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LANDSCAPE BOOLEAN FUNCTIONS

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Abstract. In this paper we define a class of generalized Boolean functions defined on $\mathbb{F}_2^n$ with values in $\mathbb{Z}_q$ (we consider $q = 2^k$, $k \geq 1$, here), which we call landscape functions (whose class contains generalized bent, semibent, and plateaued) and find their complete characterization in terms of their Boolean components. In particular, we show that the previously published characterizations of generalized plateaued Boolean functions (which includes generalized bent and semibent) are in fact particular cases of this more general setting. Furthermore, we provide an inductive construction of landscape functions, having any number of nonzero Walsh-Hadamard coefficients. We also completely characterize generalized plateaued functions in terms of the second derivatives and fourth moments.

Keywords: (Generalized) Boolean functions; landscape; plateaued; bent.
MSC2010: 06E30, 94C10, 11A07.

1. **Introduction.** Generalized Boolean functions have become an active area of research [5, 6, 7, 9, 10, 11, 16, 18, 19, 21], with most of these papers dealing with descriptions/constructions of generalized bent/plateaued functions. In this work, we show that in fact the class of generalized plateaued functions, which includes the class of generalized bent and semibent, and their characterizations in terms of their Boolean components, are in fact particular instances of the more general case of landscape functions, which are introduced in this paper.

We take $\mathbb{F}_2^n$ to be an n-dimensional vector space over the two-element field $\mathbb{F}_2$ and for an integer q, let $\mathbb{Z}_q$ be the ring of integers modulo q. By '$+$' and '$-$' we respectively denote addition and subtraction modulo q, while '$\oplus$' is the addition over $\mathbb{F}_2^n$. A *generalized Boolean function* on n variables is a function from $\mathbb{F}_2^n$ to $\mathbb{Z}_q$ ($q \geq 2$), whose set is denoted by $\mathcal{GB}_n^q$, and when $q = 2$, by $\mathcal{B}_n$. If $2^{k-1} < q \leq 2^k$ for

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some $k \geq 1$, the binary expansion of integers gives a unique decomposition of any $f \in \mathcal{GB}_n^q$ as a sequence of Boolean functions $a_i \in \mathcal{B}_n$ $(i = 0, 1, \ldots, k-1)$ such that

$$f(\mathbf{x}) = a_0(\mathbf{x}) + 2a_1(\mathbf{x}) + \cdots + 2^{k-1}a_{k-1}(\mathbf{x}), \text{ for all } \mathbf{x} \in \mathbb{F}_2^n.$$

The (*Hamming*) *weight* of $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{F}_2^n$ is denoted by $wt(\mathbf{x})$ and equals $\sum_{i=1}^n x_i$. For $f : \mathbb{F}_2^n \to \mathbb{F}_2$, we have $wt(f) = \sum_{\mathbf{x}\in\mathbb{F}_2^n} f(\mathbf{x}) = \#\{\mathbf{x} \in \mathbb{F}_2^n : f(\mathbf{x}) = 1\}$, where $\#S$ denotes the cardinality of the set S. The complement (in a universal set understood from the context) of a set S is denoted by $\overline{S}$.

For a generalized Boolean function $f : \mathbb{F}_2^n \to \mathbb{Z}_q$ we define the (unnormalized) *generalized Walsh-Hadamard transform* to be the complex valued function

$$\mathcal{H}_f^{(q)}(\mathbf{u}) = \sum_{\mathbf{x}\in\mathbb{F}_2^n} \zeta_q^{f(\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}},$$

where $\zeta_q = e^{\frac{2\pi i}{q}}$ is a q-primitive complex root of 1 and $\mathbf{u}\cdot\mathbf{x}$ denotes the conventional dot product on $\mathbb{F}_2^n$ (for simplicity, we sometimes use ζ, $\mathcal{H}_f$, instead of ζ_q, respectively, $\mathcal{H}_f^{(q)}$, when q is fixed). The map $\mathcal{F}_f(\mathbf{u}) = \sum_{\mathbf{x}\in\mathbb{F}_2^n} f(\mathbf{x})(-1)^{\mathbf{u}\cdot\mathbf{x}}$ is called the Fourier transform. For $q = 2$, we obtain the usual *Walsh-Hadamard transform*

$$\mathcal{W}_f(\mathbf{u}) = \sum_{\mathbf{x}\in\mathbb{F}_2^n} (-1)^{f(\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}.$$

For $f, g \in \mathcal{GB}_n^q$, the sum

$$\mathcal{C}_{f,g}^{(q)}(\mathbf{z}) = \sum_{\mathbf{x}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x}\oplus\mathbf{z})-g(\mathbf{x})}$$

is the *crosscorrelation* of f and g at $\mathbf{z} \in \mathbb{F}_2^n$, and the *autocorrelation* of $f \in \mathcal{GB}_n^q$ at $\mathbf{u} \in \mathbb{F}_2^n$ is $\mathcal{C}_f^{(q)}(\mathbf{u}) := \mathcal{C}_{f,f}(\mathbf{u})$ (we will drop the superscript if there is no danger of confusion). It is known (see [19]) that if $f, g \in \mathcal{GB}_n^q$, then,

$$\begin{aligned}
&\sum_{\mathbf{u}\in\mathbb{F}_2^n} \mathcal{C}_{f,g}(\mathbf{u})(-1)^{\mathbf{u}\cdot\mathbf{x}} = \mathcal{H}_f(\mathbf{x})\overline{\mathcal{H}_g(\mathbf{x})},\\
&\mathcal{C}_{f,g}(\mathbf{u}) = 2^{-n}\sum_{\mathbf{x}\in\mathbb{F}_2^n} \mathcal{H}_f(\mathbf{x})\overline{\mathcal{H}_g(\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}},\\
&\mathcal{C}_f(\mathbf{u}) = 2^{-n}\sum_{\mathbf{x}\in\mathbb{F}_2^n} |\mathcal{H}_f(\mathbf{x})|^2(-1)^{\mathbf{u}\cdot\mathbf{x}}.
\end{aligned}$$

A function $f : \mathbb{F}_2^n \to \mathbb{Z}_q$ is *generalized bent* (*gbent*) if $|\mathcal{H}_f(\mathbf{u})| = 2^{n/2}$, for all $\mathbf{u} \in \mathbb{F}_2^n$. This is a generalization of functions f for which $|\mathcal{W}_f(\mathbf{u})| = 2^{n/2}$, for all $\mathbf{u} \in \mathbb{F}_2^n$, which are called *bent* functions. In the spirit of Zheng and Zhang [24], we say that $f \in \mathcal{GB}_n^q$ is (generalized) *s-gplateaued* if $|\mathcal{H}_f(\mathbf{u})| \in \{0, 2^{(n+s)/2}\}$ for all $\mathbf{u} \in \mathbb{F}_2^n$ for a fixed integer s depending on f. If $s = 0$, we recover the (generalized) bent functions, and if $s = 1$, or $s = 2$, we obtain the f (generalized) semibent. See Mesnager's excellent survey [12] for more on (g)plateaued Boolean functions. Note that, for Boolean functions, bent functions exist only when n is even; however, when $q > 2$, generalized bent functions exist for all dimensions. A similar result holds for semibent functions, as well, since the conditions n odd for $s = 1$, and n even for $s = 2$ are no longer necessary when $q > 2$.

Given a generalized Boolean function $f : \mathbb{F}_2^n \to \mathbb{Z}_q$, the *derivative* $D_{\mathbf{u}}f$ of f with respect to a vector $\mathbf{u}$ is the generalized Boolean function $D_{\mathbf{u}}f : \mathbb{F}_2^n \to \mathbb{Z}_q$ defined by

$$D_a f(\mathbf{x}) = f(\mathbf{x}) - f(\mathbf{x} \oplus \mathbf{u}), \text{ for all } \mathbf{x} \in \mathbb{F}_2^n.$$

Certainly, if f is Boolean, then $D_a f(\mathbf{x}) = f(\mathbf{x}) \oplus f(\mathbf{x} \oplus \mathbf{u})$. For $f \in \mathcal{GB}_n^q$, the *spectrum* (or *Walsh*) *support* of f is defined by $\mathrm{supp}(\mathcal{H}_f) = \{\mathbf{u} : \mathcal{H}_f(\mathbf{u}) \neq 0\}$. In this paper, we consider the case $q = 2^k$.

2. **Landscape functions and their regularity.** As defined in [10], a gbent function $f \in \mathcal{GB}_n^q$ is *regular*, if $\mathcal{H}_f(\mathbf{u}) = 2^{n/2}\zeta_q^{f^*(\mathbf{u})}$ for some function $f^* \in \mathcal{GB}_n^q$, called the dual. We extend this definition in the following way (we let $\mathbb{N}_0 = \{k \in \mathbb{Z} : k \geq 0\}$ and, $\mathbb{N} = \{k \in \mathbb{Z} : k > 0\}$).

Definition 2.1. We call a function $f \in \mathcal{GB}_n^q$ *regular*, if for all $\mathbf{u} \in \mathrm{supp}(\mathcal{H}_f)$, $\mathcal{H}_f(\mathbf{u}) = 2^{\frac{n_{\mathbf{u}}}{2}} \ell_{\mathbf{u}} \zeta_q^{f^*(\mathbf{u})}$, for some $n_{\mathbf{u}} \in \mathbb{N}_0$, $\ell_{\mathbf{u}} \in 2\mathbb{N}_0 + 1$ and some $f^*(\mathbf{u}) \in \mathbb{Z}_q$. Extending these values outside of the spectrum support of f by $f^*(\mathbf{u}) = 0$, for all $\mathbf{u} \in \overline{\mathrm{supp}(\mathcal{H}_f)}$, we obtain a function $f^* \in \mathcal{GB}_n^q$, which we call the *dual* of f (note: the function f cannot be recovered from f^*, in general).

