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Barr, Donald R.; Slezak, Normal L.

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A Comparison of Multivariate Normal Generators

Donald R. Barr
Naval Postgraduate School
and
Norman L. Slezak
United States Navy

Three methods for generating outcomes on multivariate normal random vectors with a specified variance-covariance matrix are presented. A comparison is made to determine which method requires the least computer execution time and memory space when utilizing the IBM 360/67. All methods use as a basis a standard Gaussian random number generator. Results of the comparison indicate that the method based on triangular factorization of the covariance matrix generally requires less memory space and computer time than the other two methods.

Key Words and Phrases: random number generator, normal distribution, multivariate normal distribution, multivariate normal generator.

CR Categories: 3.65, 5.5

Introduction and Summary

The frequent use of samples from one-dimensional (univariate) normal distributions in computer simulation applications has motivated the introduction of many different schemes for their generation. Muller [5] describes several such schemes and compares them in terms of computer space requirements and the speed with which they produce samples. It is our purpose to present a similar comparison of three generation methods for multivariate normal samples. These methods are based on the use of a univariate normal generator to obtain sample vectors from a multivariate normal distribution with independent components, followed by a transformation which results in normal vectors whose components have the desired variance-covariance structure.

In what follows, we describe briefly the generation methods to be compared, and the bases of the comparison. The results of this study are summarized in Table I, where computer memory space requirements and generation times for the three methods are given. These results support the recommendation of Scheuer and Stoller [8], which is based on a subjective evaluation of the relative mathematical simplicity of one of the methods.

Procedure and Results

In what follows, we assume it is desired to generate a sample of 1,000 vectors from a k-dimensional normal distribution with mean $\mathbf{O}$ and variance-covariance matrix $\boldsymbol{\Sigma}$. Following the notation and terminology of Anderson [1], we seek a random sample of $\mathbf{X}$, where $\mathbf{X} \sim \mathbf{N}_k(\mathbf{O}, \boldsymbol{\Sigma})$.[1] The three methods of generation are described briefly as follows:

(a) *Rotation Method.* If $\mathbf{X} \sim \mathbf{N}_k(\mathbf{O}, \boldsymbol{\Sigma})$, the $k \times k$ variance-covariance matrix $\boldsymbol{\Sigma}$ is positive definite and symmetric, so there is a matrix $\mathbf{P}$ such that $\mathbf{P}'\boldsymbol{\Sigma}\mathbf{P} = \mathbf{I}$. Thus $\mathbf{Y} = \mathbf{XP} \sim \mathbf{N}_k(\mathbf{O}, \mathbf{I})$. The components of $\mathbf{Y} = (Y_1, Y_2, \ldots, Y_k)$ are independent univariate $N(0, 1)$ random variables; hence samples of $\mathbf{Y}$ can be generated using a univariate normal generator. Since $\mathbf{X} = \mathbf{YP}^{-1}$, samples of $\mathbf{X}$ are obtained by multiplying samples of $\mathbf{Y}$ by $\mathbf{P}^{-1}$. The computation of the matrix $\mathbf{P}^{-1}$ for a given

Authors addresses: Donald Barr, Naval Postgraduate School, Monterey, CA 93940; Norman L. Slezak, 98-145 Lipoa Place, Aiea, Hawaii 96701.

[1] The symbol "$\mathbf{X} \sim \mathbf{N}_k(\mathbf{O}, \boldsymbol{\Sigma})$" means that the population we wish to sample has a k-dimensional normal distribution with mean vector $\mathbf{O}$ and variance-covariance matrix $\boldsymbol{\Sigma}$.

Σ involves finding the eigenvectors of Σ. This method is described in more detail by Graybill [2].

