right)\left(\sum_{m=0}^{\infty} a_m x^{\rho+m}\right) \end{aligned}$$

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He collected the coefficient of x^ρ and obtained the so-called indicial equation

$$0 = p_0\rho(\rho-1) + q_0\rho + r_0$$

which has two possibly complex roots ρ_1, ρ_2.

Regular Singular Points

For each root by setting the coefficients of $x^{\rho+m+1}$ to zero he obtained a recurrence relation a_{m+1} in terms of $a_m, \ldots, a_0$.

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In this way he got two linearly independent solutions each determined by their first coefficient a_0 .

Hence he had found the general solution to the second order linear ODE.

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Why did Euler's method work?

- **Was it because it was a second order linear ODE?**

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Differentiation lowers the degree of a monomial by 1 and multiplying it by x restores it to its original degree.

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Differentiation lowers the degree of a monomial by 1 and multiplying it by x restores it to its original degree.

Twice differentiation lowers the degree of a monomial by 2 and multiplying it by x^2 restores it to its original degree.

Deterministic Optimal Control Problem

$$\min_{u(\cdot)} \int_0^\infty l(x,u)$$

subject to

$$\begin{aligned} \dot{x} &= f(x,u) \\ x(0) &= x^0 \end{aligned}$$

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If the optimal cost $\pi(x^0)$ and optimal feedback $u = \kappa(x)$ exist and are smooth they satisfy the Hamilton-Jacobi-Bellman PDEs (HJB)

$$\begin{aligned} 0 &= \min_u \left\{ \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \right\} \\ \kappa(x) &= \operatorname{argmin}_u \left\{ \frac{\partial \pi}{\partial x}(x) f(x,u) + l(x,u) \right\} \end{aligned}$$

Deterministic Optimal Control Problem

If the quantity to be minimized is smooth with respect to u then the HJB equations imply the simplified Hamilton-Jacobi-Bellman PDEs (sHJB)

$$
\begin{aligned}
0 &= \frac{\partial \pi}{\partial x}(x) f(x, \kappa(x)) + l(x, \kappa(x)) \\
0 &= \frac{\partial \pi}{\partial x}(x) \frac{\partial f}{\partial u}(x, \kappa(x)) + \frac{\partial l}{\partial u}(x, \kappa(x))
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If the quantity to be minimized is also strictly convex with respect to u then sHJB implies HJB.

Deterministic Optimal Control Problem

If we can solve the second simplified Hamilton-Jacobi-Bellman PDE for $\kappa(x)$ in terms of $\frac{\partial \pi}{\partial x}(x)$ then we plug it into the first simplified Hamilton-Jacobi-Bellman PDE and get a single first order nonlinear PDE for $\frac{\partial \pi}{\partial x}(x)$

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I not sure Al'brekht thought of his work in terms of regular points.

Al'brekht's Method

Al'brekht's assumptions:

1. **For some $d \geq 1$, f, l are smooth around $x = 0, u = 0$ and have Taylor polynomial expansions**

$$
\begin{aligned}
l(x,u) &= \frac{1}{2}\left(x'Qx + 2x'Su + u'Ru\right) + l^{[3]}(x,u) \\
&\quad + \ldots + l^{[d+1]}(x,u) + O(x,u)^{d+2} \\
f(x,u) &= Fx + Gu + f^{[2]}(x,u) \\
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where $^{[d]}$ indicates a homogeneous polynomial of degree d in x, u.

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where $^{[d]}$ indicates a homogeneous polynomial of degree d in x, u.

2. The quadratic part of l and the linear part of f constitute a nice LQR.

Al'brekht's Method

He assumed that the optimal cost and optimal feedback had similar Taylor polynomial expansions

$$\begin{aligned}\pi(x) &= \frac{1}{2}x'Px + \pi^{[3]}(x) + \ldots + \pi^{[d+1]}(x) + O(x)^{d+2} \\ \kappa(x) &= Kx + \kappa^{[2]}(x) + \ldots + \kappa^{[d]}(x) + O(x)^{d+1}\end{aligned}$$

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Then he plugged these expansions into the HJB equations and solved the resulting equations degree by degree.

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Then he plugged these expansions into the HJB equations and solved the resulting equations degree by degree.

At the lowest degrees he got the familiar LQR equations.

Al'brekht's Method

The HJB equations are nonlinear because the second HJB equation can be solved for $\kappa(x)$ in terms of $\frac{\partial \pi}{\partial x}(x)$ and so the first HJB equation has terms quadratic in $\frac{\partial \pi}{\partial x}(x)$.