By modifying a method of Kumar, Scholtz and Welch [7], in [10] it was shown that all gbent functions $f \in \mathcal{GB}_n^{2^k}$ are regular, except for n odd and $k = 2$, in which case one has $\mathcal{H}_f(\mathbf{u}) = 2^{\frac{n-1}{2}}(\pm 1 \pm i)$. We observe that with our definition of regularity, a function cannot be regular unless the absolute value of all nonzero Walsh-Hadamard coefficients of f are of the form $2^{\frac{m_1}{2}}\ell_1, 2^{\frac{m_2}{2}}\ell_2, \ldots$ with $m_1, m_2, \ldots \in \mathbb{N}_0$, $\ell_1, \ell_2, \ldots \in 2\mathbb{N}_0 + 1$. With that in mind, we introduce the following notion.

Definition 2.2. We call a function $f \in \mathcal{GB}_n^q$ a *landscape* function if there exist $t \geq 1$, $m_i \in \mathbb{N}_0, \ell_i \in 2\mathbb{N}_0 + 1$, $1 \leq i \leq t$, such that

$$\{|\mathcal{H}_f(\mathbf{u})|\}_{\mathbf{u} \in \mathrm{supp}(\mathcal{H}_f)} = \{2^{\frac{m_1}{2}}\ell_1, \ldots, 2^{\frac{m_t}{2}}\ell_t\}.$$

We call the set of pairs $\{(m_1, \ell_1), (m_2, \ell_2), \ldots\}$, the *levels* of f, and $t+1$ (if 0 belongs to the Walsh-Hadamard spectrum), or t (if 0 is not in the spectrum) the *length* of f.

Certainly, every classical Boolean function is a landscape function. That is not true for $q > 2$ (as the Walsh-Hadamard values are $\pm$ sums of powers of the primitive root, so the moduli of the spectra values may contain elements outside $\mathbb{Z} \cup \sqrt{2}\,\mathbb{Z}$), however, gplateaued (which includes generalized bent/semibent) functions are all examples of landscape functions.

In Theorem 4.2 we will construct, in an inductive fashion, large classes of landscape functions $: \mathbb{F}_2^n \to \mathbb{Z}_{2^k}$, for all $k \geq 2$.

First, we show the regularity of landscape functions, by modifying the proof from [7, 10]. Recall that when $q = 2^k$ is fixed, we use $\mathcal{H}_f$, in lieu of $\mathcal{H}_f^{(2^k)}$. The interested reader can consult the necessary algebraic number theory material from [15, 23] or his/her favorite book on the subject.

Theorem 2.3. *Let $f \in \mathcal{GB}_n^q$, $q = 2^k$, $k \geq 1$, be a landscape function, and $\zeta = e^{\frac{2\pi i}{2^k}}$ be a 2^k-primitive root of unity. Let $\mathbf{u} \in \mathrm{supp}(\mathcal{H}_f)$, with $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$, $m \in \mathbb{N}_0$, $\ell \in 2\mathbb{N}_0 + 1$. Then, if m is even, or m is odd and $k > 2$, we have $\mathcal{H}_f(\mathbf{u}) = 2^{\frac{m}{2}}\ell\zeta^{f^*(\mathbf{u})}$, for some value $f^*(\mathbf{u}) \in \mathbb{Z}_q$. If m is odd and $k = 2$, and $a_0 \neq 0$, $\mathcal{H}_f(\mathbf{u}) = 2^{\lfloor m/2 \rfloor}\ell(\epsilon_1 + \epsilon_2 i)$, with $\epsilon_1, \epsilon_2 \in \{\pm 1\}$, with the additional possibility,* *if ℓ is the largest component of a Pythagorean triple $\ell_1^2+\ell_2^2=\ell^2$, of $\mathcal{H}_f(\mathbf{u}) = 2^{\lfloor m/2\rfloor}\left(\ell_1\epsilon_1+\ell_2\epsilon_2 \pm i(\ell_1\epsilon_2-\ell_2\epsilon_1)\right)$. If m is odd and $k=2$, there is no function f with $a_0=0$ such that $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$. If m is odd and $k=1$, there is no function f such that $|\mathcal{H}_f(\mathbf{u})| = |\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$, $m\in\mathbb{N}_0$, $\ell\in 2\mathbb{N}_0+1$.*

Proof. If $k=1$ and m is even, the result simply states that if $\mathbf{u}\in\mathrm{supp}(\mathcal{W}_f)$ and $|\mathcal{W}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$, then $\mathcal{W}_f(\mathbf{u}) = 2^{\frac{m}{2}}\ell(-1)^{f^*(\mathbf{u})}$, which is certainly true, since the two roots of unity are ±1.

Let $k\geq 2$, $\mathbf{u}\in\mathrm{supp}(\mathcal{H}_f)$ with $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$ (recall that $m,\ell\in\mathbb{N}_0$, ℓ odd), and assume that m is even, or, m is odd and $k\neq 2$. As in [7, 10], the ideal generated by 2 is totally ramified in $\mathbb{Z}[\zeta]$ (which is the ring of algebraic integers in the cyclotomic field $\mathbb{Q}(\zeta)$), so we have the decomposition in $\mathbb{Z}[\zeta]$ of the ideal $\langle 2\rangle = \langle 1-\zeta\rangle^{2^{k-1}}$, where $\langle 1-\zeta\rangle$ is a prime ideal in $\mathbb{Z}[\zeta]$. Observe that $\mathcal{H}_f(\mathbf{u})\overline{\mathcal{H}_f(\mathbf{u})} = 2^m\ell^2$. From [7, Property 7], we observe that $\mathcal{H}_f(\mathbf{u})$ and $\overline{\mathcal{H}_f(\mathbf{u})}$ will generate the same ideal in $\mathbb{Z}[\zeta]$ and so, $2^{-m}\ell^{-2}(\mathcal{H}_f(\mathbf{u}))^2$ is a unit, and consequently, $2^{-\frac{m}{2}}\ell^{-1}\mathcal{H}_f(\mathbf{u})$ is an algebraic integer. Therefore, by Proposition 1 of [7], $2^{-\frac{m}{2}}\ell^{-1}\mathcal{H}_f(\mathbf{u})$ is a root of unity. Further, observe that the Gauss quadratic sum $G(2^k) = \displaystyle\sum_{i=0}^{2^k-1}\zeta^{i^2} = 2^{k/2}(1+i)$ and so, $\sqrt{2}\in\mathbb{Q}(\zeta)$, and so the root of unity $2^{-\frac{m}{2}}\ell^{-1}\mathcal{H}_f(\mathbf{u})$ must be in the cyclotomic field $\mathbb{Q}(\zeta)$, unless $k=2$ (since then $1+i\notin\mathbb{Q}(\zeta)$). The first assertion is shown for m even, as well as for m odd with $k\neq 2$.

When m is odd and $k=2$, then $\mathcal{H}_f(\mathbf{u}) = a_{\mathbf{u}} + b_{\mathbf{u}}\, i$, for some integers $a_{\mathbf{u}}, b_{\mathbf{u}}$.

We distinguish between two cases:

Case 1. We first consider the case $a_0\neq 0$. Since $|\mathcal{H}_f(\mathbf{u})|^2 = 2^m\ell^2$, we get the diophantine equation $a_{\mathbf{u}}^2 + b_{\mathbf{u}}^2 = 2^m\ell^2$.

Since m is odd, the solutions for $x^2+y^2=2^m$ are $(x,y) = (\epsilon_1 2^{\lfloor m/2\rfloor}, \epsilon_2 2^{\lfloor m/2\rfloor})$, with $\epsilon_1,\epsilon_2\in\{\pm1\}$. If ℓ is not the largest component of a Pythagorean triple, all solutions of $a_{\mathbf{u}}^2+b_{\mathbf{u}}^2 = 2^m\ell^2$ are of the form $(a_{\mathbf{u}}, b_{\mathbf{u}}) = (\epsilon_1 2^{\lfloor m/2\rfloor}\ell, \epsilon_2 2^{\lfloor m/2\rfloor}\ell)$. If ℓ is the largest component of a Pythagorean triple (ℓ_1,ℓ_2,ℓ), all solutions of $a_{\mathbf{u}}^2+b_{\mathbf{u}}^2 = 2^m\ell^2$ are of the form $(a_{\mathbf{u}}, b_{\mathbf{u}}) = (\epsilon_1 2^{\lfloor m/2\rfloor}\ell, \epsilon_2 2^{\lfloor m/2\rfloor}\ell)$ and $(a_{\mathbf{u}}, b_{\mathbf{u}}) = (2^{\lfloor m/2\rfloor}(\ell_1\epsilon_1+\ell_2\epsilon_2), \pm 2^{\lfloor m/2\rfloor}(\ell_1\epsilon_2-\ell_2\epsilon_1))$ (here we use the fact that the product of sums of squares is a sum of squares, that is, $(a^2+b^2)(c^2+d^2) = (ac+bd)^2+(ad-bc)^2$).
Case 2. Finally, we consider here the case when m is odd, $k=2$, and $a_0=0$. In this case, $f=2a_1$. Therefore, $\mathcal{H}_f(\mathbf{u}) = \sum_{\mathbf{x}\in\mathbb{F}_2^n} i^{f(\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}} = \sum_{\mathbf{x}\in\mathbb{F}_2^n}(-1)^{a_1(\mathbf{x})+\mathbf{u}\cdot\mathbf{x}}$. Since $\mathcal{H}_f(\mathbf{u})$ is then an integer, $|\mathcal{H}_f(\mathbf{u})|^2 = 2^{\frac{m}{2}}\ell$ is impossible, and the proof for $k=2$ is completed.

