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Finite difference approximations for the determination of dynamic instability

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ABSTRACT

Approximate forms of the vorticity and thermal equations are linearized and combined to yield a second-order partial differential equation for the amplitude of harmonic perturbations. Finite-difference approximations for the derivatives yield a homogeneous system of algebraic equations; and the condition that its determinants vanish for a non-trivial solution yields the "frequency" equation, which may be solved to give the phase velocities of the harmonic waves. Solutions are obtained for zonal currents in which the wind varies vertically and horizontally and for a variety of conditions with respect to grid distances, latitude and current width. Generally speaking, the computations showed that decreasing the latitude and shear and increasing the static stability were all destabilizing influences, not without some exceptions, however. In addition, very short waves were found to be stable; however, instability was found for very long waves, including a retrogressive unstable mode. Moreover, multiple unstable modes were found for many wavelengths.

Calculations based on actual observations of the jet stream in December show it to be dynamically unstable, both baroclinically and barotropically, with one mode of maximum instability at a wavelength of about 3000 to 4000 km and a secondary maximum at about 10,000 km.

1. Introduction

The dynamic stability of parallel currents has been a subject of interest to hydrodynamicists and meteorologists for a long time. With regard to the meteorological aspects, a classical paper by CHARNEY (1947) provided the first analysis of the stability of baroclinic currents with vertical shear. On the other hand, KUO (1949), following the work of C. C. Lin, determined the necessary and sufficient condition for dynamic instability in a non-divergent barotropic jet. The baroclinic case has received much attention including the recent work of GREEN (1960) and BURGER (1962), the latter clarifying the relationship between instability and wavelength of the perturbation. Both CHARNEY (1951) and POCINKI (1955) have considered the problem of vertical shear superimposed on a horizontal jet structure with the general conclusion that the vertical shear, with its concomitant horizontal temperature gradient and the conversion of potential energy to kinetic energy, is primarily responsible for the development of the typical large-scale unstable waves observed in the atmosphere. Very recently CHARNEY & STERN (1962) developed stability criteria for the case of an "internal jet" which is stable when the gradient of potential vorticity in isentropic surfaces does not vanish. On the other hand, if it vanishes on a closed isopleth of constant mean zonal vorticity the jet is unstable.

In all of the investigations cited above the differential boundary value problem is highly complicated and it is very difficult to obtain information concerning the solutions and phase velocities and their relationship to the characteristics of the basic current, that is, latitude wavelength, etc. To obtain such information it has frequently been necessary to resort to approximation methods. The latter have consisted mainly of utilizing finite differences for approximating certain derivatives or using a finite Fourier series to achieve this end. For example, WIIN-NIELSEN (1961) utilized the Fourier series method to determine the stability properties of the divergent quasi-barotropic model for a jet-type current. HALTINER & SONG (1963) made some comparisons between the finite-difference and finite-Fourier methods for several quasi-barotropic models with both single- and double-jet currents. On the other hand, THOMPSON (1953), WIIN-NIELSEN (1959–1962), and others have used finite-differences for vertical derivatives in two- and three-level models to examine baroclinic flows.

In this study finite-difference approximations are utilized for both vertical and lateral derivatives in order to investigate zonal flows possessing both horizontal and vertical shear. The static stability, latitude, and grid distances are varied somewhat so as to determine their effects on the dynamic stability of harmonic waves.

2. Derivation of the basic equations

The vorticity equation will be approximated by the following conventional form:

$$\frac{\partial \zeta}{\partial t} + V \cdot \nabla(\zeta + f) = \bar{f}\frac{\partial \omega}{\partial p}. \quad (1)$$

Here $V = k \times \nabla \Psi$, $\bar{f}$ is the mean coriolis parameter and $\omega = dp/dt$. Next the first law of thermodynamics for adiabatic motions may be written

$$\frac{\partial T}{\partial t} + V \cdot \nabla T - \sigma' \omega = 0, \quad (2)$$

with
$$\sigma' = \frac{RT}{pc_p} - \frac{\partial T}{\partial p}. \quad (3)$$

The further approximation

$$\frac{\partial \phi}{\partial p} = \bar{f}\frac{\partial \Psi}{\partial p}, \quad (4)$$

together with the hydrostatic equation relating geopotential ϕ and T, permits equation (2) to be expressed as

$$\frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial p}\right) + V \cdot \nabla \frac{\partial \Psi}{\partial p} + \sigma \omega = 0. \quad (5)$$

$$\sigma = \frac{R}{p\bar{f}}\left(\frac{RT}{pc_p} - \frac{\partial T}{\partial p}\right). \quad (6)$$