(b) *Conditional Method.* The joint density function $f_X(\mathbf{x})$ of X can be written as a product of univariate conditional density functions,

$$f_X(x_1, x_2, \ldots, x_k) = f_1(x_1 \mid x_2, \ldots, x_k) \cdot f_2(x_2 \mid x_3, \ldots, x_k) \cdot \ldots \cdot f_k(x_k).$$

If X is multivariate normal, then the conditional densities are univariate normal densities. Thus samples of X can be generated, using a univariate normal generator, as follows: first generate x_k from the distribution $f_k(\cdot)$, then x_{k-1} from the conditional distribution $f_{k-1}(\cdot \mid x_k)$, next x_{k-2} from the conditional distribution $f_{k-2}(\cdot \mid x_{k-1}, x_k)$, and so on. The resulting vector $(x_1, \ldots, x_k)$ is a sample of size one of X. This method is suggested by Rosenblatt [7]. The computation of $f_1, f_2, \ldots, f_k$ for given Σ involves the computation of cofactors of Σ and is discussed in detail by Scheuer and Stoller [8].

(c) *Triangular Factorization Method.* The matrix Σ can be factored into a product of a lower triangular matrix T and its transpose, $\Sigma = T'T$. Then if $Y \sim N_k(O, I)$, it follows that $X = YT \sim N_k(O, \Sigma)$. Thus, sample vectors X can be obtained by multiplying generated vectors Y by a lower triangular matrix. Computation of T for a given Σ is simple, involving only standard routines, and is discussed in detail by Graybill [3] and Pearson [6].

In order to compare these methods, programs using them were run on an IBM 360/67. Data concerning these runs is summarized in Table I. Each of the methods described involves the computation of a transformation and then repeated generation using the transformation. Thus we have considered two times in this evauluation: the "setup" time and the "repetition" time. The setup time for each method is the execution time for the portion of the program involving the determination, for a given Σ, of the transformation required, exclusive of input time. The repetition time is the execution time for the generation of 1,000 vectors of independent univariate normal variates and their transformation to desired form. The time required to generate univariate normal random numbers can thus be used to establish a lower bound on the repetition time for any method which uses such numbers as a basis. For example, if Σ is 5×5, the time required to generate 5,000 *univariate* normal random numbers is a lower bound on the repetition time required to generate 1,000 five-dimensional normal *vectors* by any of the methods considered here. These lower bounds are shown in Table I for a generator using the Marsaglia technique [4].

Similarly, the memory space requirements shown in Table I makes use of two numbers. The first provides an indication of the amount of space required for the three programs, including an eigenvalue/eiegenvector subroutine in the rotation method and the univariate normal generator in all cases. The second number is the total space required for the program plus any external functions used such as square roots, absolute values, exponentials, and input and output devices.

Based on the data obtained, it appears that the triangularization method is the best of the three methods considered. This was found to be the case for a wide variety of Σ inputs. It is anticipated that the relative ranking of the methods would remain the same with changes in program details, or with computer systems other than the IBM 360/67.

Table I. Memory Space and Execution Time Requirements for Three Multivaritate Normal Generators

Method	Matrix Size	Lower Bound Time	Setup Time	Repetition Time	Program Space	Total Space
	2 × 2	.2033	.00496	.37666	11,952	36,288
Rotation	3 × 3	.3230	.01398	.62023	16,008	40,344
	5 × 5	.5320	.06117	1.23357	24,192	48,528
	10 × 10	1.0674	.41652	3.72289	44,992	69,328
	2 × 2	.2033	.00273	.33985	11,792	37,168
Conditioning	3 × 3	.3230	.00741	.55383	15,880	41,256
	5 × 5	.5320	.03975	1.06335	24,160	49,536
	10 × 10	1.0674	.70306	2.75560	45,424	70,800
	2 × 2	.2033	.00106	.33800	9,528	33,864
Triangularization	3 × 3	.3230	.00145	.52411	13,568	37,904
	5 × 5	.5320	.00247	.99554	21,712	46,048
	10 × 10	1.0674	.00795	2.52280	42,336	66,672

Note: Each time is in seconds and each space requirement is in bytes.

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