Finally, when m is odd and $k=1$, there is no function f such that $|\mathcal{H}_f(\mathbf{u})| = |\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$, $m\in\mathbb{N}_0$, $\ell\in 2\mathbb{N}_0+1$, since $|\mathcal{H}_f(\mathbf{u})|\in\mathbb{N}$. $\square$

Remark 1. When the length of the involved landscape functions in the theorem above is 3, if 0 is in the spectrum, length 2, otherwise, that is, we have generalized plateaued functions (gplateaued), $q=2$ and $k\geq 2$, the regularity was given by Mesnager et al. [13].

3. **Characterizing landscape functions in terms of components.** In this section, we will completely characterize the landscape functions in terms of their components, by using the method of [9]. It is rather intriguing that generalized bentness does not play a role in the method, rather the modulus of values of the

Walsh-Hadamard spectrum being of the form $2^{\frac{m}{2}}\ell$ is important, independent of how many such different values occur.

We define the "canonical bijection" $\iota : \mathbb{F}_2^{k-1} \to \mathbb{Z}_{2^{k-1}}$ by $\iota(\mathbf{c}) = \sum_{j=0}^{k-2} c_j 2^j$ where $\mathbf{c} = (c_0, c_1, \ldots, c_{k-2})$. We gather in the next lemma some computations from [9, 19, 20], providing a relationship between the generalized Walsh-Hadamard transform and the classical transform.

Lemma 3.1. *For a generalized Boolean $f \in \mathcal{GB}_n^{2^k}$, $f(\mathbf{x}) = a_0(\mathbf{x}) + 2a_1(\mathbf{x}) + \cdots + 2^{k-1}a_{k-1}(\mathbf{x})$, $a_i \in \mathcal{B}_n$, we have*

$$\mathcal{H}_f(\mathbf{u}) = \frac{1}{2^{k-1}} \sum_{(\mathbf{c},\mathbf{d}) \in \mathbb{F}_2^{k-1} \times \mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot\mathbf{d}} \zeta_{2^k}^{\iota(\mathbf{d})} \mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}),$$

where $f_{\mathbf{c}}(\mathbf{x}) = c_0 a_0(\mathbf{x}) \oplus c_1 a_1(\mathbf{x}) \oplus \cdots \oplus c_{k-2}a_{k-2}(\mathbf{x}) \oplus a_{k-1}(\mathbf{x})$.

We now show the main theorem of this section. We note that the case of the generalized bent and the larger class of gplateaued functions (particular cases of our theorem below) has appeared in the works [13, 9, 19, 20].

Theorem 3.2. *Let $f : \mathbb{F}_2^n \to \mathbb{Z}_{2^k}$, $k \geq 2$, be a function given as $f(\mathbf{x}) = a_0(\mathbf{x}) + 2a_1(\mathbf{x}) + \cdots + 2^{k-1}a_{k-1}(\mathbf{x})$. Then, f is a landscape function whose spectra moduli are in $\left\{0, 2^{\frac{m_1}{2}}L_1, \ldots, 2^{\frac{m_t}{2}}L_t\right\}$ ($t \in \mathbb{N}, m_i \in \mathbb{N}_0$, $L_i \in 2\mathbb{N}_0 + 1$) if and only if for each $\mathbf{c} \in \mathbb{F}_2^{k-1}$, the Boolean function $f_{\mathbf{c}}$ defined as*

$$f_{\mathbf{c}}(\mathbf{x}) = c_0 a_0(\mathbf{x}) \oplus c_1 a_1(\mathbf{x}) \oplus \cdots \oplus c_{k-2}a_{k-2}(\mathbf{x}) \oplus a_{k-1}(\mathbf{x})$$

is a Boolean function such that (we take $\ell \in \{L_1, \ldots, L_t\}$):

- *(i)* $\mathcal{H}_f(\mathbf{u}) = 0$, *if and only if* $\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) = 0$.
- *(ii)* $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$, *$m$ even, if and only if* $\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) = (-1)^{\mathbf{c}\cdot\iota^{-1}(g(\mathbf{u}))+s(\mathbf{u})}2^{\frac{m}{2}}\ell$, *for some* $g : \mathbb{F}_2^n \to \mathbb{Z}_{2^{k-1}}$, $s : \mathbb{F}_2^n \to \mathbb{F}_2$.
- *(iii)* $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$, *$m$ odd, $k \neq 2$, if and only if*

  $$\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) = \left((-1)^{\mathbf{c}\cdot\iota^{-1}(g_1(\mathbf{u}))+s_1(\mathbf{u})} - (-1)^{\mathbf{c}\cdot\iota^{-1}(g_2(\mathbf{u}))+s_2(\mathbf{u})}\right) 2^{\lfloor\frac{m}{2}\rfloor}\ell,$$

  for some $g_j : \mathbb{F}_2^n \to \mathbb{Z}_{2^{k-1}}, s_j : \mathbb{F}_2^n \to \mathbb{F}_2$, $j = 1, 2$, *where* $g_2(\mathbf{u}) - g_1(\mathbf{u}) + 2^{k-1}(s_2(\mathbf{u}) - s_1(\mathbf{u})) = 2^{k-2}$ *in* $\mathbb{Z}_{2^k}$.
- *(iv)* $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$, *$m$ odd, $k = 2$, if and only if $a_0 \neq 0$ and (note that c is a bit)*

  $$\mathcal{W}_{f_c}(\mathbf{u}) = \begin{cases} 2^{\frac{m-1}{2}}\left(\ell_1\epsilon_1 + \ell_2\epsilon_2 \pm (-1)^c(\ell_1\epsilon_2 - \ell_2\epsilon_1)\right), \text{ or} \\ 2^{\frac{m-1}{2}}\ell\left(\epsilon_1 + \epsilon_2(-1)^c\right), \end{cases}$$

  if $\ell_1^2 + \ell_2^2 = \ell^2$; otherwise,

  $$\mathcal{W}_{f_c}(\mathbf{u}) = 2^{\frac{m-1}{2}}\ell\left(\epsilon_1 + \epsilon_2(-1)^c\right).$$

  If m is odd, $k = 2$, and $a_0 = 0$, there are no functions f such that $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$.

Consequently, $f_{\mathbf{c}}$ has nonzero spectra moduli given by $\left\{2^{\lceil\frac{m_1}{2}\rceil}L_1, \ldots, 2^{\lceil\frac{m_t}{2}\rceil}L_t\right\}$.

Proof. *(i)* First, let us treat the case of $\mathbf{u} \notin \mathrm{supp}(\mathcal{H}_f)$. Thus,

$$0 = \sum_{(\mathbf{c},\mathbf{d}) \in \mathbb{F}_2^{k-1} \times \mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot\mathbf{d}} \zeta_{2^k}^{\iota(\mathbf{d})} \mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) = \sum_{\mathbf{d} \in \mathbb{F}_2^{k-1}} \left(\sum_{\mathbf{c} \in \mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot\mathbf{d}} \mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u})\right) \zeta_{2^k}^{\iota(\mathbf{d})}, \tag{1}$$

and so, $\sum_{\mathbf{c}\in\mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot\mathbf{d}}\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) = 0$, since $\{1,\zeta_{2^k},\ldots,\zeta_{2^k}^{2^{k-1}-1}\}$ is a basis of $\mathbb{Q}(\zeta_{2^k})$. Inverting, we get

$$\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) = \frac{1}{2^{k-1}}\sum_{(\mathbf{u},\mathbf{v})\in\mathbb{F}_2^{k-1}} (-1)^{(\mathbf{u}+\mathbf{c})\cdot\mathbf{v}}\mathcal{W}_{f_{\mathbf{u}}}(\mathbf{u}) = 0.$$

The converse follows easily.

(ii) Let now $\mathbf{u}\in\mathrm{supp}(\mathcal{H}_f)$ with $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$, $m\in\mathbb{Z}$ even, ℓ odd. Then, $f:\mathbb{F}_2^n\to\mathbb{Z}_{2^k}$ satisfies $\mathcal{H}_f(\mathbf{u}) = 2^{\frac{m}{2}}\ell\,\zeta_{2^k}^{f^*(\mathbf{u})}$ for some $f^*:\mathbb{F}_2^n\to\mathbb{Z}_{2^k}$. Decompose f^* as $f^* = g + 2^{k-1}s$ with $g:\mathbb{F}_2^n\to\mathbb{Z}_{2^{k-1}}$ and $s:\mathbb{F}_2^n\to\mathbb{F}_2$ so that

$$\mathcal{H}_f(\mathbf{u}) = 2^{\frac{m}{2}}\ell\,(-1)^{s(\mathbf{u})}\zeta_{2^k}^{g(\mathbf{u})}.$$

Then,

$$\sum_{\mathbf{d}\in\mathbb{F}_2^{k-1}}\left(\frac{1}{2^{k-1}}\sum_{\mathbf{c}\in\mathbb{F}_2^{k-1}}(-1)^{\mathbf{c}\cdot\mathbf{d}}\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u})\right)\zeta_{2^k}^{\iota(\mathbf{d})} - 2^{\frac{m}{2}}\ell(-1)^{s(\mathbf{u})}\zeta_{2^k}^{g(\mathbf{u})} = 0. \tag{2}$$

Again using that $\{1,\zeta_{2^k},\ldots,\zeta_{2^k}^{2^{k-1}-1}\}$ is a basis of $\mathbb{Q}(\zeta_{2^k})$ (denoting by δ_0 the Dirac symbol $\delta_0(u,v)=1$ if $u=v$, and 0, otherwise), we infer

$$\frac{1}{2^{k-1}}\sum_{\mathbf{c}\in\mathbb{F}_2^{k-1}}(-1)^{\mathbf{c}\cdot\mathbf{d}}\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) = 2^{\frac{m}{2}}\ell\,(-1)^{s(\mathbf{u})}\delta_0\left(\iota(\mathbf{d}),g(\mathbf{u})\right). \tag{3}$$

