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Generalized bent Boolean functions and strongly regular Cayley graphs

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Abstract

In this paper we define the (edge-weighted) Cayley graph associated to a generalized Boolean function, introduce a notion of strong regularity and give several of its properties. We show some connections between this concept and generalized bent functions (gbent), that is, functions with flat Walsh-Hadamard spectrum. In particular, we find a complete characterization of quartic gbent functions in terms of the strong regularity of their associated Cayley graph.

1 (Generalized) Boolean functions background

Let $\mathbb{V}_n$ be the vector space of dimension n over the two element field $\mathbb{F}_2$, and for a positive integer q, let $\mathbb{Z}_q$ be the ring of integers modulo q. Let us denote the addition, respectively, product operators over $\mathbb{F}_2$ by "$\oplus$", respectively, "$\cdot$". A Boolean function f on n variables is a mapping from $\mathbb{V}_n$ into $\mathbb{F}_2$, that is, a multivariate polynomial over $\mathbb{F}_2$,

$$f(x_1, \ldots, x_n) = a_0 \oplus \sum_{i=1}^{n} a_i x_i \oplus \sum_{1\le i<j\le n} a_{ij} x_i x_j \oplus \ldots \oplus a_{12\ldots n} x_1 x_2 \ldots x_n, \qquad (1)$$

where the coefficients $a_0, a_i, a_{ij}, \ldots, a_{12\ldots n} \in \mathbb{F}_2$. This representation of f is called the *algebraic normal form* (ANF) of f. The number of variables in the highest order product term with nonzero coefficient is called the *algebraic degree*, or simply the degree of f.

For a Boolean function on $\mathbb{V}_n$, the *Hamming weight* of f, $wt(f)$, is the cardinality of $\Omega_f = \{\mathbf{x} \in \mathbb{V}_n : f(\mathbf{x}) = 1\}$ (this is extended to any vector, by taking its weight to be the number of nonzero components of that vector). The *Hamming distance* between two functions $f,\, g : \mathbb{V}_n \to \mathbb{F}_2$ is $d(f,g) = wt(f \oplus g)$. A Boolean function $f(\mathbf{x})$ is called an *affine function* if its algebraic degree is 1. If, in addition, $a_0 = 0$ in (1), then $f(\mathbf{x})$ is a *linear function* (see [8] for more on Boolean functions). In $\mathbb{V}_n = \mathbb{F}_2^n$, the vector space of the n-tuples over $\mathbb{F}_2$, we use the conventional dot product $\mathbf{u} \cdot \mathbf{x}$ as an inner product.

For a *generalized Boolean function* $f : \mathbb{V}_n \to \mathbb{Z}_q$ we define the *generalized Walsh-Hadamard transform* to be the complex valued function

$$\mathcal{H}_f^{(q)}(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{V}_n} \zeta_q^{f(\mathbf{x})} (-1)^{\mathbf{u} \cdot \mathbf{x}},$$

where $\zeta_q = e^{\frac{2\pi \imath}{q}}$ (we often use ζ, $\mathcal{H}_f$, instead of ζ_q, respectively, $\mathcal{H}_f^{(q)}$, when q is fixed). The inverse is given by $\zeta^{f(\mathbf{x})} = 2^{-n} \sum_{\mathbf{u}} \mathcal{H}_f(\mathbf{u})(-1)^{\mathbf{u} \cdot \mathbf{x}}$. For $q = 2$, we obtain the usual *Walsh-Hadamard transform*

$$\mathcal{W}_f(\mathbf{u}) = \sum_{\mathbf{x} \in \mathbb{V}_n} (-1)^{f(\mathbf{x})} (-1)^{\mathbf{u} \cdot \mathbf{x}},$$

which defines the coefficients of character form of f with respect to the orthonormal basis of the group characters $\chi_{\mathbf{w}}(\mathbf{x}) = (-1)^{\mathbf{w} \cdot \mathbf{x}}$. In turn, $f(\mathbf{x}) = 2^{-n} \sum_{\mathbf{w}} \mathcal{W}_f(\mathbf{w})(-1)^{\mathbf{u} \cdot \mathbf{x}}$.

We use the notation as in [10, 11, 12, 15, 16] (see also [14, 17]) and denote the set of all generalized Boolean functions by $\mathcal{GB}_n^q$ and when $q = 2$, by $\mathcal{B}_n$. A function $f : \mathbb{V}_n \to \mathbb{Z}_q$ is called *generalized bent* (*gbent*) if $|\mathcal{H}_f(\mathbf{u})| = 2^{n/2}$ for all $\mathbf{u} \in \mathbb{V}_n$. We recall that a function f for which $|\mathcal{W}_f(\mathbf{u})| = 2^{n/2}$ for all $\mathbf{u} \in \mathbb{V}_n$ is called a *bent* function, which only exist for even n since $\mathcal{W}_f(\mathbf{u})$ is an integer. Let $f \in \mathcal{GB}_n^q$, where $2^{k-1} < q \le 2^k$, then we can represent f uniquely as

$$f(\mathbf{x}) = a_0(\mathbf{x}) + 2a_1(\mathbf{x}) + \cdots + 2^{k-1} a_{k-1}(\mathbf{x})$$

for some Boolean functions a_i, $0 \le i \le k-1$ (this representation comes from the binary representation of the elements in the image set $\mathbb{Z}_{2^k}$). For results on classical bent functions and related topics, the reader can consult [5, 8, 13, 18].

2 Unweighted strongly regular graphs

A graph is *regular of degree* r (or r-regular) if every vertex has degree r (number of edges incident to it). We say that an r-regular graph G is a *strongly regular graph* (srg) with parameters (v, r, e, d) if there exist nonnegative integers e, d such that for all vertices $\mathbf{u}, \mathbf{v}$ the number of vertices adjacent to both $\mathbf{u}, \mathbf{v}$ is d, e, if $\mathbf{u}, \mathbf{v}$ are adjacent, respectively, nonadjacent (see for instance [9]). The *complementary* graph $\bar{G}$ of the strongly regular graph G is also strongly regular with parameters $(v, v-r-1, v-2r+e-2, v-2r+d)$ (see [9]).