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This fixes the leading degree of $\pi(x)$ at 2 because then the leading degree of $\frac{\partial \pi}{\partial x}(x)$ is 1 and so the leading degree of the quadratic terms are 2.

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If the leading degree of $\pi(x)$ is 2 then the leading degree of $\frac{\partial \pi}{\partial x}(x) f(x, \kappa(x))$ is also 2.

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If the leading degree of $\pi(x)$ is 2 then the leading degree of $\frac{\partial \pi}{\partial x}(x) f(x, \kappa(x))$ is also 2.

This is the analog of Euler's indicial equation.

Al'brekht's Method in Discrete Time

Al'brekht's method readily extends to discrete time optimal control problems of the form

$$\min_{u(\cdot)} \sum_{t=0}^{\infty} l(x,u)$$

subject to

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If they exist the optimal cost $\pi(x)$ and optimal feedback $u = \kappa(x)$ satisfy the Dynamic Programming Equations (DPE)

$$\begin{aligned} 0 &= \min_u \{\pi(f(x,u)) + l(x,u)\} \\ \kappa(x) &= \operatorname{argmin}_u \{\pi(f(x,u)) + l(x,u)\} \end{aligned}$$

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If we differentiate the quantity to be minimized with respect to u and set the result equal to zero we get the simplified Dynamic Programming Equations (sDPE)

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\begin{aligned}
0 &= \pi(f(x,\kappa(x))) + l(x,\kappa(x)) \\
0 &= \frac{\partial \pi}{\partial x}(f(x,\kappa(x)))\frac{\partial f}{\partial u}(x,\kappa(x)) + l(x,\kappa(x))
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As before we assume that

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\begin{aligned}
l(x,u) &= \frac{1}{2}\left(x'Qx + 2x'Su + u'Ru\right) + l^{[3]}(x,u) \\
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\pi(x) &= \frac{1}{2}x'Px + \pi^{[3]}(x) + \ldots + \pi^{[d+1]}(x) + O(x)^{d+2} \\
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If one plugs these expansions into the sDPE at the lowest degrees one obtains the equations of a discrete Linear Quadratic Regulator.

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At higher degrees one obtains a sequence of linear algebraic equations for the higher degree terms.

Al'brekht's Method in Discrete Time

If one plugs these expansions into the sDPE at the lowest degrees one obtains the equations of a discrete Linear Quadratic Regulator.

At higher degrees one obtains a sequence of linear algebraic equations for the higher degree terms.

We have written MATLAB code to solve these equations in any dimensions to any degrees.

Why is Al'brekht's Method important today?

Model Predictive Control

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It is impossible to approximately solve off-line the HJB or DPE equations on a large domain in the state space if $n \geq 3$.

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Because the Optimization Community has developed fast and robust Nonlinear Program Solvers we can use MPC techniques to solve on-line finite horizon discrete time optimal control problems given the current state $x(t_1) = x^1$.

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Because the Optimization Community has developed fast and robust Nonlinear Program Solvers we can use MPC techniques to solve on-line finite horizon discrete time optimal control problems given the current state $x(t_1) = x^1$.

$$\min_{u(\cdot)} \sum_{s=t_1}^{t_1+T-1} l(x(s), u(s)) + \pi_T(t_1 + T)$$

subject to

$$\begin{aligned} x(s+1) &= f(x(s), u(s)) \\ x(t_1) &= x^1 \end{aligned}$$

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An ideal terminal cost is the optimal cost of the corresponding infinite horizon discrete time optimal control problem.

Al'brekht's method, extended to such discrete time problems, supplies a reasonable terminal cost that is valid in some domain around $x = 0$.

But how big is that domain where the Taylor polynomial approximation to the optimal cost is a valid control Lyapunov function?

Adaptive Horizon Model Predictive Control

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We might expect that increasing the degree d we might increase the size of the domain but that is not always the case. It would be very expensive to compute off line the domain of Lyapunov validity and verify that the end of an optimal trajectory returned by the solver is in this domain.

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Instead we on-line verify that the end of the optimal trajectory returned by the solver is in this domain by projecting it an additional number of time steps using the Taylor polynomial approximation to the optimal feedback supplied by the discrete time version of Al'brekht's method.

Adaptive Horizon Model Predictive Control

$$\begin{aligned} x(s+1) &= f(x(s), \kappa(x(s)) \\ x(T) &= x^*(T) \end{aligned}$$

for $s = t_1 + T, \ldots, t_1 + T + S - 1$ where $x^*(T)$ is the end of the optimal trajectory computed by the solver

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We verify that the Lyapunov conditions and any feasibility constraints are satisfied on the extension.