Equations (1) and (5) constitute a pair of equations in the unknows Ψ and ω. Next these equations are linearized with $\Psi = \bar{\Psi}(y,p) + \Psi'(x,y,p,t)$ and $\omega = \omega'(x,y,p,t)$, with the result

$$\frac{\partial}{\partial t}\nabla^2\Psi' + U\frac{\partial}{\partial x}\nabla^2\Psi' + \left(\beta - \frac{\partial^2 U}{\partial y^2}\right)\frac{\partial \Psi'}{\partial x} = \bar{f}\frac{\partial \omega'}{\partial p}. \quad (7)$$

$$\frac{\partial}{\partial t}\frac{\partial \Psi'}{\partial p} + U\frac{\partial}{\partial x}\frac{\partial \Psi'}{\partial p} - \frac{\partial U}{\partial p}\frac{\partial \Psi'}{\partial x} + \sigma\omega' = 0, \quad (8)$$

where $U = U(y,p)$ represents the basic current. Now assuming harmonic solutions of the form $\Psi' = A(y,p)\,e^{i\mu(x-ct)}$, $\omega' = W(y,p)\,e^{i\mu(x-ct)}$ and substituting into equations (7) and (8) leads, respectively, to

$$(U-c)\frac{\partial^2 A}{\partial y^2} + \left[\beta - \frac{\partial^2 U}{\partial y^2} - \mu^2(U-c)\right]A + \frac{i\bar{f}}{\mu}\frac{\partial W}{\partial p} = 0 \quad (9)$$

$$(U-c)\frac{\partial A}{\partial p} - \frac{\partial U}{\partial p}A - \frac{i\sigma}{\mu}W = 0. \quad (10)$$

For boundary conditions it will be assumed that

$$A = 0, \quad \text{at } y = 0 \quad \text{and} \quad y = D. \quad (11)$$

$$W = 0, \quad \text{at } p = 0 \quad \text{and} \quad p = p_0 = 1000 \text{ mb}. \quad (12\text{a})$$

When substituted into equation (10), the vertical boundary conditions (12a) show that

$$(U-c)\frac{\partial A}{\partial p} - \frac{\partial U}{\partial p}A = 0, \quad \text{at} \quad p = 0 \quad \text{and} \quad p = p_0. \quad (12\text{b})$$

The function W is easily eliminated from the system (9) and (10) by differentiating the latter with respect to p, giving the following equation in A:

$$(U-c)\left(\frac{\partial^2 A}{\partial y^2} + \frac{\bar{f}}{\sigma}\frac{\partial^2 A}{\partial p^2}\right) - \frac{\bar{f}(U-c)}{\sigma^2}\frac{\partial \sigma}{\partial p}\frac{\partial A}{\partial p} + \left[\beta - \frac{\partial^2 U}{\partial y^2} - \mu^2(U-c) - \frac{\bar{f}}{\sigma}\frac{\partial^2 U}{\partial p^2} + \frac{\bar{f}}{\sigma^2}\frac{\partial \sigma}{\partial p}\frac{\partial U}{\partial p}\right]A = 0. \quad (13)$$

This equation is to be solved for A, subject to the boundary conditions (11) and (12a) for given distributions of the basic flow U and the related static stability σ.

If the vertical variations of U, A, and σ are omitted, equation (13) reduces to the familiar linearized equation for non-divergent barotropic perturbations. On the other hand, disregarding the lateral variation of U and A leads to an equation for purely baroclinic disturbances. In any case the lateral boundary conditions (11) impose a finite lateral extent.

Equation (13) may now be reduced to a system of algebraic equations by utilizing a finite Fourier series to represent A or by approximating the derivatives of A by finite differences. The latter method will be applied here, since it appears to be somewhat simpler with respect to the determination of the phase velocities. The atmosphere will be divided vertically into N layers, each of pressure thickness P, with the adjacent pressure levels designated from 0, at $p=0$, to N, corresponding to $p=1000$ mb. Similarly the lateral direction will be subdivided into increments of width d. At this point it will be further assumed that the perturbations are laterally symmetrical with respect to the center of the *basic current*. Hence only half of the channel need be considered, and it will be divided into M bands with the center of the current denoted by the index $j=0$, and the northern boundary by the index $j=M$.

Use of the usual centered difference approximations for the derivatives in (13) results in the following system of homogeneous algebraic equations:

$$\frac{(U_{j,k}-c)}{d^2}A_{j-1,k}+\left\{-(U_{j,k}-c)\left[\frac{2}{d^2}+\frac{2\bar{f}}{P^2\sigma_{j,k}}+\left(\frac{2\pi}{L}\right)^2\right]+\beta-\frac{(U_{j+1,k}-2U_{j,k}+U_{j-1,k})}{d^2}\right.$$
$$-\frac{\bar{f}}{P^2\sigma_{j,k}}(U_{j,k+1}-2U_{j,k}+U_{j,k-1})$$
$$\left.+\frac{\bar{f}}{4P^2\sigma_{j,k}^2}(\sigma_{j,k+1}-\sigma_{j,k-1})(U_{j,k+1}-U_{j,k-1})\right\}$$
$$\times A_{j,k}+\frac{(U_{j,k}-c)}{d^2}A_{j+1,k}+\frac{\bar{f}(U_{j,k}-c)}{P^2\sigma_{j,k}}$$
$$\times\left[1-\frac{(\sigma_{j,k+1}-\sigma_{j,k-1})}{4\sigma_{j,k}}\right][A_{j,k-1}+A_{j,k+1}]$$
$$=0\ (j=0,1,\dots,(M-1);\ k=1,2,\dots,(N-1)).\qquad (14)^{1}$$

The lateral boundary condition (11) and the assumption of lateral symmetric require:

$$A_{M,k}=0\quad (k=0,1,\dots,N);\quad \text{and}\quad A_{-1,k}=A_{1,k}.\qquad (15)$$

Finally the vertical boundary conditions (12a) lead to

$$\left.\begin{aligned} A_{j,N}&=A_{j,N-1}\frac{(U_{j,N}-c)}{(U_{j,N-1}-c)};\\ A_{j,0}&=A_{j,1}\frac{(U_{j,0}-c)}{U_{j,1}-c}.\end{aligned}\right\}\qquad (16)$$

Here a backward and a forward difference were used, respectively, for the derivatives at the vertical boundaries.