Since the objects of this paper are edge-weighted graphs $G = (V, E, w)$ (with vertices V, edges E and weight function w defined on E with values in some set, which in our case it will be either the set of integers modulo q, $\mathbb{Z}_q$ with $q = 2^k$, or the complex numbers set $\mathbb{C}$), we define the *weighted degree* $d(v)$ of a vertex v to be the sum of the weights of its incident edges, that is, $d(v) = \sum_{u,(u,v)\in E} w(u,v)$ (later, we will introduce yet another degree or strength concept). Certainly, one can also define the *combinatorial degree* $r(v)$ of a vertex to be the number of such incident edges. For more on graph theory the reader can consult [4, 9] or one's favorite graph theory book.

Let f be a Boolean function on $\mathbb{V}_n$. We define the *Cayley graph* of f to be the graph $G_f = (\mathbb{V}_n, E_f)$ whose vertex set is $\mathbb{V}_n$ and the set of edges is defined by

$$E_f = \{(\mathbf{w}, \mathbf{u}) \in \mathbb{V}_n \times \mathbb{V}_n \,:\, f(\mathbf{w} \oplus \mathbf{u}) = 1\}.$$

For some fixed (but understood from the context) positive integer s, let the canonical injection $\iota : \mathbb{V}_s \to \mathbb{Z}_{2^s}$ be defined by $\iota(\mathbf{c}) = \mathbf{c} \cdot (1, 2, \ldots, 2^{s-1}) = \sum_{j=0}^{s-1} c_j 2^j$, where $\mathbf{c} = (c_0, c_1, \ldots, c_{s-1})$. For easy writing, we denote by $\mathbf{j} := \iota^{-1}(j)$.

The adjacency matrix A_f is the matrix whose entries are $A_{i,j} = f(\mathbf{i} \oplus \mathbf{j})$ (here ι is defined on $\mathbb{V}_n$). It is simple to prove that A_f has the dyadic property: $A_{i,j} = A_{i+2^{k-1}, j+2^{k-1}}$. Also, from its definition, we derive that G_f is a *regular graph of degree* $wt(f) = |\Omega_f|$ (see [9, Chapter 3] for further definitions).

Given a graph f and its adjacency matrix A, the *spectrum*, with notation $Spec(G_f)$, is the set of eigenvalues of A (called also the eigenvalues of G_f). We assume throughout that G_f is connected (in fact, one can show that all connected components of G_f are isomorphic).

It is known (see [9, pp. 194–195]) that a connected r-regular graph is strongly regular iff it has exactly three distinct eigenvalues $\lambda_0 = r, \lambda_1, \lambda_2$ (so $e = r + \lambda_1\lambda_2 + \lambda_1 + \lambda_2$, $d = r + \lambda_1\lambda_2$).

The following result is known [9, Th. 3.32, p. 103] (the second part follows from a counting argument and is also well known).

Proposition 1. *The following identity holds for a strongly r-regular graph:*

$$A^2 = (d-e)A + (r-e)I + eJ,$$

where J is the all 1 matrix. Moreover, $r(r-d-1) = e(v-r-1)$.

In [1, 2] it was shown that a Boolean function f is bent if and only if the Cayley graph G_f is strongly regular with $e = d$. We shall refer to this as the Bernasconi-Codenotti correspondence.

3 The Cayley graph of a generalized Boolean function

We now let $f : \mathbb{V}_n \to \mathbb{Z}_q$ be a generalized Boolean function. We define the *(generalized) Cayley graph* G_f to be the graph where vertices are the elements of $\mathbb{V}_n$ and two vertices $\mathbf{u}, \mathbf{v}$ are connected by a weighted edge of (multiplicative) weight $\zeta^{f(\mathbf{u}\oplus\mathbf{v})}$ (respectively, additive weight $f(\mathbf{u} \oplus \mathbf{v})$). Certainly, the underlying unweighted graph is a complete pseudograph (every vertex also has a loop). We sketch in Figure 1 such an example.

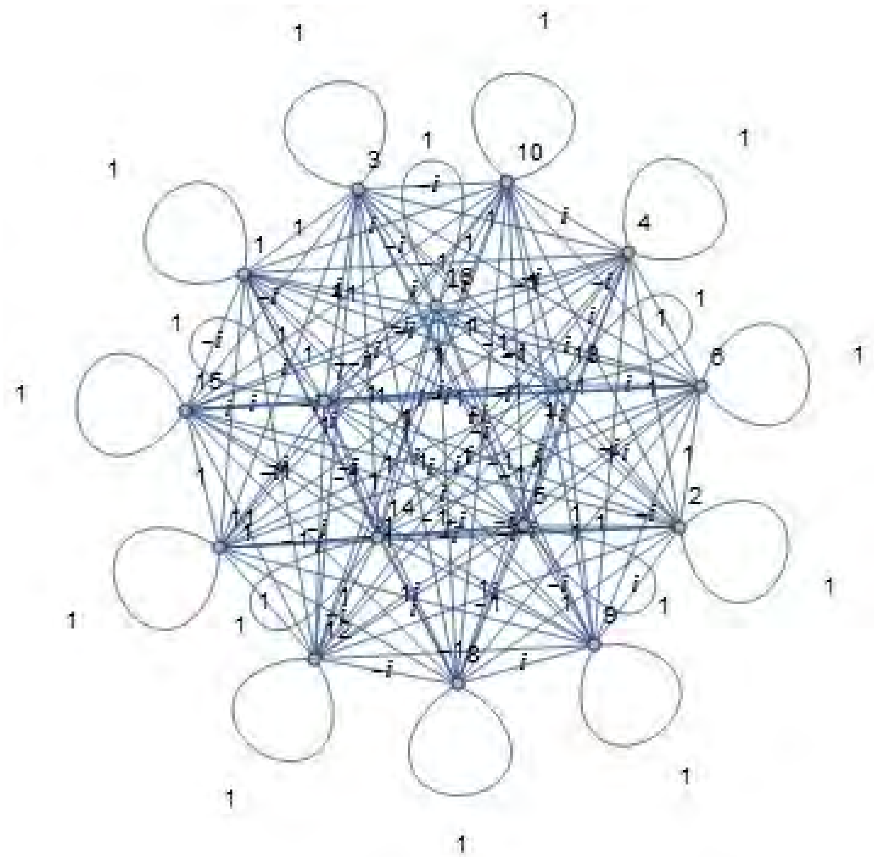

Figure 1: Cayley graph associated to the gbent $f(\mathbf{x}) = x_1 + 2(x_1x_2 \oplus x_3x_4)$

Certainly, one can define a modified (generalized) Cayley graph G'_f where two vertices are connected if and only if $f(\mathbf{u} \oplus \mathbf{v}) \neq 0$ with weights given by $\zeta^{f(\mathbf{u}\oplus\mathbf{v})}$. We sketch in Figure 2 such a graph (it is ultimately the above graph with all weight 1 edges removed).

