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LINEAR TIME-INVARIANT SPACE-VARIANT FILTERS AND THE PARABOLIC EQUATION APPROXIMATION

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Abstract. Wave propagation in a random, inhomogeneous ocean is treated as transmission through a linear, time-invariant, space-variant, random communication channel. Using the *parabolic equation approximation* of the Helmholtz wave equation, a random transfer function of the ocean volume is *derived*. The ocean volume is characterized by a *three-dimensional* random index of refraction which is decomposed into deterministic and random components. Two additional calculations are performed using the transfer function. The first involves the derivation of the equations for the random, output electrical signals at *each* element in a receive planar array of complex weighted point sources in terms of the frequency spectrum of the transmitted electrical signal, the transmit and receive arrays, and the transfer function of the ocean medium. The second involves the derivation of the coherence function.

Zusammenfassung. Die Wellenausbreitung in einem inhomogenen diffusen Ozean wird modelliert als Übertragung über einen linearen, zeitinvarianten, ortsvarianten Kommunikationskanal. Mit Hilfe der parabolischen Näherung der Helmholtz-Wellengleichung wird eine Zufalls-Übertragungsfunktion des Ozeanvolumens abgeleitet. Hierbei wird das Ozeanvolumen durch den dreidimensionalen Refraktionsindex beschrieben; dieser wird zerlegt in deterministische und stochastische Komponenten. Mit Hilfe der ermittelten Übertragungsfunktion werden zwei zusätzliche Berechnungen durchgefürhrt. Die erste Herleitung ergibt Gleichungen für die elektrischen Zufalls-Ausgangssignale an jedem Element einer ebenen Gruppe von Empfängern in Form komplexe gewichteter Punktquellen; diese Signale werden dargestellt in Abhängigkeit vom Frequenzspektrum des übertragenen elektrischen Signals, von der Sende- und Empfangsantennenanordnung sowie von der Übertragungsfunktion des Ozeans. Bei der zweiten Berechnung wird die Kohärenzfunktion abgeleitet.

Résumé. La propagation des ondes dans un ocean inhomogène et de caractéristiques aléatoires, est traitée comme un problème de transmission à travers un canal de communication linéaire invariable avec le temps mais variable avec l'espace. Une fonction de transfert aléatoire est obtenue pour le milieu océanique en utilisant l'approximation de l'équation parabolique pour les équations de propagation de Helmholtz. Le milieu océanique est caracterisé par un index de réfraction aléatoire à trois dimensions composé d'une partie deterministe et d'une partie aléatoire. Deux calculs supplementaires sont effectués à partir de la fonction de transfert. Le premier établit les équations des signaux de sortie électriques aléatoires de chaque élement d'une antenne de réception plane composée de sources pondérées par un coefficient complexe, en fonction du spectre en frequence du signal electrique transmis, des antennes d'émission et de récéption, et de la fonction de transfert du milieu océanique. La second établit la fonction de cohérence.

Keywords. Linear time-invariant space-variant filters, wave propagation in random media, parabolic equation approximation, ocean volume transfer function, planar arrays, random output electrical signals, coherence function, coherence bandwidth, spatial coherence, angular coherence.

1. Introduction

Since the wave equation for *small* amplitude acoustic signals is *linear*, we can represent the ocean medium as a *linear*, time-variant, space-variant, random filter (system or communication channel) in general. The term "time-variant" implies motion amongst targets, the ocean surface, discrete point scatterers, and the transmit and receive apertures (arrays). Discrete point scatterers in the ocean may include, for example, gas bubbles, fish, and other particulate matter. The time-variant property results in both Doppler spread and spread in time delay values. If the filter is time-*in*variant, then *no* motion is implied. As a result, there will be no Doppler spread and no spread in time delay.

The term "space-variant" implies that the sound-speed profile (index of refraction) of the ocean is a function of position. The space-variant property results in scatter or angular spread due to refraction. If the filter is space-*in*variant, then an isospeed medium is implied. As a result, there will be no refraction, and hence, no scatter or angular spread since the sound rays will be travelling in straight lines.

In addition, since any motion and/or the index of refraction can be decomposed into a sum of deterministic (average) and random (fluctuating) components, these random components can be accounted for via a random filter representation vis-à-vis a deterministic filter representation.

By using a systems theory approach, surface, volume, and/or bottom reverberation returns can be modelled as the outputs from linear filters. In addition, target returns can also be modelled as filter outputs. Furthermore, different transmit signals and transmit and receive directivity functions can easily be *coupled* to various models (i.e., transfer functions) of the random, inhomogeneous ocean medium in a *straightforward* and *logical fashion* in order to study their effects on pulse propagation in random media, underwater acoustic communication, and target detection or parameter estimation using various space-time signal processing algorithms.

The approach of treating the ocean as an isospeed medium, and hence, as a linear, time-variant, random communication channel is well established [1–19]. This linear, time-varying, random systems theory approach has also been applied to target scattering problems in radar astronomy [20] and to communication channels in general [21–23]. However, with respect to target models, past research efforts have been devoted mainly to the slowly fluctuating point target problem [24–31]. Efforts to treat more complicated target models were made by Kooij [32], Moose [10], and Ziomek and Sibul [19, 33]. Kooij [32] and Moose [10] both modelled the target as a linear, *time-invariant, deterministic* filter while Ziomek and Sibul [19, 33] modelled the target as a linear, *time-varying, random* filter. In addition, Ziomek [34] has shown that the form of the generalized ambiguity function can be derived by treating the scattered acoustic pressure field from a point target (in relative motion with respect to a bistatic transmit/receive array geometry) as the output of a linear, time-varying, random filter.

Some work has been done in treating the ocean medium as a linear, time-variant, *space-variant*, random filter by Laval [9, 35] and Laval and Labasque [36]. However, the notation used to incorporate the space-variant property is ad hoc, i.e., spatial variables are simply included in the arguments of the impulse response and transfer functions, for example, rather than having evolved from a systematic and consistent notation based upon linear, time-varying, space-varying systems theory. In addition, Laval and Labasque [36] *assume* functional forms for the ocean transfer function instead of deriving them. Middleton [37, 38] also studied underwater acoustic propagation in a random, inhomogeneous ocean, but did not concern himself directly with the derivation of random, time-variant, space-variant ocean transfer functions. He described the propagation phenomena using space-time operators.

Ziomek [39] studied underwater acoustic propagation in a random, inhomogeneous ocean by treating the ocean medium as a linear, time-variant, space-variant, random filter and published his preliminary findings in a recent technical report. The major results contained in Ziomek [39] are summarized in the following list:

1) A *consistent notation, fundamental input–output relations*, and various *time–space transformations* for both deterministic and random linear, time-variant, space-variant filters were established. The notation is

consistent in the sense that all of the various input-output relations which are based upon the general theory will reduce to the classical relations of linear, time-*in*variant filter theory. These results are a generalization of the expressions contained in Ziomek [40] and should be of interest to persons involved in the general area of linear systems theory, and not only to those involved in underwater acoustics.

2) With the use of the method of separation of variables and the *WKB* approximation [41-43], a time-invariant, space-variant, random transfer function of the ocean medium was *derived.* The transfer function was time-invariant because motion was not considered. The transfer function modelled the ocean volume between transmit and receive apertures (arrays). The ocean volume was characterized by a one-dimensional random index of refraction (sound-speed profile) which was a function of depth. The index of refraction was decomposed into deterministic and random components.

3) Besides the transfer function derivation, two additional calculations were made. The first calculation demonstrated the use of the *coupling equations* and involved the derivation of a mathematical expression for the random output electrical signals at *each* element in a receive planar array of complex weighted point sources. The output signals were expressed in terms of the frequency spectrum of the transmitted electrical signal, the transmit and receive arrays, and the transfer function of the ocean medium. This derivation demonstrated that output electrical signals could be obtained in a logical and straightforward fashion. The second calculation involved the derivation of the coherence function, i.e., the autocorrelation function of the transfer function. In order to obtain somewhat simplified results, it was necessary to assume that the random component of the index of refraction was Gaussian and wide-sense stationary.

