2014

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Elsevier B.V.

Rappold, James A., and Keenan D. Yoho. "Setting safety stocks for stable rotation cycle schedules." International Journal of Production Economics 156 (2014): 146-158.
http://hdl.handle.net/10945/57080

Int. J. Production Economics 156 (2014) 146–158

Contents lists available at ScienceDirect

Int. J. Production Economics

journal homepage: www.elsevier.com/locate/ijpe

Setting safety stocks for stable rotation cycle schedules

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ARTICLE INFO

Article history:
Received 5 January 2014
Accepted 29 May 2014
Available online 9 June 2014

Keywords:
Inventory optimization
Cyclic scheduling
Stable production cycles
Target inventory level
Process industries

ABSTRACT

In the process industries, specialized equipment and production processes often necessitate the manufacture of products in a pre-determined sequence to minimize changeover time and to simplify scheduling complexity; these types of schedules are referred to as pure rotation schedules, or product wheels, where the circumference of the wheel is the *production cycle length*. In these industries changeover times between the production of individual products can consume considerable time as well as raw materials and it is therefore often desirable to stabilize the production cycles in order to minimize unplanned changeovers as well as quote accurate lead times to customers. Materials requirements planning (MRP) systems are often used to plan and coordinate production and supply resources with demand in these environments. Central to the effectiveness of the MRP system is the dependability of the lead time parameters. In this paper, we introduce an optimization model to determine safety stock levels that minimize long run expected costs where a stable, cyclic schedule is used. Our model may be used strategically to assess inventory investment requirements as a function of capacity investment, product mix, production technology, demand volatility, and customer service levels. It may be used tactically to optimize item-level planning parameters such as lot size, safety stock and lead time in an MRP system and to support sales and operations planning (S&OP) processes where knowing the future costs associated with current decisions is highly desirable.

1. Introduction and motivation

In the process industries, specialized equipment and production processes have significant, and often sequence dependent, changeover times. The scheduling complexity of sequence dependent changeovers is commonly addressed and simplified using pure rotation schedules, or product wheels, in which products are produced in a pre-determined sequence to minimize the total changeover time. The circumference of the product wheel is the *production cycle length*. Materials requirements planning (MRP) systems are often used to plan and coordinate production and supply resources with demand in these environments. Central to the effectiveness of the MRP system is the dependability of the manufacturing lead time parameters (Koh et al., 2002; Dogui and Ould-Louly, 2002; Ould-Louly and Dogui, 2004; Mula et al., 2006). In this paper, we present a novel production policy that constrains each production cycle and an optimization model to determine safety stock levels that minimize long run expected costs.

We conducted an exploratory analysis to understand the precise nature of the production and inventory planning environment for a range of process manufacturing firms. Our exploratory analysis included site visits to 13 different process manufacturing firms where we conducted in-depth interviews with managers, production planners, inventory planners, and sales personnel to understand the nature of their production equipment, how they planned production and inventory, and the types of enterprise resource planning (ERP) and production planning systems they were using. What we discovered was that these firms were using systems that assumed manufacturing lead times were fixed or constant however the reality was that such lead times were anything but firm or stable. Here, we refer to manufacturing lead time as the span of time between beginning and completing a manufacturing order that is ready for shipment. In some cases the firms were trying to adopt lean manufacturing methods used in discrete manufacturing environments to drive production and inventory planning (see Yoho and Rappold, 2011 for a full discussion). What was clear was a gap in both how the firms thought about the production planning and inventory control problem as well as a lack of a method to plan effectively once the details of the problem were understood.

Production and inventory planning in process industries presents unique challenges that differ from discrete manufacturing environments. Because of the relative inflexibility of process manufacturing scheduling environments, adhering to a single lead time parameter in the MRP system can be a challenge. If production cycle lengths and therefore manufacturing lead times vary

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http://dx.doi.org/10.1016/j.ijpe.2014.05.020
0925-5273/

significantly what single manufacturing lead time parameter should be entered into the MRP system? Further, what lead time should be quoted for new customer orders? How will the available-to-promise (ATP) logic in these systems be affected? What safety stock levels are required to achieve a target customer service level?

Stable production cycles are highly desirable in process industries where setup times tend to be long and the production process often requires a warm-up time before finished goods are produced. With the motivation to stabilize manufacturing lead times, we restrict ourselves to a production policy that constrains the production cycle length between an upper and a lower limit. Because the cycle length is constrained by an upper and a lower limit there will be circumstances in which more inventory is produced than necessary and others where there is not enough capacity to meet the demand in the current period thereby necessitating some amount of inter-cyclic safety stock to ensure that neither inventory or backorders grow without bound. Determining the inventory levels that will act as a system "shock absorber" (Holt et al., 1960) to satisfy uncertain demand so that lead times can be stabilized is the motivation behind constraining the production cycle length and determining an optimal target inventory level which, to date, is an unsolved problem.

Inventory in this system is composed of both cycle stock and safety stock. Cycle stock is due to the lot sizing that occurs within each cycle to economize on the changeover time. Safety stock is due to the demand uncertainty both *within* the current production cycle (intra-cycle) and *between* successive production cycles (inter-cycle) as shown in Fig. 1. That is, within the current production cycle, lot sizes are adjusted dynamically, subject to an upper and a lower production cycle length, to minimize expected holding and shortage costs through the end of the current cycle. However, unless we consider the implications of the inventory levels at the start of the next production cycle, we begin the next production cycle in a cost-disadvantageous situation. Indeed, this is similar to the end-of-the-world phenomenon in finite horizon dynamic programs. Solving the intra-cycle lot-sizing problem using a finite horizon dynamic program would attempt to end the horizon with zero inventories and our model prevents this from happening.

In our case, we must consider the cycle-to-cycle evolution of inventories and the inventory levels that are left for the next cycle. We address this by making current production cycle lot sizing decisions while considering an end-of-cycle target inventory level (*TIL*). We call the end-of-cycle deviation from *TIL* the *inventory net shortfall*. The inventory net shortfall at the end of a production cycle is the amount of inventory in the system that is either short or in excess of our target inventory level, *TIL*. Because we have both a lower and an upper limit on our production cycle length, we will observe instances where there is a shortage of inventory on hand as well as in excess of our target level. Determining an optimal value of *TIL* that minimizes the long-run expected cost is the purpose of this paper.

For a single cycle, we could establish optimal beginning inventory levels for each item that minimize expected holding and backorder costs over the course of the cycle. However, the net inventory level of *each item* at the beginning of a cycle is a vector of random variables that depend on the lot sizes and realized demand of the prior cycle. Were we dealing with deterministic demand, this problem could be solved combinatorially. In this paper, we develop an approach that provides fast, scalable, and near-optimal target inventory levels.

In this paper, we develop a novel stochastic inventory model that can be used to estimate safety stock requirements to support a production policy that stabilizes cycle lengths. Our contribution to the literature is an optimization model and simple solution approach that computes the safety stock requirements in support of production and inventory policies that stabilize the production cycle length. Our solution strategy consists of two steps. First, we determine an optimal aggregate target inventory level, *TIL**, for the system that considers demand volatility, production capacity, an upper and a lower production cycle limit, and end-of-cycle expected costs associated with a particular system inventory profile. Second, we disaggregate *TIL** to determine individual item target inventory levels that will minimize costs in a future cycle. Our model may be used strategically to assess the inventory investment requirements as a function of capacity investment, product mix, production technology, demand volatility, and customer service levels. It may be used at a tactical level to optimize item-level planning parameters such as lot size, safety stock and lead time in an MRP system and to support sales and operations planning (S&OP) processes where knowing the future costs associated with current decisions is highly desirable.

The remainder of the paper is organized as follows. Section 2 is a review of the research and a discussion of our contributions to the literature. In Section 3 we characterize a stochastic process representing the cycle-to-cycle inventory dynamics. We explore how demand uncertainty, capacity utilization, lower and upper cycle limits affect expected inventory shortages. In Section 4 we present our optimization model and a fast solution approach. Section 5 describes the results of an extensive numerical study in which the optimal value of the Target Inventory Level (*TIL**) is calculated and associated costs with values of *TIL** are reported. Section 6 summarizes our work and discusses our conclusions.