In Example 2, we give an example of a generalized Cayley graph, and its spectrum.

Example 2. *Let $f : \mathbb{V}_n \to \mathbb{Z}_4$ defined by $f(x_1, x_2) = x_1x_2 + 2x_1$. The truth table is $(0\ 0\ 2\ 3)^T$ (using the lexicographical order x_1, x_2). Then, the adjacency matrix (with multiplicative weights) is*

$$A_f = \begin{pmatrix} 1 & 1 & -1 & -i \\ 1 & 1 & -i & -1 \\ -1 & -i & 1 & 1 \\ -i & -1 & 1 & 1 \end{pmatrix}.$$

A basis for its eigenspace is $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, where $\vec{v}_1 = (1\ 1\ 1\ 1)^T$ with $\chi_1(\mathbf{x}) = (-1)^0$ $\vec{v}_2 = (1\ -1\ 1\ -1)^T$ with $\chi_2(\mathbf{x}) = (-1)^{x_2}$, $\vec{v}_3 = (1\ 1\ -1\ -1)^T$ with $\chi_3(\mathbf{x}) = (-1)^{x_1}$, $\vec{v}_4 = (1\ -1\ -1\ 1)^T$ with $\chi_4(\mathbf{x}) = (-1)^{x_1+x_2}$, having respective eigenvalues $\lambda_0 =$

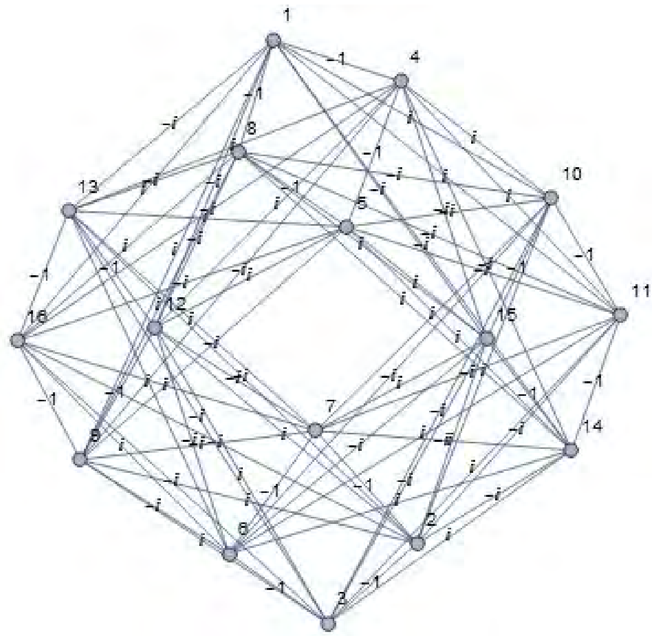

Figure 2: Modified Cayley graph associated to gbent $f(\mathbf{x}) = x_1 + 2(x_1x_2 \oplus x_3x_4)$

$1-i$, $\lambda_1 = -1+i$, $\lambda_2 = 3+i$, $\lambda_3 = 1-i$. *We can see that the eigenvalues A_f are*

$$\lambda_0 = i^0\chi_1(00) + i^0\chi_1(01) + i^2\chi_1(10) + i^3\chi_1(11) = 1+1+i^2+i^3 = 1-i = \mathcal{H}_f^{(4)}(\mathbf{0}),$$
$$\lambda_1 = i^0\chi_2(00) + i^0\chi_2(01) + i^2\chi_2(10) + i^3\chi_2(11) = 1-1+i^2-i^3 = -1+i = \mathcal{H}_f^{(4)}(\mathbf{1}),$$
$$\lambda_2 = i^0\chi_3(00) + i^0\chi_3(01) + i^2\chi_3(10) + i^3\chi_3(11) = 1+1-i^2-i^3 = 3+i = \mathcal{H}_f^{(4)}(\mathbf{2}),$$
$$\lambda_3 = i^0\chi_4(00) + i^0\chi_4(01) + i^2\chi_4(10) + i^3\chi_4(11) = 1-1-i^2+i^3 = 1-i = \mathcal{H}_f^{(4)}(\mathbf{3}).$$

Although, we do not use it in this paper, we define the *strength* of the vertex $\mathbf{a}$ in the Cayley graph G_f as the sum of the additive weights of incident edges, that is, $s(\mathbf{a}) = \sum_b f(\mathbf{a} \oplus \mathbf{b})$.

Remark 3. *If $f \in \mathcal{GB}_n^q$ and G_f is its Cayley graph, we observe that all vertices are adjacent of multiplicative (respectively, additive) weights in $\mathbb{U}_q = \{1, \zeta, \zeta^2, \ldots, \zeta^{q-1}\}$ (respectively, in $\mathbb{Z}_q = \{0, 1\ldots, q-1\}$).*

We next show that the eigenvalues of the Cayley graph G_f (with multiplicative weights) are precisely the (generalized) Walsh-Hadamard coefficients.

Theorem 4. *Let $f : \mathbb{V}_n \to \mathbb{Z}_q$, $q = 2^k$, and let $\lambda_i, 0 \le i \le 2^n - 1$ be the eigenvalues of its associated (multiplicative) edge-weighted graph G_f. Then,*

$$\lambda_i = \mathcal{H}_f^{(q)}(\mathbf{i}) \text{ (\textit{recall that} } \mathbf{i} = \iota^{-1}(i)).$$

Proof. Let $\chi : \mathbb{V}_n \to \mathbb{C}$ be a character of $\mathbb{V}_n$, and for each such character, let $\mathbf{x}_\chi = (x_j)_{0 \le j \le 2^n - 1} \in \mathbb{C}^{2^n}$, where $x_j = \chi(\mathbf{j})$. We claim (and show) that $\mathbf{x}_\chi$ is an eigenvector of $A = A_f$ (for simplicity, we use A in lieu of A_f in this proof), with eigenvalue $\sum_{k=0}^{q-1} \sum_{\mathbf{s_k} \in S_k} \zeta^k \chi(\mathbf{s_k})$, where $S_k = \{\mathbf{s}_k : f(\mathbf{s}_k) = k\}$. (Observe that the characters of $\mathbb{V}_n$ are $\chi_{\mathbf{w}}(\mathbf{x}) = (-1)^{\mathbf{u}\cdot\mathbf{x}}$, and thus the eigenvalues are exactly the Walsh–Hadamard transform coefficients).