Based upon the recent successes obtained by using a linear systems theory approach together with the *WKB* approximation to model wave propagation in a random medium [39], the purpose of this paper is to follow the same approach and to perform the same calculations as in [39], but to use the *parabolic equation approximation* of the Helmholtz wave equation in all mathematical derivations. The parabolic equation approximation allows one to handle *three-dimensional* random indices of refraction [44, 45] and to model long range wave propagation in a random SOFAR channel [46]. Random SOFAR channels will not be considered in this paper.

Finally, it should be mentioned that the material contained in the technical report by Ziomek [39] can also be found in the recent textbook by Ziomek [62].

2. Transfer function

We will now proceed to derive a transfer function which models the bistatic communication channel geometry shown in Fig. 1. The communication channel is regarded to be the ocean volume between the apertures so that surface and bottom scattering effects are not included. Both apertures are stationary (not in motion), and it is assumed that no discrete point scatterers (such as bubbles, fish, etc.) are in the volume between the apertures. No motion implies that the resulting transfer function will be time-invariant.

The propagation of *small* amplitude acoustic signals in the ocean from the transmit aperture to the receive aperture can be described by the following *linear*, inhomogeneous, scalar wave equation:

$$\nabla^2\varphi(t, \boldsymbol{r}) - \frac{1}{c^2(\boldsymbol{r})}\frac{\partial^2}{\partial t^2}\varphi(t, \boldsymbol{r}) = x_M(t, \boldsymbol{r}) \tag{2-1}$$

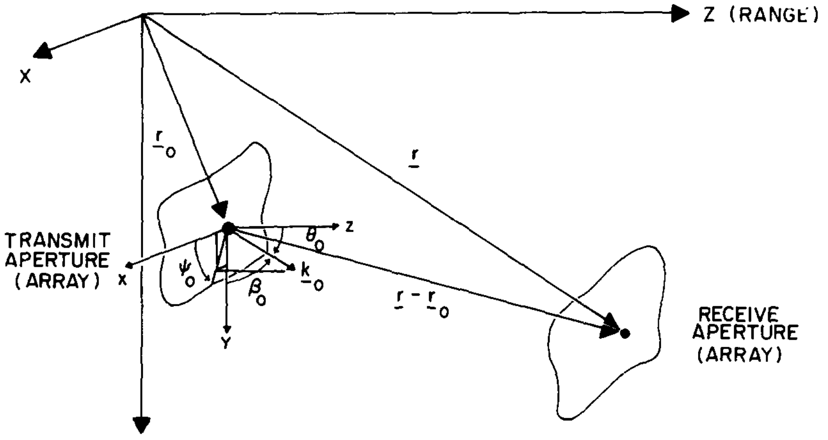

Fig. 1. Basic bistatic communication channel geometry. Also shown is the reference propagation vector $\boldsymbol{k}_0$ and associated angles.

where $\varphi(t, \boldsymbol{r})$ is the velocity potential at time t and position $\boldsymbol{r}=(x, y, z)$, $x_M(t, \boldsymbol{r})$ is the source distribution, and $c(\boldsymbol{r})$ is the speed of sound in the ocean. Since the *coupling equations* discussed in Ziomek [39, 62] already allow for an arbitrary $x_M(t, \boldsymbol{r})$ with corresponding frequency and angular spectrum $X_M(f, \boldsymbol{\nu})$, we need only find the solution to the following Helmholtz wave equation:

$$\nabla^2\varphi(\boldsymbol{r})+k_0^2n^2(\boldsymbol{r})\varphi(\boldsymbol{r})=0 \tag{2-2}$$

where

$$k_0=2\pi f/c_0=2\pi/\lambda_0 \tag{2-3}$$

is the constant reference wave number,

$$n(\boldsymbol{r})=c_0/c(\boldsymbol{r}) \tag{2-4}$$

is the random index of refraction,

$$c_0=c(\boldsymbol{r}_0)=f\lambda_0 \tag{2-5}$$

is the constant reference speed of sound at the source position $\boldsymbol{r}_0=(x_0, y_0, z_0)$, and

$$\varphi(t, \boldsymbol{r})=\varphi(\boldsymbol{r})\exp(+\mathrm{j}2\pi ft) \tag{2-6}$$

is the time-harmonic solution of eq. (2-1) when $x_M(t, \boldsymbol{r})$ is set equal to zero, and where $\varphi(\boldsymbol{r})$ is the solution of eq. (2-2).

The index of refraction is commonly written as [43, 48-51]

$$n(\boldsymbol{r})=n_D(\boldsymbol{r})+n_R(\boldsymbol{r}) \tag{2-7}$$

or

$$n(\boldsymbol{r})=n_D(\boldsymbol{r})+\sigma(\boldsymbol{r})n_{NR}(\boldsymbol{r}) \tag{2-8}$$

where $n_D(\boldsymbol{r})$ is the deterministic component and is usually close to unity in value, $n_R(\boldsymbol{r})$ is the random, zero-mean component, $\sigma(\boldsymbol{r})$ is the standard deviation of $n_R(\boldsymbol{r})$, and

$$n_{NR}(\boldsymbol{r})=n_R(\boldsymbol{r})/\sigma(\boldsymbol{r}) \tag{2-9}$$

is the *normalized* random component with zero mean and variance equal to unity. Note that the average value of $n(\boldsymbol{r})$ is equal to $n_D(\boldsymbol{r})$.

The parabolic equation approximation of the Helmholtz wave equation given by eq. (2-2) can be obtained as follows. First assume that eq. (2-2) has a solution in the form of a wave propagating in the positive Z direction, i.e., assume that

$$\varphi(\boldsymbol{r}) = g(\boldsymbol{r}) \exp(-\mathrm{j}k_0 z) \tag{2-10}$$

where $g(\boldsymbol{r})$ is to be determined. Substituting eq. (2-10) into eq. (2-2) yields

$$\nabla^2 g(\boldsymbol{r}) - \mathrm{j}2k_0 \frac{\partial}{\partial z} g(\boldsymbol{r}) + k_0^2[n^2(\boldsymbol{r}) - 1]g(\boldsymbol{r}) = 0. \tag{2-11}$$

If it is further assumed that

$$\left|\frac{\partial^2}{\partial z^2} g(\boldsymbol{r})\right| \ll 2k_0 \left|\frac{\partial}{\partial z} g(\boldsymbol{r})\right|, \tag{2-12}$$

then eq. (2-11) reduces to

$$\nabla_T^2\, g(\boldsymbol{r}) - \mathrm{j}2k_0 \frac{\partial}{\partial z} g(\boldsymbol{r}) + k_0^2[n^2(\boldsymbol{r}) - 1]g(\boldsymbol{r}) = 0 \tag{2-13}$$

which is the *parabolic equation approximation* of the Helmholtz wave equation, where

$$\nabla_T^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{2-14}$$

is referred to as the *transverse Laplacian.* The assumption represented by eq. (2-12) is the "small angle approximation" or the "parabolic approximation" since it leads to a parabolic partial differential equation, i.e., eq. (2-13). Equation (2-12) is valid whenever [44, 52, 53]

$$\lambda \ll s_Z \tag{2-15}$$

where s_Z is the scale size of the random medium in the Z direction. We will use eq. (2-13) as the starting point for the transfer function derivation.

From eq. (2-8)

$$n^2(\boldsymbol{r}) = n_D^2(\boldsymbol{r}) + 2n_D(\boldsymbol{r})\sigma(\boldsymbol{r})n_{NR}(\boldsymbol{r}) + \sigma^2(\boldsymbol{r})n_{NR}^2(\boldsymbol{r}). \tag{2-16}$$

If the medium is "weakly irregular" or "weakly scattering", i.e., if $\sigma(\boldsymbol{r})$ is very small compared to unity so that [52]

$$k_0\sigma(\boldsymbol{r})s_Z \ll 1, \tag{2-17}$$

then terms involving $\sigma^2(\boldsymbol{r})$ can be neglected [49]. If it is further assumed that the deterministic component $n_D(\boldsymbol{r})$ of the index of refraction is close to unity [43–45, 47–51], then eq. (2-16) reduces to

$$n^2(\boldsymbol{r}) \approx 1 + 2\sigma(\boldsymbol{r})n_{NR}(\boldsymbol{r}). \tag{2-18}$$

Substituting eq. (2-18) into eqs. (2-2) and (2-13) yields

$$\nabla^2\varphi(\boldsymbol{r}) + k_0^2[1 + 2\sigma(\boldsymbol{r})n_{NR}(\boldsymbol{r})]\varphi(\boldsymbol{r}) = 0 \tag{2-19}$$

and

$$\frac{\partial}{\partial z} g(\boldsymbol{r}) = [-\mathrm{j}/(2k_0)]\nabla_T^2 g(\boldsymbol{r}) - \mathrm{j}k_0\sigma(\boldsymbol{r})n_{NR}(\boldsymbol{r})g(\boldsymbol{r}) \tag{2-20}$$

respectively, where $\varphi(\boldsymbol{r})$ and $g(\boldsymbol{r})$ are related by eq. (2-10). The first term on the right-hand side of eq. (2-20) accounts for free space propagation in a uniform medium, while the second term accounts for scattering by the random medium [44, 47].