The condition that the homogeneous system (14), (15) and (16) possess a solution for the A's is that the determinant of the coefficients vanish. This yields the "frequency equation" for the phase velocity c, the value of which can be calculated for given distributions of wind, static stability and also latitude, wavelength, etc. These phase velocities, in turn, can then be used with the values of the A's to construct the complete solution. In this study the stability characteristics, as indicated by these wave velocities, are of primary interest. A series of computations was made for a number of different distributions of zonal wind and static stability, as well as varying wavelength, latitude and grid length.

3. Results

The first zonal wind field considered is represented by the following matrices of zonal velocity (Table 1) and static stability (Table 2).

[1] LORENZ (1960) has shown by application of the Gauss divergence theorem that for the system (1), (2) to conserve energy (excluding the kinetic energy of the vertical motion), the static stability parameter σ must be treated as a function of p at most. To conform with these results, the subscript j on σ in (14) must be considered as superfluous.

TABLE 1. *Zonal wind velocity U in m/sec with pressure and latitude increasing with the vertical and horizontal indices k and j.*

0	0	0	0	0	0	0	0
1	0	2.00	7.50	15.00	22.50	28.00	30.00
2	0	4.00	15.00	30.00	45.00	56.00	60.00
3	0	1.80	6.75	13.50	20.25	25.00	27.00
4	0	1.20	4.50	9.00	13.50	16.80	18.00
5	0	0.80	3.00	6.00	9.00	11.20	12.00
6	0	0.47	1.75	3.50	5.25	6.50	7.00
7	0	0.20	0.75	1.50	2.50	2.80	3.00
8	0	0	0	0	0	0	0
k/j	6	5	4	3	2	1	0

In addition, $\bar{f} = 1.03 \times 10^{-4}\,\text{sec}^{-1}$ and $\beta = 1.63 \times 10^{-14}\,\text{sec}^{-1}\,\text{cm}^{-1}$ (corresponding to 45 degrees latitude), $d = 250$ km, $P = 125$ mb, and the total width of the basic current D is 3000 km. This particular velocity distribution was chosen to correspond to the function $U = B(1 - \cos 2\pi y/D)$, with $B = 30$ m/sec at the level of maximum wind. This field satisfies Kuo's condition for barotropic instability and has been investigated to some extent previously (WIIN-NIELSEN, 1962; HALTINER, 1963). The temperature field was determined first by taking the 1000 mb values as well as a vertical sounding near the center of the jet from some mean values for 12 days in December (1946). The remainder of the temperature distribution was determined by the geostrophic thermal wind relation, and finally the stability parameter was calculated from the temperature field. The computed static stability field showed a few slightly super-adiabatic values just below the tropopause surmounted by a sharp inversion. To avoid these abnormalities, the σ field was smoothed somewhat in the vicinity of the tropopause, with the results shown in Table 2. The last column shows the horizontally averaged values $\bar{\sigma}$ for each pressure level.

TABLE 3. *Complex values of wave velocity as a function of wavelength for Case 1, M = 3 (d = 500 km), N = 4 (P = 250 mb).*

Zonal wind is given by Table 1 and mean static stability $\bar{\sigma}$ in the last column of Table 2 (alternate values only). Case 2 is similar to Case 1 except that the zonal wind is a linear function of pressure.

	c(m/s)	
$L(10^3\,\text{km})$	Case 1	Case 2
1	none	none
2	$7.30 \pm 0.31\,i$	$16.43 \pm 1.47\,i$
3	none	$21.78 \pm 1.51\,i$; $6.18 \pm 1.43\,i$
4	none	$21.11 \pm 1.79\,i$
5	$17.62 \pm 2.05\,i$	$20.49 \pm 2.63\,i$
6	$15.78 \pm 2.70\,i$	$19.22 \pm 3.97\,i$
7	$14.29 \pm 2.13\,i$	$17.89 \pm 4.57\,i$
8	none	$16.75 \pm 4.70\,i$
9	none	$15.78 \pm 4.58\,i$
10	none	$14.98 \pm 4.29\,i$
11	none	$14.29 \pm 3.90\,i$
12	none	$13.69 \pm 3.42\,i$
13	$-0.22 \pm 0.41\,i$	$13.15 \pm 2.84\,i$
14	$-0.84 \pm 0.83\,i$	$12.60 \pm 2.08\,i$
15	$-1.41 \pm 0.73\,i$	$10.80 \pm 1.09\,i$
16–25	none	no computation

Next, wave velocities were determined for a number of different cases with several values of P and d. Table 3 gives only the *complex values* ($c_i \neq 0$) for Case 1 corresponding to $d = 500$ km ($M = 3$); and $P = 250$ mb ($N = 4$), and for wavelengths ranging from 1000 to 25,000 km. Progressive unstable waves are found for wavelengths of 5000 through 7000 km, while slowly retrogressive waves with considerably