2. Literature review

Much of the academic research for the management and planning of production resources has been based upon either the

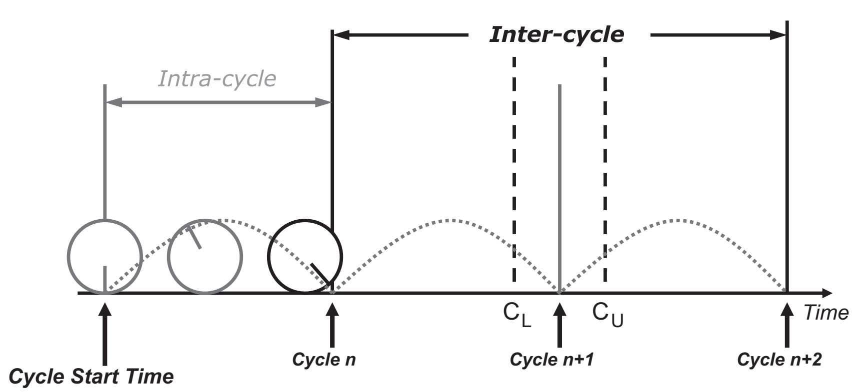

Fig. 1. Intra-cyclic dynamics of successive production cycles.

economic lot size problem (ELSP) in which demand is treated as deterministic, or the stochastic economic lot size problem (SELSP) in which demand is considered to be uncertain. When demand is deterministic, the ELSP works well. However, demand uncertainty is so significant in many process industry manufacturing environments that it cannot be ignored. While mathematically attractive, the SELSP paradigm for dealing with demand uncertainty has a serious flaw that hinders its wide-spread applicability and use: it adjusts lot sizes dynamically without regard for the resulting variability in the production cycle length. This leads to variability in manufacturing lead times and lost capacity due to disruptions in the production sequence and unplanned changeovers. If production cycle lengths were stabilized explicitly, an unsolved question is how to optimize safety stock levels so inventory may absorb random demand and allow production cycles to be stabilized on a cycle-to-cycle basis.

Our problem *is not a lot sizing problem* but rather a method of determining a target level of inventory that allows a production facility to maintain a stable production cycle length and acts as a buffer against highly variable demand. We assume a production environment that operates according to a cyclic schedule with upper and lower cycle limits. This type of cyclic scheduling problem differs significantly from other types of lot sizing problems such as the economic lot sizing problem (ELSP) and the stochastic economic lot sizing problem (SELSP). The ELSP determines lot sizes for multiple items produced on a single production resource with setup times where demand is deterministic and therefore cycle lengths are constant. The objective of the ELSP is to minimize long-run average inventory holding costs, setup costs and production costs and has been given ample consideration in the operations research and management science literature (see Delporte and Thomas, 1977; Elmaghraby, 1978; Graves, 1979; Jones and Inman, 1989; Roundy, 1989; Schweitzer and Silver, 1983; Van Hoesel et al., 2005 for a full treatment of the problem)

The stochastic economic lot sizing problem (SELSP) was developed to address random demand by *fluctuating the production cycle length* – or, in the case where polling systems were considered, to adjust cycle times – in order to minimize one or more of the following: customer waiting times, inventory holding costs or backorders (see, for example, Wagner and Whitin, 1958; Sarkar and Zangwill, 1989, 1991; Federgruen and Katalan, 1996a, 1996b, 1998; Sox and Muckstadt, 1996, 1997; Lan and Olsen, 2006). While the stochastic lot sizing problem considers random demand, stabilizing production cycles is not an objective. The SELSP adjusts individual item lot sizes, and therefore fluctuates production cycle lengths considerably, to meet intra-cyclic demand in order to reduce inventory holding and backorder costs. In our model, we assume a production environment in which management has made stabilizing the cycle length a priority. We adjust lot sizes but the overall cycle length is constrained. As mentioned previously, the cycle length is constrained by an upper and a lower limit and therefore periodically there will be instances where more inventory is produced than necessary and others where there is not enough capacity to meet the demand in the current period. In order to ensure that neither inventory or backorders grow without bound it is necessary to determine some amount of inter-cyclic safety stock.

Other approaches that have similar qualities to the one we present here but also differ in significant, nontrivial ways. None of the previous work in cyclic scheduling explicitly considers both capacity utilization and demand uncertainty in the calculation of an inventory level where cycle stability is a stated managerial objective. Leachman and Gascon (1988) formulate a “dynamic cycle length policy” whose name alone distinguishes it from the object of our current research. Our work fundamentally differs from theirs in that we seek to stabilize production cycles whereas their policy fluctuates cycles. Demand for the period immediately following the production decision is known with certainty. In their numerical study, the authors calculate the item safety stocks to allow for three standard deviations of forecast errors of demand during a changeover time where forecast errors of demand are normally distributed. In our study, we assume a stable rather than dynamic cycle length policy, and our contribution is to determine an inventory level that enables a stable production cycle where demand may be highly variable. Gallego (1994) shows that a base stock policy is always successful in recovering a cyclic schedule after a disruption with minimal excess over average costs. In this work, the production cycles are fluctuated to meet target inventory levels and the capacity utilization is not an explicit feature of the control mechanism.

Bourland and Yano (1994) consider the use of capacity slack in the form of overtime in a production environment using a pure rotation schedule under stochastic demand. Our work differs from theirs in several ways. First, their work is primarily concerned with the production plan and its execution whereas our work assumes that the production will follow a stable cyclic schedule and our contribution is to determine an appropriate inventory level to allow management to maintain a stable production cycle without inventories growing or depleting without limit. Second, in a production setting that runs 24 h per day, 7 days per week, there is no overtime available to act as capacity slack. Third, the expected shortage of a part is determined entirely by its reorder point and the initial target inventory level set by the authors does not take into consideration the variability in demand nor the utilization of the production capacity both of which have a significant impact on how long it will take to work off backorders resulting from insufficient capacity or inventory on-hand. Fourth, the authors simulate that the performance of their policy assuming demand is approximately normally distributed whereas our policy considers demand environments that are much more volatile and consistent with recent field observations of large-scale production operations in process industries.

Fransoo et al. (1995) discuss a two-level hierarchical model similar to that previously introduced by Leachman and Gascon (1988). The authors propose a complex, nonlinear objective function to maximize profit where demand is considered to be normally distributed. Our present contribution differs from this work in several ways. First, we introduce a model that specifically stabilizes the production cycle in the objective function – it is a single-step approach whose implementation is far simpler than that proposed by the authors. Second, because the authors assume that demand is greater than the production capacity, and that any orders not filled by inventory are lost, they are able to ignore the most difficult aspects of determining a target inventory level that will allow for the stabilization of production cycles in a continuous process environment that runs 24 h per day, 7 days per week: that of having far too much inventory (as a result of not being able to turn off production) or far too little inventory (as a result of highly variable demand that may exceed capacity in any given period or cycle). Finally, the deterministic partial expectation function from Brown (1963) used to determine the expected inventory at the end of a cycle is not realistic for use in the large-scale industrial environments observed during field work. Our model determines a target inventory level needed to stabilize production cycles where the variability of demand, the capacity utilization and the production cycle length are all considered to minimize backorders and holding costs in an environment that, in the absence of the appropriate inventory to buffer stochastic demand, could easily become unstable whereby backorders or inventory grows to unmanageable levels.

Eisenstein (2005) introduces an augmentation to traditional produce-up-to policies to recover a cyclic schedule when a shock in demand is experienced. In this work, the target inventory is a function of the production rate, the setup time and normalized, smooth demand observed over the cycle. Our research differs in that we are not introducing a production control mechanism – we assume a stable cycle length bounded by upper and lower limits. Further, our contribution does not assume normalized, smooth demand over the production cycle. Another crucial distinction is that the author assumes that the inventory level for each item begins no greater than its produce-up-to point at the start of a cycle. However, in the environment we consider, where the production cycle is bounded by a lower as well as an upper limit, inventory may be produced in excess of the target in any period depending upon the variability of the demand.

Our model integrates both the planning and scheduling aspects of production and inventory control and readily allows inventory to be considered not just in terms of units but also in terms of time. Our approach respects the practice of stabilizing production cycles in process manufacturing environments and provides a mechanism to enable their implementation and ensure robustness in the overall production and inventory planning and scheduling system.

3. The inventory net shortfall process

We begin by structuring the problem as though we are solving for a single item (which will represent the aggregate target inventory level) and express inventory in *production run time units of capacity* (e.g., hours of production). Our target inventory level, *TIL*, is the amount of stored capacity that is necessary to buffer the cycle-to-cycle dynamics, resulting from the underlying demand uncertainty. We wish to find the *TIL** that minimizes the long-run expected holding and backorder costs per period.

First, we calculate the steady-state probability distribution of the inventory at the beginning of a production cycle using an *inventory net shortfall process*. We then employ a simplified version of a production execution model from Rappold and Yoho (2008) to determine an expected cost associated with beginning a production cycle given a current inventory, capacity, utilization and demand variance-to-mean ratio (VTMR) state. Using the variance-to-mean ratio (VTMR) allows us to model demand using a negative binomial distribution and scale the uncertainty easily; we may also sum individual negative binomial distributions into an aggregate negative binomial distribution with the same VTMR which is an attractive characteristic for calculation purposes. Additionally, much of the demand we observed during field visits to process industries is well-represented using a negative binomial distribution where the variance-to-mean ratio of demand is greater than 1.