An approximate solution of eq. (2-20) is given by [44, 47]

$$g(\boldsymbol{r}) \approx g_D(\boldsymbol{r}) g_R(\boldsymbol{r}) \tag{2-21}$$

where $g_D(\boldsymbol{r})$ and $g_R(\boldsymbol{r})$ are the deterministic and random components, respectively, of the solution. The form of eq. (2-21) was chosen to conform with the general practice of representing a field in a random medium as the product of a deterministic and a random function [43]. In order to determine $g_D(\boldsymbol{r})$, we set [44, 47]

$$\sigma(\boldsymbol{r})n_{NR}(\boldsymbol{r}) = n_R(\boldsymbol{r}) = 0$$

in eq. (2-20) and, as a result, we obtain

$$\frac{\partial}{\partial z} g(\boldsymbol{r}) = [-\mathrm{j}/(2k_0)]\nabla_T^2 g(\boldsymbol{r}). \tag{2-22}$$

An *exact* solution of eq. (2-22) is given by $g(\boldsymbol{r}) = g_D(\boldsymbol{r})$ where

$$g_D(\boldsymbol{r}) = \exp[-\mathrm{j}2\pi f_X(x-x_0)]\exp[-\mathrm{j}2\pi f_Y(y-y_0)]\exp\{+\mathrm{j}[(2\pi)^2(f_X^2+f_Y^2)/(2k_0)](z-z_0)\} \tag{2-23}$$

where

$$f_X = u_0/\lambda_0, \qquad f_Y = v_0/\lambda_0 \qquad \text{and } f_Z = w_0/\lambda_0 \tag{2-24-26}$$

are the *transmitted spatial frequencies* (with units of cycles/m) in the X, Y, and Z directions, respectively, where k_0 and λ_0 are given by eqs. (2-3) and (2-5), respectively; and where

$$u_0 = \sin\theta_0\cos\psi_0, \qquad v_0 = \sin\theta_0\sin\psi_0 = \cos\beta_0 \qquad \text{and } w_0 = \cos\theta_0 \tag{2-27-29}$$

are the direction cosines w.r.t. the positive X, Y, and Z axes, respectively. The angles (θ_0, ψ_0) [see Fig. 1] are the vertical and azimuthal spherical angles measured w.r.t. the positive Z and X axes, respectively, representing the *initial* directions of wave propagation. The angle β_0 (see Fig. 1) is measured w.r.t. the positive Y axis. The reference or transmitted propagation vector $\boldsymbol{k}_0$ is given by (see Fig. 1)

$$\boldsymbol{k}_0 = k_X\hat{x} + k_Y\hat{y} + k_Z\hat{z} \tag{2-30}$$

where

$$k_X = k_0u_0 = 2\pi f_X, \qquad k_Y = k_0v_0 = 2\pi f_Y \qquad \text{and } k_Z = k_0w_0 = 2\pi f_Z. \tag{2-31-33}$$

In order to determine the random component of the solution $g_R(\boldsymbol{r})$, we ignore the first term in eq. (2-20) [47], or equivalently, we make the following additional assumptions which are analogous to eq. (2-12) [44]:

$$\left|\frac{\partial^2}{\partial x^2} g(\boldsymbol{r})\right| \ll 2k_0\left|\frac{\partial}{\partial z} g(\boldsymbol{r})\right| \qquad \text{and } \left|\frac{\partial^2}{\partial y^2} g(\boldsymbol{r})\right| \ll 2k_0\left|\frac{\partial}{\partial z} g(\boldsymbol{r})\right|. \tag{2-34, 35}$$

Using eqs. (2-34) and (2-35), eq. (2-20) reduces to

$$\frac{\partial}{\partial z} g(\boldsymbol{r}) = -\mathrm{j}k_0\sigma(\boldsymbol{r})n_{NR}(\boldsymbol{r})g(\boldsymbol{r}) \tag{2-36}$$

which has an *exact* solution $g(\boldsymbol{r}) = g_R(\boldsymbol{r})$ where [44, 47]

$$g_R(x, y, z) = g(x, y, z_0) \exp\left\{-\mathrm{j}k_0 \int_{z_0}^{z} \sigma(x, y, \zeta)n_{NR}(x, y, \zeta)\,\mathrm{d}\zeta\right\}. \tag{2-37}$$

However, as was mentioned previously, since the *coupling equations* [39, 62] already account for an arbitrary *source distribution* as a function of time and space, the initial field distribution term $g(x, y, z_0)$ can be dropped from eq. (2-37). Thus, for our purposes, we will use the following expression for the random component $g_R(\boldsymbol{r})$:

$$g_R(x, y, z) = \exp\left\{-\mathrm{j}k_0 \int_{z_0}^{z} \sigma(x, y, \zeta)n_{NR}(x, y, \zeta)\,\mathrm{d}\zeta\right\}. \tag{2-38}$$

Therefore, in summary, eq. (2-21) is an approximate solution of the parabolic equation given by eq. (2-20), where $g_D(\boldsymbol{r})$ and $g_R(\boldsymbol{r})$ are given by eqs. (2-23) and (2-38), respectively, provided that the assumptions represented by eqs. (2-12), (2-18), (2-34), and (2-35) are valid; and in addition, that the cross product terms

$$\frac{1}{k_0}\frac{\partial}{\partial x} g_D(\boldsymbol{r}) \frac{\partial}{\partial x} g_R(\boldsymbol{r}) \ll 1 \qquad \text{and} \; \frac{1}{k_0}\frac{\partial}{\partial y} g_D(\boldsymbol{r}) \frac{\partial}{\partial y} g_R(\boldsymbol{r}) \ll 1 \tag{2-39, 40}$$

so that they too can be neglected. The additional requirements imposed by eqs. (2-39) and (2-40) are a consequence of substituting eqs. (2-21), (2-23), and (2-38) into eq. (2-20) and determining which additional terms must be small in order to approximate equality. We are now in a position to obtain an expression for the ocean medium transfer function.

The solution of the Helmholtz wave equation given by eq. (2-19) is approximately equal to

$$\varphi(\boldsymbol{r}) \approx g_D(\boldsymbol{r})g_R(\boldsymbol{r}) \exp(-\mathrm{j}k_0 z) \tag{2-41}$$

where eq. (2-21) was substituted into eq. (2-10). The corresponding time-harmonic solution is given by

$$\varphi(t, \boldsymbol{r}) \approx g_D(\boldsymbol{r})g_R(\boldsymbol{r}) \exp(-\mathrm{j}k_0 z) \exp(+\mathrm{j}2\pi f t) \tag{2-42}$$

where eq. (2-41) was substituted into eq. (2-6). Substituting eqs. (2-23) and (2-38) into eq. (2-42), and multiplying the result by

$$\exp[-\mathrm{j}2\pi f_Z(z - z_0)] \exp[+\mathrm{j}2\pi f_Z(z - z_0)] = 1 \tag{2-43}$$

yields

$$\varphi(t, \boldsymbol{r}) \approx \exp[-\mathrm{j}2\pi(\hat{f}_Z - f_Z)(z - z_0)] \exp\left\{-\mathrm{j}k_0 \int_{z_0}^{z} \sigma(x, y, \zeta)n_{NR}(x, y, \zeta)\,\mathrm{d}\zeta\right\}$$
$$\cdot \exp(+\mathrm{j}2\pi f t) \exp[-\mathrm{j}2\pi \boldsymbol{\nu} \cdot (\boldsymbol{r} - \boldsymbol{r}_0)] \tag{2-44}$$