We now invert the above identity, so, for any $\mathbf{c}\in\mathbb{F}_2^{k-1}$,

$$\begin{aligned}
\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) &= \frac{1}{2^{k-1}}\sum_{(\mathbf{u},\mathbf{v})\in\mathbb{F}_2^{k-1}}(-1)^{(\mathbf{u}+\mathbf{c})\cdot\mathbf{v}}\mathcal{W}_{f_{\mathbf{u}}}(\mathbf{u})\\
&= \sum_{\mathbf{v}\in\mathbb{F}_2^{k-1}}(-1)^{\mathbf{c}\cdot\mathbf{v}}\left(\frac{1}{2^{k-1}}\sum_{\mathbf{u}\in\mathbb{F}_2^{k-1}}(-1)^{\mathbf{u}\cdot\mathbf{v}}\mathcal{W}_{f_{\mathbf{u}}}(\mathbf{u})\right)\\
&= (-1)^{\mathbf{c}\cdot\iota^{-1}(g(\mathbf{u}))+s(\mathbf{u})}2^{\frac{m}{2}}\ell,
\end{aligned}$$

which shows that $f_{\mathbf{c}}$ satisfies the imposed conditions on the Walsh-Hadamard coefficient at $\mathbf{u}$.

Conversely, suppose that there exist $g:\mathbb{F}_2^n\to\mathbb{Z}_{2^{k-1}}$ and $s:\mathbb{F}_2^n\to\mathbb{F}_2$ such that, for every $\mathbf{c}\in\mathbb{F}_2^{k-1}$, $\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) = 2^{\frac{m}{2}}\ell(-1)^{\mathbf{c}\cdot\iota^{-1}(g(\mathbf{u}))+s(\mathbf{u})}$. By Lemma 3.1, we can write

$$\begin{aligned}
\mathcal{H}_f(\mathbf{u}) &= \frac{1}{2^{k-1}}\sum_{(\mathbf{c},\mathbf{d})\in\mathbb{F}_2^{k-1}\times\mathbb{F}_2^{k-1}}(-1)^{\mathbf{c}\cdot\mathbf{d}}\zeta_{2^k}^{\iota(\mathbf{d})}\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u})\\
&=2^{\frac{m}{2}}\ell\cdot\frac{1}{2^{k-1}}\sum_{(\mathbf{c},\mathbf{d})\in\mathbb{F}_2^{k-1}\times\mathbb{F}_2^{k-1}}(-1)^{\mathbf{c}\cdot\mathbf{d}+\mathbf{c}\cdot\iota^{-1}(g(\mathbf{u}))+s(\mathbf{u})}\zeta_{2^k}^{\iota(\mathbf{d})}\\
&=2^{\frac{m}{2}}\ell\,(-1)^{s(\mathbf{u})}\sum_{\mathbf{d}\in\mathbb{F}_2^{k-1}}\left(\frac{1}{2^{k-1}}\sum_{\mathbf{c}\in\mathbb{F}_2^{k-1}}(-1)^{\mathbf{c}\cdot(\mathbf{d}+\iota^{-1}(g(\mathbf{u})))}\right)\zeta_{2^k}^{\iota(\mathbf{d})}\\
&=2^{\frac{m}{2}}\ell\,(-1)^{s(\mathbf{u})}\zeta_{2^k}^{g(\mathbf{u}))},
\end{aligned}$$

proving that f satisfies $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$.

(*iii*) Now, let $\mathbf{u} \in \text{supp}(\mathcal{H}_f)$, with $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$, $m, \ell \in \mathbb{Z}^{\geq 1}$ odd, $k \neq 2$. By Theorem 2.3, then

$$\mathcal{H}_f(\mathbf{u}) = 2^{\frac{m}{2}}\,\ell\,\zeta_{2^k}^{f^*(\mathbf{u})} = 2^{\frac{m-1}{2}}\,\sqrt{2}\,\ell\,\zeta_{2^k}^{f^*(\mathbf{u})},$$

for some power $f^*(\mathbf{u}) \in \mathbb{Z}_{2^k}$. Recall now that $\mathbb{Q}(\sqrt{2}) \subset \mathbb{Q}(\zeta_{2^k})$, since $\sqrt{2} = \zeta_8 + \bar{\zeta_8} = \zeta_8 - \zeta_8^3 = \zeta_{2^k}^{2^{k-3}} - \zeta_{2^k}^{3\cdot 2^{k-3}}$. Thus,

$$\mathcal{H}_f(\mathbf{u}) = 2^{\frac{m-1}{2}}\,\ell\,\left(\zeta_{2^k}^{f^*(\mathbf{u})+2^{k-3}} - \zeta_{2^k}^{f^*(\mathbf{u})+3\cdot 2^{k-3}}\right).$$

As in [9], we let $f^*(\mathbf{u}) + 2^{k-3} = g_1(\mathbf{u}) + 2^{k-1}s_1(\mathbf{u}) + 2^k t_1(\mathbf{u})$ and $f^*(\mathbf{u}) + 3\cdot 2^{k-3} = g_2(\mathbf{u}) + 2^{k-1}s_2(\mathbf{u}) + 2^k t_2(\mathbf{u})$, where $g_i : \mathbb{F}_{2^n} \to \mathbb{Z}_{2^{k-1}}$ and $s_i, t_i : \mathbb{F}_{2^n} \to \mathbb{F}_2$, so that

$$\mathcal{H}_f(\mathbf{u}) = 2^{\frac{m-1}{2}}\,\ell\,(-1)^{s_1(\mathbf{u})}\zeta_{2^k}^{g_1(\mathbf{u})} - 2^{\frac{m-1}{2}}\,\ell\,(-1)^{s_2(\mathbf{u})}\zeta_{2^k}^{g_2(\mathbf{u})}. \tag{4}$$

Observe that from their definition, we have $g_2(\mathbf{u}) - g_1(\mathbf{u}) + 2^{k-1}(s_2(\mathbf{u}) - s_1(\mathbf{u})) = 2^{k-2}$ in $\mathbb{Z}_{2^k}$. If $g_2(\mathbf{u}) = g_1(\mathbf{u})$, then $2^{k-1}(s_2(\mathbf{u}) - s_1(\mathbf{u})) = 2^{k-2}$ in $\mathbb{Z}_{2^k}$, which is impossible, since $s_2(\mathbf{u}), a_1(\mathbf{u}) \in \{0, 1\}$.

Recall that ι is the canonical bijection from $\mathbb{F}_2^{k-1}$ to $\mathbb{Z}_{2^{k-1}}$, $\iota(c_0, \ldots, c_{k-2}) = \sum_{j=0}^{k-2} c_j 2^j$. Using Lemma 3.1, we write

$$\mathcal{H}_f(\mathbf{u}) = \sum_{\mathbf{d}\in\mathbb{F}_2^{k-1}} \left(\frac{1}{2^{k-1}} \sum_{\mathbf{c}\in\mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot\mathbf{d}}\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u})\right)\zeta_{2^k}^{\iota(\mathbf{d})},$$

which, when combined with equation (4) (recall that $g_1(\mathbf{u}) \neq g_2(\mathbf{u})$), implies that for all $\mathbf{d} \in \mathbb{Z}_{2^{k-1}}$,

$$\begin{aligned}\frac{1}{2^{k-1}} \sum_{\mathbf{c}\in\mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot\mathbf{d}}\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) &= 2^{\frac{m-1}{2}}\,\ell\,(-1)^{s_1(\mathbf{u})}\delta_0\left(\iota(\mathbf{d}), g_1(\mathbf{u})\right)\\ &\quad - 2^{\frac{m-1}{2}}\,\ell\,(-1)^{s_2(\mathbf{u})}\delta_0\left(\iota(\mathbf{d}), g_2(\mathbf{u})\right).\end{aligned}$$

Thus,

$$\begin{aligned}\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) &= \frac{1}{2^{k-1}} \sum_{(\mathbf{u},\mathbf{v})\in\mathbb{F}_2^{k-1}\times\mathbb{F}_2^{k-1}} (-1)^{(\mathbf{u}+\mathbf{c})\cdot\mathbf{v}}\mathcal{W}_{f_{\mathbf{u}}}(\mathbf{u})\\ &= \sum_{\mathbf{v}\in\mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot\mathbf{v}}\frac{1}{2^{k-1}} \sum_{\mathbf{u}\in\mathbb{F}_2^{k-1}} (-1)^{\mathbf{u}\cdot\mathbf{v}}\mathcal{W}_{f_{\mathbf{u}}}(\mathbf{u})\\ &= \frac{(-1)^{\mathbf{c}\cdot\iota^{-1}(g_1(\mathbf{u}))+s_1(\mathbf{u})} - (-1)^{\mathbf{c}\cdot\iota^{-1}(g_2(\mathbf{u}))+s_2(\mathbf{u})}}{2}\,2^{\frac{m+1}{2}}\ell.\end{aligned}$$

By the definition of the g_i, s_i, we have that $g_2(\mathbf{u}) - g_1(\mathbf{u}) + 2^{k-1}(s_2(\mathbf{u}) - s_1(\mathbf{u})) = 2^{k-2}$ in $\mathbb{Z}_{2^k}$. Since

$$\frac{(-1)^{\mathbf{c}\cdot\iota^{-1}(g_1(\mathbf{u}))+s_1(\mathbf{u})} - (-1)^{\mathbf{c}\cdot\iota^{-1}(g_2(\mathbf{u}))+s_2(\mathbf{u})}}{2} \in \{-1, 0, 1\},$$

for the fixed $\mathbf{u} \in \mathbb{F}_{2^n}$, the claim is proven.