Combining the inventory net shortfall and expected costs allows a *TIL** to be calculated that explicitly considers the long-run *cost consequences* of an upper cycle limit, C_U, a lower cycle limit, C_L, a capacity utilization, and a demand VTMR.

Let $d_i(t)$ be non-negative random variables representing the demand for item $i=1,\ldots,M$ over $t>0$ time periods, where M is the number of items, and time period is some suitable base time period such as a shift, day, or week. Demand is assumed to be expressed in units of product per base time period. We assume demand for each item is independent, and identically distributed (i.i.d.), and that $\Pr\{d_i(t)>0\}>0$ for all i and $t>0$. Let $\mu_i(t)$ and $\sigma_i^2(t)$ be the mean and the variance, respectively, of demand for item i over t time periods. Let r_i be the unit production rate (units of product/base time period) for item i; we will assume the run rates for each item are the same however this assumption is not necessary for our results to hold, as we will aggregate the demand. Define $\rho=\sum_i \mu_i/r_i$ to be the run time capacity utilization where $\mu_i=\mu_i(1)$ is the expected demand for item i per base time period. We assume that $0<\rho<1$ so that the production system is stable, there exists adequate production capacity to keep up with mean demand, and backorders do not grow without bound. Let k_i be the setup time of item i and $K=\sum_i k_i$ be the sum of all item setup times; for discussion purposes we will assume setup times for each item are the same however it is not necessary to make this assumption for the results to hold. Let $D(t)$ represent the aggregate demand (in *run time* units) over t time periods, or $D(t)=\sum_i (1/r_i)d_i(t)$. This translates the demand from units of product into demand for production capacity (units of time).

As illustrated in Fig. 1, a production cycle is the total amount of time, on average, necessary to produce the mean demand during the cycle plus setup times for each item. Let $n=1,2,\ldots$ be the index for the production cycle, and C_n be the length of the nth production cycle, in units of time. In practice, each production cycle length C_n is uncertain, and a function of demand uncertainty and the individual item lot sizing decisions within the nth cycle, as determined by a master scheduler. As discussed in Silver et al. (1998), the expected *cycle length*, $\overline{C}=E(C_n)$, is the expected time required to satisfy demand during the cycle plus the sum of all setup times. It is defined as

$$\overline{C}=\sum_i k_i+\sum_i(\mu_i/r_i)\overline{C} \Rightarrow \overline{C}=K/(1-\rho). \tag{3.1}$$

Note that the production cycle length is a function of the run time utilization. The higher the utilization, the longer the production cycle length. The production cycle length will be constrained between a lower cycle limit, C_L, and an upper cycle limit, C_U, such that $K<C_L<\overline{C}<C_U$. We assume that at least some production will take place in each cycle and require this condition for system stability so that neither inventories nor backorders grow without bound. In practice, C_L and C_U are management parameters determined typically by a master scheduler, and are based upon the particular economics of the strategic environment and on the level of desired schedule stability.

To capture the cycle-to-cycle dynamics of the system we define a stochastic process that represents the evolution of the end-of-cycle system net inventory level, expressed as a deviation from a Target Inventory Level, *TIL*, expressed in units of time. This deviation depends on ρ, C_L, C_U, as well as the mean and the variance of aggregate demand, D. We call the end-of-cycle deviation from *TIL* the *inventory net shortfall*. The net shortfall is related to the work of the dam models developed by Prabhu (1965) and it allows the inventory position to be modeled as a Markov chain – independent of *TIL* – when using target inventory levels. The separation of the evolution of inventory levels from the explicit value of *TIL* greatly simplifies the optimization model.

Recall that because we have a lower limit on our production cycle length, we observe instances where there is inventory on hand (i.e., stored time) in excess of our target level. The inventory net shortfall at the end of cycle n, denoted V_n, is the amount of time that the system is either short or *in excess* of our target, *TIL*. A positive shortfall, $V_n>0$, indicates that we are *below* the target inventory level, while a negative shortfall, $V_n<0$, indicates that we are above the target inventory level.

To determine the value of stored inventory necessary to minimize expected costs while stabilizing our production cycle length, we must calculate the stationary probability distribution of the inventory net shortfall. Let the aggregate demand for capacity (*time*) in the current production cycle be the total time needed to setup each product plus the expected time required for production to satisfy the mean demand, or $K+\rho\overline{C}$. The sequence of events with respect to recognizing the inventory net shortfall and making the aggregate production decision for the current cycle is given in Fig. 2.

At the beginning of cycle n, we observe the inventory net shortfall from the last cycle, V_{n-1}. Given the observed inventory

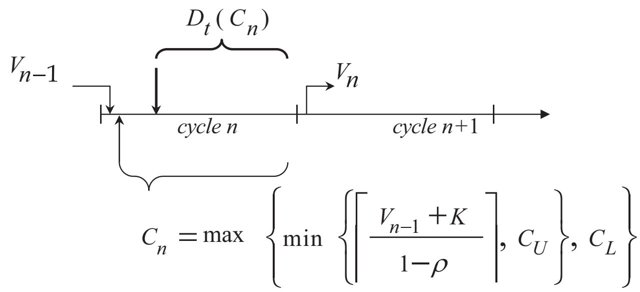

Fig. 2. Cycle-to-cycle sequence of events considering the inventory net shortfall in the production decision.

Table 1
Factors and factor levels used to calculate inventory net shortfall distributions.

Lower cycle limit, C_L (a percentage of $E(C)$)	Upper cycle limit, C_U (a percentage of $E(C)$)	Capacity utilization	Demand VTMR
0.75	1.05	0.90	1.01
0.85	1.15	0.95	5.00
0.95	1.25	0.99	10.00

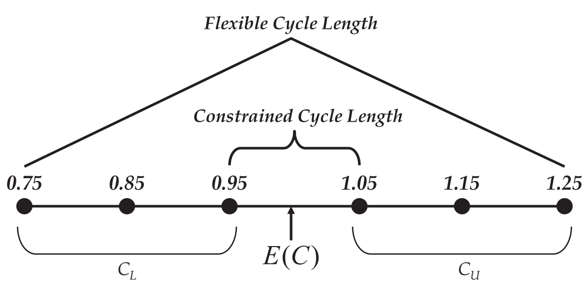

Fig. 3. Illustration of production cycle lengths as a percentage of the planned cycle length.

shortfall, capacity utilization, demand (VTMR), and production cycle lower and upper limits, C_L and C_U, we have a certain likelihood of restoring the aggregate inventory to *TIL* at the end of the current cycle. This depends on the lot size decisions of cycle n and how they affect the current cycle length. Second, we compute a cycle length, $\tilde{C}_n$, that in expectation would restore the system to *TIL* (i.e., $V_n=0$) as follows:

$$\tilde{C}_n = V_{n-1} + K + \rho \cdot \tilde{C}_n \Leftrightarrow \tilde{C}_n = \frac{V_{n-1}+K}{1-\rho}. \tag{3.2}$$

That is, $\tilde{C}_n$ considers the starting shortfall plus the expected time required to setup and produce the expected demand. However, for our actual aggregate production decision, C_n, to be feasible, we must have that $C_n \in [C_L, C_U]$. Thus, considering the minimum and maximum limits on the production cycle length, we set C_n as

$$C_n = \begin{cases} C_U & \text{if } \tilde{C}_n > C_U, \\ C_L & \text{if } \tilde{C}_n < C_L, \quad \text{or} \\ \tilde{C}_n & \text{otherwise.} \end{cases} \tag{3.3}$$

Alternatively, we have

$$C_n = \max\left\{\min\left\{\left\lceil \frac{V_{n-1}+K}{1-\rho} \right\rceil, C_U\right\}, C_L\right\}, \quad n > 0, \tag{3.4}$$

where $\lceil x \rceil$ is the integer ceiling of $x \in \mathcal{R}$. Third, we observe customer demand throughout the course of the cycle, $D(C_n)$. Fourth, and finally, the production cycle concludes and the observed shortfall for the current cycle is V_n.

Based on this sequence of events, the inventory net shortfall process can be described as follows:

$$V_0 = 0,$$
$$V_n = V_{n-1} + K + D(C_n) - C_n, \quad n > 0, \tag{3.5}$$

where C_n is given by (3.4). It is important to observe that $\{V_n\}_{n \ge 0}$ does *not* depend on *TIL*. This is a crucial modeling attribute and we will use it when optimizing *TIL*. Note that $V_n > 0$ when $\tilde{C}_n > C_U$ and the cycle cannot be extended to restore the inventories to *TIL*. Also note that $V_n < 0$ when $\tilde{C}_n < C_L$ and we are compelled to extend the production cycle beyond that which is desired. Thus a positive shortfall occurs when system inventories are below *TIL*, and a negative shortfall occurs when the system inventories are above *TIL*.