where

$$\boldsymbol{\nu} = (f_X, f_Y, f_Z), \quad \hat{f}_Z = \hat{w}_0/\lambda_0 = f\hat{w}_0/c_0 \qquad \text{and } \hat{w}_0 = 1 - [(u_0^2 + v_0^2)/2], \quad \text{or } \hat{w}_0 = 1 - [(\sin^2\theta_0)/2], \tag{2-45–48}$$

which is an approximation or estimate of the direction cosine w_0. Equations (2-47) and (2-48) are, in fact, binomial expansions of w_0, valid as long as

$$u_0^2 + v_0^2 = \sin^2\theta_0 \ll 1, \tag{2-49}$$

or

$$\sin^2\theta_0 \leqslant 0.1, \tag{2-50}$$

which implies

$$\theta_0 \leqslant 18.0°. \tag{2-51}$$

Thus, $\varphi(t, \boldsymbol{r})$ as given by eq. (2-44) is based upon the assumption that the initial angles of transmission θ_0 are small, i.e., that θ_0 satisfies eq. (2-51). Compare eq. (2-51) with $\theta_0 \leqslant 20°$ as recommended by McDaniel [54] and with $\theta_0 \leqslant 16°$ as recommended by Tappert [61]. Figure 2 is a plot of w_0 and $\hat{w}_0$ as given by eqs. (2-29) and (2-48), respectively, versus θ_0. Table 1 is a tabulation of the approximation error $\Delta = \hat{w}_0 - w_0$ versus θ_0 and the percent error $\Delta\%$ w.r.t. w_0.

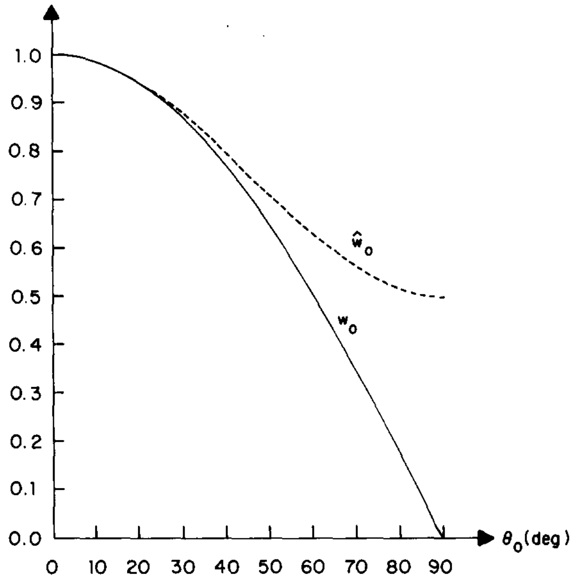

Fig. 2. Plot of the direction cosine w_0 (solid curve) given by Eq. (2-29) and its approximation $\hat{w}_0$ (dashed curve) given by Eq. (2-48) versus the initial angle of transmission θ_0 in degrees.

Since the input to the communication channel is the time-harmonic plane wave

$$\exp(+j2\pi ft)\exp[-j2\pi\boldsymbol{\nu}\cdot(\boldsymbol{r}-\boldsymbol{r}_0)],$$

and since eq. (2-44) represents the output from the channel at any time t and position $\boldsymbol{r}$, then (see eq. (2.1-21) of Ziomek [39, 62])

$$H_M(f, \boldsymbol{\nu}; t, \boldsymbol{r}) = H_M(f, \boldsymbol{\nu}; \boldsymbol{r}) = H_M(f, f_Z; \boldsymbol{r}) \tag{2-52}$$

where the random, time-invariant, space-variant transfer function of the ocean medium is given by

$$H_M(f, f_Z; \boldsymbol{r}) = \exp[+j\theta_{MD}(f, f_Z; \boldsymbol{r})]\exp[+j\theta_{MR}(f, f_Z; \boldsymbol{r})], \quad \text{where}$$

$$\theta_{MD}(f, f_Z; \boldsymbol{r}) = -2\pi(\hat{f}_Z - f_Z)(z - z_0) \tag{2-53, 54}$$

Table 1
Tabulation of the approximation error $\Delta = \hat{w}_0 - w_0$ versus θ_0 (deg) and the percent error Δ% with respect to w_0

θ_0 (deg)	Δ ($\times 10^{-3}$)	Δ%
0	0.0	0.0
5	0.007	0.001
10	0.115	0.012
15	0.581	0.06
20	1.818	0.194
25	4.389	0.484
30	8.975	1.036
35	16.353	1.996
40	27.368	3.573
45	42.893	6.066
50	63.8	9.926

is the deterministic phase component, and

$$\theta_{MR}(f, f_Z; \boldsymbol{r}) = -k_0 \int_{z_0}^{z} \sigma(x, y, \zeta) n_{NR}(x, y, \zeta) \, d\zeta \qquad (2\text{-}55)$$

is the random phase component.

The medium transfer function given by eq. (2-53) indicates that the major effect of the medium is to *angle modulate* the transmitted field. Furthermore, if $\hat{f}_Z - f_Z \approx 0$, then $\theta_{MD}(f, f_Z; \boldsymbol{r}) \approx 0$ [see eq. (2-54)] and, as a result, the angle modulation is due strictly to the random fluctuations $\sigma(\boldsymbol{r}) n_{NR}(\boldsymbol{r})$ of the index of refraction [see eq. (2-55)]. Note that the angle modulation process is often referred to as "scattering" [55]. Also note that the transfer function *derived* in this section is written as the product of two functions, one deterministic and the other random which agrees with the *assumed* transfer function expression of Laval and Labasque [36] and with the general practice of representing a field in a random medium as the product of a deterministic and a random function [43]. Note that the form of eq. (2-53) is a direct consequence of the assumed form of eq. (2-21).

Finally, it should be emphasized that the transfer function specified by eqs. (2-53) through (2-55) was based, in part, upon the assumption given by eq. (2-17) which allowed neglect of terms involving $\sigma^2(\boldsymbol{r})$. Equation (2-17) restricts the upper limit of integration z appearing in eq. (2-55) to be on the order of the scale size s_Z of the random medium in the Z direction. Although the random component of the index of refraction angle modulates or "scatters" the propagating wave by only a small amount (i.e., the wave is "weakly scattered" or "singly scattered") within a distance s_Z, this does not mean that the theory is restricted to weak scatter only [60]. For longer distances of wave propagation, one need only divide the overall distance z into "elementary layers" or "phase screens" of thickness s_Z and then integrate from z_0 to s_Z, s_Z to $2s_Z$, etc. Some authors suggest that the thickness of the elementary layers should be $10s_Z$ [44].

3. Output electrical signal

Now that we have derived an ocean medium transfer function, let us demonstrate the use of the coupling equations presented in Ziomek [39, 62] by calculating the output electrical signal $y(t, \boldsymbol{r})$ from the receive aperture (array). In the case of a time-invariant, space-variant transfer function, $y(t, \boldsymbol{r})$ is given by (see eq. (4.2-7) in Ziomek [39] or eq. (7.2-57) in Ziomek [62])

$$y(t, \boldsymbol{r}) = \int_{-\infty}^{\infty} X(f) \int_{-\infty}^{\infty} D_T(f, \boldsymbol{\nu}) H_M(f, \boldsymbol{\nu}; \boldsymbol{r}) \exp(-\mathrm{j}2\pi\boldsymbol{\nu}\cdot\boldsymbol{r})\, \mathrm{d}\boldsymbol{\nu}\, A_R(f, \boldsymbol{r}) \exp(+\mathrm{j}2\pi f t)\, \mathrm{d}f \tag{3-1}$$

where $X(f)$ is the frequency spectrum of the transmitted electrical signal, $D_T(f, \boldsymbol{\nu})$ is the far-field transmit directivity function (beam pattern), $H_M(f, \boldsymbol{\nu}; \boldsymbol{r})$ is the random, time-invariant, space-variant transfer function of the ocean medium, and $A_R(f, \boldsymbol{r})$ is the complex receive aperture function.