TABLE 2. *Static stability σ in MTS units.*

0		10^3	10^3	10^3	10^3	10^3	10^3	10^3
1	—	107.9	107.8	108.5	112.4	113.5	112.1	110.0
2	—	54.50	52.70	48.70	41.40	31.70	21.9	42.54
3	—	0.20	0.60	1.25	2.40	3.75	5.50	2.17
4	—	4.74	4.79	4.91	5.12	5.37	5.65	5.08
5	—	3.13	3.12	3.11	3.12	3.12	3.15	3.12
6	—	2.00	2.00	2.02	2.15	2.22	2.01	2.08
7	—	1.08	1.10	1.14	1.18	1.22	1.58	1.19
8	—	1.00	1.05	1.11	1.15	1.20	1.25	1.13
k/j	6	5	4	3	2	1	0	$\bar{\sigma}$

TABLE 4. *Wave velocities (m/s) for Case 3, $M = 3$ ($d = 500$ km), $N = 4$ ($P = 250$ mb) for zonal wind of Table 1 (alternate values only), $\bar{\sigma}$ of Table 2 and a wavelength of 600 km.*

Case 1 (*a*)	Case 3 (*b*)
60.00	60.00
45.00	45.00
41.14	40.74
15.00	15.00
$15.78 \pm 2.70\,i$	$16.38 \pm 2.35\,i$
8.12	9.16
7.00	7.00
5.97	6.65
5.25	5.25
3.25	$2.27 \pm 0.11\,i$
1.88	
1.75	1.75
−0.27	−0.83
−22.51	−22.67

smaller amplification factors are found for wavelengths of 13,000 through 15,000 kilometers. For each of the above wavelengths only one of the 15 eigenvalues was complex and the real values of c ranged from 60 m/s to −22.5 m/s.

Most previous studies of baroclinic instability assumed a simple zonal wind profile in the vertical; for example, U has been taken as a linear function of pressure or height. In order to get some indication of the effect of the vertical jet-type structure, the computations were repeated for wavelengths 1 through 15 with conditions identical to Case 1 save that the zonal wind was made to vary linearly with pressure from zero at 1000 mb and reaching the same values as the row, $k = 2$, in Table 1, and then continuing to increase at the same rate up to $k = 0$. Thus the zonal winds for the column $j = 0$, 20, 40, 60, and 80 m/sec at $k =$

TABLE 5. *Wave velocities (m/s) for a wavelength of 6000 km as a function of horizontal grid index for zonal winds of Tables 1 and 6 and static stability $\bar{\sigma}$ of Table 2.*

Those values marked with an asterisk are exactly equal to the zonal wind velocities at certain mesh points, namely at $k = 2$ and 6 in Table 1 and at $k = 1$ and 3 in Table 6.

$M = 2, N = 4$	$M = 3, N = 4$	$M = 4, N = 4$	$M = 5, N = 4$	$M = 6, N = 4$
60.00*	60.00*	60.00*	60.00*	60.00*
		51.21*	54.27*	56.00*
		47.76	51 59	53.93
	45.00*			45.00*
	41.14		39 27*	42.79
30.00*		30.00*	37.24	30.00*
			20.95	
29.35		29.12	20.93*	29.33
	15.00*			15.00*
	$15.78 \pm 2.70\,i$	$12.94 \pm 3.78\,i$	$11.53 \pm 2.43\,i$	$14.39 \pm 2.53\,i$
13.99			10.88	12.79
8.51	8.12	8.79*	7.94	8.50
		$7.91 \pm 0.28\,i$		$8.01 \pm 0.39\,i$
7.00*	7.00*	7.00*	7.00*	7.00*
		6.80	6.93	7.02
			6.33*	6.50*
	5.97	5.98*	5.82	5.66
	5.25*		5.73*	5.25*
			5.12	4.45
				4.00*
4.08		3.80	4.58*	3.82
3.50*	3.25	3.50*	2.54	3.50*
		2.645	2.42*	
	1.88		1.90	1.81
	1.75*			1.75*
		1.05	$0.72 \pm 0.03\,i$	$1.15 \pm 0.17\,i$
		1.026*		0.47*
		0.44	0.67*	0.46
−4.02	−0.27			
−23.01	−22.51	−22.33	−22.25	−22.21

TABLE 6. *Zonal wind velocity U (m/s) for the cases, n = 4, N = 4 (upper), and M = 5, N = 4 (lower).*

0	0	0	0	0	0
1	0	8.790	30.0	51.210	60.0
2	0	2.640	9.0	15.360	18.0
3	0	1.026	3.5	5.980	8.0
4	0	0	0	0	0
k/j	4	3	2	1	0

0	0	0	0	0	0	0
1	0	5.730	20.730	39.270	54.270	60
2	0	1.720	6.220	11.780	16.280	18
3	0	0.669	2.420	4.580	6.330	7
4	0	0	0	0	0	0
j/k	5	4	3	2	1	0

8, 6, 4, 2, and 0, respectively. The results, which are labeled Case 2 in Table 3, are markedly different in several respects. Firstly there is more instability including two unstable modes for $L = 3000$ km; and secondly the instability is shifted toward longer wavelengths. This seems to suggest that the vertical jet-like structure is perhaps as important as the horizontal jet characteristics with respect to dynamic instability.