From the definition (3.5), and from the independence of demand between periods, and therefore cycles, the inventory net shortfall process may be modeled as a Markov chain. Assuming the existence of V in steady-state, we define its steady-state probability distribution to be $\pi(v)$, $v \in \mathcal{Z}$. The transition probabilities of $\{V_n\}_{n>0}$ for states $i,j \in \mathcal{Z}$ are $P_{ij} = \Pr\{V_n = j | V_{n-1} = i\}$, where

$$P_{ij} = \begin{cases} \Pr\{D(C_n) = j-(i+K)+C_U\} \\ \quad \text{for } j>0,\ i \le C_U + j - K, \\ \Pr\{C_L-(i+K) \le D(C_n) \le C_U-(i+K)\} \\ \quad \text{for } j=0,\ i \le C_U - K, \\ \Pr\{D = j-(i+K)+C_L\} \\ \quad \text{for } j<0,\ i \le C_L + j - K, \\ 0 \quad \text{otherwise.} \end{cases} \tag{3.6}$$

We will later use these transition probabilities to compute the steady-state, *TIL*-independent net shortfall distribution, and subsequently the optimal *TIL** to minimize costs while supporting a stable production cycle.

3.1. An illustration of the inventory net shortfall distribution

Based upon four factors with three factor levels each (see Table 1) we evaluate 81 different scenarios for which we calculate the net shortfall distribution where demand is modeled as a negative binomial distribution and the demand variance-to-mean ratio (VTMR) is determined on a per period basis. The capacity and demand variance parameters used in the simulation are based upon direct observation of the production and inventory operations of nine process manufacturing firms over 2 years. One organizational phenomenon we observed in several of these firms was the construction of new production facilities in developing countries to produce material that would serve an entire continent or global hemisphere. These new, large plants would therefore be serving multiple national markets characterized by asynchronous, seasonal demand as well as financing and credit environments that drove ordering behavior that was far from level or smooth.

In all cases, there are five items being produced in a pure rotation schedule, or "product wheel," and the setup time per item i is $k_i=60$ min, with the total setup time per cycle being $K = \sum_{i=1}^{5} k_i = 300$ min. All products are assumed to have identical production rates. The lower and upper limits of the production cycle, C_L and C_U, are expressed as percentages of the planned cycle length $E(C)$ given in (3.1) and illustrated in Fig. 3.

For illustration purposes, consider the steady-state probability distribution of the inventory net shortfall, $\pi(v)$, when the production cycle length is least flexible (C_L=0.95 and C_U=1.05), the demand variance-to-mean ratio is 10 and the capacity utilization is at 99%. Fig. 4 shows the net shortfall in days of inventory where there are 480 min in a day; though the curve appears continuous in the figure it is, in fact, discrete. The x, or horizontal, axis represents the amount of inventory short of *TIL*, represented by *time* and given in number of days; zero on the *x*-axis indicates that there is no deviation from *TIL*. The *y*-axis is the probability mass function (pmf) of the steady-state net shortfall distribution. The right side of the curve represents the positive shortfall, or the amount of inventory that is *below TIL*. The left side of the curve represents the negative shortfall, or the amount of inventory that is *above* or in excess, with respect to *TIL*. The expected cycle length, $E(C)$, in Fig. 4 is 62.50 days. The expected shortfall, $E(V)$, is 5.84 days with a standard deviation, $StdDev(V)$, of 141.19 days and the probability that our shortfall will be equal to zero, $P(V=0)$, is less than 0.01. The shortfall distribution for all scenarios was calculated to support the numerical study that appears in Section 5.

3.2. The effect of reducing demand uncertainty

Fig. 5 shows the effect of reducing the demand VTMR on the inventory net shortfall when the production cycle is constrained so that $C_L=0.95$ and $C_U=1.05$. As the demand VTMR is reduced from 10 to 1.01 the expected shortfall, $E(V)$, is reduced from 5.84 to 0.27 days. More importantly, the standard deviation of the net shortfall is reduced from 141.19 to 15.51 days so that as the demand variance decreases there is less risk associated with the amount of inventory that may or may not exceed our target, *TIL*, in any given cycle. This illustration of the effect of reducing demand uncertainty provides evidence and support for collaboration between customers and suppliers in order to reduce inventory risk, overall inventory investment, and a desire to lower working capital.

If we decrease the capacity utilization to 90%, Fig. 6 demonstrates that the effect of reducing demand uncertainty is even more pronounced. The expected cycle length, $E(C)$, for each scenario is 6.25 days. The expected inventory net shortfall is 0.40 days with a standard deviation of 12.99 days when the demand VTMR is 1.01. But when the demand VTMR is increased to 5 and 10 the expected net shortfall is 23.92 and 216.21 days with standard deviations of 76.01 and 293.66 days, respectively. When the production cycle length is constrained we observe a clear relationship between the demand VTMR and the expected inventory net shortfall: as demand uncertainty increases so does the expected net shortfall, and while lengthening the production cycle may help to mitigate some of the risk associated with the demand uncertainty (allowing demand to be "pooled" over a longer time horizon), it will also necessitate additional cycle stocks which, in turn, creates another level of risk over an even longer cycle length.

3.3. The effect of relaxing the cycle limits: bounded flexibility

When there are lower and upper production cycle limits imposed, the net shortfall, and therefore the net inventory level, is much more difficult to predict in any given cycle. If we constrain the production cycle length so that $C_L=0.95$ and $C_U=1.05$, and make the demand VTMR 10, we observe, what at first appear to be, some counterintuitive results. Fig. 7 shows that as the capacity utilization increases, the expected net shortfall, $E(V)$, as well as its standard deviation, $StdDev(V)$, *decreases*.

The expected net shortfall when the capacity utilization is at 90% is 216.21 days. As the utilization is increased to 99%, $E(V)$ drops to 5.84 days and the standard deviation is cut by more than half. It is important to note, however, that the *relative* demand uncertainty in each of the scenarios in Fig. 7 is not the same. When the average cycle length is 6.25 days (such as the case when capacity utilization is at 90%) the high demand uncertainty is over a shorter cycle length. Increasing the capacity utilization will increase the average cycle length, and the high demand uncertainty will subsequently be spread over a longer time horizon that may allow the production facility an opportunity to recover and potentially reduce the expected net shortfall. However, there will be a significant cost in the form of cycle stock over longer production cycles.

4. Optimizing the target inventory level

To determine the value of *TIL* that minimizes the long-run expected inventory holding and backorder costs per period, we

VTMR	Capacity Utilization	C_L	C_U	E(C) in days	E(V)	StdDev(V)	P(V=0)
10	0.99	0.95	1.05	62.50	5.84	141.18	0.00

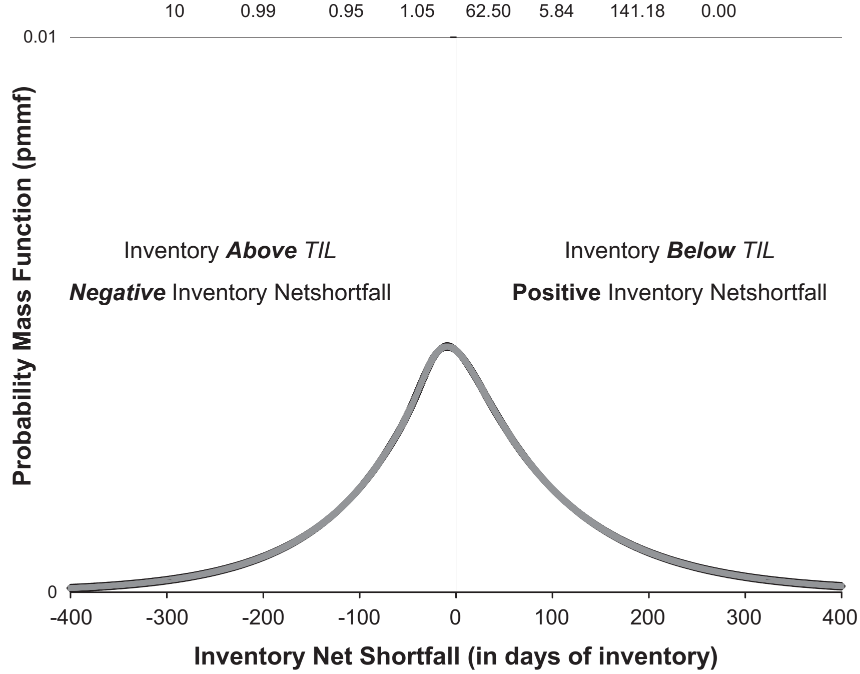

Fig. 4. Inventory net shortfall distribution with VTMR=10, utilization=99%, $C_L=0.95$ and $C_U=1.05$.