Assume that the transmit aperture depicted in Fig. 1 is a planar array of $M \times N$ (odd) complex weighted point sources, centered at (x_0, y_0, z_0) and parallel to the XY plane. In addition, assume that the complex weights are separable. Therefore

$$D_T(f, \boldsymbol{\nu}) = \sum_{m=-(M-1)/2}^{(M-1)/2} \sum_{n=-(N-1)/2}^{(N-1)/2} c_m d_n \exp(+\mathrm{j}2\pi f_X m d_X) \exp(+\mathrm{j}2\pi f_Y n d_Y) \exp(+\mathrm{j}2\pi f_X x_0)$$
$$\cdot \exp(+\mathrm{j}2\pi f_Y y_0) \exp(+\mathrm{j}2\pi f_Z z_0) \tag{3-2}$$

is the far-field beam pattern of the transmit array, c_m and d_n are complex weights, d_X and d_Y are the interelement spacings in the X and Y directions, respectively, and the last three exponentials are phase factors which account for the array being centered at (x_0, y_0, z_0) instead of at the origin (0, 0, 0).

Next assume that the receive aperture depicted in Fig. 1 is a planar array of $M' \times N'$ (odd) complex weighted point sources, centered at (x_R, y_R, z_R) and parallel to the XY plane. In addition, assume that the complex weights are separable. Therefore, the receive aperture function is given by

$$A_R(f, \boldsymbol{r}) = \sum_{i=-(M'-1)/2}^{(M'-1)/2} \sum_{q=-(N'-1)/2}^{(N'-1)/2} c'_i d'_q \delta(x - [x_R + i d'_X]) \delta(y - [y_R + q d'_Y]) \delta(z - z_R) \tag{3-3}$$

where c'_i and d'_q are complex weights and d'_X and d'_Y are the interelement spacings in the X and Y directions, respectively.

Upon substituting eqs. (2-45), (2-52), (3-2), and (3-3) into eq. (3-1) and recalling that $\boldsymbol{r} = (x, y, z)$, one obtains

$$y(t, x, y, z) = \sum_{i=-(M'-1)/2}^{(M'-1)/2} \sum_{q=-(N'-1)/2}^{(N'-1)/2} c'_i d'_q \int_{-\infty}^{\infty} X(f) \sum_{m=-(M-1)/2}^{(M-1)/2} \sum_{n=-(N-1)/2}^{(N-1)/2} c_m d_n$$
$$\cdot \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} H_M(f, f_Z; x_R + i d'_X, y_R + q d'_Y, z_R)$$
$$\cdot \exp(-\mathrm{j}2\pi[f_X \Delta X_{im} + f_Y \Delta Y_{qn} + f_Z \Delta Z])\, \mathrm{d}f_X\, \mathrm{d}f_Y\, \mathrm{d}f_Z$$
$$\cdot \exp(+\mathrm{j}2\pi f t)\, \mathrm{d}f \delta(x - [x_R + i d'_X]) \delta(y - [y_R + q d'_Y]) \delta(z - z_R) \tag{3-4}$$

where

$$\Delta X_{im} = (x_R - x_0) + (i d'_X - m d_X),$$
$$\Delta Y_{qn} = (y_R - y_0) + (q d'_Y - n d_Y) \qquad \text{and } \Delta Z = z_R - z_0. \tag{3-5-7}$$

In order to proceed further we make the following observations. From eq. (2-30) we can write that

$$k_Z^2 = k_0^2 - (k_X^2 + k_Y^2) \tag{3-8}$$

where $k_0^2 = |\boldsymbol{k}_0|^2$, and upon substituting eq. (2-3) and eqs. (2-31) through (2-33) into eq. (3-8), we obtain

$$f_Z = \pm[(f/c_0)^2 - (f_X^2 + f_Y^2)]^{1/2}. \tag{3-9}$$

Thus, the spatial frequency f_Z is seen to be *dependent* upon f, f_X, and f_Y. As a result, we can *eliminate* the integration w.r.t. f_Z in eq. (3-4) (for example, see Stratton [56]). Equation (3-9) can also be expressed in terms of the direction cosines u_0 and v_0 by using eqs. (2-24) and (2-25), i.e.,

$$f_Z = \pm(f/c_0)[1 - (u_0^2 + v_0^2)]^{1/2}, \qquad \text{or } f_Z = f w_0/c_0 \tag{3-10,11}$$

where

$$w_0 = \pm[1 - (u_0^2 + v_0^2)]^{1/2} \tag{3-12}$$

is the direction cosine w.r.t. the positive Z axis. If $u_0^2 + v_0^2 \leq 1$, then

$$w_0 = +[1 - (u_0^2 + v_0^2)]^{1/2}; \quad u_0^2 + v_0^2 \leq 1 \tag{3-13}$$

which corresponds to an acoustic field propagating in the positive Z direction [57]. However, if $u_0^2 + v_0^2 > 1$, then

$$w_0 = -[1 - (u_0^2 + v_0^2)]^{1/2}; \quad u_0^2 + v_0^2 > 1 \tag{3-14}$$

or

$$w_0 = -\mathrm{j}[(u_0^2 + v_0^2) - 1]^{1/2}; \quad u_0^2 + v_0^2 > 1 \tag{3-15}$$

which corresponds to an *evanescent wave*, i.e., a decaying exponential [57].

If we now 1) eliminate the integration w.r.t. f_Z in eq. (3-4), 2) change variables from the spatial frequencies f_X and f_Y to the direction cosines u_0 and v_0 by substituting eqs. (2-24) and (2-25) into eq. (3-4) and treat the frequency variable f as a constant so that

$$\mathrm{d}f_X = \mathrm{d}(fu_0/c_0) = (f/c_0)\,\mathrm{d}u_0 \qquad \text{and } \mathrm{d}f_Y = \mathrm{d}(fv_0/c_0) = (f/c_0)\,\mathrm{d}v_0$$

and 3) substitute eqs. (3-11) and (3-13) into $\exp(-\mathrm{j}2\pi f_Z \Delta Z)$ which appears in the integrand of eq. (3-4), then eq. (3-4) becomes

$$\begin{aligned} y(t, x, y, z) = {} & (1/c_0)^2 \sum_{i=-(M'-1)/2}^{(M'-1)/2} \sum_{q=-(N'-1)/2}^{(N'-1)/2} c_i' d_q' \int_{-\infty}^{\infty} f^2 X(f) \\ & \cdot \int_{u_0=-1}^{1} \int_{v_0=-v_{0_{\max}}}^{v_{0_{\max}}} \sum_{m=-(M-1)/2}^{(M-1)/2} \sum_{n=-(N-1)/2}^{(N-1)/2} c_m d_n \\ & \cdot H_M(f, [f/c_0][1-(u_0^2+v_0^2)]^{1/2}; x_R + id'_X, y_R + qd'_Y, z_R) \\ & \cdot \exp\{-\mathrm{j}[2\pi f/c_0]([1-(u_0^2+v_0^2)]^{1/2})\Delta Z\} \\ & \cdot \exp(-\mathrm{j}[2\pi f/c_0][u_0 \Delta X_{im} + v_0 \Delta Y_{qn}])\,\mathrm{d}v_0\,\mathrm{d}u_0 \\ & \cdot \exp(+\mathrm{j}2\pi f t)\,\mathrm{d}f\,\delta(x - [x_R + id'_X])\delta(y - [y_R + qd'_Y])\delta(z - z_R); \\ & u_0^2 + v_0^2 \leq 1 \end{aligned} \tag{3-16}$$

where

$$v_{0_{\max}} = \sin\theta_{0_{\max}}, \quad \text{and [see eq. (2-51)] } \theta_{0_{\max}} \leq 18.0°. \tag{3-17,18}$$

The limits of integration w.r.t. v_0 were obtained by noting that

$$\frac{\pi}{2} - \theta_{0_{max}} \leq \beta_0 \leq \frac{\pi}{2} + \theta_{0_{max}}$$

and since $v_0 = \cos\beta_0$ [see eq. (2-28) and Fig. 1],

$$\cos\left(\frac{\pi}{2} + \theta_{0_{max}}\right) = -\sin\theta_{0_{max}} \qquad \text{and} \cos\left(\frac{\pi}{2} - \theta_{0_{max}}\right) = \sin\theta_{0_{max}},$$

and therefore

$$-\sin\theta_{0_{max}} \leq v_0 \leq \sin\theta_{0_{max}}.$$

The transfer function H_M is given by eqs. (2-53) through (2-55) where $\hat{f}_Z$ is given by eq. (2-46) and $\hat{w}_0$ by eq. (2-47).