In order to determine the effect of a change in mean latitude, and also as partial check on the numerical method, the mean latitude was changed from 45° to 60°. The comparatively minor changes in the parameters $\bar{f}$ and β constitute an indirect indication of the effect of round-off errors during matrix inversion. The phase velocities for a wavelength of 6000 km are shown in Table 4 with column (*a*) corresponding to 45° and (*b*) 60°. In general the values correspond very closely. The higher latitude yields an additional unstable mode with the phase velocity $(2.27 + 0.11i)$ m/s. Although the imaginary part is rather small, there is an indication here of greater instability corresponding to the higher latitude. On the other hand, the primary unstable mode shows a slightly smaller imaginary part, namely a decrease from $2.70i$ to $2.36i$. In this connection, some other computations with a decreased β parameter corresponding to 75° gave larger c_i's, a shift in the maximum c_i toward longer wavelengths and slightly faster eastward propagation.

The purpose of the next series of computations was to determine the effects of decreasing the horizontal and vertical mesh size while maintaining the same overall current width and depth, as well as the other parameters. Table 5 gives the phase velocities for a wavelength of 6000 km, mean latitude of 45°, vertical index, $N = 4$, and horizontal indices from $M = 2$ ($d = 10^3$ km) to $M = 6$, ($d = 250$ km). The zonal winds for the $M = 2$, 3, and 6 cases are to be found in Table 1, corresponding to every third column, every second column, and all columns, respectively. The zonal velocity fields for $M = 4$ and 5 which are analytically determined with respect to the lateral direction, are given in Table 6. These are included because a number of the phase velocities (indicated with an asterisk) given in Table 5 are trivial roots corresponding to zonal velocities at mesh points at levels $k = 1$ and 3 in Table 6, and $k = 2$ and 6 in Table 1.

Table 5 shows the expected increased number of roots with an increased number of subdivisions in the horizontal. Of particular interest is the appearance of several unstable modes for this particular wavelength. The most unstable mode has an eastward propagation speed of approximately 13 to 14 m/s, while the other unstable modes have much smaller amplification factors. There is quite good continuity of the various roots with increasing number of subdivisions; however, indications of convergence are not always apparent, a feature which is

TABLE 7. *Complex values of wave velocity (m/s) for zonal winds of Tables 1 and 6 and mean static stability $\bar{\sigma}$ of Table 2 for a wavelength of 6000 km as a function of horizontal and vertical grid indices.*

$M = 2, N = 4$	$M = 3, N = 4$	$M = 4, N = 4$	$M = 5, N = 4$	$M = 6, N = 4$	$M = 3, N = 8$	$M = 6, N = 8$
none	$15.75 \pm 2.78i$	$12.94 \pm 3.78i$	$11.53 \pm 2.43i$	$14.39 \pm 2.35i$	$15.85 \pm 3.15i$	$14.44 \pm 3.21i$
		$7.91 \pm 0.28i$		$8.01 \pm 0.39i$	$7.86 \pm 2.34i$	$7.62 \pm 2.35i$
					$5.31 \pm 1.23i$	$4.77 \pm 0.71i$
						$3.04 \pm 0.58i$
			$0.72 \pm 0.03i$	$1.15 \pm 0.17i$	$1.58 \pm 0.53i$	$1.42 \pm 0.29i$

TABLE 8. *Similar to Table 7, but for special conditions as indicated.*

Case 4 ($\partial\sigma/\partial p = 0$, $M = 3$, $N = 8$, $L = 6$)		Case 5 ($\partial\sigma/\partial p = \partial U/\partial p = 0$, $M = 3$, $N = 8$, $L = 6$)		Case 6 ($\partial U/\partial y = 0$, $M = 3$, $N = 8$)
$\sigma = 20$ MTS units	$\sigma = 2$ MTS units	$\sigma = 20$ MTS units	$\sigma = 2$ MTS units	$L = 6$
$10.39 \pm 0.52\,i$	$28.93 \pm 8.81\,i$	$20.55 \pm 6.20\,i$	$16.65 \pm 6.77\,i$	$9.21 \pm 5.57\,i$
$9.33 \pm 2.47\,i$	$10.88 \pm 2\ 60\,i$	$16.65 \pm 6.77\,i$		$7.53 \pm 1.57\,i$
$4.88 \pm 0.19\,i$	$5.50 \pm 4.17\,i$			$6.57 \pm 1.24\,i$
	$1.71 \pm 0.123\,i$			

somewhat clouded by the addition of five roots for each unit increase of M.

Table 7 gives only the complex phase velocities for the examples of Table 5 and in addition several cases with greater vertical resolution, namely, for $M = 3$, $N = 8$ and $M = 6$, $N = 8$. Again it may be noted that the number of unstable modes increases with increasing number of mesh points, culminating in five conjugate pairs of complex eigenvalues for the $M = 6$, $N = 8$ case. There appears to be fair continuity with respect to the propagation speeds of roughly 14, 8, 5 and 1 m/s; however, the imaginary part of the phase velocity appears to fluctuate somewhat more, particularly in the rather large leap from $N = 4$ to $N = 8$.