Scenario	VTMR	Capacity Utilization	C_L	C_U	E(C) in days	E(V)	StdDev(V)	P(V=0)
1	1.01	0.99	0.95	1.05	62.50	0.27	15.51	0.03
2	5	0.99	0.95	1.05	62.50	2.69	71.29	0.01
3	10	0.99	0.95	1.05	62.50	5.84	141.19	0.00

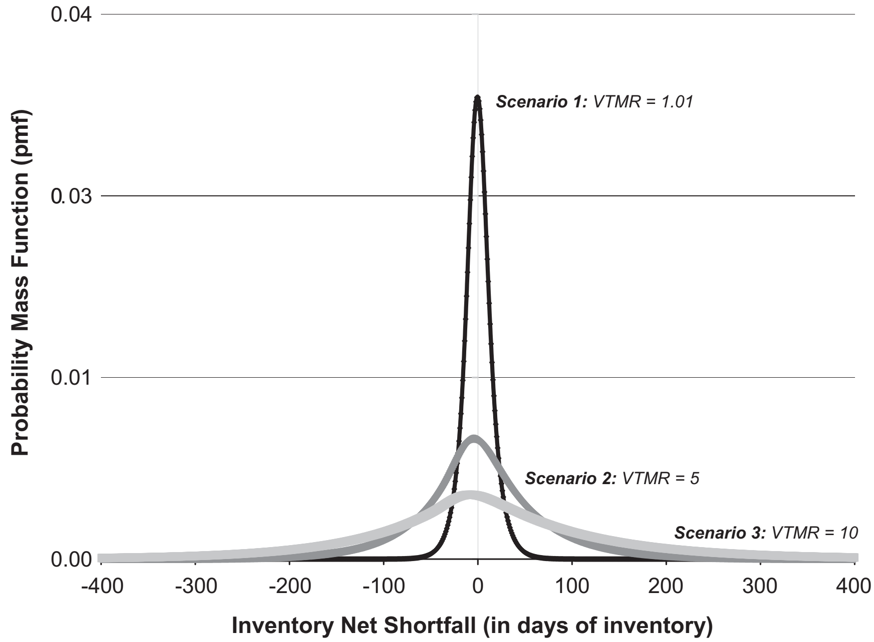

Fig. 5. Inventory net shortfall distribution when the production cycle is constrained, $C_L = 0.95$ and $C_U = 1.05$, and the demand uncertainty increases.

Scenario	VTMR	Capacity Utilization	C_L	C_U	E(C) in days	E(V)	StdDev(V)	P(V=0)
4	1.01	0.90	0.95	1.05	6.25	0.40	12.99	0.05
5	5	0.90	0.95	1.05	6.25	23.92	76.01	0.01
6	10	0.90	0.95	1.05	6.25	216.21	293.66	0.00

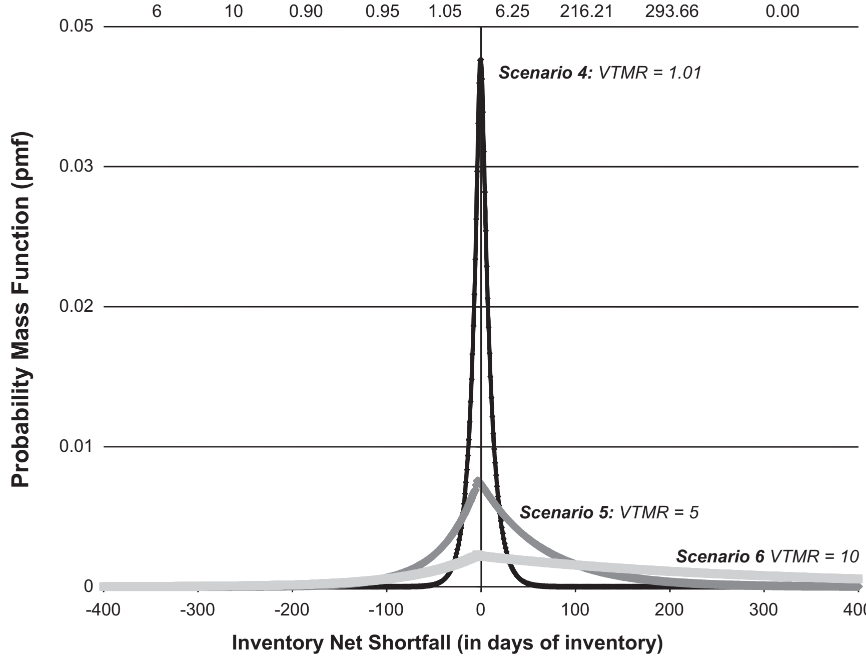

Fig. 6. Inventory net shortfall distribution when the production cycle is constrained, $C_L = 0.95$ and $C_U = 1.05$, capacity utilization is at 90% and the demand uncertainty increases.

must first estimate the expected cost per period associated with beginning a production cycle with a *specific level of inventory*, which depends on our choice of *TIL* and the previous period's inventory net shortfall, ν. Let *INV* be the random variable representing the net inventory level in aggregate at the beginning of a production cycle in steady-state. That is, $INV = TIL - \nu$. First, calculating the expected cost associated with beginning a production cycle with a particular *INV* requires that we make some

Scenario	VTMR	Capacity Utilization	C_L	C_U	E(C) in days	E(V)	StdDev(V)	P(V=0)
6	10	0.90	0.95	1.05	6.25	216.21	293.66	0.00
7	10	0.95	0.95	1.05	12.50	9.77	135.62	0.00
8	10	0.99	0.95	1.05	62.50	5.84	141.19	0.00

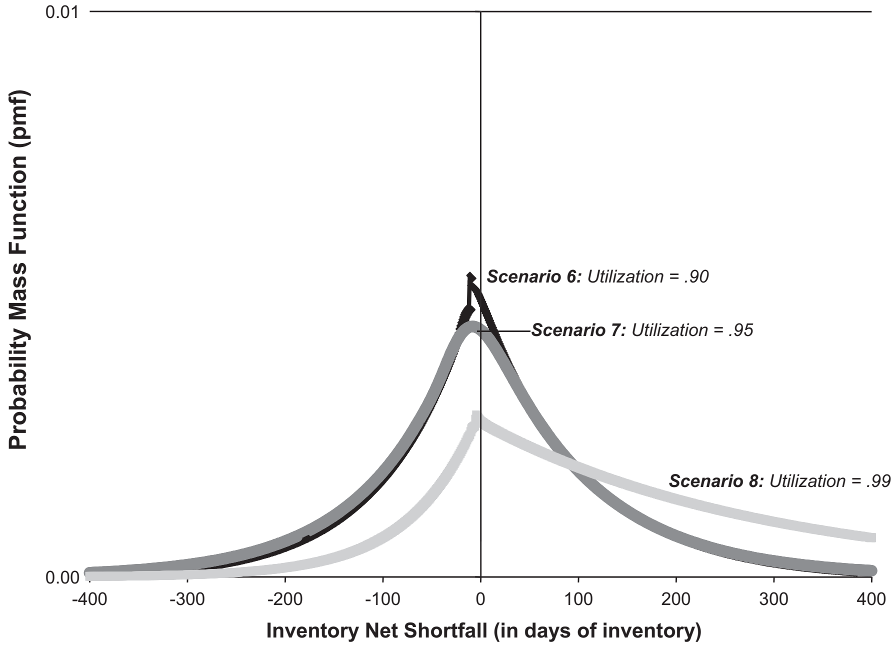

Fig. 7. Inventory net shortfall distribution when the cycle length is constrained, $C_L=0.95$ and $C_U=1.05$, and the capacity utilization increases.

assumptions about how the inventory is distributed among the items. Second, we estimate the lowest possible cost that the system can achieve in expectation over the next cycle through the use of a lot sizing model that explicitly attempts to stabilize production cycle lengths. Finally, we take advantage of the cost function convexity and optimize *TIL*.