Equation (3-16) is the desired result, i.e., it represents the random output electrical signal at *each* element in the receive array in terms of the frequency spectrum of the transmitted electrical signal, the transmit and receive arrays, and the random transfer function of the ocean medium. If $y(t, x, y, z)$ is complex, then simply take the real part to obtain the real output electrical signals. Note that Eq. (3-16) pertains to the case of an acoustic field propagating in the positive Z direction since $u_0^2 + v_0^2 \leq 1$. When $u_0^2 + v_0^2 > 1$, then

$$y(t, x, y, z) = 0; \quad u_0^2 + v_0^2 > 1 \tag{3-19}$$

since only rapidly decaying evanescent waves emanate from the transmit array. Note that eq. (4.2-13) for $y(t, x, y, z)$ appearing in Ziomek [39] is not completely correct. The steps which were outlined above in going successfully from eq. (3-4) to eq. (3-16) should also be used in going from eq. (4.2-9) (which is correct) to eq. (4.2-13) in Ziomek [39]. The correct equation for $y(t, x, y, z)$ that incorporates the *WKB* approximation is given by eq. (7.2-71) in Ziomek [62].

Finally, consider the following two terms which appear in the integrand of eq. (3-16):

$$H_M(f, [f/c_0][1-(u_0^2+v_0^2)]^{1/2}; x_R + id'_X, y_R + qd'_Y, z_R) \cdot \exp\{-j[2\pi f/c_0]([1-(u_0^2+v_0^2)]^{1/2})\Delta Z\}. \tag{3-20}$$

If eqs. (2-53), (2-54), (2-46), (2-47), (3-11), and (3-13) are substituted into eq. (3-20), we obtain

$$\exp\{-j(2\pi f/c_0)(1-[(u_0^2+v_0^2)/2])\Delta Z\} \cdot \exp[+j\theta_{MR}(f, [f/c_0][1-(u_0^2+v_0^2)]^{1/2}; x_R + id'_X, y_R + qd'_Y, z_R)] \tag{3-21}$$

where θ_{MR} is given by eq. (2-55). If we now assume that the medium is homogeneous, i.e., space-*in*variant, and non-random, then the random component of the index of refraction is zero and, as a result [see eqs. (2-9) and (2-55)], $\theta_{MR} = 0$. With $\theta_{MR} = 0$, eq. (3-21) becomes

$$\exp\{-j(2\pi f/c_0)(1-[(u_0^2+v_0^2)/2])\Delta Z\}. \tag{3-22}$$

It is interesting to note that eq. (3-22) is equivalent to the expression that Goodman [58] obtains for a space-*in*variant transfer function which describes the effects of propagation in the region of Fresnel diffraction close to the Z axis, i.e., where θ_0 does not exceed 18° [59].

4. Coherence function

Upon inspecting eqs. (3-1) and (2-52), it can be seen that the autocorrelation function of the output electrical signal will depend upon the autocorrelation function of the transfer function

$$R_{H_M}(f,f',f_Z,f'_Z;\boldsymbol{r},\boldsymbol{r}')=E\{H_M(f,f_Z;\boldsymbol{r})H^*_M(f',f'_Z;\boldsymbol{r}')\} \tag{4-1}$$

which is also known as the *coherence function.* Substituting eqs. (2-53) and (2-55) into eq. (4-1) yields

$$R_{H_M}(f,f',f_Z,f'_Z;\boldsymbol{r},\boldsymbol{r}')=\exp(+\mathrm{j}[\theta_{MD}(f,f_Z;\boldsymbol{r})-\theta_{MD}(f',f'_Z;\boldsymbol{r}')]) \cdot E\{\exp(+\mathrm{j}[KI(\boldsymbol{r})+K'I(\boldsymbol{r}')])\} \tag{4-2}$$

where

$$K=-k_0=-2\pi f/c_0, \qquad K'=+k'_0=+2\pi f'/c_0, \qquad I(\boldsymbol{r})=\int_{z_0}^{z}\sigma(x,y,\zeta)n_{NR}(x,y,\zeta)\,\mathrm{d}\zeta \tag{4-3-5}$$

and

$$I(\boldsymbol{r}')=\int_{z_0}^{z'}\sigma(x',y',\zeta')n_{NR}(x',y',\zeta')\,\mathrm{d}\zeta' \tag{4-6}$$

where θ_{MD} is given by eq. (2-54) and the random component of the index of refraction $n_R(\boldsymbol{r})=\sigma(\boldsymbol{r})n_{NR}(\boldsymbol{r})$ was assumed to be real. The expectation appearing in eq. (4-2) is the *characteristic function* of the random quantity $[KI(\boldsymbol{r})+K'I(\boldsymbol{r}')]$. If it is assumed that $I(\boldsymbol{r})$ is a real *Gaussian* random process, which implies that $n_R(\boldsymbol{r})$, and hence, $n_{NR}(\boldsymbol{r})$ is a real *Gaussian* random process, then eq. (4-2) can be written as

$$R_{H_M}(f,f',f_Z,f'_Z;\boldsymbol{r},\boldsymbol{r}')=\exp(+\mathrm{j}[\theta_{MD}(f,f_Z;\boldsymbol{r})-\theta_{MD}(f',f'_Z;\boldsymbol{r}')])\exp(-E\{\theta^2_{MR}(f,f_Z;\boldsymbol{r})\}/2) \cdot\exp(+E\{\theta_{MR}(f,f_Z;\boldsymbol{r})\theta_{MR}(f',f'_Z;\boldsymbol{r}')\}) \cdot\exp(-E\{\theta^2_{MR}(f',f'_Z;\boldsymbol{r}')\}/2) \tag{4-7}$$

where

$$E\{\theta^2_{MR}(f,f_Z;\boldsymbol{r})\}=k_0^2\int_{z_0}^{z}\int_{z_0}^{z}\sigma(x,y,\zeta)\sigma(x,y,\zeta')R_{n_{NR}}(x,x,y,y,\zeta,\zeta')\,\mathrm{d}\zeta\,\mathrm{d}\zeta', \tag{4-8}$$

$$E\{\theta^2_{MR}(f',f'_Z;\boldsymbol{r}')\}=(k'_0)^2\int_{z_0}^{z'}\int_{z_0}^{z'}\sigma(x',y',\zeta')\sigma(x',y',\zeta'') \cdot R_{n_{NR}}(x',x',y',y',\zeta',\zeta'')\,\mathrm{d}\zeta'\,\mathrm{d}\zeta'', \tag{4-9}$$

$$E\{\theta_{MR}(f,f_Z;\boldsymbol{r})\theta_{MR}(f',f'_Z;\boldsymbol{r}')\}=k_0k'_0\int_{z_0}^{z}\int_{z_0}^{z'}\sigma(x,y,\zeta)\sigma(x',y',\zeta') \cdot R_{n_{NR}}(x,x',y,y',\zeta,\zeta')\,\mathrm{d}\zeta\,\mathrm{d}\zeta' \tag{4-10}$$

and

$$R_{n_{NR}}(x,x',y,y',z,z')=E\{n_{NR}(x,y,z)n_{NR}(x',y',z')\}. \tag{4-11}$$

If it is further assumed that the random component $n_R(\boldsymbol{r})$, and hence, the normalized random component $n_{NR}(\boldsymbol{r})$ is wide-sense stationary, i.e.,

$$R_{n_{NR}}(x, x', y, y', z, z') = R_{n_{NR}}(\Delta x, \Delta y, \Delta z) \tag{4-12}$$

where $\Delta x = x - x'$, $\Delta y = y - y'$, and $\Delta z = z - z'$; then eqs. (4-8) through (4-10) become, respectively,

$$E\{\theta^2_{MR}(f, f_Z\,;\,\boldsymbol{r})\} = (k_0\sigma)^2(z - z_0) \int_{-(z-z_0)}^{(z-z_0)} \left[1 - \frac{|\zeta|}{(z - z_0)}\right] R_{n_{NR}}(0, 0, \zeta)\, \mathrm{d}\zeta \tag{4-13}$$

$$E\{\theta^2_{MR}(f', f'_Z\,;\,\boldsymbol{r}')\} = (k'_0\sigma)^2(z' - z_0) \int_{-(z'-z_0)}^{(z'-z_0)} \left[1 - \frac{|\zeta|}{(z' - z_0)}\right] R_{n_{NR}}(0, 0, \zeta)\, \mathrm{d}\zeta \tag{4-14}$$

and

$$\begin{aligned} &E\{\theta_{MR}(f, f_Z\,;\,\boldsymbol{r})\theta_{MR}(f', f'_Z\,;\,\boldsymbol{r}')\} \\ &= k_0k'_0\sigma^2\left\{(z - z') \int_{(z-z')}^{(z-z_0)} R_{n_{NR}}(\Delta x, \Delta y, \zeta)\, \mathrm{d}\zeta + (z' - z_0) \int_{-(z'-z_0)}^{(z-z_0)} R_{n_{NR}}(\Delta x, \Delta y, \zeta)\, \mathrm{d}\zeta \right. \\ &\quad \left. + \int_{-(z'-z_0)}^{0} \zeta R_{n_{NR}}(\Delta x, \Delta y, \zeta)\, \mathrm{d}\zeta - \int_{(z-z')}^{(z-z_0)} \zeta R_{n_{NR}}(\Delta x, \Delta y, \zeta)\, \mathrm{d}\zeta \right\} \end{aligned} \tag{4-15}$$

where σ is the constant standard deviation of the real Gaussian, zero mean, wide-sense stationary, random component $n_R(\boldsymbol{r})$ of the index of refraction.