Conversely, for a fixed $\mathbf{u} \in \mathbb{F}_2^n$, assume that for all $\mathbf{c} \in \mathbb{F}_2^{k-1}$, $f_{\mathbf{c}}$ have their Walsh-Hadamard transforms of the form

$$\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{u}) = \left((-1)^{\mathbf{c}\cdot\iota^{-1}(g_1(\mathbf{u}))+s_1(\mathbf{u})} - (-1)^{\mathbf{c}\cdot\iota^{-1}(g_2(\mathbf{u}))+s_2(\mathbf{u})}\right)2^{\frac{m-1}{2}}\ell,$$

for some $g_j : \mathbb{F}_2^n \to \mathbb{Z}_{2^{k-1}}, s_j : \mathbb{F}_2^n \to \mathbb{F}_2$, $j = 1, 2$, with $g_2(\mathbf{u}) - g_1(\mathbf{u}) + 2^{k-1}(s_2(\mathbf{u}) - s_1(\mathbf{u})) = 2^{k-2}$ in $\mathbb{Z}_{2^k}$, and m, ℓ odd integers.

Observe that (we use the fact that $g_1(\mathbf{u}) \neq g_2(\mathbf{u})$, which follows easily from the identity above),

$$\sum_{\mathbf{c}\in\mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot(\mathbf{d}\oplus\iota^{-1}(g_1(\mathbf{u}))+s_1(\mathbf{u})} - \sum_{\mathbf{c}\in\mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot(\mathbf{d}\oplus\iota^{-1}(g_2(\mathbf{u}))+s_2(\mathbf{u})}$$
$$= \begin{cases} 0 & \text{if } \iota(\mathbf{d}) \notin \{g_1(\mathbf{u}), g_2(\mathbf{u})\} \\ 2^{k-1}(-1)^{s_1(\mathbf{u})} & \text{if } \iota(\mathbf{d}) = g_1(\mathbf{u}) \neq g_2(\mathbf{u}) \\ -2^{k-1}(-1)^{s_2(\mathbf{u})} & \text{if } \iota(\mathbf{d}) = g_2(\mathbf{u}) \neq g_1(\mathbf{u}). \end{cases}$$

Further, using this identity and Lemma 3.1, we get

$$\begin{aligned}\mathcal{H}_f(\mathbf{u}) &= 2^{\frac{m+1}{2}-k}\ell \sum_{\mathbf{d}\in\mathbb{F}_2^{k-1}} \zeta_{2^k}^{\iota(\mathbf{d})} \left(\sum_{\mathbf{c}\in\mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot(\mathbf{d}\oplus\iota^{-1}(g_1(\mathbf{u}))+s_1(\mathbf{u})} \right.\\ &\qquad \left. - \sum_{\mathbf{c}\in\mathbb{F}_2^{k-1}} (-1)^{\mathbf{c}\cdot(\mathbf{d}\oplus\iota^{-1}(g_2(\mathbf{u}))+s_2(\mathbf{u})} \right)\\ &= 2^{\frac{m-1}{2}}\ell\left((-1)^{s_1(\mathbf{u})}\zeta_{2^k}^{g_1(\mathbf{u})} - (-1)^{s_2(\mathbf{u})}\zeta_{2^k}^{g_2(\mathbf{u})}\right)\\ &= 2^{\frac{m-1}{2}}\ell\,(-1)^{s_1(\mathbf{u})}\zeta_{2^k}^{g_1(\mathbf{u})}\left(1-\zeta_{2^k}^{g_2(\mathbf{u})-g_1(\mathbf{u})+2^{k-1}(s_2(\mathbf{u})-s_1(\mathbf{u}))}\right)\\ &= 2^{\frac{m-1}{2}}\ell\,(-1)^{s_1(\mathbf{u})}\zeta_{2^k}^{g_1(\mathbf{u})}(1-\zeta_{2^k}^{2^{k-2}}) = 2^{\frac{m}{2}}\ell\,(-1)^{s_1(\mathbf{u})}\zeta_{2^k}^{g_1(\mathbf{u})}\bar{\zeta}_8,\end{aligned}$$

and so, $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$. The claim is shown.

(iv) First, let $\mathbf{u} \in \mathrm{supp}(\mathcal{H}_f)$, with $|\mathcal{H}_f(\mathbf{u})| = 2^{\frac{m}{2}}\ell$, $m, \ell \in \mathbb{N}$ odd, $k = 2$ (observe now that $\zeta_{2^k} = i$), $a_0 \neq 0$. Let us consider the case where ℓ is not the largest component of a Pythagorean triple. From Theorem 2.3, we infer that $\mathcal{H}_f(\mathbf{u}) = 2^{\frac{m-1}{2}}\ell\,(\epsilon_1 + \epsilon_2 i)$.

Using Lemma 3.1, we write

$$\mathcal{H}_f(\mathbf{u}) = \sum_{d\in\mathbb{F}_2} \left(\frac{1}{2}\sum_{c\in\mathbb{F}_2} (-1)^{cd}\mathcal{W}_{f_c}(\mathbf{u}) \right) i^d.$$

Together with the previous value of $\mathcal{H}_f(\mathbf{u})$, this renders

$$\frac{1}{2}\sum_{c\in\mathbb{F}_2} (-1)^{cd}\mathcal{W}_{f_c}(\mathbf{u}) = \pm 2^{\frac{m-1}{2}}\ell \ \text{ for } d = 0, 1.$$

Thus,

$$\begin{aligned}\mathcal{W}_{f_c}(\mathbf{u}) &= \frac{1}{2}\sum_{(u,v)\in\mathbb{F}_2\times\mathbb{F}_2} (-1)^{(u+c)v}\mathcal{W}_{f_u}(\mathbf{u}) = \sum_{v\in\mathbb{F}_2}(-1)^{cv}\frac{1}{2}\sum_{u\in\mathbb{F}_2}(-1)^{uv}\mathcal{W}_{f_u}(\mathbf{u})\\ &= 2^{\frac{m-1}{2}}\ell\,(\epsilon_1 + \epsilon_2(-1)^c).\end{aligned}$$

Conversely, let $\mathcal{W}_{f_c}(\mathbf{u}) = 2^{\frac{m-1}{2}}\ell\,(\epsilon_1 + \epsilon_2(-1)^c)$. Then, using Lemma 3.1, we write

$$\begin{aligned}\mathcal{H}_f(\mathbf{u}) &= \sum_{d\in\mathbb{F}_2} \left(\frac{1}{2}\sum_{c\in\mathbb{F}_2} (-1)^{cd}\mathcal{W}_{f_c}(\mathbf{u}) \right) i^d\\ &= \frac{1}{2}2^{\frac{m-1}{2}}\ell \sum_{d\in\mathbb{F}_2} \left((-1)^0(\epsilon_1+\epsilon_2) + (-1)^d(\epsilon_1-\epsilon_2)\right) i^d = 2^{\frac{m-1}{2}}\ell\,(\epsilon_1+\epsilon_2 i).\end{aligned}$$

Assume that ℓ is the largest component of a Pythagorean triple, $\ell_1^2 + \ell_2^2 = \ell^2$. By Theorem 2.3, we obtain either the previous case, or that $\mathcal{H}_f(\mathbf{u}) = 2^{\frac{m-1}{2}}\left(\ell_1\epsilon_1 + \ell_2\epsilon_2 \pm (\ell_1\epsilon_2 - \ell_2\epsilon_1)\, i\right)$. In the latter case, using Lemma 3.1, we write

$$\mathcal{H}_f(\mathbf{u}) = \sum_{d\in\mathbb{F}_2}\left(\frac{1}{2}\sum_{c\in\mathbb{F}_2}(-1)^{cd}\mathcal{W}_{f_c}(\mathbf{u})\right) i^d.$$

Together with the previous identity, this renders

$$\frac{1}{2}\sum_{c\in\mathbb{F}_2}(-1)^{cd}\mathcal{W}_{f_c}(\mathbf{u}) = 2^{\frac{m-1}{2}}(\ell_1\epsilon_1 + \ell_2\epsilon_2) \text{ for } d = 0,$$

$$\frac{1}{2}\sum_{c\in\mathbb{F}_2}(-1)^{cd}\mathcal{W}_{f_c}(\mathbf{u}) = \pm 2^{\frac{m-1}{2}}(\ell_1\epsilon_2 - \ell_2\epsilon_1) \text{ for } d = 1.$$

Thus,

$$\begin{aligned}\mathcal{W}_{f_c}(\mathbf{u}) &= \frac{1}{2}\sum_{(u,v)\in\mathbb{F}_2\times\mathbb{F}_2}(-1)^{(u+c)v}\mathcal{W}_{f_u}(\mathbf{u})\\ &= \sum_{v\in\mathbb{F}_2}(-1)^{cv}\frac{1}{2}\sum_{u\in\mathbb{F}_2}(-1)^{uv}\mathcal{W}_{f_u}(\mathbf{u})\\ &= 2^{\frac{m-1}{2}}\left(\ell_1\epsilon_1 + \ell_2\epsilon_2 \pm (-1)^c(\ell_1\epsilon_2 - \ell_2\epsilon_1)\right).\end{aligned}$$

Conversely, let $\mathcal{W}_{f_c}(\mathbf{u}) = 2^{\frac{m-1}{2}}\left(\ell_1\epsilon_1 + \ell_2\epsilon_2 \pm (-1)^c(\ell_1\epsilon_2 - \ell_2\epsilon_1)\right)$. Then, using Lemma 3.1, we write

$$\begin{aligned}\mathcal{H}_f(\mathbf{u}) &= \sum_{d\in\mathbb{F}_2}\left(\frac{1}{2}\sum_{c\in\mathbb{F}_2}(-1)^{cd}\mathcal{W}_{f_c}(\mathbf{u})\right) i^d\\ &= \frac{1}{2}2^{\frac{m-1}{2}}\sum_{d\in\mathbb{F}_2}\big((-1)^0(\ell_1\epsilon_1 + \ell_2\epsilon_2 \pm (\ell_1\epsilon_2 - \ell_2\epsilon_1))\\ &\qquad\qquad + (-1)^d(\ell_1\epsilon_1 + \ell_2\epsilon_2 \mp (\ell_1\epsilon_2 - \ell_2\epsilon_1))\big)\, i^d\\ &= 2^{\frac{m-1}{2}}(\ell_1\epsilon_1 + \ell_2\epsilon_2) \pm (\ell_1\epsilon_2 - \ell_2\epsilon_1)i.\end{aligned}$$

Finally, the case m odd, $k = 2$, $a_0 = 0$, is stated in Theorem 2.3. □

The following corollary is then immediate.