4.1. Assuming balanced inventories and equal days of supply

For an aggregate amount of inventory at the beginning of a cycle, *INV*, we could assume that the items have an equal number days-of-supply *at the beginning of a production cycle*; however, this would not be reasonable since each item will begin its production run at different times throughout the cycle. That is, at the beginning of a cycle, production will start for item 1. We would therefore expect to have a net inventory level for item 1 that is low, relative to the other items. The net inventory at the beginning of the cycle for item *M* (the last item in the cycle) needs to satisfy demand until the *start of its production run* – much later in the cycle. Thus, by balanced beginning-of-cycle inventories we mean that the distribution of aggregate net inventory at the beginning of the cycle for each item will have an equal number of days of supply *at the beginning of its projected production start time*. While there are alternate ways of distributing the aggregate inventory at the beginning of the cycle, this method is simple and provides a reasonable estimate for our purposes, namely to estimate the expected system cost at the beginning of the next production cycle and to optimize *TIL*.

Days-of-supply is an expected value, and while not the only measure of inventory, is familiar to firms because it is actually used in enterprise resource planning (ERP) systems, and is a standard in financial reporting as required by the Generally Accepted Accounting Principles (GAAP) set forth by the U.S. Federal Accounting Standards Advisory Board. The days-of-supply (DOS) for item i at time instant τ is defined as

$$DOS_i(\tau)=\frac{E(X_i(\tau))}{\mu_i} \quad \text{for } i=1,2,\dots,M, \tag{4.7}$$

where $X_i(\tau)$ is the net inventory random variable at time instant $\tau>0$ and μ_i is the mean demand per period for item i. Assuming the inventory days-of-supply is the same for all items at their projected production start time in the cycle allows us to reduce the problem to a single parameter search to determine the relative distribution of inventory across all items at the beginning of the cycle for any value of *INV*.

Recall that $E(C)=\overline{C}=K/(1-\rho)$ is the expected cycle length in days where ρ is the capacity utilization. Let $\overline{q}_i=\mu_i E(C)$ be the planned lot size of item i. Define *INV* to be the aggregate net inventory level (in units of time) and x_{ni} to be the net inventory (in physical units) at the start of a production cycle n for item i where $\sum_i x_{ni}/r_i=INV$. Our goal is to determine x_{ni}, a physical allocation of *INV* that is balanced, as we have defined. Let $\tau_i\geq 0$ be the start time of item i in cycle n. Without loss of generality, let the production start time of item 1 begin at time zero, $\tau_1=0$. Based on the planned lot sizes $\overline{q}_i$, the expected start times of the remaining items in the cycle are given by

$$\tau_i=\tau_{i-1}+\frac{\overline{q}_{i-1}}{r_{i-1}}+k_i \quad \text{for } i=2,\dots,M. \tag{4.8}$$

Given *INV*, to balance the net inventory levels x_{ni} at the beginning of cycle n, we search for the single value of $\alpha\in\mathcal{R}$ that equates the days of supply of each item at its respective production start time. That is,

$$\alpha=DOS_1(\tau_1)=DOS_2(\tau_2)=\cdots=DOS_N(\tau_N). \tag{4.9}$$

Since $E[X_i(\tau_i)]=x_{ni}-\mu_i\tau_i$ is the expected net inventory of item i at time τ_i in the cycle, we can equalize the days-of-supply by solving for the x_{ni} such that

$$\frac{x_{ni}-\mu_i\tau_i}{\mu_i}=\alpha \quad \text{for all } i, \tag{4.10}$$

and

$$\sum_i^M \frac{x_{ni}}{r_i}=INV. \tag{4.11}$$

We can solve for α that satisfies (4.10) and (4.11) via a binary search. Thus, for every value of INV, there is a unique vector $\mathbf{x}^{INV}$ that equalizes the days-of-supply across items at their respective production start time (see Fig. 8 for a graphical illustration).

4.2. A simplified execution model

In Rappold and Yoho (2008) the authors propose solving a production execution model, **EM**, which is a convex optimization problem that stabilizes production cycles, however, it does *not* solve for the inventory level necessary to support the stable production cycle policy. We will use a modified version of **EM** as the model lends itself nicely to solving for the minimum expected cost of beginning a cycle with a given inventory net shortfall. While we will use this model to create an environment of stabilized cycles other models or methods to stabilize the production cycle could be used. The **EM** we will discuss minimizes a three-part cost function consisting of a time-dependent newsvendor function, $G(\cdot)$, a cost function that penalizes holding inventory of an item with uncertain customer demand, $Q(\cdot)$, and a cost penalty, $H(\cdot)$, that is incurred for deviating from the target inventory level.

The cost function, $G(\cdot)$, is a time-dependent newsvendor function that considers the trade-offs between holding inventory versus backordering demand as the length of the cycle varies. The time-dependent version of the newsvendor function in cycle n for item i is defined as

$$\begin{aligned} G_i(s_{ni},C_n) &= h_iE[s_{ni}-d_i(C_n)]^+ + b_iE[d_i(C_n)-s_{ni}]^+ \\ &= b_i(\mu_iC_n-s_{ni}) \\ &\quad +(h_i+b_i)\sum_{j=0}^{s_{ni}-1}\Pr\{d_i(C_n)\le j\}, \end{aligned} \tag{4.12}$$

where s_{ni} is the stock level of item i at the start of the cycle n, and $[x]^+=\max\{0,x\}$ for $x\in\mathcal{R}$. The cost function $Q(\cdot)$ penalizes the storage of inventory in an item that is unlikely to be ordered in the short-term due to a high degree of uncertainty in customer demand. We define the expected cost function for item i in cycle n as

$$Q_i(s_{ni},C_n)=\begin{cases} h_i\sum_{t>C_n}E[s_{ni}-d_i(t)]^+ & \text{for } s_{ni},\ C_n>0, \\ 0 & \text{otherwise.}\end{cases} \tag{4.13}$$

The $Q(\cdot)$ function estimates the total future expected cost of holding s_{ni} units of inventory in item i *beyond* current cycle n of length C_n. $Q(\cdot)$ is an infinite sum and is an approximation of future holding cost risks that a dynamic programming formulation would normally consider. Finally, $H(\cdot)$ induces the system to restore the aggregate inventory to a target level, T, at the end of the current production cycle where T is in physical units of inventory. Whenever we are below this inventory target we will assume that we are able to purchase product on the spot market at a cost of ϕ per unit (regardless of item type) to restore the system to T. Likewise, whenever our system inventory is above T, we will assume that we are able to sell our excess stock at a discount of δ per unit. We define the end-of-cycle expected cost function as

$$\begin{aligned} H(S_n,C_n,T) &= \phi E[T-(S_n-D(C_n))]^+ \\ &\quad +\delta E[(S_n-D(C_n))-T]^+. \end{aligned} \tag{4.14}$$

where $D(C_n)=\sum_i d_i(C_n)$ is the total demand over cycle n, $S_n=\sum_i s_{ni}$ is the sum of all starting item stock levels s_{ni} in cycle n, and T is the aggregate target inventory level. We assemble all three cost functions in the following **EM**. Let x_{ni} be the beginning net inventory level for item i in cycle n, and $\mathbf{x}_n$ be the corresponding vector of beginning net inventory levels in cycle n. We denote the objective function value of **EM** as $Z(\mathbf{x}_n,T)$, representing the minimum expected cost per cycle given C_U, C_L, a vector of beginning inventory levels $\mathbf{x}_n$, lot sizes for item i in cycle n denoted by q_{ni}, and where a target inventory level is defined in *units* of inventory, T:

(**EM**)

$$\begin{aligned} Z(\mathbf{x}_n,T) = \min_{q_{ni}\ge 0}\frac{1}{C_n}\Big\{\sum_{i=1}^M (G_i(s_{ni},C_n) \\ +Q_i(s_{ni},C_n))+H(S_n,C_n,T)\Big\} \end{aligned} \tag{4.15}$$

$$\text{s.t.}\quad s_{ni}=x_{ni}+q_{ni},\quad S_n=\sum_i s_{ni}, \tag{4.16}$$

$$C_n=K+\sum_{i=1}^M\frac{q_{ni}}{r_i}, \tag{4.17}$$

$$C_L\le C_n\le C_U. \tag{4.18}$$

We now propose solving a simplified version of the execution model, **EM**, which we call **SEM**, and whose optimal objective function value, $\tilde{Z}(v,TIL)$, represents the minimum expected cost of beginning a cycle with an inventory net shortfall of v time units, given that the end-of-cycle target inventory level (in time) is TIL.