Next some special cases were computed with uniformity of U and/or σ in the lateral or vertical direction. Case 4 in Table 8 gives the complex values of c for $L = 6000$ km and the velocity field of Table 1, but with constant static stability values of 20 and 2 MTS units, respectively. The lower static stability has one more (dynamically) unstable mode and appears generally more dynamically unstable, in conformance with earlier studies. This is also brought out by comparison to Table 7 with $\bar{\sigma}$. Perhaps the most unusual aspect here is the appearance of the root $c = 28.93 \pm 8.81i$, which is considerably larger than any other value, both with respect to propagation speed and amplification factor.

Case 5 corresponds to uniformity of U and σ in the vertical direction (barotropy) with the U values taken at the level of maximum wind, i.e., the third row ($k = 2$) of Table 1. Here the number of unstable modes is sharply decreased as compared to the baroclinic cases. There appears to be no simple explanation for the appearance of an additional phase velocity of $20.55 \pm 6.20i$ in the barotropic example with higher static stability. Perhaps, the reader should be reminded here that even though the basic current is barotropic, the disturbances are not assumed vertically uniform, nor are vertical motions neglected *a priori* as in some of the previous barotropic studies mentioned earlier.

Case 6 considers what might be termed pure baroclinic instability, that is, without lateral shear. The values of U and σ were taken at the center of the jet, corresponding to $j = 0$ in Tables 1 and 2. Nevertheless, the lateral boundary conditions still prevail so that the disturbances are of finite width. The results for $L = 6$ are given in Table 8. Comparison to Table 7 ($M = 3$, $N = 8$) shows one more unstable mode in the latter, where lateral shear is also present. It should be noted, however, that the stability parameter differed somewhat in the two cases, primarily near the level of maximum wind.

With respect to the reliability of the method used here, there are perhaps two main aspects. One concerns the replacement of the differential equation by a system of algebraic equations and thus involves truncation errors originating from the finite difference approximations to derivatives, while the second pertains to errors resulting from the iterative methods used for the inversion of the large-order matrices associated with the algebraic system. A brief discussion of the second point is given in an appendix at the end of the paper.

In general it is difficult to estimate the reliability of numerical methods because analytical solutions are not available for the models considered here. A check was made for one of the low order barotropic cases by comparing the roots obtained by the method used herein to those gotten by an entirely different method. The comparison was very good indeed. Also WIIN-NIELSEN (1962) provided an ana-

TABLE 9. *Average zonal wind (m/s) and static stability parameter (MTS units), U/σ for 12 cases in December 1946 as taken from "Jet Streams in the Atmosphere" published by the U.S. Navy.*

0	10/23.0	16/23.0	20/23.2	40/20.0	22.7
1	5/22.8	11/22.9	25/23.2	67/20.9	22.7
2	4/4.25	6/4.25	17/4.55	49/4.10	4.32
3	3/2.62	5/2.41	13/2.12	16/2.20	2.31
4	3/1.95	4/1.74	5/1.44	5/1.10	1.57
k/j	3	2	1	0	$\bar{\sigma}$

lytical solution for the case of a constant zonal current $U=10$ m/sec and variable static stability, $\sigma = 2\ (p_0/p)$ MTS units. The wave velocities he obtained are as follows: $c = 4.57$, 7.80, 8.86, and 9.31 m/s for $L = 6000$ km; and 1.51, 7.42, 8.77, and 9.28 m/sec^{-1} for $L =$ 28,000 km. The above perturbations, however, were of infinite lateral extent whereas in the present model, they are of finite width. Nevertheless, the results were very similar. All roots computed by the present method were real as above, and for $M = 3$, $N = 4$, the 15 roots ranged from 2.53 to 10 m/sec^{-1}; while for $M = 3$, $N = 8$, there were 27 real roots with exactly the same range. It should be mentioned that there were some multiple roots; for example, the value, $c = 10$ m/sec $= U$, had a multiplicity of six in both runs. It may be recalled here that Haurwitz waves of finite lateral width move somewhat slower than the corresponding Rossby waves of infinite lateral extent, which may account for the slightly lower propagation speed.

Next some computations were made with data based entirely on observation. Table 9 gives the zonal wind and stability parameter for an average jet derived from twelve observation times in December 1946 extending from the center of the jet at 45 to 60° N. The zonal winds and stabilities for $k = 0$ are taken from about 100 to 150 mb. This data was obtained from "Jet Streams of the Atmosphere", NAVWEPS 50-1P-549, published by the Chief of Naval Operations, U.S. Navy. Table 10, Case 7, gives the phase velocities for unstable waves for wavelengths from 1000 to 15,000 km utilizing these zonal winds and $\bar{\sigma}$ values. Two unstable modes appear for most wavelengths. The first mode has a maximum c_i near values of L of 4 and 5; then c_i decreases steadily up through $L = 15$. The eastward propagation speeds here appear somewhat large compared to the normal movement of the upper air troughs. The propagation speeds of the second unstable mode are more realistic, but the amplification rates are definitely smaller up through $L = 9000$ km. It is interesting to compare Table 10 to Table 3. Firstly, Case 7 shows greater instability than both Cases 1 and 2, but especially the former. From the various examples studied this appears to be partly due to the greater shear exhibited in Table 9 and also due to the assumed velocities at $k = 0$, at least in comparison to Case 1, where negative vertical shear is very marked above the level of maximum wind. This negative shear appears to have a stabilizing influence.