Then we adjust the horizon T as needed.

Stochastic Optimal Control Problem

$$\min_{u(\cdot)} \mathbf{E} \left\{ \int_0^\infty l(x,u)\ dt \right\}$$

subject to the Ito equation

$$dx = f(x,u)dt + \sum_{k=1}^{r} \gamma_k(x,u)dw_k$$

where $w = [w_1; w_2; \ldots; w_r]$ is a standard r dimensional Wiener process.

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This is probably true if $\gamma_k(x,u) = O(1)$ or, in other words, if $\gamma_k(0,0) \neq 0$.

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But what happens if $\gamma_k(x,u) = O(x,u)$ or, in other words, if $\gamma_k(0,0) = 0$.

ISS Example

Here is an example where $\gamma_k(0, 0) = 0$.

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Consider a pendulum of length $1\ m$ and mass $1\ kg$ orbiting approximately 400 kilometers above Earth on the International Space Station (ISS). The "gravity constant" at this height is approximately $g = 8.7\ m/sec^2$. The pendulum can be controlled by a torque u that can be applied at the pivot and there is damping at the pivot with linear damping constant $c_1 = 0.1\ kg/sec$ and cubic damping constant $c_3 = 0.05\ kg\ sec/m^2$.

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Let x_1 denote the angle of pendulum measured counter clockwise from the outward pointing ray from the center of the Earth and let x_2 denote its angular velocity. The determistic equations of motion are

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= lg \sin x_1 - c_1 x_2 - c_3 x_2^3 + u \end{aligned}$$

Example

But the shape of the earth is not a perfect sphere and its density is not uniform so there are fluctuations in the "gravity constant". We set these fluctuations at around one percent although they are probably smaller. There might also be fluctuations in the damping constants of around one percent. Further assume that the commanded torque is not always realized and the relative error of the actual torque fluctuates around one percent.

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We model these stochastically by three white noises

$$
\begin{aligned}
dx_1 &= x_2 \, dt \\
dx_2 &= \left(lg \sin x_1 - c_1 x_2 - c_3 x_2^3 + u\right) \, dt \\
&\quad +0.01 lg \sin x_1 \, dw_1 - 0.01(c_1 x_2 + c_3 x_2^3) \, dw_2 \\
&\quad +0.01 u \, dw_3
\end{aligned}
$$

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Example

This is an example about how stochastic models with noise coefficients of order $O(x, u)$ can arise.

If the noise is modeling an uncertain environment then its coefficients are likely to be $O(1)$.

But if it is the model that is uncertain then noise coefficients are likely to be $O(x, u)$.

Example

The goal is to find a feedback $u = \kappa(x)$ that stabilizes the pendulum to straight up in spite of the noises so we take the criterion to be

$$\min_u \mathbf{E}\left\{\frac{1}{2}\int_0^\infty \|x\|^2 + u^2 \ dt\right\}$$

subject to

$$dx_1 = x_2 \ dt$$
$$dx_2 = \left(lg\sin x_1 - c_1 x_2 - c_3 x_2^3 + u\right) \ dt$$
$$+0.01 lg \sin x_1 \ dw_1 - 0.01(c_1 x_2 + c_3 x_2^3) \ dw_2 + 0.01 u \ dw_3$$

We shall return to this example in a moment but first we consider the general case.

Stochastic Optimal Control Problem

$$\min_{u(\cdot)} \mathbf{E} \left\{ \int_0^\infty l(x,u)\ dt \right\}$$

subject to the Ito equation

$$\begin{aligned} dx &= f(x,u)\ dt + \sum_{k=0}^{r} \gamma_k(x,u)\ dw_k \\ x(0) &= x^0 \end{aligned}$$

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If the optimal cost $\pi(x^0)$ and optimal feedback $u = \kappa(x)$ exist and are smooth then they satisfies the simplified stochastic Hamilton-Jacobi-Bellman PDEs (sSHJB)

$$\begin{aligned} 0 &= \frac{\partial \pi}{\partial x}(x) f(x,\kappa(x)) + l(x,\kappa(x)) \\ &\quad + \frac{1}{2}\sum_k \gamma_k'(x,u)\frac{\partial^2 \pi}{\partial x^2}(x)\gamma_k(x,u) \\ 0 &= \frac{\partial \pi}{\partial x}(x)\frac{\partial f}{\partial u}(x,\kappa(x)) + \frac{\partial l}{\partial u}(x,\kappa(x)) \end{aligned}$$