The i-th entry of $A\mathbf{x}$ is

$$(A\mathbf{x})_i = \sum_j A_{i,j} x_j = \sum_j A_{i,j} \chi(\mathbf{j}) = \sum_{k=0}^{q-1} \sum_{\mathbf{i}\oplus\mathbf{j} \in S_k} \zeta^k \chi(\mathbf{j})$$

If $\mathbf{i} \oplus \mathbf{j} \in S_k$, then $\mathbf{i} \oplus \mathbf{j} = \mathbf{s}_k$, for some $\mathbf{s}_k \in S_k$, and so, $\mathbf{j} = \mathbf{i} \oplus \mathbf{s}_k$. Since χ is a character,

$$\chi(\mathbf{j}) = \chi(\mathbf{i} \oplus \mathbf{s}_k) = \chi(\mathbf{i})\chi(\mathbf{s}_k) = x_i \chi(\mathbf{s}_k)$$

Then,

$$(A\mathbf{x})_i = \sum_{k=0}^{q-1} \sum_{\mathbf{s}_k \in S_k} \zeta^k x_i \chi(\mathbf{s}_k) = x_i \sum_{k=0}^{q-1} \sum_{\mathbf{s}_k \in S_k} \zeta^k \chi(\mathbf{s}_k),$$

which shows our theorem. □

4 Generalized bents and their Cayley graphs

We recall that a *q-Butson Hadamard matrix* [6] (q-BH) of dimension d is a $d \times d$ matrix H with all entries q-th roots of unity such that $HH^* = dI_d$, where H^* is the conjugate transpose of H. When $q = 2$, q-BH matrices are called Hadamard matrices (where the entries are ± 1). Recall that the *crosscorrelation* function is defined by

$$\mathcal{C}_{f,g}(\mathbf{z}) = \sum_{\mathbf{x} \in \mathbb{V}_n} \zeta^{f(\mathbf{x}) - g(\mathbf{x} \oplus \mathbf{z})},$$

and the *autocorrelation* of $f \in \mathcal{GB}_n^q$ at $\mathbf{u} \in \mathbb{V}_n$ is $\mathcal{C}_{f,f}(\mathbf{u})$ above, which we denote by $\mathcal{C}_f(\mathbf{u})$.

Theorem 5. *Let $f \in \mathcal{GB}_n^q$. Then f is gbent if and only if the adjacency matrix A_f of the (multiplicative) edge-weighted Cayley graph associated to f is a q-Butson Hadamard matrix.*

Proof. Let $A_f = \left(\zeta^{f(\mathbf{a}+\mathbf{b})}\right)_{\mathbf{a},\mathbf{b}}$. Then, the $(\mathbf{a}, \mathbf{b})$-entry of $A_f \cdot \bar{A}_f$ is

$$\left(A_f \cdot \bar{A}_f\right)_{\mathbf{a},\mathbf{b}} = \sum_{\mathbf{c} \in \mathbb{V}_n} \zeta^{f(\mathbf{a}\oplus\mathbf{c})} \zeta^{\bar{f}(\mathbf{c}\oplus\mathbf{b})} = \sum_{\mathbf{c} \in \mathbb{V}_n} \zeta^{f(\mathbf{a}\oplus\mathbf{c}) - f(\mathbf{c}\oplus\mathbf{b})} = \mathcal{C}_f(\mathbf{a} \oplus \mathbf{b}). \qquad (2)$$

Now, recall from [15] that if $f, g \in \mathcal{GB}_n^q$, then

$$\sum_{\mathbf{u} \in \mathbb{V}_n} \mathcal{C}_{f,g}(\mathbf{u})(-1)^{\mathbf{u}\cdot\mathbf{x}} = 2^{-n} \mathcal{H}_f(\mathbf{x}) \overline{\mathcal{H}_g(\mathbf{x})},$$

$$\mathcal{C}_{f,g}(\mathbf{u}) = 2^{-n} \sum_{\mathbf{x} \in \mathbb{V}_n} \mathcal{H}_f(\mathbf{x}) \overline{\mathcal{H}_g(\mathbf{x})} (-1)^{\mathbf{u}\cdot\mathbf{x}}.$$

Thus, equation (2) becomes

$$\left(A_f \cdot \bar{A}_f\right)_{\mathbf{a},\mathbf{b}} = 2^{-n} \sum_{\mathbf{x}\in\mathbb{V}_n} \mathcal{H}_f(\mathbf{x})\overline{\mathcal{H}_f(\mathbf{x})}(-1)^{(\mathbf{a}\oplus\mathbf{b})\cdot\mathbf{x}} = 2^{-n} \sum_{\mathbf{x}\in\mathbb{V}_n} \|\mathcal{H}_f(\mathbf{x})\|^2(-1)^{(\mathbf{a}\oplus\mathbf{b})\cdot\mathbf{x}}.$$

By Parseval's identity, if $\mathbf{a} = \mathbf{b}$, then $\left(A_f \cdot \bar{A}_f\right)_{\mathbf{a},\mathbf{a}} = 2^n$. Assume now that $\mathbf{a} \neq \mathbf{b}$ and we shall show that $\left(A_f \cdot \bar{A}_f\right)_{\mathbf{a},\mathbf{b}} = 0$ for some $\mathbf{a} \neq \mathbf{b}$ if and only if f is gbent. Certainly, if f is gbent then $\|\mathcal{H}_f(\mathbf{x})\|^2 = 2^n$, and since $\sum_{\mathbf{x}\in\mathbb{V}_n}(-1)^{(\mathbf{a}\oplus\mathbf{b})\cdot\mathbf{x}} = 0$, for $\mathbf{a}\oplus\mathbf{b} \neq 0$, we have that implication. We can certainly show it directly, but the converse follows from [15, Theorem 1 (iv)]. □

For the remaining of the paper, for simplicity, we shall only consider additive weights, namely, our edge-weighted graphs (V, E, w) will have the weight function $w : E \to \mathbb{Z}_q$, $q = 2^k$.

Next, we say that a weighted graph $G = (V, E, w)$, $V \subseteq \mathbb{V}_n$, $w : E \to \mathbb{Z}_q$, $q = 2^k$, is a *weighted regular graph* (wrg) of parameters $(v; r_0, r_1, \ldots, r_{q-1})$ if every vertex will have exactly r_j neighbors of edge weight j. We denote by $N_j(\mathbf{a})$ the set of all neighbors of a vertex $\mathbf{a}$ of corresponding edge weight j.