Corollary 1. *Let $f : \mathbb{F}_2^n \to \mathbb{Z}_{2^k}$, $k \geq 1$, be a function given as $f(\mathbf{x}) = a_0(\mathbf{x}) + 2a_1(\mathbf{x}) + \cdots + 2^{k-1}a_{k-1}(\mathbf{x})$. Let $s \geq 0$ be an integer. Then f is s-gplateaued if and only if for each $\mathbf{c} \in \mathbb{F}_2^{k-1}$, the Boolean function $f_{\mathbf{c}}$ defined as in Theorem 3.2 is an s-plateaued (if $n + s$ is even), respectively, an $(s+1)$-plateaued function (if $n + s$ is odd) with the extra conditions on the Walsh-Hadamard coefficients, as in Theorem 3.2.*

We will derive other characterizations of gplateaued functions in the last section.

4. **Some constructions of landscape functions.** First, we start with some examples of five valued spectra (certainly, landscape) functions, and later on, we shall give constructions of arbitrary length landscape functions. In [8, Theorem 19], a class of functions with five valued spectra was constructed. These are (see [2, 4] for the definitions of these notions) n-variable, m-resilient, degree $(n-m-1)$ functions with nonlinearity $nl(f) = 2^{n-1} - 2^{(n+m-2)/2}$, if $n-m+1$ is odd, and $nl(f) = 2^{n-1} - 2^{(n+m-1)/2}$, if $n-m+1$ is even. They are also five valued Walsh spectrum (under $n-m \geq 5$), namely, $\{\pm 2^{(n+m)/2}, 2^{(n+m)/2} - 2^{m+2}, -2^{m+2}, 0\}$, if $n-m+1$ is odd, respectively, $\{\pm 2^{(n+m+1)/2}, 2^{(n+m+1)/2} - 2^{m+2}, -2^{m+2}, 0\}$, if $n-m+1$ is even. To generate landscape functions that have five valued spectra, we take $m := n-5$, and so, $n-m+1 = 6$, $n+m+1 = 2n-4$, and the spectrum will be $\{\pm 2^{n-2}, \pm 2^{n-3}, 0\}$; also, $n-m = 6$, so $n-m+1 = 7$, and the spectrum will be $\{\pm 2^{n-3}, \pm 2^{n-4}, 0\}$.

We do not need it here, but using Catalan's Conjecture (now known as Mihăilescu's Theorem [14]), which states that the only nontrivial (that is, $a, b > 1, x, y > 0$) diophantine solution to $x^a - y^b = 1$ is $x = 3, a = 2, y = 2, b = 3$), we can infer that we can only get these examples of landscape functions from the specific construction of [8, Theorem 19].

Starting with the existence of generalized bent functions in any dimension, it is not very difficult to show that landscape functions of *any level* exist for every dimension, as our next proposition shows. We adapt some classical inductive plateaued construction (see, for instance, [7, 10, 11, 16, 19] for the construction of generalized Boolean bent functions), as well as the paper [17], which contains some constructions of semibent and even more general plateaued in the spirit of Maiorana-McFarland construction of bent functions. There are certainly quite a few works on the analysis of the spectrum of a Boolean functions and we point out here [3, 8, 22], just to mention a few.

Proposition 1. *Let $a \in \mathbb{F}_2$, $q = 2^k$, $k \geq 1$, f be a generalized Boolean function in $\mathcal{GB}_n^q$ and $g : \mathbb{F}_2^{n+1} \to \mathbb{Z}_q$ be defined by $g(\mathbf{x}, y) = f(\mathbf{x}) + 2^{k-1}ay$.*

(i) If f is s-gplateaued, $s \geq 0$, then g is $(s+1)$-gplateaued (in particular, if f is generalized bent, then g is 1-gplateaued).

(ii) If f is a landscape function of length t and levels $\{(m_1, \ell_1), (m_2, \ell_2), \ldots\}$, then g is a landscape function of length t and levels $\{(m_1+1, \ell_1), (m_2+1, \ell_2), \ldots\}$.

Proof. Let f be a landscape function of levels $\{(m_1, \ell_1), (m_2, \ell_2), \ldots, (m_t, \ell_t)\}$. We compute the Walsh-Hadamard transform of g at $(\mathbf{u}, \mathbf{v}) \in \mathbb{F}_2^n \times \mathbb{F}_2$, and get

$$\begin{aligned}
\mathcal{H}_g(\mathbf{u}, \mathbf{v}) &= \sum_{(\mathbf{x}, y) \in \mathbb{F}_2^n \times \mathbb{F}_2} \zeta^{f(\mathbf{x}) + 2^{k-1}ay} (-1)^{\mathbf{u} \cdot \mathbf{x} + vy} \\
&= \sum_{\mathbf{x} \in \mathbb{F}_2^n, y=0} \zeta^{f(\mathbf{x})} (-1)^{\mathbf{u} \cdot \mathbf{x}} + \sum_{\mathbf{x} \in \mathbb{F}_2^n, y=1} \zeta^{f(\mathbf{x}) + 2^{k-1}a} (-1)^{\mathbf{u} \cdot \mathbf{x} + v} \\
&= \mathcal{H}_f(\mathbf{u}) + (-1)^{a+v} \mathcal{H}_f(\mathbf{u}) = \left(1 + (-1)^{a+v}\right) \mathcal{H}_f(\mathbf{u}).
\end{aligned}$$

Thus, $|\mathcal{H}_g(\mathbf{u}, v)| \in \{0, 2|\mathcal{H}_f(\mathbf{u})|\}$, from which we can infer all of our claims. $\square$

Carlet [1] introduced a secondary construction (often called the "indirect sum"), as follows. Let $n = r + s$, where r and s are positive integers, and $f_1, f_2 \in \mathcal{B}_r$, $g_1, g_2 \in \mathcal{B}_s$. Define h as the concatenation of the four functions f_1, $\bar{f}_1$, f_2, $\bar{f}_2$, in an order controlled by $g_1(\mathbf{y})$ and $g_2(\mathbf{y})$,

$$h(\mathbf{x}, \mathbf{y}) = f_1(\mathbf{x}) \oplus g_1(\mathbf{y}) \oplus (f_1(\mathbf{x}) \oplus f_2(\mathbf{x}))(g_1(\mathbf{y}) \oplus g_2(\mathbf{y})). \tag{5}$$

It is known that in the Boolean case, if r, s are even and f_1, f_2 are semibent and g_1, g_2 are bent, then h is semibent. In fact, a more general result is true as we shall show next. The following lemma is known and easy to show.

Lemma 4.1. *For $s \in \mathbb{F}_2$ and $z \in \mathbb{C}$, it holds that*
$$z^s = \frac{1+(-1)^s}{2} + \frac{1-(-1)^s}{2} z.$$

Theorem 4.2. *Let $q = 2^k$, $k \geq 1$, and $h : \mathbb{F}_2^r \times \mathbb{F}_2^s \to \mathbb{Z}_q$ be given by $h(\mathbf{x}, \mathbf{y}) = f_1(\mathbf{x}) + 2^{k-1} g_1(\mathbf{y}) + (f_2(\mathbf{x}) - f_1(\mathbf{x}))(g_1(\mathbf{y}) + g_2(\mathbf{y}))$ (all operations are in $\mathbb{Z}_q$), where $f_1, f_2 \in \mathcal{GB}_r^q$, $g_1, g_2 \in \mathcal{B}_s$, $q = 2^k$, with g_1, g_2 bent (thus, s is even). The following hold:*

(i) *If f_1, f_2 are t-gplateaued, then h is t-gplateaued (hence, of length 2).*

(ii) *If $t_1 \neq t_2$ and $g_1 \neq g_2, g_1 \neq \overline{g}_2$, then h is a landscape function of length 3, namely, the moduli of its spectra are $\{0, 2^{\frac{n+t_1}{2}}, 2^{\frac{n+t_2}{2}}\}$. In particular, if $q = 2$, then h has five valued spectra.*

(iii) *If f_1, f_2 are landscape functions such that*
$$\{|\mathcal{H}_{f_1}(\mathbf{u})|\}_{\mathbf{u} \in \mathrm{supp}(\mathcal{H}_{f_1})} = \{2^{\frac{p_1}{2}} \ell_{11}^2, \ldots, 2^{\frac{p_t}{2}} \ell_{1t}^2\},$$
$$\{|\mathcal{H}_{f_2}(\mathbf{u})|\}_{\mathbf{u} \in \mathrm{supp}(\mathcal{H}_{f_2})} = \{2^{\frac{q_1}{2}} \ell_{21}^2, \ldots, 2^{\frac{q_f}{2}} \ell_{2f}^2\},$$
and $g_1 \neq g_2, g_1 \neq \overline{g}_2$, then h is a landscape function such that
$$\{|\mathcal{H}_h(\mathbf{u}, \mathbf{v})|\}_{\mathbf{u},\mathbf{v}} = \{0, 2^{\frac{s+p_1}{2}} \ell_{11}^2, \ldots, 2^{\frac{s+p_t}{2}} \ell_{1t}^2, 2^{\frac{s+q_1}{2}} \ell_{21}^2, \ldots, 2^{\frac{s+q_f}{2}} \ell_{2f}^2\}.$$