LQR with Bilinear Noise

Let's look at a simple case.

$$\min_{u(\cdot)} \mathbf{E}\left\{\frac{1}{2}\int_0^\infty x'Qx + 2x'Su + u'Ru \; dt\right\}$$

subject to the Ito equation

$$\begin{aligned} dx &= (Fx + Gu)\,dt + \sum_{k=1}^{r} (C_k x + D_k u)\,dw_k \\ x(0) &= x^0 \end{aligned}$$

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$$\begin{aligned} dx &= (Fx + Gu)\, dt + \sum_{k=1}^{r} (C_k x + D_k u)\, dw_k \\ x(0) &= x^0 \end{aligned}$$

We suspect that optimal cost and optimal feedback are of the form

$$\begin{aligned} \pi(x) &= \frac{1}{2} x'Px \\ \kappa(x) &= Kx \end{aligned}$$

and plug these expressions into the sSHJB equations.

Stochastic Algebraic Riccati Equations (SARE)

$$
\begin{aligned}
\mathbf{0} &= PF + F'P + Q - K'RK \\
&\quad + \sum_{k=1}^{r} \left(C_k' + K'D_k'\right) P \left(C_k + D_k K\right) \\
K &= -\left(R + \sum_{k=1}^{r} D_k' P D_k\right)^{-1} \left(G'P + S' + \sum_{k=1}^{r} D_k' P C_k\right)
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Does SARE have a nonnegative definite solution P and how do we find it?

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Does SARE have a nonnegative definite solution P and how do we find it?

Here is an iterative method for solving SARE. Let $P_{(0)}$ and $K_{(0)}$ be be the solutions of the deterministic ARE

$$
\begin{aligned}
0 &= P_{(0)}F + F'P_{(0)} + Q - K_{(0)}'RK_{(0)} \\
K_{(0)} &= -R^{-1}(G'P + S')
\end{aligned}
$$

SARE Iteration

Given $P_{(\tau-1)}$ define

$$
\begin{aligned}
Q_{(\tau)} &= Q + \sum_{k=1}^{r} C_k' P_{(\tau-1)} C_k \\
R_{(\tau)} &= R + \sum_{k=1}^{r} D_k' P_{(\tau-1)} D_k \\
S_{(\tau)} &= S + \sum_{k=1}^{r} C_k' P_{(\tau-1)} D_k
\end{aligned}
$$

SARE Iteration

Given $P_{(\tau-1)}$ define

$$
\begin{aligned}
Q_{(\tau)} &= Q + \sum_{k=1}^{r} C'_k P_{(\tau-1)} C_k \\
R_{(\tau)} &= R + \sum_{k=1}^{r} D'_k P_{(\tau-1)} D_k \\
S_{(\tau)} &= S + \sum_{k=1}^{r} C'_k P_{(\tau-1)} D_k
\end{aligned}
$$

Let $P_{(\tau)}$ and $K_{(\tau)}$ be the solutions of the ARE

$$
\begin{aligned}
0 &= P_{(\tau)} F + F' P_{(\tau)} + Q_{(\tau)} - K'_{(\tau)} R_{(\tau)} K_{(\tau)} \\
K_{(\tau)} &= -R_{(\tau)}^{-1} \left(G' P_{(\tau)} + S'_{(\tau)} \right)
\end{aligned}
$$

SARE Iteration

Given $P_{(\tau-1)}$ define

$$\begin{aligned} Q_{(\tau)} &= Q + \sum_{k=1}^{r} C_k' P_{(\tau-1)} C_k \\ R_{(\tau)} &= R + \sum_{k=1}^{r} D_k' P_{(\tau-1)} D_k \\ S_{(\tau)} &= S + \sum_{k=1}^{r} C_k' P_{(\tau-1)} D_k \end{aligned}$$

Let $P_{(\tau)}$ and $K_{(\tau)}$ be the solutions of the ARE

$$\begin{aligned} 0 &= P_{(\tau)} F + F' P_{(\tau)} + Q_{(\tau)} - K_{(\tau)}' R_{(\tau)} K_{(\tau)} \\ K_{(\tau)} &= -R_{(\tau)}^{-1} \left(G' P_{(\tau)} + S_{(\tau)}' \right) \end{aligned}$$

We have found using MATLAB's `are.m`, that if matrices C_k and D_k are not too big then the iteration conveges. But it can diverge if C_k and D_k are large. Further study of this is needed.