Proposition 6. *Given a generalized Boolean function $f \in \mathcal{GB}_n^q$, the associated Cayley graph is weighted regular (of some parameters), that is, every vertex will have the same number of incident edges with a fixed weight.*

Proof. Fix a weight j and a vertex $\mathbf{x}_0$, and consider the equation $f(\mathbf{x}_0 \oplus \mathbf{y}) = j$ with solutions $\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t$, say. For any other vertex $\mathbf{x}_1$, the equation $f(\mathbf{x}_1 \oplus \mathbf{y}) = j$ will have solutions $\mathbf{y}_1 \oplus \mathbf{x}_1 \oplus \mathbf{x}_0, \mathbf{y}_2 \oplus \mathbf{x}_1 \oplus \mathbf{x}_0, \ldots, \mathbf{y}_t \oplus \mathbf{x}_1 \oplus \mathbf{x}_0$. The proof of the lemma is done. □

We will define our first concept of strong regularity here. Let $X, \bar{X}$ be a fixed bisection of the weights $\mathbb{Z}_q = X \cup \bar{X}, X \cap \bar{X} = \emptyset, |X| = |\bar{X}| = 2^{k-1}$, and let $Y \subseteq \mathbb{Z}_q$. We say that a weighted regular (of parameters $(v; r_0, r_1, \ldots, r_{q-1})$) graph $G = (V, E, w)$, $V \subseteq \mathbb{V}_n$, $w : E \to \mathbb{Z}_q$, $q = 2^k$, is a (generalized) $(X; Y)$-*strongly regular* (srg) of parameters $(v; r_0, r_1, \ldots, r_{q-1}; e_X, d_X)$ if and only if the number of vertices $\mathbf{c}$ adjacent to both $\mathbf{a}, \mathbf{b}$, with $w(\mathbf{a}, \mathbf{c}) \in Y, w(\mathbf{b}, \mathbf{c}) \in Y$, is exactly e_X if $w(\mathbf{a}, \mathbf{b}) \in X$, respectively, d_X if $w(\mathbf{a}, \mathbf{b}) \in \bar{X}$. One can weaken the condition and define a $(X_1, X_2; Y)$-srg notion, where $X_1 \cap X_2 = \emptyset$, not necessarily a bisection, and require the number of vertices $\mathbf{c}$ adjacent to both $\mathbf{a}, \mathbf{b}$, with $w(\mathbf{a}, \mathbf{c}) \in Y, w(\mathbf{b}, \mathbf{c}) \in Y$, to be exactly e_X if $w(\mathbf{a}, \mathbf{b}) \in X_1$, respectively, d_X if $w(\mathbf{a}, \mathbf{b}) \in X_2$; or even allowing a multi-section, and all of these variations can be fresh areas of research for graph theory experts.

Note that our definition (see also [7] for an alternative concept, which we mention in the last section) is a natural extension of the classical definition: Let $q = 2$, and $X = \{1\}$. A classical strongly regular graph is then equivalent to an $(X; X)$-strongly regular graph.

We first show that (part of) Proposition 1 can be adapted to this notion, as well, in some cases, and we deal below with one such instance.

Proposition 7. *Let $G = (V, E, w)$ be a weighted $(X; X)$-strongly regular graph of parameters $(v; r_0, r_1, \ldots, r_{q-1}; e_X, d_X)$, where $X \subseteq \mathbb{Z}_q$, $v = |V|$. Then,*

$$r_X(r_X - e_X - 1) = d_X(v - r_X - 1),$$

where $r_X = \sum_{i \in X} r_i$.

Proof. Without loss of generality we assume that the weights are additive, that is, they belong to $\mathbb{Z}_q$. Fix a vertex $\mathbf{u} \in V$ and let A be the set of vertices adjacent to $\mathbf{u}$ with connecting edges of weight in X, and $B = V \setminus \{A, \mathbf{u}\}$. Observe that $|A| = \sum_{i \in X} r_i = r_X$ and $|B| = v - r_X - 1$. We somewhat follow the combinatorial method of the classical case, and we shall count the number of vertices between A and B in two different ways. For any vertex $\mathbf{a} \in A$, there are exactly e_X vertices in A adjacent to both $\mathbf{u}, \mathbf{a}$ of edge weights in X, and so, exactly $r_X - e_X - 1$ neighbors in B whose connecting edges have weight in X. Therefore, the number of edges of weight in X between A and B is $r_X(r_X - e_X - 1)$.

On the other hand, any vertex $\mathbf{b} \in B$ is adjacent to d_X vertices in A of connecting edge with weight in X (since $\mathbf{u}, \mathbf{b}$ must share d_X common vertices of connecting edges of weight in X) and so, the total number of edges of weight in X between A and B is $d_X(v - r_X - 1)$. The proposition follows. $\square$

Let $G = (V, E, w)$ $(w : E \to \mathbb{Z}_q)$ be a weighted graph, where $w(E) \subseteq \mathbb{Z}_q$ (or $w(E) \subseteq \mathbb{U}_q$). We define the *complement* of G, denoted by $\bar{G}$ the graph of vertex set V with an edge between two vertices $\mathbf{a}, \mathbf{b}$ having weight $q-1-f(\mathbf{a} \oplus \mathbf{b})$ (or, multiplicatively, $\zeta^{q-1-f(\mathbf{a} \oplus \mathbf{b})}$. This is a natural definition, since if G is the Cayley graph associated to $f = a_0 + 2a_1, a_0, a_i \in \mathcal{B}_n$, then we observe that $\bar{G}$ is the Cayley graph associated to $\bar{f} = \bar{a}_0 + 2\bar{a}_1 + \cdots + 2^{k-1}\bar{a}_{k-1}$, where $\bar{a}_i$ is the binary complement of a_i (that follows from $2^k - 1 - f = (1 - a_0) + 2(1 - a_1) + \cdots + 2^{k-1}(1 - a_{k-1}) = \bar{a}_0 + 2\bar{a}_1 + \cdots + 2^{k-1}\bar{a}_{k-1}$).