TABLE 10. *Wave velocities for unstable waves for the zonal wind U and mean static stability σ of Table 9, $M = 3$, $N = 4$.*

$L(10^3$ km)	Case 7 c(m/s)	
1	none	
2	$22.93 \pm 4.03\,i$	
3	$33.26 \pm 8.01\,i$	$18.23 \pm 5.16\,i$
4	$29.52 \pm 10.05\,i$	$14.94 \pm 2.66\,i$
5	$26.57 \pm 10.04\,i$	$5.73 \pm 1.88\,i$
6	$24.24 \pm 9.24\,i$	
7	$22.42 \pm 8.22\,i$	
8	$21.05 \pm 6.63\,i$	
9	$20.37 \pm 4.54\,i$	$14.68 \pm 2.42\,i$
10	$20.96 \pm 2.61\,i$	$13.57 \pm 3.53\,i$
11	$22.18 \pm 1.93\,i$	$12.48 \pm 3.84\,i$
12	$22.75 \pm 1.84\,i$	$11.64 \pm 3.83\,i$
13	$23.06 \pm 1.77\,i$	$10.98 \pm 3.67\,i$
14	$23.28 \pm 1.71\,i$	$10.46 \pm 3.44\,i$
15	$23.43 \pm 1.66\,i$	$10.07 \pm 3.17\,i$
16–25	none	

The computations were also carried out including the lateral variation of static stability as given by Table 9. Although the permission of lateral variation of static stability results in an energetically inconsistent model, it was felt that some useful information might be inferred. In any event, the phase velocities were very nearly equal to those of Table 10.

Next the computations were conducted with a vertically uniform zonal wind with values corresponding to $k = 1$ in Table 9. The static stability parameter is constant at $\sigma = 10$ MTS units and other parameters remain fixed. The

TABLE 11. *Wave velocities for unstable waves for a vertically uniform zonal wind with horizonta shear corresponding to the level k = 1 in Table.* 9.

$L(10^3\,\text{km})$	c(m/s)		
1	none		
2	none		
3	$33.13 \pm 9.39\,i$	$35.29 \pm 6.90\,i$	
4	$32.67 \pm 9.73\,i$	$36.55 \pm 4.04\,i$	$28.64 \pm 11.21\,i$
5	$30.82 \pm 10.71\,i$	$35.78 \pm 6.02\,i$	$24.71 \pm 10.69\,i$
6	$29.52 \pm 11.09\,i$	$35.30 \pm 6.89\,i$	$21.23 \pm 8.70\,i$
7	$28.59 \pm 11.22\,i$	$34.97 \pm 7.38\,i$	$17.05 \pm 4.79\,i$
8	$27.91 \pm 11.25\,i$	$34.76 \pm 7.68\,i$	$12.00 \pm 4.20\,i$
9	$24.40 \pm 11.24\,i$	$34.60 \pm 7.89\,i$	$9.51 \pm 2.37\,i$
10	$27.02 \pm 11.21\,i$	$34.49 \pm 8.03\,i$	
11	$26.71 \pm 11.18\,i$	$34.40 \pm 8.13\,i$	
12	$26.48 \pm 11.14\,i$	$34.34 \pm 8.21\,i$	
13	$26.29 \pm 11.11\,i$	$34.28 \pm 8.27\,i$	
14	$26.13 \pm 11.08\,i$	$34.24 \pm 8.32\,i$	
15	$26.01 \pm 11.06\,i$	$34.21 \pm 8.36\,i$	

results in Table 11 show pronounced barotropic instability at the level of maximum wind and maximum horizontal shear. The computations were then repeated with the larger static stability of 22 MTS units. The results generally showed a marked reduction in the degree of instability with mostly smaller values of c_i and a maximum of two unstable modes. It hardly need be stated that computations with winds corresponding to other levels (other than the maximum wind), where the horizontal shear is less, give less instability. Thus the so-called "barotropic instability" resulting from horizontal shear may be expected to be most prominent near the level of maximum wind. On the other hand vertical shear is large throughout a sizable portion of the troposphere.

Concluding remarks

The dynamic stability characteristics of zonal currents have been obtained by a numerical evaluation of the phase velocities of harmonic waves based upon finite-difference approximations of the linearized thermal and vorticity equations. The results bear out some of the conclusions of earlier models and indicate that a decrease in the β parameter, static stability, horizontal and vertical shear are generally destabilizing influences, but not without some exceptions.

The particular examples studied here, including observed data on the jet stream in December, exhibited both baroclinic and barotropic instability; and the latter is by no means negligible, particularly at the level of maximum wind. Perturbations of short wavelength, about 1000 to 2000 km, were found to be dynamically stable; however, at other wavelengths, there were rather wide differences among the various cases. In Case 1 maximum instability was found at a wavelength of about 6000 km, then there was a band with no unstable waves between 8000 and 12,000 km, followed by some very slow retrogressive waves with slight instability range 13,000 to 15,000 km. An interesting feature was the striking change in the character of the dynamic instability when the vertical jet structure was eliminated by allowing the zonal wind to increase linearly with pressure. Also the stability properties of the more or less hypothetical Case 1 differed considerably from those based strictly on observational data.