Stochastic Nonlinear Optimal Control

Suppose the problem is not linear-quadratic, the dynamics is given by an Ito equation

$$dx = f(x,u)\ dt + \sum_{k=1}^{r} \gamma_k(x,u)\ dw_k$$

and the criterion to be minimized is

$$\min_{u(\cdot)} \mathbf{E} \int_0^\infty l(x,u)\ dt$$

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We assume that $f(x,u), \gamma_k(x,u), l(x,u)$ are smooth functions that have Taylor polynomial expansions around $x = 0, u = 0$,

$$\begin{aligned} f(x,u) &= Fx + Gu + f^{[2]}(x,u) + \ldots + f^{[d]}(x,u) + O(x,u)^{d+1} \\ \gamma_k(x,u) &= C_k x + D_k u + \gamma_k^{[2]}(x,u) + \ldots + \gamma_k^{[d]}(x,u) + O(x)^{d+1} \\ l(x,u) &= \frac{1}{2}\left(x'Qx + 2x'Su + u'Ru\right) + l^{[3]}(x,u) + \ldots + l^{[d+1]}(x \end{aligned}$$

Stochastic HJB Equations

The sSHJB equations are

$$
\begin{aligned}
0 &= \frac{\partial \pi}{\partial x}(x) f(x, \kappa(x)) + l(x, \kappa(x)) \\
&\quad + \frac{1}{2} \sum_{k=1}^{r} \gamma_k'(x, \kappa(x)) \frac{\partial^2 \pi}{\partial x^2}(x) \gamma_k(x, \kappa(x)) \\
0 &= \frac{\partial \pi}{\partial x}(x) \frac{\partial f}{\partial u}(x, \kappa(x)) + \frac{\partial l}{\partial u}(x, \kappa(x)) \\
&\quad + \sum_{k=1}^{r} \gamma_k'(x, \kappa(x)) \frac{\partial^2 \pi}{\partial x^2}(x) \frac{\partial \gamma_k}{\partial u}(x, \kappa(x))
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&\quad + \sum_{k=1}^{r} \gamma_k'(x,\kappa(x)) \frac{\partial^2 \pi}{\partial x^2}(x) \frac{\partial \gamma_k}{\partial u}(x,\kappa(x))
\end{aligned}
$$

The simplified Stochastic HJB equations are second order and have a regular singular point at $x = 0$.

Stochastic HJB Equations

Following Euler and Al'brekht we assume that the optimal cost and the optimal feedback have Taylor polynomial expansions

$$\begin{aligned}
\pi(x) &= \frac{1}{2}x'Px + \pi^{[3]}(x) + \ldots + \pi^{[d+1]}(x) + O(x)^{d+2} \\
\kappa(x) &= Kx + \kappa^{[2]}(x) + \ldots + \kappa^{[d]}(x) + O(x)^{d+1}
\end{aligned}$$

We plug all these expansions into the simplified SHJB equations At lowest degrees, we get the familiar SARE.

$$\begin{aligned}
0 &= PF + F'P + Q - K'RK \\
&\quad + \sum_{k=1}^{r} \left(C_k' + K'D_k'\right) P \left(C_k + D_k K\right) \\
K &= -\left(R + \sum_{k=1}^{r} D_k' P D_k\right)^{-1} \left(G'P + S' + \sum_{k=1}^{r} D_k' P C_k\right)
\end{aligned}$$

Stochastic HJB Equations

If SARE are solvable then we may proceed to the next degrees

$$\begin{aligned} 0 &= \frac{\partial \pi^{[3]}}{\partial x}(x)(F+GK)x + x'Pf^{[2]}(x,Kx) + l^{[3]}(x,Kx) \\ &+ \frac{1}{2}\sum_k x'(C_k'+K'D_k')\frac{\partial^2 \pi^{[3]}}{\partial x^2}(x)(C_k+D_kK)x \\ &+ \sum_k x'(C_k'+K'D_k)P\gamma_k^{[2]}(x,Kx) \end{aligned}$$

$$\begin{aligned} 0 &= \frac{\partial \pi^{[3]}}{\partial x}(x)G + x'P\frac{\partial f^{[2]}}{\partial u}(x,Kx) + \frac{\partial l^{[3]}}{\partial u}(x,Kx) \\ &+ \sum_k x'(C_k+D_kK)'\left(P\frac{\partial \gamma_k^{[2]}}{\partial u}(x,Kx) + \frac{\partial^2 \pi^{[3]}}{\partial x^2}(x)D_k\right) \\ &+ \sum_k \gamma_k^{[2]}(x,Kx)PD_k + (\kappa^{[2]}(x))'\left(R + \sum_k D_k'PD_k\right) \end{aligned}$$

Stochastic HJB Equations

The unknowns in these linear equations are $\pi^{[3]}(x)$ and $\kappa^{[2]}(x)$ nt Notice that the first equation does not contain $\kappa^{[2]}(x)$ and $\pi^{[3]}(x)$ appears twice.