Lemma 8. *Let $G = (V, E, w)$ $(w : E \to \mathbb{Z}_q)$ be a weighted regular graph of parameters $(v; r_0, r_1, \ldots, r_{q-1})$. Then the complement $\bar{G}$ is a weighted regular graph of parameters $(v; \bar{r}_0, \ldots, \bar{r}_{q-1})$, where $\bar{r}_{q-1-j} = r_j$.*

Proof. Let $\mathbf{a}$ be an arbitrary vertex. Recall that we denote by $N_j(\mathbf{a})$ the set of all neighbors of a vertex $\mathbf{a}$ of corresponding edge weight j. Since G is weighted regular, then $|N_j(\mathbf{a})| = r_j$. In the graph $\bar{G}$, the weight j will transform into $q - 1 - j$, therefore $\bar{r}_{q-1-j} = r_j$ and the lemma is shown. $\square$

Let $A \subset B$ and $x \in B$. As it is customary, we will denote by $x + A$ the set $\{x + a : a \in A\}$.

Theorem 9. *Let $G = (V, E, w)$ ($V \subseteq \mathbb{F}_2^n$, $w : E \to \mathbb{Z}_q$) be an $(X; Y)$-strongly regular, for some $X, Y \subseteq \mathbb{Z}_q$ with $|X| = 2^{k-1}, q = 2^k$, of parameters $(v; r_0, r_1, \ldots, r_{q-1}; e_X, d_X)$ such that $q - 1 - X = X$ or $\bar{X}$, and $q - 1 - Y = Y$. Then, the complement $\bar{G}$ is a $(q - 1 - X; Y)$-strongly regular graph of parameters $(v; \bar{r}_0, \ldots, \bar{r}_{q-1}; \bar{e}_{q-1-X}, \bar{d}_{q-1-X})$, where $\bar{r}_{q-1-j} = r_j$, $\bar{e}_{q-1-X} = e_X$ and $\bar{d}_{q-1-X} = d_X$, if $q - 1 - X = X$, respectively, $\bar{r}_{q-1-j} = r_j$, $\bar{e}_{q-1-X} = d_X$ and $\bar{d}_{q-1-X} = e_X$, if $q - 1 - X = \bar{X}$.*

Proof. The first claim follows from Lemma 8. We consider the two cases $q-1-X=X$, or $\bar{X}$, separately. As before, for any two vertices $\mathbf{a},\mathbf{b}$ we denote by $N_Y(\mathbf{a},\mathbf{b})$ the set of all vertices $\mathbf{c}$ adjacent to both $\mathbf{a},\mathbf{b}$ such that $w(\mathbf{a},\mathbf{c})\in Y, w(\mathbf{b},\mathbf{c})\in Y$.
Case 1. Let $q-1-X=X$. For any two vertices $\mathbf{a},\mathbf{b}$ with $w(\mathbf{a},\mathbf{b})\in X$, then $|N_Y(\mathbf{a},\mathbf{b})|=e_X$, since the weight of the edge between $\mathbf{a},\mathbf{b}$ remains in X. Similarly, for two vertices $\mathbf{a},\mathbf{b}$ with $w(\mathbf{a},\mathbf{b})\in\bar{X}$, then $|N_Y(\mathbf{a},\mathbf{b})|=d_X$.
Case 2. Let $q-1-X=X$. For any two vertices $\mathbf{a},\mathbf{b}$ with $w(\mathbf{a},\mathbf{b})\in X$, then the weight of the edge between $\mathbf{a},\mathbf{b}$ in $\bar{G}$ is now in $\bar{X}$, and we know that in that case $N_Y(\mathbf{a},\mathbf{b})=d_X$. Similarly, for two vertices $\mathbf{a},\mathbf{b}$ with $w(\mathbf{a},\mathbf{b})\in\bar{X}$, then $|N_Y(\mathbf{a},\mathbf{b})|=e_X$. □

In the next theorem, we shall show a strong regularity theorem (a Bernasconi-Codenotti correspondence) for gbents $f\in\mathcal{GB}_n^4$ when n even and $k=2$. For two vertices $\mathbf{a},\mathbf{b}$ of the associated Cayley graph, for $i,j\in\{0,1,2,3\}$, let $N_{\{i,j\}}(\mathbf{a},\mathbf{b})$ be the set of all "neighbor" vertices $\mathbf{w}$ to both $\mathbf{a},\mathbf{b}$ such that the edges have additive weights $f(\mathbf{w}\oplus\mathbf{a})\in\{i,j\}, f(\mathbf{w}\oplus\mathbf{b})\in\{i,j\}$.

Theorem 10. *Let $f\in\mathcal{GB}_n^4$, n even. Then f is gbent if and only if the associated generalized Cayley graph is $(X;\bar{X})$-strongly regular with $e_X=d_X$, for both $X=\{0,1\}$, and $X=\{0,3\}$, that is, if and only if the following two conditions are satisfied:*

(*i*) *For any two pairs of vertices $\{\mathbf{a},\mathbf{b}\}$, $\{\mathbf{c},\mathbf{d}\}$, then $|N_{\{2,3\}}(\mathbf{a},\mathbf{b})|=|N_{\{2,3\}}(\mathbf{c},\mathbf{d})|$.*

(*ii*) *For any two pairs of vertices $\{\mathbf{a},\mathbf{b}\}$, $\{\mathbf{c},\mathbf{d}\}$, then $|N_{\{1,2\}}(\mathbf{a},\mathbf{b})|=|N_{\{1,2\}}(\mathbf{c},\mathbf{d})|$.*

Proof. We know that $f=a_0+2a_1$, where $a_0,a_1\in\mathcal{B}_n$, is gbent if and only if $a_1,a_1\oplus a_0$ are both bent (see [14, 15]). Let $\mathbf{u}\in\mathbb{V}_n$. We have that:

1. $f(\mathbf{u})=0\Leftrightarrow a_0(\mathbf{u})=0,(a_1\oplus a_0)(\mathbf{u})=0$
2. $f(\mathbf{u})=1\Leftrightarrow a_0(\mathbf{u})=1,(a_1\oplus a_0)(\mathbf{u})=1$
3. $f(\mathbf{u})=2\Leftrightarrow a_0(\mathbf{u})=0,(a_1\oplus a_0)(\mathbf{u})=1$
4. $f(\mathbf{u})=3\Leftrightarrow a_0(\mathbf{u})=1,(a_1\oplus a_0)(\mathbf{u})=1$

If f is gbent, then $a_1,a_1\oplus a_0$ are both bent. Then, by [1], their respective graphs are srg with respective parameters $e=d, e'=d'$. We consider the following cases:

(*a*) Let any $\mathbf{a},\mathbf{b},\mathbf{c}$ such that $f(\mathbf{a}\oplus\mathbf{c})\in\{1,2\}$ and $f(\mathbf{b}\oplus\mathbf{c})\in\{1,2\}$, then $(a_1\oplus a_0)(\mathbf{a}\oplus\mathbf{c})=1=(a_1\oplus a_0)(\mathbf{b}\oplus\mathbf{c})$. Since the graph corresponding to $a_1\oplus a_0$ is srg with $e'=d'$, then $|\{\mathbf{c}:(a_1\oplus a_0)(\mathbf{a}\oplus\mathbf{c})=1=(a_1\oplus a_0)(\mathbf{b}\oplus\mathbf{c})\}|=e'$. Therefore, $|N_{\{1,2\}}(\mathbf{a},\mathbf{b})|=e'$.