Some of the computations displayed multiple unstable modes with various wavelengths. In general the number of roots increased with an increasing number of subdivisions of the fundamental region. This is expected, of course, since the order of the matrix increases with an increasing horizontal and vertical indices; however, the number of roots is not of necessity equal to the order of the matrix as may be readily shown by example. There is some obvious physical significance to such an increase in the number of roots. Firstly, there is frequently an infinite number of eigenvalues associated with the analytic solution of a boundary-value problem; and it would appear that increasing the number of subdivisions permits more roots to appear, particularly as additional nodal points. Secondly, many of the roots may be the more or less trivial solutions corresponding to phase velocities equal to the zonal wind at the mesh points. Nevertheless, this numerical approach may introduce some eigenvalues extraneous to the original boundary value problem.

Another obvious disadvantage of such purely numerical methods is that the phase velocities are not expressed explicitly in terms of the properties of the basic current such as shear, latitude, static stability, etc., so that the role of each of these parameters is not so readily apparent. Of course, successive runs may be made with all parameters held constant save one, thus determining the influence of that particular parameter. But even this is obviously not as satisfactory as an explicit analytical solution; moreover, it may require extensive computational effort. On the other hand, the numerical method does provide approximate phase velocities for harmonic waves in rather complex zonal wind fields.

Appendix

The system of equations given by (14), (15) and (16) can be written in the following form:

$$\begin{bmatrix} (B_{0,0}-cD_{0,0}) & (B_{0,1}-cD_{0,1})\dots(B_{0,MN+M-1}-cD_{0,MN+M-1}) \\ (B_{1,0}-cD_{1,0}) & \dots \\ \vdots & \\ & \\ (B_{MN+M1,0}-cD_{0,MN+M-1})\dots & (B_{MN+M-1,MN+M-1}-cD_{MN+M-1,MN+M-1}) \end{bmatrix} \begin{bmatrix} A_{0,0} \\ A_{0,1} \\ \vdots \\ A_{0,N} \\ A_{0,0} \\ \vdots \\ A_{M-1,N} \end{bmatrix} = 0$$

or briefly $(B-cD)A=0,$

where c is the phase velocity, B and D are square matrices involving the velocity and static stability fields, latitude, etc., and A is a column vector. The necessary and sufficient condition for a non-zero solution of A is that the matrix $(B-cD)$ be singular. The values of c which permit this condition to be fulfilled are, of course, the phase velocities of the harmonic waves of the meteorological system.

If D is non-singular we can write

$$(B-cD)A=(BD^{-1}-cI)DA=0,$$

where I is the unit matrix and D^{-1} is the inverse of D. The condition for a non-zero solution of A may now be seen to depend on the singularity of the matrix $(BD^{-1}-cI)$. Thus the phase velocities of the system (14), (15) and (16) are just the eigenvalues of the matrix BD^{-1}, and similarly its eigenvectors DA will directly provide the amplitude functions A merely by multiplication with D^{-1}; i.e., $A = D^{-1}(DA)$.

The problem of determining the eigenvalues and eigenvectors of a matrix is classical and there is considerable literature on the subject. Iterative methods, particularly suited to modern electronic computers, have been devised for application to large-order matrices. The method used here is described by OSBORNE (1958) and consists essentially of three parts. Briefly, the so-called power method is applied first until certain criteria are met, then the inverse power method of Wieland is utilized until further conditions are met. Finally, on finding an eigenvalue, the matrix is deflated. Then the process repeated until the continued deflation leads to a one-by-one matrix, which is the final eigenvalue.

Osborne reports test cases which show an accuracy for the eigenvalues to five significant figures for a matrix of order 50. Another case of order 70 was being tested and was reported to be accurate to six places in the first 19 eigenvalues, the remainder were yet to be computed at the time of writing. These reports attest the accuracy of the method for highorder matrices. The maximum size tried thus far in these meteorological experiments was for the ($M = 6$, $N = 8$) case of Table 7, which had $6 \times 9 = 54$ eigenvalues. The required computing time was slightly over one hour on a Control Data Corporation (CDC) 1604 computer having a core storage of 32,768 words of 48 bits each and a typical add time of 7.2 microseconds.

As an illustration of the manner of convergence of the iterative scheme, the successive approximations for one of the complex eigenvalues is given below:

$1.9727168 + 0.25197500i$
$2.7299276 + 0.32692392i$
$2.8093565 + 0.37778137i$
$2.9266281 + 0.39321148i$
$2.9388445 + 0.45056607i$
$2.9744696 + 0.49454242i$
$3.0049183 + 0.53366869i$
$3.0326779 + 0.56017892i$
$3.0516001 + 0.57844203i$
$3.0617377 + 0.59076983i$
$3.0646291 + 0.60018088i$
$3.0627056 + 0.60859656i$
$3.0582198 + 0.61712984i$
$3.0529552 + 0.62621076i$
$3.0481466 + 0.63581022i$
$3.0445522 + 0.64662653i$
$3.0425255 + 0.65521790i$
$3.0420947 + 0.66411370i$
$3.0430370 + 0.67190399i$
$3.0449639 + 0.67830924i$
$3.0537138 + 0.70134526i$
$3.0536066 + 0.70149176i$
$3.0536067 + 0.70149177i$

Some roots showed somewhat greater fluctuations, at least initially, in converging toward the final value.

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ERRATA

In the Abstract of this article, line 10, the word destabilizing should be stabilizing.