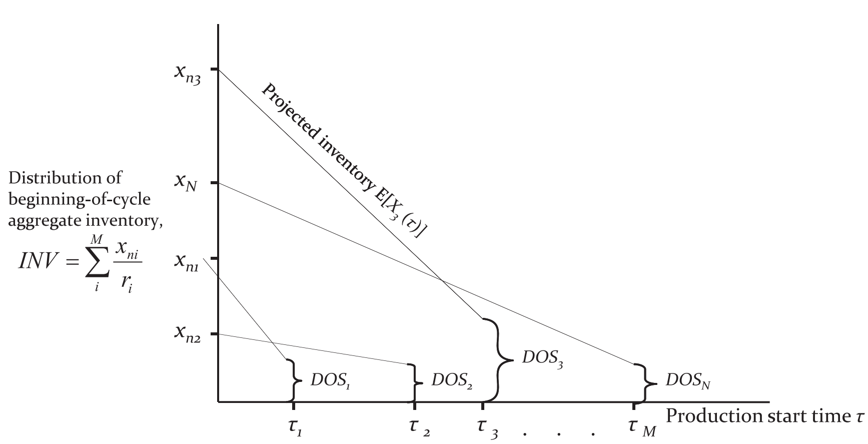

Fig. 8. Equal days-of-supply at the beginning of the production of item i.

We call it a simplified version because it does not use the actual state vector $\mathbf{x}_n$ of net inventory levels for each item. Instead, it uses only the starting inventory net shortfall v (a scalar value representing units of time), and makes the assumption that the corresponding net inventory levels among all items are given by the balanced vector $\mathbf{x}^{TIL-v}$ that uniquely satisfies (4.8), (4.10) and (4.11) for $INV = TIL - v$. Because TIL is in units of time, we reformulate the end-of-cycle $H(\cdot)$ function in terms of time. From (4.14), we have

$$H(S_n, C_n, T) = \phi E[T-(S_n - D(C_n))]^+ + \delta E[(S_n - D(C_n)) - T]^+. \tag{4.19}$$

Note that S_n may be translated into units of time as

$$S_n = \sum_i x_{ni}/r_i + \sum_i q_{ni}/r_i.$$

The aggregate net inventory level expressed in units of time at the beginning of the cycle n is $\sum_i x_{ni}/r_i = TIL - V_{n-1}$. Similarly, we can express the production time used in cycle n as $\sum_i q_{ni}/r_i = C_n - K$. Combining terms, we have

$$S_n = (TIL - V_{n-1}) + (C_n - K).$$

From (3.5) note that $V_n = V_{n-1} + K + D(C_n) - C_n$ for $n > 0$. Let $\hat{\phi}$ and $\hat{\delta}$ be the cost of underage and overage, respectively, per unit of inventory time, instead of per inventory unit. We now have enough to define a reformulated end-of-cycle expected cost as

$$\begin{aligned}\tilde{H}(V_{n-1}, C_n) &= \hat{\phi} E[TIL - S_n + D(C_n)]^+ \\ &\quad + \hat{\delta} E[S_n - D(C_n) - TIL]^+ \\ &= \hat{\phi} E[V_{n-1} - C_n + K + D(C_n)]^+ \\ &\quad + \hat{\delta} E[C_n - K - D(C_n) - V_{n-1}]^+ \\ &= \hat{\phi} E[V_n]^+ + \hat{\delta} E[-V_n]^+.\end{aligned}$$

With this reformulated end-of-cycle expected cost function, we state the simplified execution model (**SEM**) as

$$(\mathbf{SEM})\ \tilde{Z}(V_{n-1}, TIL) = \min_{q_{ni} \ge 0} \frac{1}{C_n} \left\{ \sum_{i=1}^{M} (G_i(s_{ni}, C_n) + Q_i(s_{ni}, C_n)) + \tilde{H}(V_{n-1}, C_n) \right\} \tag{4.20}$$

$$\text{s.t.} \quad s_{ni} = x_i^{TIL - V_{n-1}} + q_{ni}, \tag{4.21}$$

$$C_n = K + \sum_{i=1}^{M} \frac{q_{ni}}{r_i}, \tag{4.22}$$

$$C_L \le C_n \le C_U. \tag{4.23}$$

Constraints (4.21) reflect the cumulative supply available of each product, s_{ni}, and $x_i^{TIL-V_{n-1}}$ is value for item i in vector $\mathbf{x}^{TIL-V_{n-1}}$. Constraint (4.22) relates the lot sizing decisions to the production cycle length. Constraint (4.23) ensures that the start time for the next cycle is between C_L and C_U time periods. Our estimate of the expected cost per period over the next cycle associated with a starting inventory level of $INV = TIL - v$ is $\tilde{Z}(v, TIL) \cong Z(\mathbf{x}^{INV}, TIL)$. We solve $\tilde{Z}(v, TIL)$ using the algorithm presented in Rappold and Yoho (2008) and take the objective function value as an expected cost estimate for all cycles that have a starting net inventory of $INV = TIL - v$. For $TIL = 0$, Fig. 9 illustrates the corresponding expected cost curve $\tilde{Z}(v, 0)$ along with the net shortfall distribution. Specifically, $\tilde{Z}(v, 0)$ represents the expected operational inventory and backorder costs and $\pi(v)$ represents the steady-state percentage of cycles which will begin with an inventory net shortfall of v. The inventory net shortfall, $\pi(v)$, was calculated using a VTMR $=5$, capacity utilization of 90%, $C_L = .85$ and $C_U = 1.15$.

Fig. 10 illustrates the expected cost curve associated with $\tilde{Z}(v, 0)$, and the net shortfall distribution, $\pi(v)$; it is the *sum* of both curves. Point *A* shows the expected cost associated with $TIL = 0$ in days of inventory and point *B* shows the expected cost

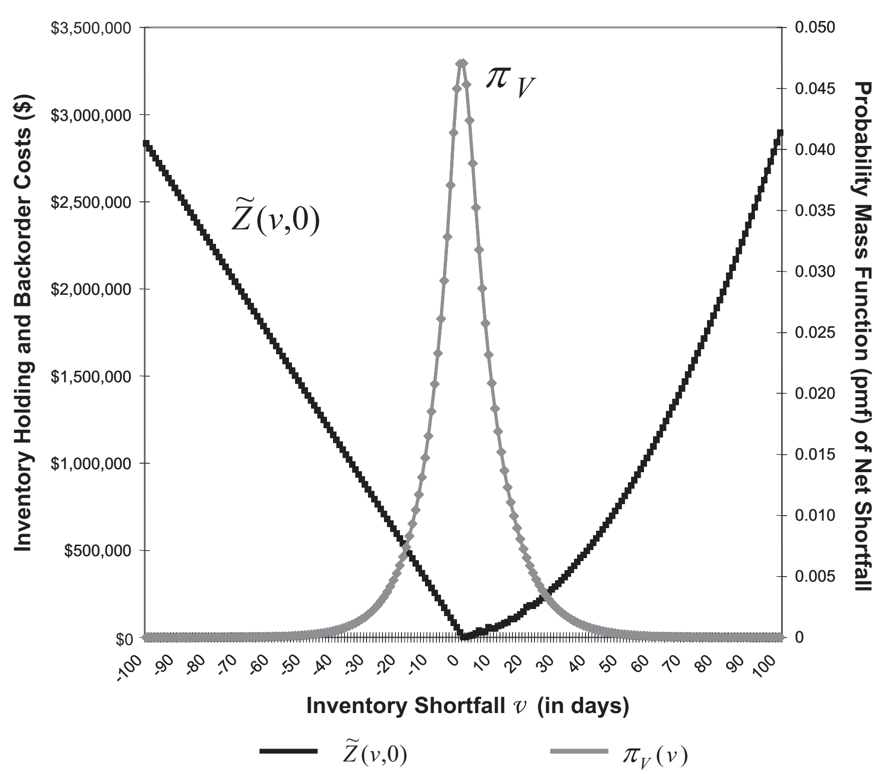

Fig. 9. Cost associated with $TIL = 0$, $\tilde{Z}(v, 0)$, and the net shortfall distribution, $\pi(v)$.