(*b*) Let any $\mathbf{a},\mathbf{b},\mathbf{c}$ such that $f(\mathbf{a}\oplus\mathbf{c})\in\{2,3\}$ and $f(\mathbf{b}\oplus\mathbf{c})\in\{2,3\}$, then $a_1(\mathbf{a}\oplus\mathbf{c})=1=a_1(\mathbf{b}\oplus\mathbf{c})$. Since the graph corresponding to a_1 is srg with $e=d$, then $|\{\mathbf{c}:a_1(\mathbf{a}\oplus\mathbf{c})=1=a_1(\mathbf{b}\oplus\mathbf{c})\}|=e$. Therefore, $|N_{\{2,3\}}(\mathbf{a},\mathbf{b})|=e'$.

Conversely, let the generalized Cayley graph be such that, for any two pairs of vertices $\{\mathbf{a}, \mathbf{b}\}$, $\{\mathbf{c}, \mathbf{d}\}$, then $|N_{\{2,3\}}(\mathbf{a}, \mathbf{b})| = |N_{\{2,3\}}(\mathbf{c}, \mathbf{d})|$, and $|N_{\{1,2\}}(\mathbf{a}, \mathbf{b})| = |N_{\{1,2\}}(\mathbf{c}, \mathbf{d})|$. As seen in the first part of the proof, $|N_{\{2,3\}}(\mathbf{a}, \mathbf{b})| = |\{\mathbf{c} : a_1(\mathbf{a} \oplus \mathbf{c}) = 1 = a_1(\mathbf{b} \oplus \mathbf{c})\}|$. This number is a constant, regardless of the value of $a_1(\mathbf{a} \oplus \mathbf{b})$. This implies that the Cayley graph corresponding to a_1 is srg with $e = d$, where $e = |N_{\{2,3\}}(\mathbf{a}, \mathbf{b})|$.

Similarly, $|N_{\{1,2\}}(\mathbf{a}, \mathbf{b})| = |\{\mathbf{c} : (a_1 \oplus a_0)(\mathbf{a} \oplus \mathbf{c}) = 1 = (a_1 \oplus a_0)(\mathbf{b} \oplus \mathbf{c})\}|$. This number is a constant, regardless of the value of $(a_1 \oplus a_0)(\mathbf{a} \oplus \mathbf{b})$. This implies that the Cayley graph corresponding to $a_1 \oplus a_0$ is srg with $e' = d'$, where $e' = |N_{\{1,2\}}(\mathbf{a}, \mathbf{b})|$. Since both a_1 and $a_1 \oplus a_0$ are therefore bent, we conclude that f is gbent. $\square$

It is not hard to show that in some instances a "uniform" strong regularity will hold.

Corollary 11. *Let S be a bent set (see [3]), that is, every element of S is a bent function and the sum of any two such is also a bent function. Let $a_0, a_1 \in S$. Then, the generalized edge-weighted Cayley graph of $f = a_0 + 2a_1$ is $(X; \bar{X})$-strongly regular for any X with $|X| = 2$.*

Remark 12. *One certainly could inquire whether a similar result holds for a gbent for n odd. Since the answer depends on a characterization (not currently known) of classical semibents in terms of their Cayley graphs, we leave that question for a subsequent project of an interested reader.*

While we cannot find a necessary and sufficient condition on a gbent in $\mathcal{GB}_n^q$, $q = 2^k$, we can follow a similar approach as in Theorem 10 to find a necessary condition on the Cayley graph of a generalized bent in $\mathcal{GB}_n^q$. As in the previous result, for $X \subseteq \mathbb{Z}_q$ and two vertices $\mathbf{u}, \mathbf{v}$, let $N_X(\mathbf{u}, \mathbf{v})$ be the set of vertices $\mathbf{w}$ such that $f(\mathbf{u} \oplus \mathbf{w}) \in X$ and $f(\mathbf{v} \oplus \mathbf{w}) \in X$. As usual, $\bar{\mathbf{c}}$ is the complement of the vector $\mathbf{c}$, and for two vectors $\mathbf{a} = (a_1, \ldots, a_t), \mathbf{b} = (b_1, \ldots, b_t)$, the notation $\mathbf{a} \preceq \mathbf{b}$ means that $a_i \le b_i$, for all $1 \le i \le t$. Recall that the canonical injection $\iota : \mathbb{V}_s \to \mathbb{Z}_{2^s}$, $\iota(\mathbf{c}) = \mathbf{c} \cdot (1, 2, \ldots, 2^{s-1}) = \sum_{j=0}^{s-1} c_j 2^j$, where $\mathbf{c} = (c_0, c_1, \ldots, c_{s-1})$.

Theorem 13. *Let n be even, and $f = a_0 + 2a_1 + \cdots + 2^{k-1} a_{k-1}$, $k \ge 2$, $a_i \in \mathcal{B}_n$, be a generalized Boolean function. If f is gbent then the associated edge-weighted Cayley graph is $(X_{\mathbf{c}}^0; X_{\mathbf{c}}^1)$-strongly regular with $e_{X_{\mathbf{c}}^0} = d_{X_{\mathbf{c}}^0}$, where $X_{\mathbf{c}}^i = \{\iota(\tilde{\mathbf{c}}) + \iota(\mathbf{d}) : \tilde{\mathbf{c}} \preceq (\mathbf{c}, 1), wt(\tilde{\mathbf{c}}) \equiv i \pmod 2, \mathbf{d} \preceq \bar{\mathbf{c}}\}$, $i = 0, 1$, for all $\mathbf{c} \in \mathbb{V}_{k-1}$; that is, for all $\mathbf{c} \in \mathbb{V}_{k-1}$, and for any two pairs of vertices $(\mathbf{u}, \mathbf{v}), (\mathbf{x}, \mathbf{y})$,*

$$\left| N_{X_{\mathbf{c}}^1}(\mathbf{u}, \mathbf{v}) \right| = \left| N_{X_{\mathbf{c}}^1}(\mathbf{x}, \mathbf{y}) \right|.$$