The coherence function given by eq. (4-1) represents the *correlation* between the two *output* fields $H_M(f, f_Z\,;\,\boldsymbol{r})$ and $H_M(f', f'_Z\,;\,\boldsymbol{r}')$ at two different spatial locations, $\boldsymbol{r}$ and $\boldsymbol{r}'$, when two small amplitude time-harmonic plane wave *input* fields are transmitted at two different frequencies, f and f', and in two different *initial* directions, f_Z and f'_Z.

The effects of the time-invariant, space-variant, random medium considered in this paper on small amplitude wave propagation can be described in terms of *coherence bandwidth*, *spatial coherence*, and *angular coherence*. This information can be obtained from the coherence function given by eq. (4-2) or, in the case of Gaussian statistics, from eq. (4-7).

The value of the frequency difference $\Delta f = f - f'$ at which the autocorrelation function

$$R_{H_M}(f, f', f_Z, f_Z\,;\,\boldsymbol{r}, \boldsymbol{r})$$

decreases to a specified level is called the *coherence bandwidth*. It is a measure of the frequency correlation of a field at a spatial location $\boldsymbol{r}$ when the initial direction of propagation is f_Z. The inverse of the coherence bandwidth is a measure of the amount of spread to be expected in time delay values.

The value of the spatial difference $\Delta\boldsymbol{r} = \boldsymbol{r} - \boldsymbol{r}' = (\Delta x, \Delta y, \Delta z)$ at which the autocorrelation function

$$R_{H_M}(f, f, f_Z, f_Z\,;\,\boldsymbol{r}, \boldsymbol{r}')$$

decreases to a specified level is called the *spatial coherence*. It is a measure of the spatial correlation of a field at a frequency f when the initial direction of propagation is f_Z. The quantities $\Delta x = x - x'$, $\Delta y = y - y'$, and $\Delta z = z - z'$ are referred to as the *scale sizes* of the medium in the X, Y, and Z directions, respectively. The inverse of the scale sizes are measures of the amount of spatial frequency spreading (in cycles/m) to be expected. Since spatial frequencies are related to direction cosines, and hence, to the directions of wave propagation, spatial frequency spread corresponds to angular spread or scatter.

And in conclusion, the value of the angular difference $\Delta f_Z = f_Z - f'_Z$ at which the autocorrelation function

$$R_{H_M}(f, f, f_Z, f'_Z;\ \boldsymbol{r}, \boldsymbol{r})$$

decreases to a specified level is called the *angular coherence.* It is a measure of the correlation between two waves at a frequency f, and at a spatial location $\boldsymbol{r}$ when the initial directions of propagation are f_Z and f'_Z. The inverse of the quantity $\Delta f_Z = f_Z - f'_Z$ is a measure of the spatial spread or physical dimensions of the source.

The result of all this spreading is, of course, distortion of the shape of the transmitted pulse.

5. Summary

A time-invariant, space-variant, random transfer function of the ocean medium was *derived* based upon the *parabolic equation approximation* of the Helmholtz wave equation. The transfer function was time-invariant instead of time-variant because motion was not considered in the present derivation. The transfer function modelled the ocean volume between transmit and receive apertures (arrays). The ocean volume was characterized by a *three-dimensional* random index of refraction (sound-speed profile) which was decomposed into deterministic and random components.

In addition to the transfer function derivation, two example calculations were made. The first example demonstrated the use of the coupling equations and involved the derivation of a mathematical expression for the random output electrical signal at *each* element in a receive planar array of complex weighted point sources. The output signals were expressed in terms of the frequency spectrum of the transmitted electrical signal, the transmit and receive arrays, and the transfer function of the ocean medium. The first example demonstrated that an output electrical signal could be derived in a *logical* and *straightforward* fashion. The second example involved the derivation of the coherence function, i.e., the autocorrelation function of the transfer function. In order to obtain somewhat simplified results, it was necessary to assume that the random component of the index of refraction was Gaussian and wide-sense stationary.

The effects of the time-invariant, space-variant, random medium considered in this paper on small amplitude wave propagation can be described in terms of *coherence bandwidth, spatial coherence,* and *angular coherence.* This information can be obtained from the coherence function.

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References

[1] A.W. Ellinthorpe and A. H. Nuttall, "Theoretical and empirical results on the characterization of undersea acoustic channels", *IEEE First Annual Communication Convention* (1965), pp. 585–591.

[2] D. Middleton, "A statistical theory of reverberation and similar first-order scattered fields. Part I: Waveforms and the general process", *IEEE Trans. Inform. Theory,* Vol. 13, 372–392 (1967).

[3] K.A. Sostrand, "Mathematics of the time-varying channel", in: *Proceedings of NATO Advanced Study Institute on Signal Processing with emphasis on underwater acoustics* (Enschede, The Netherlands, (1968), Vol. II, pp. (25-1)–(25-20).