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The eigenvalues of the linear operator

$$\pi^{[3]}(x) \quad \mapsto \quad \frac{\partial \pi^{[3]}}{\partial x}(x)(F+GK)x$$

are sums of three eigenvalues of $F+GK$ in the open left half plane and hence never zero.

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The eigenvalues of the linear operator

$$\pi^{[3]}(x) \quad \mapsto \quad (C_k' + K'D_k')\frac{\partial^2 \pi^{[3]}}{\partial x^2}(x)(C_k+D_kK)$$

are sums of three products of two eigenvalues of C_k+D_kK and are small if C_k+D_kK is small.

Stochastic HJB Equations

We have found that these equations are solvable if C_k and D_k are not too big.

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We have written general purpose MATLAB code to solve these equations to any degree in any dimensions.

Stochastic HJB Equations

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The higher degree terms are found in a similar fashion.

We have written general purpose MATLAB code to solve these equations to any degree in any dimensions.

The code is fast but in high degrees and/or high dimensions requires considerable memory.

ISS Example Revisited

We return to the example of the noisy inverted pendulum on the ISS.

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We return to the example of the noisy inverted pendulum on the ISS.

Because the Lagrangian is an even function and the dynamics is an odd function of x, u, we know that $\pi(x)$ is an even function and $\kappa(x)$ is an odd function.

$$
\begin{aligned}
\pi(x) &= 26.7042x_1^2 + 17.4701x_1x_2 + 2.9488x_2^2 \\
&\quad -4.6153x_1^4 - 2.9012x_1^3x_2 - 0.5535x_1^2x_2^2 \\
&\quad -0.0802x_1x_2^3 - 0.0157x_2^4 \\
&\quad 0.3361x_1^6 + 0.1468x_1^5x_2 - 0.0015x_1^4x_2^2 - 0.0077x_1^3x_2^3 \\
&\quad -0.0022x_1^2x_2^4 - 0.0003x_1x_2^5 + 0.0000x_2^6 + \ldots \\
\kappa(x) &= -17.4598x_1 - 5.8941x_2 \\
&\quad +2.9012x_1^3 + 1.1071x_1^2x_2 + 0.2405x_1x_2^2 + 0.0628x_2^3 \\
&\quad -0.1468x_1^5 + 0.0031x_1^4x_2 + 0.0232x_1^3x_2^2 \\
&\quad +0.0089x_1^2x_2^3 + 0.0014x_1x_2^4 - 0.0002x_2^5 + \ldots
\end{aligned}
$$

ISS Example Revisited

Notice the some of quartic terms have negative signs. Why?

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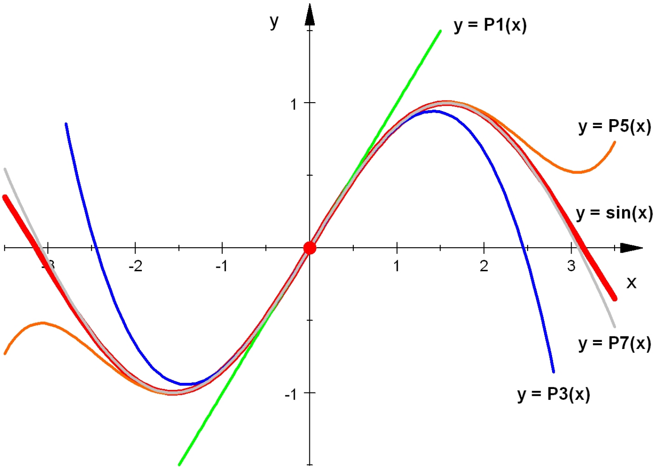

Figure: Taylor approximations of sin(x)

Finite Horizon Continuous Time Extension

This method readily extends to finite horizon stochastic optimal control problems.

$$\min_{u(\cdot)} \mathbf{E}\left\{\int_0^T l(t,x,u)dt + \pi_T(x(T))\right\}$$

subject to

$$\begin{aligned} dx &= f(t,x,u)dt + \sum_{k=1}^{r} \gamma_k(t,x,u)dw_k \\ x(0) &= x^0 \end{aligned}$$

We assume that f, l, γ_k, π_T are smooth and $\gamma_k(t,0,0) = 0$.