Proof. The weighted regularity of f follows from Proposition 6. If f is gbent then by [10, Theorem 8], we know that for each $\mathbf{c} \in \mathbb{V}_{k-1}$, the Boolean function $f_{\mathbf{c}}$ defined as

$$f_{\mathbf{c}}(\mathbf{x}) = c_0 a_0(\mathbf{x}) \oplus c_1 a_1(\mathbf{x}) \oplus \cdots \oplus c_{k-2} a_{k-2}(\mathbf{x}) \oplus a_{k-1}(\mathbf{x})$$

is a bent function with $\mathcal{W}_{f_{\mathbf{c}}}(\mathbf{a}) = (-1)^{\mathbf{c} \cdot \iota^{-1}(g(\mathbf{a})) + s(\mathbf{a})} 2^{\frac{n}{2}}$, for some $g : \mathbb{V}_n \to \mathbb{Z}_{2^{k-1}}$, $s : \mathbb{V}_n \to \mathbb{F}_2$.

While we cannot control in a simple manner the Walsh-Hadamard spectra conditions of $f_{\mathbf{c}}$ on the Cayley graph of a gbent f, we can derive some necessary conditions for f to be gbent. Let $\mathbf{c} \in \mathbb{V}_{k-1}$ and $f_{\mathbf{c}}$ bent. Consider $\mathbf{u} \in \mathbb{V}_n$. Certainly, the condition that $f_{\mathbf{c}}(\mathbf{u}) = 1$ means that an odd number of functions a_j, occurring (that is, the corresponding coefficient is nonzero) in $f_{\mathbf{c}}$ will output 1 at $\mathbf{u}$. The a_j's corresponding to entries that are 0 in $\mathbf{c}$ can be taken either 0 or 1 (hence the condition in the definition of $X^i_{\mathbf{c}}$ that $\mathbf{d} \preceq \bar{\mathbf{c}}$). We see that the set of values of f when $f_{\mathbf{c}}(\mathbf{u}) = 1$ is exactly $X^1_{\mathbf{c}} = \{\iota(\tilde{\mathbf{c}}) + \iota(\mathbf{d}) \,:\, \tilde{\mathbf{c}} \preceq (\mathbf{c}, 1), wt(\tilde{\mathbf{c}}) \equiv 1 \pmod 2, \mathbf{d} \preceq \bar{\mathbf{c}}\}$. Similarly, the set of values for f when $f_{\mathbf{c}}(\mathbf{u}) = 0$ is $X^0_{\mathbf{c}} = \{\iota(\tilde{\mathbf{c}}) + \iota(\mathbf{d}) \,:\, \tilde{\mathbf{c}} \preceq (\mathbf{c}, 1), wt(\tilde{\mathbf{c}}) \equiv 0 \pmod 2, \mathbf{d} \preceq \bar{\mathbf{c}}\}$.

Since $f_{\mathbf{c}}$ is bent, then any two vertices, $\mathbf{u}, \mathbf{v}$, will have the same number of adjacent $\mathbf{w}$ with $f_{\mathbf{c}}(\mathbf{u} \oplus \mathbf{w}) = f_{\mathbf{c}}(\mathbf{v} \oplus \mathbf{w}) = 1$, regardless of the value of $f_{\mathbf{c}}(\mathbf{u} \oplus \mathbf{v})$. This implies that $\left|N_{X^1_{\mathbf{c}}}(\mathbf{u}, \mathbf{v})\right|$ is constant for all $\mathbf{u}, \mathbf{v}$. □

5 Further comments

We follow the notation of [7] and define yet another strong regularity concept here. Let Γ be an edge-weighted graph (with no loops) with vertices V, edges E, and weight set W (in [7], W was taken to be $\mathbb{Z}_q^*$, although it could be arbitrary). As before, for each $\mathbf{u} \in V$ and $a \in W \cup \{0\}$, the weighted a-neighborhood of u, $N_a(\mathbf{u})$, is defined as follows:

- $N_a(\mathbf{u})$ = the set of all neighbors $\mathbf{v}$ of $\mathbf{u}$ in Γ for which the edge $(\mathbf{u}, \mathbf{v}) \in E$ has weight a (for each $a \in W$).
- $N^0(\mathbf{u})$ = the set of all nonadjacent $\mathbf{v}$ of $\mathbf{u}$ in Γ (i.e., the set of $\mathbf{v}$ such that $(\mathbf{u}, \mathbf{v}) \notin E$), that is, $N^0(\mathbf{u}) = V \setminus \cup_{a \in W} N_a(\mathbf{u})$. In particular, $\mathbf{u} \in N^0(\mathbf{u})$.

In [7], the following definition of weighted strongly regular graph is given. Let Γ be a connected edge-weighted graph which is regular as a simple (unweighted) graph. Let W be the set of edge-weights of Γ. The graph Γ is called an *edge-weighted local strongly regular* (to distinguish it from our definition we inserted the adjective "local") with parameters v, $k = (k_a)_{a \in W}$, $\lambda = (\lambda_a)_{a \in W^3}$, and $\mu = (\mu_a)_{a \in W^2}$, denoted $SRG_W(v, k, \lambda, \mu)$, if Γ has v vertices, and there are constants k_a, λ_{a_1,a_2,a_3}, and μ_{a_1,a_2}, for a, a_1, a_2, $a_3 \in W$, such that

$$|N_a(\mathbf{u})| = k_a \text{ for all vertices } \mathbf{u},$$

and for vertices $\mathbf{u}_1 \neq \mathbf{u}_2$ we have

$$|N_{a_1}(\mathbf{u}_1) \cap N_{a_2}(\mathbf{u}_2)| = \begin{cases} \lambda_{a_1,a_2,a_3} & \text{if } \exists\, a_3 \in W \text{ with } \mathbf{u}_1 \in N_{a_3}(\mathbf{u}_2); \\ \mu_{a_1,a_2} & \text{if } \mathbf{u}_1 \notin N_{a_3}(\mathbf{u}_2) \text{ for all } a_3. \end{cases}$$

As was observed in [7] for functions $f : \mathbb{F}_{p^n} \to \mathbb{F}_p$, where several questions were posed, it is not clear what the connection between this concept and generalized (or p-ary) bentness is. Our strong regularity definition does allow us to show such a connection and in the case $k = 2$, we have a complete Bernasconi–Codenotti correspondence [1, 2].

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