[4] H.L. Van Trees, *Detection, Estimation and Modulation Theory* (John Wiley and Sons, New York, 1971), Vol. III, Chap. 13, pp. 459-460.
[5] A.N. Venetsanopoulos and F.B. Tuteur, "Stochastic filter modeling for the sea-surface scattering channel", *J. Acoust. Soc. Am.* Vol. 49, 1100-1107 (1971).
[6] J.F. McDonald and R.C. Spindel, "Implications of Fresnel corrections in a non-Gaussian surface scatter channel", *J. Acoust. Soc. Am.* Vol. 50, 746-757 (1971).
[7] L. Fortuin, "The sea surface as a random filter for underwater sound waves", *J. Acoust. Soc. Am.* Vol. 52, 302-315 (1972).
[8] J. Johnson, "Spectrum analysis of reverberation", in: *Signal Processing*, edited by J.W.R. Griffiths, P.L. Stocklin and C. Van Schooneveld (Academic Press, New York, 1973), pp. 97-115.
[9] R. Laval, "Sound propagation effects on signal processing", in: *Signal Processing*, edited by J.W.R. Griffiths, P.L. Stocklin and C. Van Schooneveld (Academic Press, New York, 1973), pp. 223-241.
[10] P.H. Moose, "Signal processing in reverberant environments", in: *Signal Processing*, edited by J.W.R. Griffiths, P.L. Stocklin and C. Van Schooneveld (Academic Press, New York, 1973), pp. 413-428.
[11] J.F. McDonald, "Fresnel-corrected second-order interfrequency correlations for a surface-scatter channel", *IEEE Trans. Commun.* Vol. COM-22, 138-145 (1974).
[12] J.F. McDonald and F.B. Tuteur, "Calculation of the range-Doppler plot for a doubly spread surface-scatter channel at high Rayleigh parameters", *J. Acoust. Soc. Am.* Vol. 57, 1025-1029 (1975).
[13] J.G. Zornig and J.F. McDonald, "Experimental measurement of the second-order interfrequency correlation function of the random surface scatter channel", *IEEE Trans. Commun.* Vol. COM-23, 341-347 (1975).
[14] F.B. Tuteur, J.F. McDonald and H. Tung, "Second-order statistical moments of a surface scatter channel with multiple wave direction and dispersion", *IEEE Trans. Commun.* Vol. COM-24, 820-831 (1976).
[15] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic Press, New York, 1978), Vol. I, Chap. 5, pp. 93-94.
[16] A.B. Baggeroer, "Sonar signal processing", in: *Applications of Digital Signal Processing*, edited by A.V. Oppenheim (Prentice-Hall, Englewood Cliffs, New Jersey, 1978), Chap. 6, pp. 365-366.
[17] L.J. Ziomek, "Generalized Kirchhoff approach to the ocean surface-scatter communication channel. Part I: Transfer function of the ocean surface", *J. Acoust. Soc. Am.* Vol. 71, 116-126 (1982).
[18] L.J. Ziomek, "Generalized Kirchhoff approach to the ocean surface-scatter communication channel. Part II: Second-order functions", *J. Acoust. Soc. Am.* Vol. 71, 1487-1495 (1982).
[19] L.J. Ziomek and L.H. Sibul, "Broadband and narrowband signal-to-interference ratio expressions for a doubly spread target", *J. Acoust. Soc. Am.* Vol. 72, 804-819 (1982).
[20] P.E. Green, Jr., "Radar measurements of target scattering properties", in: *Radar Astronomy*, edited by J.V. Evans and T. Hagfors (McGraw-Hill, New York, 1968), Chap. 1, pp. 1-77.
[21] T. Kailath, "Channel characterization: Time-variant dispersive channels", in: *Lectures on Communication System Theory*, edited by E.J. Baghdady (McGraw-Hill, New York, 1961), pp. 95-123.
[22] P.A. Bello, "Characterization of randomly time-variant linear channels", *IEEE Trans. Commun. Systems*, Vol. 11, 360-393 (1963).
[23] R.S. Kennedy, *Fading Dispersive Communication Channels* (Wiley-Interscience, New York, 1969).
[24] W.D. Rummler, "Clutter suppression by complex weighting of coherent pulse trains", *IEEE Trans. Aerospace Electron. Systems*, Vol. 2, 689-699 (1966).
[25] D.F. DeLong, Jr. and E.M. Hofstetter, "On the design of optimum radar waveforms for clutter rejection", *IEEE Trans. Inform. Theory*, Vol. 13, 454-463 (1967).
[26] W.D. Rummler, "A technique for improving the clutter performance of coherent pulse train signals", *IEEE Trans. Aerospace Electron. Systems*, Vol. 3, 898-906 (1967).
[27] J.S. Thompson and E.L. Titlebaum, "The design of optimal radar waveforms for clutter rejection using the maximum principle", *Supplement to IEEE Trans. Aerospace Electron. Systems*, Vol. 3(6), 581-589 (1967).
[28] L.J. Spafford, "Optimum radar signal processing in clutter", *IEEE Trans. Inform. Theory*, Vol. 14, 734-743 (1968).
[29] C.A. Stutt and L.J. Spafford, "A best mismatched filter response for radar clutter discrimination", *IEEE Trans. Inform. Theory*, Vol. 14, 280-287 (1968).
[30] D.F. DeLong, Jr. and E.M. Hofstetter, "The design of clutter-resistant radar waveforms with limited dynamic range", *IEEE Trans. Inform. Theory*, Vol. 15, 376-385 (1969).
[31] L.H. Sibul and E.L. Titlebaum, "Signal design for detection of targets in clutter", *Proc. IEEE*, Vol. 69, 481-482 (1981).
[32] T. Kooij, "Optimum signals in noise and reverberation", in: *Proceedings of NATO Advanced Study Institute on Signal Processing with emphasis on underwater acoustics* (Enschede, The Netherlands, 1968), Vol. I, pp. (17-1)-(17-12).
[33] L.J. Ziomek and L.H. Sibul, "Maximization of the signal-to-interference ratio for a doubly spread target: Problems in nonlinear programming", *Signal Processing*, Vol. 5, No. 4, 355-368 (1983).
[34] L.J. Ziomek, "Comments on the generalized ambiguity function", *IEEE Trans. Acoust., Speech, Signal Proc.*, Vol. ASSP-30, 117-119 (1982).
[35] R. Laval, "Time-frequency-space generalized coherence and scattering functions", in: *Aspects of Signal Processing*, edited by G. Tacconi (D. Reidel Publishing Company, Dordrecht, Holland, 1977), Vol. I, pp. 69-87.

[36] R. Laval and Y. Labasque, "Medium inhomogeneities and instabilities: Effects on spatial and temporal processing", in: *Underwater Acoustics and Signal Processing*, edited by L. Bjorno (D. Reidel Publishing Company, Dordrecht, Holland, 1981), pp. 41-70.
[37] D. Middleton, "A statistical theory of reverberation and similar first-order scattered fields. Part III: Waveforms and fields", *IEEE Trans. Inform. Theory*, Vol. 18, 35-67 (1972).
[38] D. Middleton, "The underwater medium as a generalized communication channel", in: *Underwater Acoustics and Signal Processing*, edited by L. Bjorno (D. Reidel Publishing Company, Dordrecht, Holland, 1981), pp. 589-612.
[39] L.J. Ziomek, "Linear time-variant space-variant filters and the *WKB* approximation", NPS Technical Report 62-83-058, October 1983, 93 pages. Also available from DTIC, AD Number A 134886.
[40] L.J. Ziomek, *A Scattering Function Approach to Underwater Acoustic Detection and Signal Design*, Ph.D. Dissertation, The Pennsylvania State University, University Park, PA (1981), Chap. 2, pp. 10-49.
[41] L.M. Brekhovskikh, *Waves in Layered Media* (1st Edition, Academic Press, New York, 1960), pp. 193-198.
[42] I. Tolstoy and C.S. Clay, *Ocean Acoustics* (McGraw-Hill, New York, 1966), pp. 48-52.
[43] R.H. Clarke, "Sound Propagation in a variable ocean", *J. Sound and Vibration*, Vol. 34, 457-477 (1974).
[44] R.H. Clarke, "Acoustic and electromagnetic waves propagating in a tenuous random medium", in: *Aspects of Signal Processing*, edited by G. Tacconi (D. Reidel Publishing Company, Dordrecht, Holland, 1977), Vol. I, pp. 61-67.
[45] B.J. Uscinski, *The Elements of Wave Propagation in Random Media* (McGraw-Hill, New York, 1977).
[46] C. Macaskill, B.J. Uscinski and N. Freedman, "Acoustic propagation in the upper sound channel", *J. Acoust. Soc. Am.*, Vol. 72, 1544-1555 (1982).
[47] D.L. Knepp, "Multiple phase-screen calculation of the temporal behavior of stochastic waves", *Proc. IEEE*, Vol. 71, 722-737 (1983).
[48] see [45], p. 2.
[49] D. Mintzer, "Wave propagation in a randomly inhomogeneous medium. I", *J. Acoust. Soc. Am.*, Vol. 25, 922-927 (1953).
[50] D. Mintzer, "Wave propagation in a randomly inhomogeneous medium. II", *J. Acoust. Soc. Am.*, Vol. 25, 1107-1111 (1953).
[51] D. Mintzer, "Wave propagation in a randomly inhomogeneous medium. III", *J. Acoust. Soc. Am.*, Vol. 26, 186-190 (1954).
[52] see [45], p. 20.
[53] A. Ishimaru, *Wave Propagation and Scattering in Random Media* (Academic Press, New York, 1978), Vol. II, p. 408.
[54] S.T. McDaniel, "Application of the parabolic approximation to predict acoustical propagation in the ocean", *Am. J. Phys.*, Vol. 47 (1), 63-68 (1979).
[55] see [45], pp. 4-6.
[56] J.A. Stratton, *Electromagnetic Theory*, (McGraw-Hill, New York, 1941), pp. 361-364.
[57] J.W. Goodman, *Introduction to Fourier Optics*, (McGraw-Hill, New York, 1968), pp. 48-51.
[58] see [57], p. 60, eq. (4-11).
[59] see [57], p. 58.
[60] see [45], p. 6.
[61] F.D. Tappert, "The parabolic approximation method", in: *Wave Propagation and Underwater Acoustics*, edited by J.B. Keller and J.S. Papadakis, Springer-Verlag, Berlin (1977), pp. 224-287.
[62] L.J. Ziomek, *Underwater Acoustics—A Linear Systems Theory Approach* (Academic Press, Orlando, Florida, 1985).