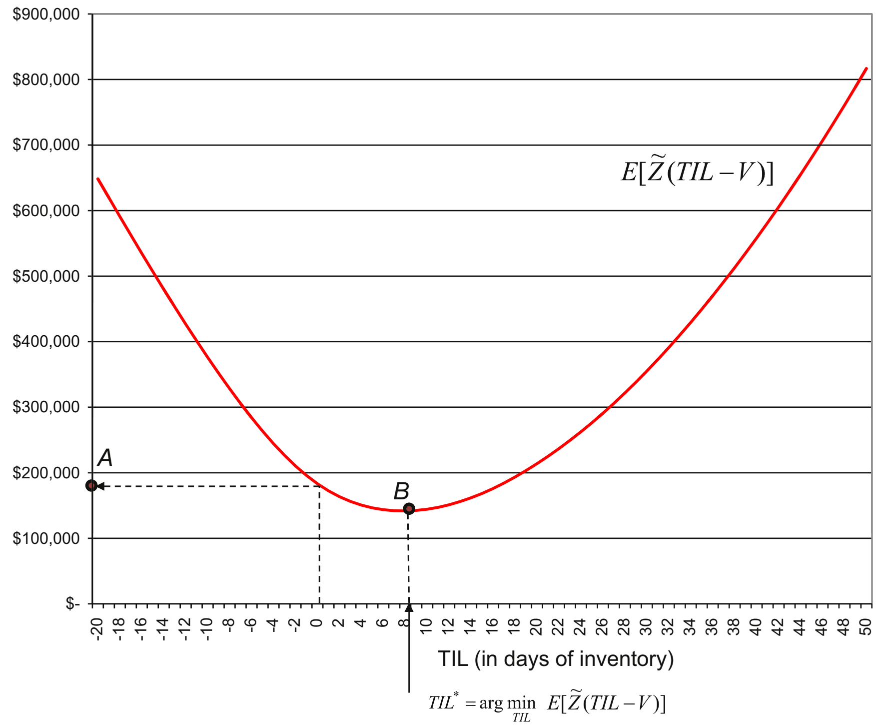

Fig. 10. Expected cost curve associated with $\tilde{Z}(v,0)$, and the net shortfall distribution, $\pi(v)$.

Table 2
Optimal *TIL* (expressed in days of inventory) determined using the net shortfall.

Capacity utilization	Demand VTMR	Lower cycle limit 0.75 Upper cycle limit			Lower cycle limit 0.85 Upper cycle limit			Lower cycle limit 0.95 Upper cycle limit		
		1.05	1.15	1.25	1.05	1.15	1.25	1.05	1.15	1.25
0.9	1.01	12	5	3	11	4	2	6	0	−1
	5	26	12	7	23	9	3	0	−13	−18
	10	8	13	7	0	7	1	0	−39	−50
0.95	1.01	15	6	4	14	5	3	9	2	0
	5	34	17	11	30	13	7	2	−7	−12
	10	28	21	13	17	12	5	0	−34	−44
0.99	1.01	26	16	14	22	12	9	16	5	2
	5	63	34	24	57	28	18	31	8	0
	10	63	45	33	51	35	23	0	0	−10

associated with the optimal *TIL* of 8 which minimizes the expected costs.

4.3. Solving for the optimal target inventory level

The steady-state probability of the inventory net shortfall, $\pi(v) = \Pr\{V=v\}$ for v integer, represents the long-run fraction of cycles that will begin with a system inventory net shortfall of v period's worth of inventory. To determine the value of *TIL* that minimizes the long-run expected system costs, we solve

$$TIL^* = \arg\min_{TIL} E_V[\tilde{Z}(V, TIL)] \quad (4.24)$$

$$TIL^* = \arg\min_{TIL} \sum_{v \in \mathcal{Z}} \tilde{Z}(v, TIL)\pi(v). \quad (4.25)$$

The value of TIL^* is an approximation, yet it integrates both the inventory requirements within each cycle as well as the consequences of carrying inventory between successive cycles. A particularly attractive characteristic of $\tilde{Z}(v, TIL)$ is that it is *unimodal* in v for a fixed *TIL* and may be solved very quickly in a spreadsheet. Taking the expectation with respect to V, we are taking the convex combination of $\tilde{Z}(\cdot)$. Note that $E_V[\tilde{Z}(V, TIL)]$ is also unimodal in *TIL* with a unique minimizer. Using TIL^*, we may solve (**SEM**) to disaggregate TIL^* and compute the individual item target inventory levels.

5. Numerical experiments and managerial insights

The TIL^* for each of the 81 scenarios, previously discussed in Section 4.3, is summarized numerically in Table 2 (expressed in days of inventory). We assumed inventory holding costs to be $1 and backorder costs to be $9 per unit per period. In all cases, as the upper limit is relaxed, TIL^* decreases. The TIL^* tends to increase as the demand VTMR increases except when the lower cycle limit is 0.95 and inventory is being pushed into the system. We note that in some

Table 3
Expected cost per period during a production cycle with *TIL** targeted days of inventory for each of the 81 scenarios.

Capacity utilization	Demand VTMR	Lower cycle limit 0.75 Upper cycle limit			Lower cycle limit 0.85 Upper cycle limit			Lower cycle limit 0.95 Upper cycle limit		
		1.05	1.15	1.25	1.05	1.15	1.25	1.05	1.15	1.25
0.9	1.01	$17,073	$5824	$4159	$17,443	$6846	$4912	$22,093	$12,746	$11,573
	5	$91,366	$39,673	$25,837	$93,080	$44,384	$31,438	$232,372	$125,011	$117,060
	10	$57,227	$79,438	$57,340	$381,466	$88,623	$69,119	$1,053,007	$315,739	$294,824
0.95	1.01	$16,190	$6237	$5031	$15,676	$6267	$5110	$19,083	$10,722	$9828
	5	$76,137	$35,929	$24,723	$76,968	$39,015	$28,662	$118,255	$99,144	$91,990
	10	$87,870	$68,966	$52,143	$92,752	$79,244	$64,661	$410,483	$286,668	$271,892
0.99	1.01	$15,273	$10,666	$11,596	$14,722	$9591	$10,272	$15,968	$10,972	$11,428
	5	$61,122	$32,344	$26,546	$60,841	$33,310	$27,618	$75,750	$54,429	$50,287
	10	$75,439	$56,871	$46,296	$76,811	$60,583	$51,201	$291,019	$135,593	$122,516

instances TIL^* is less than zero which suggests that backorders are targeted for some or all items. Combined with $E(V)$, the system still has positive average inventories in those cases where the TIL^* is negative; that is, $TIL^* - E(V) > 0$.

The expected costs, $E_V[\tilde{Z}(V, TIL^*)]$, associated with having an ending target of TIL^* days of inventory are given in Table 3. This is the expected cost per period during the production cycle given the current demand uncertainty, capacity utilization, upper and lower cycle limits and assuming that inventories have an equal number of days of supply. Executives are often upbraided by their accountants and financial officers for having a capacity utilization that is "too low" and to fix the problem by increasing output to spread the equipment investment costs over more units of output. We now have an expected cost associated with increasing the capacity utilization and can use it to make a management decision that considers more than activity-based depreciation.

We are now able to draw several important managerial insights based on the analysis of the expected costs, $\tilde{Z}(\cdot)$, and on the steady-state net shortfall distribution:

1. when the upper cycle limit is large, thereby allowing the production cycle length to be flexible, increasing the capacity utilization may increase costs whereas decreasing the utilization may introduce both flexibility *and* cost reduction;
2. when the capacity utilization is high there is less flexibility to respond to uncertainty within the management-determined lower and upper cycle lengths, C_L and C_U;
3. as the capacity utilization increases, so does the production cycle length thereby pooling the higher demand uncertainty over longer cycles and potentially reducing the expected net shortfall, and;
4. when the upper cycle limit is constrained, it may be necessary to collaborate with customers to reduce demand uncertainty if management wishes to reduce inventory.

We have developed a novel modeling approach to link tactical system dynamics with execution-level decision-making. We exploit the structure of the expected cost to model the system inventory in aggregate, thus dramatically reducing the size of the system's inventory state space. The inventory net shortfall allows us to calculate the steady-state distribution of inventory in the system given assumptions with respect to demand uncertainty, capacity utilization, and an upper and a lower production cycle limit. The importance of the decision of how to set the lower and upper cycle limit parameters as well as capacity utilization has been illustrated. We have seen that as demand uncertainty increases so does the inventory net shortfall, all other things held equal.

6. Conclusions

We define a system with balanced inventory to be one that has an equal number of days of supply on hand for each item at the start of each item's production run. Many firms find it desirable, and often necessary as a result of implementing enterprise resource planning (ERP) systems, to describe an entire complex system with a single number, and we have shown how to arrive at this number with respect to an inventory level that allows production cycles to be stabliized in Eqs. (.8)–(.11). We also show the expected cost at the end of a future production cycle that is associated with the current production state and environment. Given the current capacity utilization, demand variation, and planned production cycle length, management may have an estimate of future costs associated with their current production decision; this type of information is extremely useful for strategic as well as tactical planning such as the type that takes place during a sales and operations planning cycle.

Finally, we have determined the optimal target inventory level, TIL^*, for the system given demand uncertainty, capacity utilization and upper and lower production cycle limit constraints. We propose a method for disaggregating the inventory using a method to equalize the days of supply for all items. In some instances, the optimal *TIL* will be negative. Using the net shortfall distribution we are able to calculate an expected cost of future production cycles given a set *TIL*. When the expected production cycle length is more than 60 days this framework is particularly valuable for developing managerial insights, and may be used for both operational and financial planning as well as customer relationship management.

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