Proof. Using Lemma 4.1 and the indicators of the sets $\{\mathbf{y} : g_1(\mathbf{y}) + g_2(\mathbf{y}) = j\}$, $j = 0, 1$, being $\dfrac{1 + (-1)^j (-1)^{g_1(\mathbf{y}) + g_2(\mathbf{y})}}{2}$, we compute the Walsh–Hadamard transform of h and obtain
$$\begin{aligned}
2\mathcal{H}_h(\mathbf{u}, \mathbf{v}) &= 2 \sum_{\mathbf{x} \in \mathbb{F}_2^r, \mathbf{y} \in \mathbb{F}_2^s} \zeta^{f_1(\mathbf{x}) + 2^{k-1} g_1(\mathbf{y}) + (f_2(\mathbf{x}) - f_1(\mathbf{x}))(g_1(\mathbf{y}) + g_2(\mathbf{y}))} (-1)^{\mathbf{u}\cdot\mathbf{x} + \mathbf{v}\cdot\mathbf{y}} \\
&= 2 \sum_{\mathbf{x} \in \mathbb{F}_2^r} \sum_{\substack{\mathbf{y} \in \mathbb{F}_2^s \\ g_1(\mathbf{y}) + g_2(\mathbf{y}) = 0}} \zeta^{f_1(\mathbf{x}) + 2^{k-1} g_1(\mathbf{y})} (-1)^{\mathbf{u}\cdot\mathbf{x} + \mathbf{v}\cdot\mathbf{y}} \\
&\quad + 2 \sum_{\mathbf{x} \in \mathbb{F}_2^r} \sum_{\substack{\mathbf{y} \in \mathbb{F}_2^s \\ g_1(\mathbf{y}) + g_2(\mathbf{y}) = 1}} \zeta^{f_2(\mathbf{x}) + 2^{k-1} g_1(\mathbf{y})} (-1)^{\mathbf{u}\cdot\mathbf{x} + \mathbf{v}\cdot\mathbf{y}} \\
&= 2 \sum_{\mathbf{x} \in \mathbb{F}_2^r} \zeta^{f_1(\mathbf{x})} (-1)^{\mathbf{u}\cdot\mathbf{x}} \sum_{\mathbf{y} \in \mathbb{F}_2^s} (-1)^{g_1(\mathbf{y})} \frac{1 + (-1)^{g_1(\mathbf{y}) + g_2(\mathbf{y})}}{2} (-1)^{\mathbf{v}\cdot\mathbf{y}} \\
&\quad + 2 \sum_{\mathbf{x} \in \mathbb{F}_2^r} \zeta^{f_2(\mathbf{x})} (-1)^{\mathbf{u}\cdot\mathbf{x}} \sum_{\mathbf{y} \in \mathbb{F}_2^s} (-1)^{g_1(\mathbf{y})} \frac{1 - (-1)^{g_1(\mathbf{y}) + g_2(\mathbf{y})}}{2} (-1)^{\mathbf{v}\cdot\mathbf{y}} \\
&= \mathcal{H}_{f_1}(\mathbf{u}) \sum_{\mathbf{y} \in \mathbb{F}_2^s} \left((-1)^{g_1(\mathbf{y})} + (-1)^{g_2(\mathbf{y})} \right) (-1)^{\mathbf{v}\cdot\mathbf{y}} \\
&\qquad + \mathcal{H}_{f_2}(\mathbf{u}) \sum_{\mathbf{y} \in \mathbb{F}_2^s} \left((-1)^{g_1(\mathbf{y})} - (-1)^{g_2(\mathbf{y})} \right) (-1)^{\mathbf{v}\cdot\mathbf{y}} \\
&= \mathcal{H}_{f_1}(\mathbf{u}) \left(\mathcal{W}_{g_1}(\mathbf{v}) + \mathcal{W}_{g_2}(\mathbf{v}) \right) + \mathcal{H}_{f_2}(\mathbf{u}) \left(\mathcal{W}_{g_1}(\mathbf{v}) - \mathcal{W}_{g_2}(\mathbf{v}) \right).
\end{aligned}$$

If f_1, f_2 are, respectively, t_1, t_2-gplateaued, and g_1, g_2 are bent, and observing that

$$(\mathcal{W}_{g_1}(\mathbf{v})+\mathcal{W}_{g_2}(\mathbf{v}))\,(\mathcal{W}_{g_1}(\mathbf{v})-\mathcal{W}_{g_2}(\mathbf{v}))=\mathcal{W}_{g_1}^2(\mathbf{v})-\mathcal{W}_{g_2}^2(\mathbf{v})=0,$$

then either $(\mathcal{W}_{g_1}(\mathbf{v})+\mathcal{W}_{g_2}(\mathbf{v})) = \pm 2^{\frac{s}{2}+1}$ and $(\mathcal{W}_{g_1}(\mathbf{v})-\mathcal{W}_{g_2}(\mathbf{v})) = 0$, rendering $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| \in \left\{0, 2^{-1}2^{\frac{r+t_1}{2}}\cdot 2^{\frac{s}{2}+1}\right\} = \left\{0, 2^{\frac{n+t_1}{2}}\right\}$, or $(\mathcal{W}_{g_1}(\mathbf{v})+\mathcal{W}_{g_2}(\mathbf{v})) = 0$ and $(\mathcal{W}_{g_1}(\mathbf{v})-\mathcal{W}_{g_2}(\mathbf{v})) = \pm 2^{\frac{s}{2}+1}$, rendering $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| \in \left\{0, 2^{-1}2^{\frac{r+t_2}{2}}\cdot 2^{\frac{s}{2}+1}\right\} = \left\{0, 2^{\frac{n+t_2}{2}}\right\}$. If $t_1 \neq t_2$, we have the cases (both occurring, since $g_1 \neq g_2, \overline{g}_2$), $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| \in \left\{0, 2^{\frac{n+t_1}{2}}\right\}$ and $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| \in \left\{0, 2^{\frac{n+t_2}{2}}\right\}$. Consequently, $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| \in \left\{0, 2^{\frac{n+t_1}{2}}, 2^{\frac{n+t_2}{2}}\right\}$. Certainly, if $t_1 = t_2 = t$, then $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| \in \left\{0, 2^{\frac{n+t}{2}}\right\}$, so h is t-gplateaued. Claim (i) and (ii) are shown.

Next, assume that f_1, f_2 are landscape functions, and pick $\mathbf{u} \in \mathrm{supp}(\mathcal{H}_{f_1}) \cap \mathrm{supp}(\mathcal{H}_{f_2})$, and so, $|\mathcal{H}_{f_1}(\mathbf{u})| = 2^{\frac{p}{2}}\ell_1$, $|\mathcal{H}_{f_2}(\mathbf{u})| = 2^{\frac{q}{2}}\ell_2$, for some $p, q \in \mathbb{Z}$ and odd ℓ_1, ℓ_2. The argument above shows that either $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| \in \left\{0, 2^{\frac{s+p}{2}}\ell_1\right\}$, or $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| \in \left\{0, 2^{\frac{s+q}{2}}\ell_2\right\}$ (both occurring since $g_1 \neq g_2, g_1 \neq \overline{g}_2$). If $\mathbf{u} \in \mathrm{supp}(\mathcal{H}_{f_1}) \cap \overline{\mathrm{supp}(\mathcal{H}_{f_2})}$ (respectively, $\mathbf{u} \in \overline{\mathrm{supp}(\mathcal{H}_{f_1})} \cap \mathrm{supp}(\mathcal{H}_{f_2})$), then $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| \in \left\{0, 2^{\frac{s+p}{2}}\ell_1\right\}$ (respectively, $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| \in \left\{0, 2^{\frac{s+q}{2}}\ell_2\right\}$). If $\mathbf{u} \in \overline{\mathrm{supp}(\mathcal{H}_{f_1})} \cap \overline{\mathrm{supp}(\mathcal{H}_{f_2})}$, then $|\mathcal{H}_h(\mathbf{u},\mathbf{v})| = 0$. Therefore, if

$$\begin{aligned}\{|\mathcal{H}_{f_1}(\mathbf{u})|\}_{\mathbf{u}\in \mathrm{supp}(\mathcal{H}_{f_1})} &= \{2^{\frac{p_1}{2}}\ell_{11}, \ldots, 2^{\frac{p_t}{2}}\ell_{1t}\},\\ \{|\mathcal{H}_{f_2}(\mathbf{u})|\}_{\mathbf{u}\in \mathrm{supp}(\mathcal{H}_{f_2})} &= \{2^{\frac{q_1}{2}}\ell_{21}, \ldots, 2^{\frac{q_f}{2}}\ell_{2f}\},\end{aligned}$$

then $\{|\mathcal{H}_h(\mathbf{u},\mathbf{v})|\}_{\mathbf{u},\mathbf{v}} = \{0, 2^{\frac{s+p_1}{2}}\ell_{11}, \ldots, 2^{\frac{s+p_t}{2}}\ell_{1t}, 2^{\frac{s+q_1}{2}}\ell_{21}, \ldots, 2^{\frac{s+q_f}{2}}\ell_{2f}\}$. □

Remark 2. One might critique the previous theorem that in some of its claims we "recursively" construct landscape functions from landscape functions, with no initial conditions. However, Proposition 1 shows that one can increase the lengths of the landscape functions, and so we can easily start from generalized bent or, more generally, from existing constructions of gplateaued, thus obtaining, using our theorem, either higher lengths or different levels landscape functions, albeit with no precise controlled on the spectra.