Finite Horizon Continuous Time Extension

At the lowest degrees we get a stochastic differential Riccati equation that is well-known.

$$\begin{aligned} 0 &= \dot{P}(t) + P(t)F(t) + F'(t)P(t) + Q(t) - K'(t)R(t)K(t) \\ &\quad + \sum_k \left(C_k'(t) + K'(t)D_k'(t)\right) P(t) \left(C_k(t) + D_k(t)K(t)\right) \\ K(t) &= -\left(R(t) + \sum_{k=1}^{r} D_k'(t)P(t)D_k(t)\right)^{-1} \left(G'(t)P(t) + S(t)\right) \\ P(T) &= P_T \end{aligned}$$

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K(t) &= -\left(R(t) + \sum_{k=1}^{r} D_k'(t)P(t)D_k(t)\right)^{-1} \left(G'(t)P(t) + S(t)\right) \\
P(T) &= P_T
\end{aligned}$$

What is new is that we can find the higher degree terms of the optimal cost and the optimal feedback by solving a series of time varying linear differential equations.

Finite Horizon Continuous Time Extension

$$
\begin{aligned}
0 \quad = \quad & \frac{\partial \pi^{[3]}}{\partial t}(t,x) + \frac{\partial \pi^{[3]}}{\partial x}(t,x)(F(t)+G(t)K(t))x \\
& + x' P(t) f^{[2]}(t,x,K(t)x) + l^{[3]}(t,x,Kx) \\
& + \frac{1}{2}\sum_k x' C_k'(t) \frac{\partial^2 \pi^{[3]}}{\partial x^2}(t,x)\left(C_k + D_k(t)K(t)\right)(t)x \\
& + \sum_k x' \left(C_k'(t) + K'(t)D_k'(t)\right) P(t)\gamma_k^{[2]}(t,x)
\end{aligned}
$$

$$
\begin{aligned}
0 \quad = \quad & \frac{\partial \pi^{[3]}}{\partial x}(t,x)G(t) + x'P(t)\frac{\partial f^{[2]}}{\partial u}(t,x,K(t)x) + \frac{\partial l^{[3]}}{\partial u}(t,x,K(t)x) \\
+ \quad & \sum_k x'(C_k(t) + D_k(t)K(t))' \left(P(t)\frac{\partial \gamma_k^{[2]}}{\partial u}(x,K(t)x) + \frac{\partial^2 \pi^{[3]}}{\partial x^2}(x)D_k(t)\right) \\
+ \quad & \sum_k \gamma_k^{[2]}(x,K(t)x)P(t)D_k(t) + (\kappa^{[2]}(t,x))' \left(R(t) + \sum_k D_k'(t)PD_k(t)\right)
\end{aligned}
$$

Infinite Horizon Discrete Time Extension

This method readily extends to infinite horizon stochastic optimal control problems in discrete time.

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$$\min_{u(\cdot)} \mathbf{E}\left\{\sum_{t=0}^{\infty} l(x,u)\right\}$$

subject to

$$\begin{aligned} x^+ &= f(x,u) + \sum_{k=1}^{r} \gamma_k(x,u) w_k \\ x(0) &= x^0 \end{aligned}$$

where $w(t) = [w_1(t); w_2(t); \ldots; w_r(t)]$ **is a sequence of independent standard normal** r **vectors.**

Infinite Horizon Discrete Time Extension

At the lowest degrees we get new Stochastic Discrete Time Algebraic Riccati Equation (SDARE)

$$
\begin{aligned}
P &= Q + K'RK + (F+GK)'P(F+GK) \\
&\quad + \sum_{k=1^r} (C_k + D_k K)'P(C_k + D_k K) \\
K &= -\left(R + G'PG + \sum_{k=1}^{r} D_k' P D_k\right)^{-1} \\
&\quad \times \left(G'PF + S' + \sum_{k=1}^{r} D_k' P C_k\right)
\end{aligned}
$$

Infinite Horizon Discrete Time Extension

At degrees three and two we get the square linear equations

$$\pi^{[3]}(x) = \mathbf{E}\left\{\pi^{[3]}\left((F+GK)x + \sum_k w_k(C_k + D_k K)x\right)\right\}$$

$$+x'(F+GK)'Pf^{[2]}(x,Kx) + \sum_k x'(C_k + D_k K)'P\gamma_k^{[2]}(x,Kx) + l^{[3]}(x,Kx)$$

$$0 = \mathbf{E}\left\{\frac{\partial \pi^{[3]}}{\partial x}\left((F+GK)x + \sum_k w_k(C_k + D_k K)x\right)\left(G + \sum_k w_k D_k\right)\right\}$$

$$+x'P\frac{\partial f^{[2]}}{\partial u}(x,Kx) + \frac{\partial l^{[3]}}{\partial u}(x,Kx) + (\kappa^{[2]}(x))'\left(R + G'PG + \sum_k D_k' P D_k\right)$$