5. Characterizing gplateaued functions in terms of second derivatives and fourth moments.

Theorem 5.1. Let $f : \mathbb{F}_2^n \to \mathbb{Z}_{2^k}$, $k \geq 2$, s be an integer with $0 \leq s \leq n$, and $\zeta := \zeta_{2^k} = e^{\frac{2\pi i}{2^k}}$ be the primitive root of 1. Then f is s-gplateaued if and only if for all $\mathbf{x} \in \mathbb{F}_2^n$,

$$\sum_{\mathbf{u},\mathbf{b}\in\mathbb{F}_2^n} \zeta^{D_{\mathbf{b}}D_{\mathbf{u}}f(\mathbf{x})} = 2^{n+s}.$$

Furthermore, f is s-gplateaued if and only if

$$\sum_{\mathbf{d}\in\mathbb{F}_2^n} |\mathcal{H}_f(\mathbf{d})|^4 = 2^{3n+s}.$$

Proof. Let $\mathbf{x} \in \mathbb{F}_2^n$ be fixed. First observe that

$$\sum_{\mathbf{u},\mathbf{b}\in\mathbb{F}_2^n} \zeta^{D_\mathbf{b} D_\mathbf{u} f(\mathbf{x})} = 2^{n+s} \text{ is equivalent to}$$
$$F_1(\mathbf{x}) := \sum_{\mathbf{u},\mathbf{b}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x}\oplus\mathbf{u}\oplus\mathbf{b})-f(\mathbf{x}\oplus\mathbf{b})-f(\mathbf{x}\oplus\mathbf{u})} = 2^{n+s}\zeta^{-f(\mathbf{x})} =: F_2(\mathbf{x}),$$

which is further equivalent to their Fourier transforms being equal at all $\mathbf{u} \in \mathbb{F}_2^n$, that is,

$$\sum_{\mathbf{x},\mathbf{u},\mathbf{b}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x}\oplus\mathbf{u}\oplus\mathbf{b})-f(\mathbf{x}\oplus\mathbf{b})-f(\mathbf{x}\oplus\mathbf{u})}(-1)^{\mathbf{u}\cdot\mathbf{x}} = 2^{n+s} \sum_{\mathbf{x}\in\mathbb{F}_2^n} \zeta^{-f(\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}}. \tag{6}$$

We compute the two expressions in (6), separately. Now considering, the left hand side of (6), and setting $\mathbf{u}_1 := \mathbf{x} \oplus \mathbf{u}, \mathbf{b}_1 := \mathbf{x} \oplus \mathbf{b}$, we obtain

$$\begin{aligned}
&\sum_{\mathbf{x},\mathbf{u},\mathbf{b}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x}\oplus\mathbf{u}\oplus\mathbf{b})-f(\mathbf{x}\oplus b)-f(\mathbf{x}\oplus\mathbf{u})}(-1)^{u\cdot\mathbf{x}}\\
=&\sum_{\mathbf{x},\mathbf{u}_1,\mathbf{b}_1\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x}\oplus\mathbf{u}_1\oplus\mathbf{b}_1)-f(\mathbf{b}_1)-f(\mathbf{u}_1)}(-1)^{\mathbf{u}\cdot\mathbf{x}}\\
=&\sum_{\mathbf{b}_1\in\mathbb{F}_2^n} \zeta^{-f(\mathbf{b}_1)}(-1)^{\mathbf{u}\cdot\mathbf{b}_1} \sum_{\mathbf{u}_1\in\mathbb{F}_2^n} \zeta^{-f(\mathbf{u}_1)}(-1)^{\mathbf{u}\cdot\mathbf{u}_1}\\
&\qquad\cdot\sum_{\mathbf{x}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x}\oplus\mathbf{u}_1\oplus\mathbf{b}_1)}(-1)^{\mathbf{u}\cdot(\mathbf{x}\oplus\mathbf{b}_1\oplus\mathbf{u}_1)}\\
=&\overline{\mathcal{H}_f(\mathbf{u})}\;\overline{\mathcal{H}_f(\mathbf{u})}\,\mathcal{H}_f(\mathbf{u}) = |\mathcal{H}_f(\mathbf{u})|^2\,\overline{\mathcal{H}_f(\mathbf{u})}.
\end{aligned}$$

The right hand side of (6) can be written as

$$2^{n+s} \sum_{\mathbf{x}\in\mathbb{F}_2^n} \zeta^{-f(\mathbf{x})}(-1)^{\mathbf{u}\cdot\mathbf{x}} = 2^{n+s}\overline{\mathcal{H}_f(\mathbf{u})},$$

therefore (6) is equivalent to $|\mathcal{H}_f(\mathbf{u})|^2\,\overline{\mathcal{H}_f(\mathbf{u})} = 2^{n+s}\,\overline{\mathcal{H}_f(\mathbf{u})}$, that is, $|\mathcal{H}_f(\mathbf{u})| \in \{0, 2^{(n+s)/2}\}$. Our first claim is shown.

Next, using [19, Theorem 1], we compute

$$\begin{aligned}
\sum_{\mathbf{u},\mathbf{b}\in\mathbb{F}_2^n} \zeta^{D_b D_a f(\mathbf{x})} &= \sum_{\mathbf{u},\mathbf{b}\in\mathbb{F}_2^n} \zeta^{f(x\oplus\mathbf{u}\oplus\mathbf{b})-f(x\oplus\mathbf{b})-f(\mathbf{x}\oplus\mathbf{u})+f(\mathbf{x})}\\
&= \sum_{\mathbf{u}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x})-f(\mathbf{x}\oplus\mathbf{u})} \sum_{\mathbf{b}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x}\oplus\mathbf{u}\oplus\mathbf{b})-f(\mathbf{x}\oplus\mathbf{b})}\\
&\overset{\substack{\mathbf{c}:=\mathbf{x}\oplus\mathbf{b}\\ \downarrow}}{=} \sum_{\mathbf{u}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x})-f(\mathbf{x}\oplus\mathbf{u})} \sum_{\mathbf{c}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{c}\oplus\mathbf{u})-f(\mathbf{c})}\\
&= \sum_{\mathbf{u}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x})-f(\mathbf{x}\oplus\mathbf{u})}\mathcal{C}_f(\mathbf{u}), \text{ since } \mathcal{C}_f \text{ is always real}\\
&= 2^{-n} \sum_{\mathbf{u}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x})-f(\mathbf{x}\oplus\mathbf{u})} \sum_{\mathbf{d}\in\mathbb{F}_2^n} |\mathcal{H}_f(\mathbf{d})|^2(-1)^{\mathbf{u}\cdot\mathbf{d}}\\
&= 2^{-n} \sum_{\mathbf{d}\in\mathbb{F}_2^n} |\mathcal{H}_f(\mathbf{d})|^2 \sum_{\mathbf{u}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x})-f(\mathbf{x}\oplus\mathbf{u})}(-1)^{\mathbf{u}\cdot\mathbf{d}}
\end{aligned}$$

$$= 2^{-n} \sum_{\mathbf{d}\in\mathbb{F}_2^n} |\mathcal{H}_f(\mathbf{d})|^2 \zeta^{f(\mathbf{x})}(-1)^{\mathbf{x}\cdot\mathbf{d}} \sum_{\mathbf{c}\in\mathbb{F}_2^n} \zeta^{-f(\mathbf{c})}(-1)^{\mathbf{c}\cdot\mathbf{d}}$$
$$= 2^{-n} \sum_{\mathbf{d}\in\mathbb{F}_2^n} |\mathcal{H}_f(\mathbf{d})|^2 \zeta^{f(\mathbf{x})}(-1)^{\mathbf{x}\cdot\mathbf{d}}\overline{\mathcal{H}_f(\mathbf{d})}.$$

Thus,

$$2^{2n+s} = \sum_{\mathbf{x},\mathbf{u},\mathbf{b}\in\mathbb{F}_2^n} \zeta^{D_\mathbf{b}D_\mathbf{u}f(\mathbf{x})} = 2^{-n} \sum_{\mathbf{x},\mathbf{d}\in\mathbb{F}_2^n} |\mathcal{H}_f(\mathbf{d})|^2 \overline{\mathcal{H}_f(\mathbf{d})}\zeta^{f(\mathbf{x})}(-1)^{\mathbf{x}\cdot\mathbf{d}}$$
$$= 2^{-n} \sum_{\mathbf{d}\in\mathbb{F}_2^n} |\mathcal{H}_f(\mathbf{d})|^2 \overline{\mathcal{H}_f(\mathbf{d})} \sum_{\mathbf{x}\in\mathbb{F}_2^n} \zeta^{f(\mathbf{x})}(-1)^{\mathbf{x}\cdot\mathbf{d}} = 2^{-n} \sum_{\mathbf{d}\in\mathbb{F}_2^n} |\mathcal{H}_f(\mathbf{d})|^4,$$

and the second claim is shown. □

Our next corollary (see [1] for the classical counterpart) is immediate since a generalized bent function corresponds to a 0-gplateaued function.

Corollary 2. *A function $f \in \mathcal{GB}_n^q$, $q = 2^k$, is generalized bent if and only if $\sum_{\mathbf{u},\mathbf{b}\in\mathbb{F}_2^n} \zeta^{D_\mathbf{b}D_\mathbf{u}f(\mathbf{x})} = 2^n$ if and only if $\sum_{\mathbf{d}\in\mathbb{F}_2^n} |\mathcal{H}_f(\mathbf{d})|^4 = 2^{3n}$.*

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