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$$\min_{u(\cdot)} \mathbf{E} \left\{ \sum_{t=0}^{T-1} l(t,x,u) + \pi_T(x(T)) \right\}$$

subject to

$$\begin{aligned} x^+ &= f(t,x,u) + \sum_{k=1}^{r} \gamma_k(t,x,u) w_k \\ x(0) &= x^0 \end{aligned}$$

where $w(t) = [w_1(t); w_2(t); \ldots; w_r(t)]$ is a sequence of independent standard normal r vectors.

Finite Horizon Discrete Time Extension

At the lowest degrees we get a familiar Stochastic Discrete Time Riccati Difference Equation (SDRDE)

$$
\begin{aligned}
P(t) &= Q(t) + K'(t)S(t) + S(t)K'(t) + K'(t)R(t)K(t) \\
&\quad + (F(t) + G(t)K(t))'P(t+1)(F(t) + G(t)K(t)) \\
&\quad + \sum_{k=1^r} (C_k(t) + D_k(t)K(t))'P(t+1)(C_k(t) + D_k(t)K(t)) \\
K(t) &= -\left(R(t) + G'(t)P(t+1)G(t) + \sum_{k=1}^{r} D_k'(t)P(t+1)D_k(t)\right)^{-1} \\
&\quad \times \left(G'(t)P(t+1)F(t) + S'(t) + \sum_{k=1}^{r} D_k'(t)P(t+1)C_k(t)\right) \\
P(T) &= P_T
\end{aligned}
$$

Finite Horizon Discrete Time Extension

At the next degrees we get the linear difference equations

$$\pi^{[3]}(t,x) = \mathbf{E}\left\{\pi^{[3]}\left(t+1, z(t,x,w)\right)\right\}$$

$$+x'(F(t)+G(t)K(t))'P(t+1)f^{[2]}(t,x,Kx)$$

$$+\sum_k x'(C_k(t)+D_k(t)K(t))'P(t+1)\gamma_k^{[2]}(t,x,Kx)+l^{[3]}(t,x,Kx)$$

$$0 = \mathbf{E}\left\{\frac{\partial \pi^{[3]}}{\partial x}\left(t, z(t,x,w)\right)\left(G(t)+\sum_k w_k D_k(t)\right)\right\}$$

$$+x'P(t+1)\frac{\partial f^{[2]}}{\partial u}(t,x,K(t)x)+\frac{\partial l^{[3]}}{\partial u}(t,x,K(t)x)$$

$$+(\kappa^{[2]}(t,x))'\left(R(t)+G'(t)P(t+1)G(t)+\sum_k D_k'(t)P(t+1)D_k(t)\right)$$

$$\pi^{[3]}(T,x) = \pi_T^{[3]}(x)$$

where

$$z(t,x,w) \quad = \quad F(t)+G(t)K(t))x+\sum_k w_k(C_k(t)+D_k(t)K(t)x$$

Finite Horizon Discrete Time Extension

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$$+\sum_k x'(C_k(t)+D_k(t)K(t))'P(t+1)\gamma_k^{[2]}(t,x,Kx) + l^{[3]}(t,x,Kx)$$
$$0 = \mathbf{E}\left\{\frac{\partial \pi^{[3]}}{\partial x}\left(t, z(t,x,w)\right)\left(G(t)+\sum_k w_k D_k(t)\right)\right\}$$
$$+x'P(t+1)\frac{\partial f^{[2]}}{\partial u}(t,x,K(t)x) + \frac{\partial l^{[3]}}{\partial u}(t,x,K(t)x)$$
$$+(\kappa^{[2]}(t,x))'\left(R(t)+G'(t)P(t+1)G(t)+\sum_k D_k'(t)P(t+1)D_k(t)\right)$$
$$\pi^{[3]}(T,x) = \pi_T^{[3]}(x)$$

where

$$z(t,x,w) \quad = \quad F(t)+G(t)K(t))x + \sum_k w_k(C_k(t)+D_k(t)K(t)x$$

We are in the process of writing MATLAB code to solve these equations to any degree in any dimensions.

Our Slogan

Think Mathematically

Our Slogan

Think Mathematically

Act Computationally

Thank You

Thank You

Questions?